Robotics I

January 11, 2017

Exercise 1

Consider the 4-dof planar RPRP robot in Fig. 1 and assume that every joint has an unlimited range.

- Assign the link frames according to the Denavit-Hartenberg convention. Place the origin of the last frame coincident with point P. Make the free choices that are available so as to eliminate (i.e., "zeroing out") as many unnecessary constant parameters as possible. Draw the chosen frames directly on the robot in Fig. 1.
- Provide the Denavit-Hartenberg table of parameters associated to the frames that have been assigned. Draw the robot in the configuration $\boldsymbol{q} = \begin{pmatrix} q_1 & q_2 & q_3 & q_4 \end{pmatrix}^T = \begin{pmatrix} 0 & 1 & 0 & 1 \end{pmatrix}^T$.
- A task requires to place the end-effector frame at a desired position $\boldsymbol{p}_d = \begin{pmatrix} p_{dx} & p_{dy} \end{pmatrix}^T$, with a given orientation α_d of axis $\boldsymbol{x}_4$ w.r.t. axis $\boldsymbol{x}_0$ of the base. For the RPRP robot, define the analytic Jacobian associated to this three-dimensional task and determine all its singular configurations.

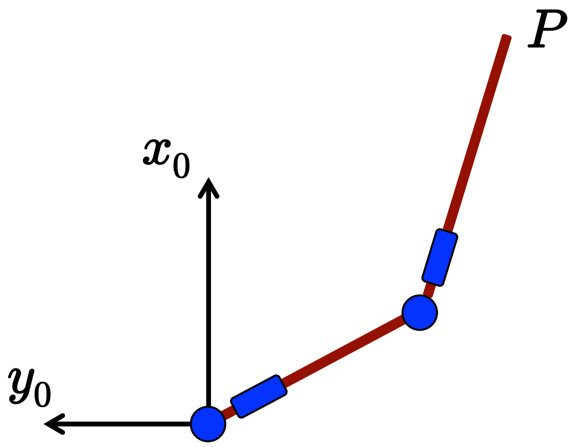

Figure 1: A 4-dof planar RPRP robot.

Exercise 2

A planar 2R robot, with link lengths $\ell_1 = 1$ and $\ell_2 = 0.5$ [m], has its end-effector placed in the Cartesian position $\boldsymbol{p}_0 = \begin{pmatrix} 0.7 & 0.7 \end{pmatrix}^T$ [m] and is at rest at time $t = 0$. Using separation in space and time, plan a Cartesian trajectory for the robot end-effector in order to pick an object in the position $\boldsymbol{p}_d = \begin{pmatrix} 0 & 1 \end{pmatrix}^T$ [m] at a given time $T > 0$ (to be treated symbolically in this problem). The object is on a conveyor belt, moving with a constant velocity $\boldsymbol{v}_d = V \cdot \begin{pmatrix} -1 & 0 \end{pmatrix}^T$, where $V = 1$ [m/s] is the speed. The robot end effector should match this velocity at the final position. Moreover, the motion task should be executed with joint velocities $\dot{\boldsymbol{q}}(t)$ that are continuous for all $t \in [0, T]$.

- Provide the parametric expression $\boldsymbol{p}(s)$ of the chosen Cartesian path, and of its first and second derivative with respect to the path parameter s.
- Provide the expression of a timing law $s(t)$ that satisfies the required conditions.
- Assuming a motion time $T = 1.6$ [s], compute the joint velocity $\dot{\boldsymbol{q}}_{mid} = \dot{\boldsymbol{q}}(T/2)$ at $t = T/2$, when the robot is executing the planned Cartesian trajectory. How many solutions are there?

Exercise 3

The kinematics of a spatial 3R robot is defined by the Denavit-Hartenberg parameters in Tab. 1, where the three constant parameters d_1, a_2, and a_3 are all strictly positive.

i	α_i	a_i	d_i	θ_i
1	$\pi/2$	0	d_1	q_1
2	0	a_2	0	q_2
3	0	a_3	0	q_3

Table 1: DH parameters of a spatial 3R robot.

The 6×3 geometric Jacobian matrix $\boldsymbol{J}(\boldsymbol{q})$ of this robot (expressed in frame 0) has the following expression, which is only partly specified:

$$\boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} \boldsymbol{J}_L(\boldsymbol{q}) \\ \boldsymbol{J}_A(\boldsymbol{q}) \end{pmatrix} = \begin{pmatrix} -\sin q_1\,(a_2\cos q_2 + a_3\cos(q_2+q_3)) & -\cos q_1\,(a_2\sin q_2 + a_3\sin(q_2+q_3)) & J_{13}(\boldsymbol{q}) \\ \cos q_1\,(a_2\cos q_2 + a_3\cos(q_2+q_3)) & -\sin q_1\,(a_2\sin q_2 + a_3\sin(q_2+q_3)) & J_{23}(\boldsymbol{q}) \\ 0 & a_2\cos q_2 + a_3\cos(q_2+q_3) & J_{33}(\boldsymbol{q}) \\ & \boldsymbol{J}_A(\boldsymbol{q}) & \end{pmatrix}. \tag{1}$$

- Provide the missing expressions of all remaining terms in eq. (1).
- Show that this geometric Jacobian has always full (column) rank.
- With the robot in the zero configuration, $\boldsymbol{q} = \boldsymbol{0}$, determine the joint torque $\boldsymbol{\tau} \in \mathbb{R}^3$ that balances statically a force $\boldsymbol{F}$, applied to the robot tip, and a moment $\boldsymbol{M}$, applied to the third link, given by

  $$\boldsymbol{F} = \begin{pmatrix} 0 & 1 & -1 \end{pmatrix}^T \text{ [N]}, \qquad \boldsymbol{M} = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}^T \text{ [Nm]}.$$

  Give the expression of $\boldsymbol{\tau}$, and then its numerical value using the robot kinematic data $d_1 = a_2 = a_3 = 1$ [m].
- Is it possible to apply simultaneously a force $\boldsymbol{F} \neq \boldsymbol{0}$ and a moment $\boldsymbol{M} \neq \boldsymbol{0}$ so that the robot remains in static equilibrium, without the need of an extra joint torque ($\boldsymbol{\tau} = \boldsymbol{0}$) for balancing the force/moment pair? Motivate your answer in general, and illustrate it with a supporting example when the robot is in the configuration $\boldsymbol{q} = \boldsymbol{0}$.

[240 minutes, open books but no computer or smartphone]

Solution

January 11, 2017

Exercise 1

Figure 2 shows a possible assignment of the Denavit-Hartenberg frames, as well as the definition of the joint variables. The associated Tab. 2 highlights that this assignment has zeroed all constant parameters that could be freely chosen. Figure 3 shows the robot in the configuration $\boldsymbol{q} = \begin{pmatrix} 0 & 1 & 0 & 1 \end{pmatrix}^T$, as requested.

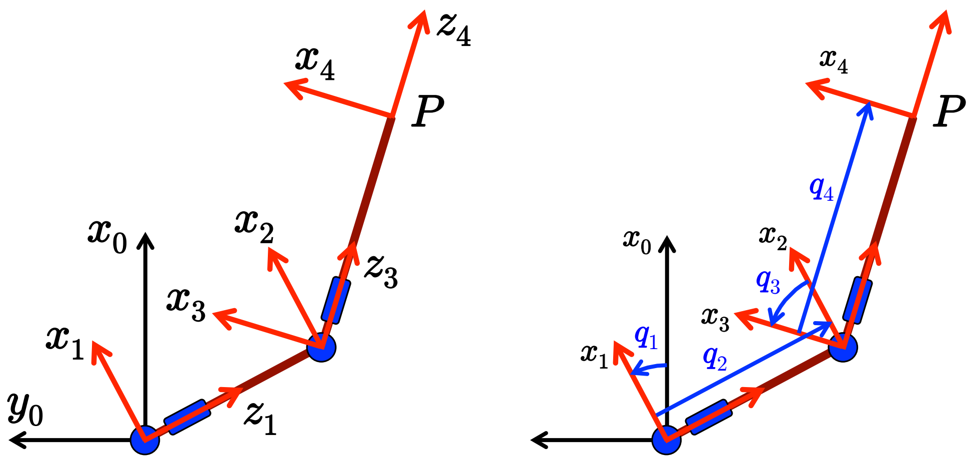

Figure 2: Assigned DH frames for the planar RPRP robot (left) and configuration variables (right).

i	α_i	a_i	d_i	θ_i
1	$\pi/2$	0	0	q_1
2	$-\pi/2$	0	q_2	0
3	$\pi/2$	0	0	q_3
4	0	0	q_4	0

Table 2: DH parameters of the planar RPRP robot.

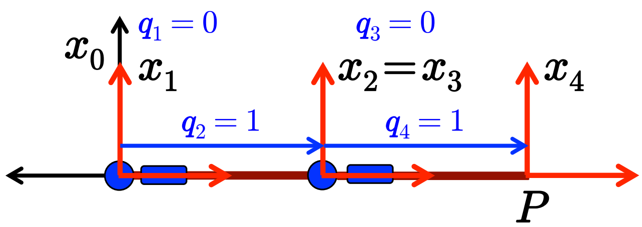

Figure 3: The planar RPRP robot in the configuration $\boldsymbol{q} = (0\ \ 1\ \ 0\ \ 1)^T$.

From Tab. 2, we evaluate the four homogeneous transformation matrices

$$
{}^0\boldsymbol{A}_1(q_1) = \begin{pmatrix} \cos q_1 & 0 & \sin q_1 & 0 \\ \sin q_1 & 0 & -\cos q_1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad {}^1\boldsymbol{A}_2(q_2) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & q_2 \\ 0 & 0 & 0 & 1 \end{pmatrix},
$$

$$
{}^2\boldsymbol{A}_3(q_3) = \begin{pmatrix} \cos q_3 & 0 & \sin q_3 & 0 \\ \sin q_3 & 0 & -\cos q_3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad {}^3\boldsymbol{A}_4(q_4) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & q_4 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
$$

Performing computations we obtain[1]

$$
{}^0\boldsymbol{T}_4(\boldsymbol{q}) = \begin{pmatrix} \cos(q_1+q_3) & 0 & \sin(q_1+q_3) & q_2 \sin q_1 + q_4 \sin(q_1+q_3) \\ \sin(q_1+q_3) & 0 & -\cos(q_1+q_3) & -(q_2 \cos q_1 + q_4 \cos(q_1+q_3)) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
$$

Therefore, the task vector of interest is

$$
\boldsymbol{r} = \begin{pmatrix} p_x \\ p_y \\ \alpha \end{pmatrix} = \begin{pmatrix} q_2 \sin q_1 + q_4 \sin(q_1+q_3) \\ -(q_2 \cos q_1 + q_4 \cos(q_1+q_3)) \\ q_1 + q_3 \end{pmatrix} = \boldsymbol{f}_r(\boldsymbol{q}). \tag{2}
$$

The analytic Jacobian for this task is the 3×4 matrix obtained by differentiating (2):

$$
\boldsymbol{J}(\boldsymbol{q}) = \frac{\partial \boldsymbol{f}_r(\boldsymbol{q})}{\partial \boldsymbol{q}} = \begin{pmatrix} q_2 \cos q_1 + q_4 \cos(q_1+q_3) & \sin q_1 & q_4 \cos(q_1+q_3) & \sin(q_1+q_3) \\ q_2 \sin q_1 + q_4 \sin(q_1+q_3) & -\cos q_1 & q_4 \sin(q_1+q_3) & -\cos(q_1+q_3) \\ 1 & 0 & 1 & 0 \end{pmatrix}. \tag{3}
$$

The singular configurations of this matrix are those where the rank drops down (to 2 or 1) from its maximum possible value (3). Equivalently, they are defined by the values of $\boldsymbol{q}$ such that all four 3×3 minors extracted from $\boldsymbol{J}(\boldsymbol{q})$ by deleting one column are equal to zero. Let $\boldsymbol{J}_{-i}(\boldsymbol{q})$ be the square matrix obtained from (3) by deleting the ith column, for $i = 1, \ldots, 4$. We have

$$
\det \boldsymbol{J}_{-1}(\boldsymbol{q}) = -\sin q_3, \quad \det \boldsymbol{J}_{-2}(\boldsymbol{q}) = q_2 \cos q_3, \quad \det \boldsymbol{J}_{-3}(\boldsymbol{q}) = \sin q_3, \quad \det \boldsymbol{J}_{-4}(\boldsymbol{q}) = -q_2.
$$

Therefore, the Jacobian $\boldsymbol{J}(\boldsymbol{q})$ is singular if and only if $q_2 = \sin q_3 = 0$. For instance, when $q_2 = q_3 = 0$, we obtain

$$
\bar{\boldsymbol{J}}(\boldsymbol{q}) := \boldsymbol{J}(q_1, 0, 0, q_4) = \begin{pmatrix} q_4 \cos q_1 & \sin q_1 & q_4 \cos q_1 & \sin q_1 \\ q_4 \sin q_1 & -\cos q_1 & q_4 \sin q_1 & -\cos q_1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \quad \Rightarrow \quad \text{rank}\left(\bar{\boldsymbol{J}}(\boldsymbol{q})\right) = 2.
$$

Exercise 2

We have to interpolate the two Cartesian positions $\boldsymbol{p}_0$ and $\boldsymbol{p}_d$ with a sufficiently smooth path, having also a prescribed tangent direction $\boldsymbol{v}_{du} = \boldsymbol{v}_d / \|\boldsymbol{v}_d\|$ (of unitary norm) at the end position $\boldsymbol{p}_d$. It is required (and also useful) to work in the Cartesian space and to keep separation between space (geometry) and time.

Note first that, because of the required continuity of the joint velocity and since $(\boldsymbol{p}_d - \boldsymbol{p}_0) \nparallel \boldsymbol{v}_d$, the Cartesian path cannot be chosen as a straight segment (linear in the path parameter s), Since the robot end effector has to reach the correct direction of the conveyor belt motion at the end of the transfer, we would have then a discontinuity in the path tangent at $\boldsymbol{p}_d$. This translates into a discontinuity for the Cartesian velocity at the final instant T, since the end effector is not allowed to stop in $\boldsymbol{p}_d$ (rather, it

[1] The expressions in ${}^0\boldsymbol{T}_4(\boldsymbol{q})$ could have been obtained also from a simple inspection of Fig. 2).

should have the speed $V > 0$ of the conveyor belt). Accordingly, a jump in the Cartesian velocity will result in a discontinuity of the joint velocity.

Therefore, when planning the path, we have to approach $\boldsymbol{p}_d$ from the right direction, and this can be done is several ways (without leaving the robot workspace). For instance, one could concatenate two linear segments, from $\boldsymbol{p}_0$ to $\boldsymbol{p}_{int}$ and then from $\boldsymbol{p}_{int}$ to $\boldsymbol{p}_d$, by introducing an intermediate position $\boldsymbol{p}_{int}$ such that $(\boldsymbol{p}_d - \boldsymbol{p}_{int}) \parallel \boldsymbol{v}_d$ and by forcing the motion to stop there (to preserve a continuous velocity in time). Another possibility is to design an arc of a circle (of suitable center and radius) that interpolates $\boldsymbol{p}_0$ with $\boldsymbol{p}_d$, having also the right tangent (in the direction of $\boldsymbol{v}_d$) at the end position $\boldsymbol{p}_d$.

We present here another viable solution that considers a quadratic function of s for the path $\boldsymbol{p}(s)$. Working on the Cartesian (x, y)-plane in vector terms (with $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \mathbb{R}^2$), we choose

$$\boldsymbol{p}(s) = \boldsymbol{a} + \boldsymbol{b}s + \boldsymbol{c}s^2, \qquad s \in [0, 1] \tag{4}$$

and impose the boundary conditions

$$\boldsymbol{p}(0) = \boldsymbol{a} = \boldsymbol{p}_0, \qquad \boldsymbol{p}(1) = \boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c} = \boldsymbol{p}_d, \qquad \boldsymbol{p}'(1) = (\boldsymbol{b} + 2\boldsymbol{c}s)_{s=1} = \boldsymbol{b} + 2\boldsymbol{c} = \boldsymbol{v}_{du}. \tag{5}$$

In this way, the 6 scalar conditions in eq. (5) —3 on the x-coordinates and 3 on the y-coordinates— will be satisfied using the equal number of 6 scalar components in the 3 bi-dimensional vectors $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$. Solving from (5), we obtain the unique values

$$\boldsymbol{a} = \boldsymbol{p}_0, \qquad \boldsymbol{b} = 2\left(\boldsymbol{p}_d - \boldsymbol{p}_0\right) - \boldsymbol{v}_{du}, \qquad \boldsymbol{c} = \boldsymbol{v}_{du} - \left(\boldsymbol{p}_d - \boldsymbol{p}_0\right), \tag{6}$$

to be replaced in (4). The first and second derivatives of $\boldsymbol{p}(s)$ w.r.t. s are then

$$\boldsymbol{p}'(s) = 2\left(\boldsymbol{p}_d - \boldsymbol{p}_0\right)(1 - s) + \boldsymbol{v}_{du}(2s - 1), \qquad \boldsymbol{p}''(s) = 2\left(\boldsymbol{v}_{du} - \left(\boldsymbol{p}_d - \boldsymbol{p}_0\right)\right). \tag{7}$$

In particular, the path tangent direction at the starting position $\boldsymbol{p}_0$ will be $\boldsymbol{p}'(0) = \boldsymbol{b} = 2\left(\boldsymbol{p}_d - \boldsymbol{p}_0\right) - \boldsymbol{v}_{du}$.

As for the timing law $s(t)$, to be defined for $t \in [0, T]$, we need to satisfy the four boundary conditions

$$s(0) = 0, \qquad s(T) = 1, \qquad \dot{s}(0) = 0, \qquad \dot{s}(T) = V, \tag{8}$$

being $V > 0$ the final speed along the path at the final instant $t = T$. Thus, a cubic polynomial in the normalized time $\tau = t/T$ will be sufficient:

$$s(\tau) = c_0 + c_1\tau + c_2\tau^2 + c_3\tau^3, \qquad \tau = t/T \in [0, 1].$$

Imposing (8), we obtain the timing law

$$s(t) = (3 - TV)\left(\frac{t}{T}\right)^2 + (TV - 2)\left(\frac{t}{T}\right)^3, \qquad t \in [0, T], \tag{9}$$

and its speed profile

$$\dot{s}(t) = \frac{1}{T}\left(2(3 - TV)\left(\frac{t}{T}\right) + 3(TV - 2)\left(\frac{t}{T}\right)^2\right), \qquad t \in [0, T]. \tag{10}$$

Note that at the motion midtime $t = T/2$, it is

$$s\left(\frac{T}{2}\right) = \frac{1}{2} - \frac{TV}{8}, \qquad \dot{s}\left(\frac{T}{2}\right) = \frac{1.5}{T} - \frac{TV}{4}. \tag{11}$$

Replacing now the problem data, we obtain from (4) and (6)

$$\boldsymbol{p}(s) = \begin{pmatrix} 0.7 \\ 0.7 \end{pmatrix} + \begin{pmatrix} -0.4 \\ 0.6 \end{pmatrix} s + \begin{pmatrix} -0.3 \\ -0.3 \end{pmatrix} s^2, \qquad s \in [0, 1],$$

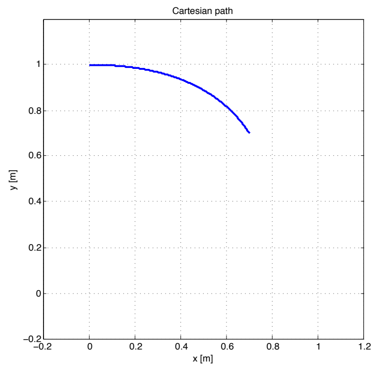

Figure 4: The Cartesian path $\boldsymbol{p}(s)$ interpolating $\boldsymbol{p}_0 = (0.7\ \ 0.0)^T$ to $\boldsymbol{p}_d = (0\ \ 1)^T$.

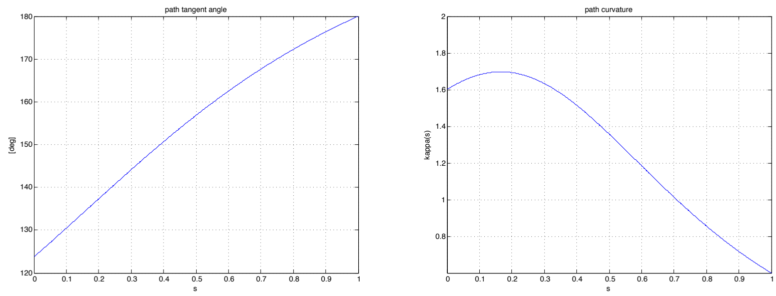

Figure 5: The angle $\alpha(s)$ of the tangent to the path (left) and the path curvature $\kappa(s)$ (right).

and from (7)

$$\boldsymbol{p}'(s) = \begin{pmatrix} -0.4 \\ 0.6 \end{pmatrix} + \begin{pmatrix} -0.6 \\ -0.6 \end{pmatrix} s, \qquad \boldsymbol{p}''(s) = \begin{pmatrix} -0.6 \\ -0.6 \end{pmatrix} s.$$

The planned Cartesian path $\boldsymbol{p}(s)$ is shown in Fig. 4. Note that the path remains always in the primary workspace of the robot ($0.5 = |\ell_1 - \ell_2| \leq \|\boldsymbol{p}\| \leq \ell_1 + \ell_2 = 1.5$). Just for completeness of illustration, Figure 5 reports, as functions of the path parameter, also the angle $\alpha(s)$ of the tangent to the path w.r.t. the $\boldsymbol{x}$-axis and the path curvature $\kappa(s)$. These have been evaluated as

$$\alpha(s) = \text{ATAN2}\left\{p'_y(s), p'_x(s)\right\}, \qquad \kappa(s) = \frac{\|\boldsymbol{p}'(s) \times \boldsymbol{p}''(s)\|}{\|\boldsymbol{p}'(s)\|^3}, \qquad s \in [0,1].$$

For the cross product in the second formula, vectors were embedded in 3D (with a zero z-component).

Similarly, using $V = 1$ [m/s] and the given motion time $T = 1.6$ [s], it follows from (9) and (10)

$$s(t) = 1.4\left(\frac{t}{1.6}\right)^2 - 0.4\left(\frac{t}{1.6}\right)^3, \qquad \dot{s}(t) = 1.75\left(\frac{t}{1.6}\right) - 0.75\left(\frac{t}{1.6}\right)^2, \qquad t \in [0, 1.6].$$

The cubic timing law $s(t)$ is shown in Fig. 6, while Figure 7 reports its first and second time derivatives. Note that $\dot{s}(0) = 0$ and $\dot{s}(T) = V = 1$ [m/s].

From the separate profiles in space and time, we can recombine the Cartesian trajectory and its time derivatives as

$$\boldsymbol{p}(t) = \boldsymbol{p}(s(t)), \qquad \dot{\boldsymbol{p}}(t) = \boldsymbol{p}'(s(t))\,\dot{s}(t), \qquad \ddot{\boldsymbol{p}}(t) = \boldsymbol{p}'(s(t))\,\ddot{s}(t) + \boldsymbol{p}''(s(t))\,\dot{s}^2(t), \qquad t \in [0, T].$$

Figure 8 shows the two components of the obtained Cartesian trajectory, in position $\boldsymbol{p}(t)$, velocity $\dot{\boldsymbol{p}}(t)$, and acceleration $\ddot{\boldsymbol{p}}(t)$.

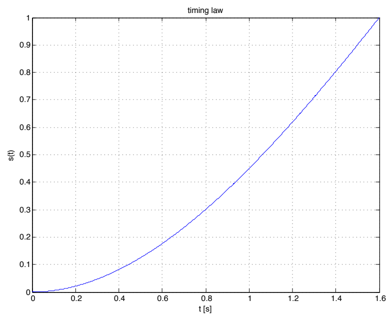

Figure 6: The timing law $s(t)$ for $T = 1.6$ [s].

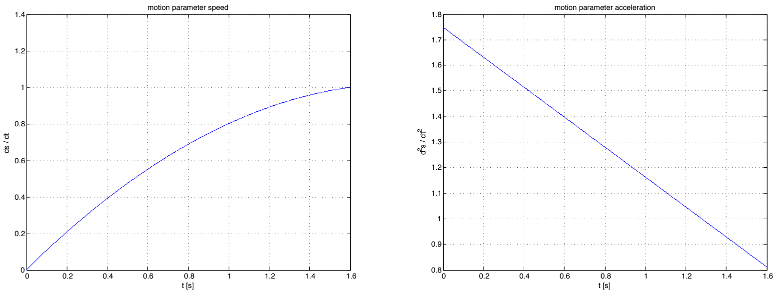

Figure 7: The timing speed $\dot{s}(t)$ (left) and acceleration $\ddot{s}(t)$ (right) for $T = 1.6$ [s].

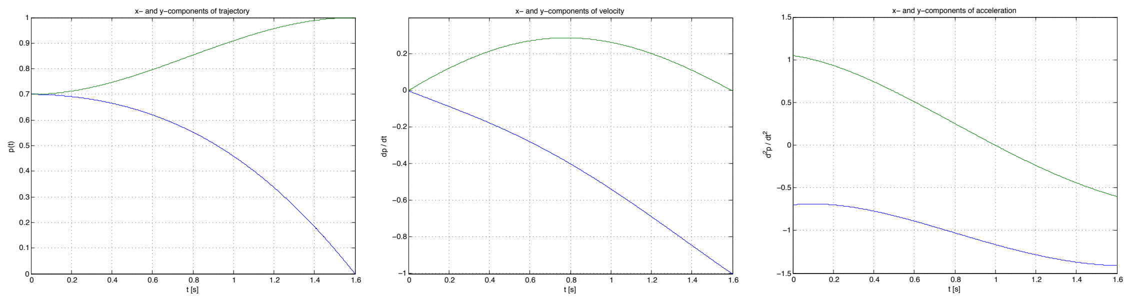

Figure 8: The components (x in blue, y in green) of the Cartesian trajectory $\boldsymbol{p}(t)$ and of its first and second time derivatives $\dot{\boldsymbol{p}}(t)$ and $\ddot{\boldsymbol{p}}(t)$ for $T = 1.6$ [s].

Finally, at the midtime $t = T/2 = 0.8$ [s] of motion, we evaluate from the previous formulas the quantities of interest:

$$s(0.8) = 0.3, \qquad \dot{s}(0.8) = 0.6875, \qquad \Rightarrow \qquad \begin{array}{l} \boldsymbol{p}(t = 0.8) = \boldsymbol{p}(s = 0.3) = \begin{pmatrix} 0.5530 \\ 0.8530 \end{pmatrix} =: \boldsymbol{p}_{mid} \\ \dot{\boldsymbol{p}}(t = 0.8) = \boldsymbol{p}'(s = 0.3)\dot{s}(0.8) = \begin{pmatrix} -0.3987 \\ 0.2888 \end{pmatrix} =: \dot{\boldsymbol{p}}_{mid}. \end{array}$$

Up to now, we have not involved the actual robot in the trajectory planning. At this stage, we need to solve an inverse kinematics problem for $\boldsymbol{p}_{mid}$, and then an inverse differential kinematic problem for $\dot{\boldsymbol{p}}_{mid}$.

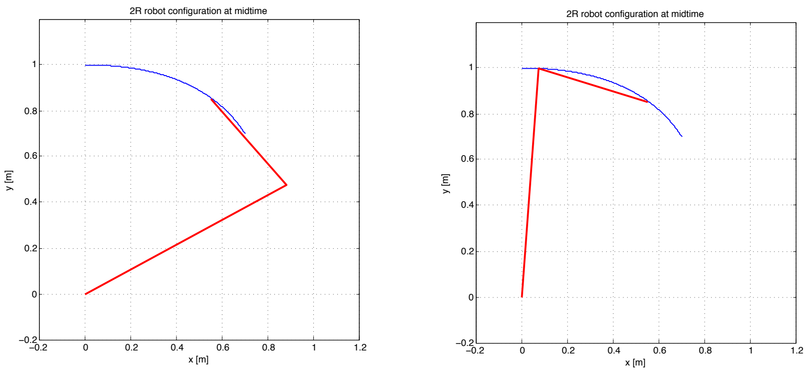

Figure 9: The two solutions to the inverse kinematics at the trajectory midtime $t = T/2 = 0.8$ [s].

By using the usual formulas for a planar 2R robot

$$c_2 = \frac{p_{mid,x}^2 + p_{mid,y}^2 - \ell_1^2 - \ell_2^2}{2\ell_1\ell_2}, \quad s_2 = \pm\sqrt{1-c_2^2} \quad \Rightarrow \quad q_2 = \text{ATAN2}\,\{s_2, c_2\}$$

and

$$s_1 = (\ell_1 + \ell_2 c_2)\, p_{mid,y} - \ell_2 s_2\, p_{mid,x}, \quad c_1 = (\ell_1 + \ell_2 c_2)\, p_{mid,x} + \ell_2 s_2\, p_{mid,y} \quad \Rightarrow \quad q_1 = \text{ATAN2}\,\{s_1, c_1\},$$

we obtain the two solutions to the inverse kinematics problem (see Fig. 9)

$$\boldsymbol{q}_{mid,A} = \begin{pmatrix} 0.4948 \\ 1.7891 \end{pmatrix} \text{[rad]} = \begin{pmatrix} 28.35° \\ 102.51° \end{pmatrix}, \qquad \boldsymbol{q}_{mid,B} = \begin{pmatrix} 1.4965 \\ -1.7891 \end{pmatrix} \text{[rad]} = \begin{pmatrix} 85.74° \\ -102.51° \end{pmatrix}.$$

Evaluating the robot Jacobian for the 2R robot

$$\boldsymbol{J}(\boldsymbol{q}) = \begin{pmatrix} -(\ell_1 \sin q_1 + \ell_2 \sin(q_1+q_2)) & -\ell_2 \sin(q_1+q_2) \\ \ell_1 \cos q_1 + \ell_2 \cos(q_1+q_2)) & \ell_2 \cos(q_1+q_2) \end{pmatrix},$$

we have in the first case

$$\boldsymbol{J}(\boldsymbol{q}_{mid,A}) = \begin{pmatrix} 0.8530 & 0.3782 \\ 0.5530 & 0.3271 \end{pmatrix} \quad \Rightarrow \quad \dot{\boldsymbol{q}}_{mid.A} = \boldsymbol{J}^{-1}(\boldsymbol{q}_{mid,A})\, \dot{\boldsymbol{p}}_{mid} = \begin{pmatrix} 0.4909 \\ -0.0528 \end{pmatrix} \text{[rad/s]},$$

and in the second case

$$\boldsymbol{J}(\boldsymbol{q}_{mid,B}) = \begin{pmatrix} -0.8530 & 0.1442 \\ 0.5530 & 0.4787 \end{pmatrix} \quad \Rightarrow \quad \dot{\boldsymbol{q}}_{mid.B} = \boldsymbol{J}^{-1}(\boldsymbol{q}_{mid,B})\, \dot{\boldsymbol{p}}_{mid} = \begin{pmatrix} 0.4764 \\ 0.0528 \end{pmatrix} \text{[rad/s]}.$$

We obtained thus two different (though quite similar) joint velocity solutions to the given problem.

Exercise 3

From Tab. 1, we evaluate the three homogeneous transformation matrices

$${}^0\boldsymbol{A}_1(q_1) = \begin{pmatrix} \cos q_1 & 0 & \sin q_1 & 0 \\ \sin q_1 & 0 & -\cos q_1 & 0 \\ 0 & 1 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad {}^1\boldsymbol{A}_2(q_2) = \begin{pmatrix} \cos q_2 & -\sin q_2 & 0 & a_2 \cos q_2 \\ \sin q_2 & \cos q_2 & 0 & a_2 \sin q_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$${}^2\boldsymbol{A}_3(q_3) = \begin{pmatrix} \cos q_3 & -\sin q_3 & 0 & a_3 \cos q_3 \\ \sin q_3 & \cos q_3 & 0 & a_3 \sin q_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Performing computations in an efficient way, we have for the position of the origin of the last frame

$$
\begin{aligned}
\boldsymbol{p}_{hom}(\boldsymbol{q}) = \begin{pmatrix} \boldsymbol{p}(\boldsymbol{q}) \\ \\ 1 \end{pmatrix} &= {}^0\boldsymbol{A}_1(q_1)\left({}^1\boldsymbol{A}_2(q_2)\left({}^2\boldsymbol{A}_3(q_3)\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right)\right) = {}^0\boldsymbol{A}_1(q_1)\left({}^1\boldsymbol{A}_2(q_2)\begin{pmatrix} a_3\cos q_3 \\ a_3\sin q_3 \\ 0 \\ 1 \end{pmatrix}\right) \\
&= {}^0\boldsymbol{A}_1(q_1)\begin{pmatrix} a_2\cos q_2 + a_3\cos(q_2+q_3) \\ a_2\sin q_2 + a_3\sin(q_2+q_3) \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \cos q_1\,(a_2\cos q_2 + a_3\cos(q_2+q_3)) \\ \sin q_1\,(a_2\cos q_2 + a_3\cos(q_2+q_3)) \\ d_1 + a_2\sin q_2 + a_3\sin(q_2+q_3) \\ 1 \end{pmatrix}.
\end{aligned}
$$

The last column of the 3×3 matrix $\boldsymbol{J}_L(\boldsymbol{q})$ in eq. (1) is then easily computed as

$$
\begin{pmatrix} J_{13}(\boldsymbol{q}) \\ J_{23}(\boldsymbol{q}) \\ J_{33}(\boldsymbol{q}) \end{pmatrix} = \frac{\partial \boldsymbol{p}(\boldsymbol{q})}{\partial q_3} = \begin{pmatrix} -a_3\cos q_1\sin(q_2+q_3) \\ -a_3\sin q_1\sin(q_2+q_3) \\ a_3\cos(q_2+q_3) \end{pmatrix}. \tag{12}
$$

As for the 3×3 matrix $\boldsymbol{J}_A(\boldsymbol{q})$, its general expression becomes in the present case

$$
\begin{aligned}
\boldsymbol{J}_A(\boldsymbol{q}) &= \begin{pmatrix} \boldsymbol{z}_0 & \boldsymbol{z}_1 & \boldsymbol{z}_2 \end{pmatrix} = \left(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad {}^0\boldsymbol{R}_1(q_1)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad {}^0\boldsymbol{R}_1(q_1)\,{}^1\boldsymbol{R}_2(q_2)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right) \\
&= \begin{pmatrix} 0 & \sin q_1 & \sin q_1 \\ 0 & -\cos q_1 & -\cos q_1 \\ 1 & 0 & 0 \end{pmatrix}.
\end{aligned} \tag{13}
$$

The singularity analysis of $\boldsymbol{J}(\boldsymbol{q})$ can be performed in different ways. We start by observing that matrix $\boldsymbol{J}_A(\boldsymbol{q})$ has always rank exactly equal to 2, since *i)* the second and third columns are identical (thus its rank is always less than 3), and *ii)* sine and cosine of the same angle never vanish simultaneously (thus the rank never drops below 2). The first observation suggests a transformation on the matrix columns, such that the study of the rank of the complete matrix $\boldsymbol{J}(\boldsymbol{q})$ is reduced to the analysis of the last column of $\boldsymbol{J}_L(\boldsymbol{q})$. In fact, substituting to the third column the difference between the second and the third one, we have

$$
\begin{aligned}
\boldsymbol{J}'(\boldsymbol{q}) &= \boldsymbol{J}(\boldsymbol{q})\,\boldsymbol{T} = \boldsymbol{J}(\boldsymbol{q})\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix} \\
&= \begin{pmatrix} -\sin q_1\,(a_2\cos q_2 + a_3\cos(q_2+q_3)) & -\cos q_1\,(a_2\sin q_2 + a_3\sin(q_2+q_3)) & -a_2\cos q_1\sin q_2 \\ \cos q_1\,(a_2\cos q_2 + a_3\cos(q_2+q_3)) & -\sin q_1\,(a_2\sin q_2 + a_3\sin(q_2+q_3)) & -a_2\sin q_1\sin q_2 \\ 0 & a_2\cos q_2 + a_3\cos(q_2+q_3) & a_2\cos q_2 \\ 0 & \sin q_1 & 0 \\ 0 & -\cos q_1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.
\end{aligned}
$$

From the obtained internal structure, it is easy to see that the rank of matrix $\boldsymbol{J}'(\boldsymbol{q})$, which is the same as the rank ρ of $\boldsymbol{J}(\boldsymbol{q})$, will be full if and only if, in any given configuration, at least one element of its last column is different from zero or, equivalently, if the last column never vanishes. This is immediate to see, again because sine and cosine of the same angle never vanish simultaneously. Thus, $\boldsymbol{J}(\boldsymbol{q})$ has always full rank $\rho = 3$, i.e., is never singular.

In alternative, one could resort to the basic definition of full rank for a matrix that has less columns than rows. Let $\boldsymbol{c}_i$ be the ith column of the matrix, with $i = 1,2,3$, for the present case of our $\boldsymbol{J}(\boldsymbol{q})$. The matrix will have full (column) rank when, for all possible scalars λ_i, $i = 1,2,3$,

$$
\lambda_1\boldsymbol{c}_1 + \lambda_2\boldsymbol{c}_2 + \lambda_3\boldsymbol{c}_3 = \boldsymbol{0} \qquad \Longleftrightarrow \qquad \boldsymbol{\lambda} = \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \end{pmatrix}^T = \boldsymbol{0}. \tag{14}
$$

Using the obtained expression (12) and (13) inside the Jacobian in eq. (1), the condition on the left-hand side of (14) results in six scalar equations. From the last one, it follows necessarily that $\lambda_1 = 0$. From the fourth and fifth equations, it follows also that $\lambda_2 = -\lambda_3$. Squaring and summing the first three equations in the remaining $\lambda_2 (\boldsymbol{c}_2 - \boldsymbol{c}_3) = \boldsymbol{0}$ yields

$$\lambda_2 a_2^2 \left(\sin^2 q_2 \left(\cos^2 q_1 + \sin^2 q_1\right) + \cos^2 q_2\right) = \lambda_2 a_2^2 = 0 \qquad \Longleftrightarrow \qquad \lambda_2 = 0.$$

As a result, $\boldsymbol{\lambda} = \boldsymbol{0}$ is the only solution, and the matrix $\boldsymbol{J}(\boldsymbol{q})$ has always full rank (for all $\boldsymbol{q}$!).

Another approach, which is however computationally heavier to perform by hand, is to check that the following determinant[2]

$$\begin{aligned}
\det\left[\boldsymbol{J}^T(\boldsymbol{q})\,\boldsymbol{J}(\boldsymbol{q})\right] &= \det \boldsymbol{T}^T \cdot \det\left[\boldsymbol{J}^T(\boldsymbol{q})\,\boldsymbol{J}(\boldsymbol{q})\right] \cdot \det \boldsymbol{T} = \det\left[\boldsymbol{J}'^T(\boldsymbol{q})\,\boldsymbol{J}'(\boldsymbol{q})\right] \\
&= \det \begin{pmatrix} 1 + (a_2 \cos q_2 + a_3 \cos(q_2+q_3))^2 & 0 & 0 \\ 0 & 1 + a_2^2 + a_3^2 + 2\,a_2 a_3 \cos q_3 & a_2^2 + a_2 a_3 \cos q_3 \\ 0 & a_2^2 + a_2 a_3 \cos q_3 & a_2^2 \end{pmatrix} \\
&= \left(1 + (a_2 \cos q_2 + a_3 \cos(q_2+q_3))^2\right) a_2^2 \left(1 + a_3^2 \sin^2 q_3\right)
\end{aligned}$$

is in fact never zero —a necessary and sufficient condition for the 6×3 matrix $\boldsymbol{J}(\boldsymbol{q})$ to have full rank $\rho = 3$.

Evaluating now $\boldsymbol{J}^T(\boldsymbol{q})$ for $\boldsymbol{q} = \boldsymbol{0}$ yields

$$\boldsymbol{J}_0^T := \boldsymbol{J}^T(\boldsymbol{0}) = \begin{pmatrix} 0 & a_2 + a_3 & 0 & 0 & 0 & 1 \\ 0 & 0 & a_2 + a_3 & 0 & -1 & 0 \\ 0 & 0 & a_3 & 0 & -1 & 0 \end{pmatrix}.$$

The joint torque $\boldsymbol{\tau}$ that balances statically in the configuration $\boldsymbol{q} = \boldsymbol{0}$ the assigned force $\boldsymbol{F} = \begin{pmatrix} 0 & 1 & -1 \end{pmatrix}^T$ and moment $\boldsymbol{M} = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}^T$ is given by

$$\boldsymbol{\tau} = -\boldsymbol{J}_0^T \begin{pmatrix} \boldsymbol{F} \\ \boldsymbol{M} \end{pmatrix} = \begin{pmatrix} -(1 + a_2 + a_3) \\ 1 + a_2 + a_3 \\ 1 + a_3 \end{pmatrix} \qquad \Rightarrow \qquad \text{for } a_2 = a_3 = 1, \quad \boldsymbol{\tau} = \begin{pmatrix} -3 \\ 3 \\ 2 \end{pmatrix}.$$

In any robot configuration, there will always be some non-zero force/moment pairs that require no balancing joint torque to keep the robot in its equilibrium state. This is independent from having the Jacobian matrix $\boldsymbol{J}(\boldsymbol{q})$ full rank or not. In fact, the 3×6 matrix $\boldsymbol{J}^T(\boldsymbol{q})$ will always have a null space $\mathcal{N}\left\{\boldsymbol{J}^T\right\}$ of dimension $6 - \rho \geq 3$. Since the Jacobian has constant full rank $\rho = 3$ (in all configurations), the null space will be of dimension $6 - 3 = 3$, and there will be always ∞^3 such force/moment pairs at any robot configuration. For example, at $\boldsymbol{q} = \boldsymbol{0}$ the pair

$$\boldsymbol{F}_0 = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}^T, \qquad \boldsymbol{M}_0 = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}^T,$$

yields

$$\boldsymbol{\tau}_0 = -\boldsymbol{J}_0^T \begin{pmatrix} \boldsymbol{F}_0 \\ \boldsymbol{M}_0 \end{pmatrix} = \boldsymbol{0}.$$

* * * * *

[2] Since $\det \boldsymbol{T} = \det \boldsymbol{T}^T = -1$, the product of these two determinants is equal to 1.