Robotics I

Midterm test in classroom – November 18, 2016

Exercise 1 [10 points]

Figure 1 shows the 6R Universal Robot UR5, with a non-spherical wrist, and two axes of the reference frame RF_0 placed at the robot base. The Denavit-Hartenberg parameters are given in Tab. 1, together with the numerical values for the constant parameters and the current values that the joint variables assume in the shown configuration.

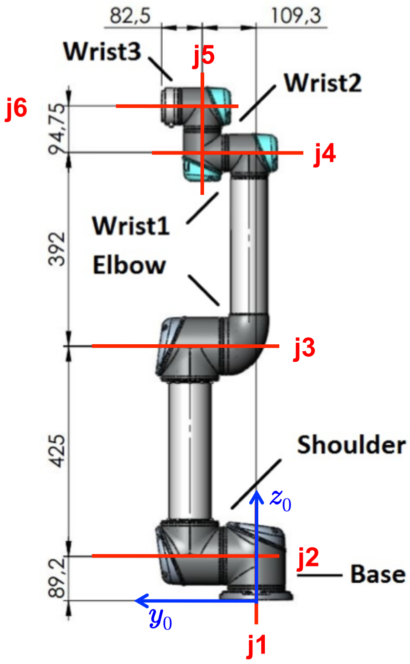

Figure 1: The 6R Universal Robot UR5 and the chosen base frame.

i	α_i	a_i	d_i	θ_i
1	$-\pi/2$	0	$d_1 = 89.2$	$\theta_1 = 0$
2	0	$a_2 = -425$	0	$\theta_2 = \pi/2$
3	0	$a_3 = -392$	0	$\theta_3 = 0$
4	$\pi/2$	0	$d_4 = 109.3$	$\theta_4 = -\pi/2$
5	$-\pi/2$	0	$d_5 = 94.75$	$\theta_5 = 0$
6	0	0	$d_6 = 82.5$	$\theta_6 = 0$

Table 1: DH parameters (in mm or rad), with the value of $\boldsymbol{\theta} \in \mathbb{R}^6$ in the shown configuration.

Using the provided sheet (please write your full name there!), draw all the Denavit-Hartenberg frames associated to the robot links according to Tab. 1.

Exercise 2 [5 points]

A frame $RF_B = \{\boldsymbol{O}_B, \boldsymbol{x}_B, \boldsymbol{y}_B, \boldsymbol{z}_B\}$ is displaced and rotated with respect to a fixed reference frame $RF_A = \{\boldsymbol{O}_A, \boldsymbol{x}_A, \boldsymbol{y}_A, \boldsymbol{z}_A\}$. The displacement is represented by the vector

$$ {}^{A}\boldsymbol{p}_{\boldsymbol{O}_A\boldsymbol{O}_B} = \begin{pmatrix} 3 & 7 & -1 \end{pmatrix}^T \qquad [\mathrm{m}], $$

while the orientation of RF_B with respect to RF_A is represented by the following sequence of three Euler $ZY'X''$ angles

$$ \alpha = \frac{\pi}{4}, \qquad \beta = -\frac{\pi}{2}, \qquad \gamma = 0 \qquad [\mathrm{rad}]. $$

For a given point P, provide the value of vector ${}^{A}\boldsymbol{p}_{\boldsymbol{O}_A P}$ knowing that its position with respect to frame RF_B is given by

$$ {}^{B}\boldsymbol{p}_{\boldsymbol{O}_B P} = \begin{pmatrix} 1 & 1 & 0 \end{pmatrix}^T \qquad [\mathrm{m}]. $$

Exercise 3 [10 points]

Consider the 2-dof robot in Fig. 2, with two revolute joints having axes (the first vertical and the second horizontal) that do not intercept.

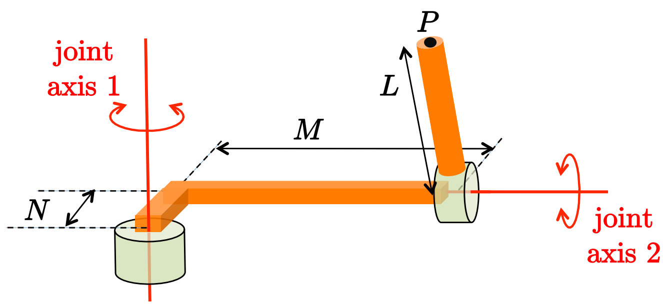

Figure 2: A 2R robot moving in the 3D space.

- Assign the frames according to the Denavit-Hartenberg convention and define the associated table of parameters. Provide the specific expression of the homogenous transformation matrices between the successive frames that you have assigned.
- Determine the symbolic expression of the position vector ${}^{0}\boldsymbol{p}_{OP}$ of point P in the chosen frame RF_0, and find its numerical value when the kinematic quantities are $L = 1$, $M = 2$, $N = 0.3$ [m] and the robot configuration is $\boldsymbol{q} = \begin{pmatrix} 90^\circ & -45^\circ \end{pmatrix}^T$.

Exercise 4 [5 points]

Given the following matrix

$$ \boldsymbol{A} = \begin{pmatrix} -0.5 & -a & 0 \\ 0 & 0 & -1 \\ a & -0.5 & 0 \end{pmatrix} $$

determine, if possible, a value $a > 0$ such that the identity $\boldsymbol{R}(\boldsymbol{r}, \theta) = \boldsymbol{A}$ holds, where $\boldsymbol{R}(\boldsymbol{r}, \theta)$ is the rotation matrix associated to an axis-angle representation of the orientation. Provide then all unit vectors $\boldsymbol{r}$ and associated angles $\theta \in (-\pi, +\pi]$ that are solutions to this equation.

[180 minutes (open books, but NO computer or internet)]

Solution of Midterm Test

November 18, 2016

Exercise 1

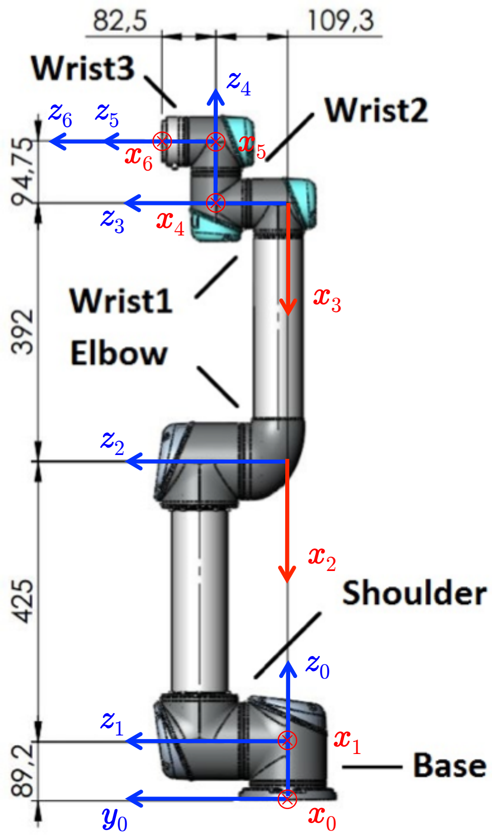

Figure 3: Assignment of DH frames for the UR5 robot associated to Tab. 1. Except for $\boldsymbol{x}_2$ and $\boldsymbol{x}_3$, all other $\boldsymbol{x}_i$ point inside the sheet. *Warning: We are not using this type of DH frame assignment for the UR10 available in the DIAG Robotics Lab.*

Exercise 2

We just need to build the homogeneous transformation matrix that relates frame RF_B to frame RF_A. The linear displacement is already represented by the given vector ${}^A\boldsymbol{p}_{\boldsymbol{O}_A\boldsymbol{O}_B}$. As for the angular part, the rotation matrix ${}^A\boldsymbol{R}_B$ is specified from the sequence of three Euler $ZY'X''$ angles. Since these are defined around moving axes, we compute

$$\boldsymbol{R}_z(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \boldsymbol{R}_y(\beta) = \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix}, \quad \boldsymbol{R}_x(\gamma) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & -\sin\gamma \\ 0 & \sin\gamma & \cos\gamma \end{pmatrix},$$

and multiply them in the suitable order to obtain

$$^{A}\boldsymbol{R}_B = \boldsymbol{R}_z(\alpha)\boldsymbol{R}_y(\beta)\boldsymbol{R}_x(\gamma).$$

Replacing the numerical values (with $\boldsymbol{R}_x(\gamma = 0) = \boldsymbol{I}$), we have

$$^{A}\boldsymbol{T}_B = \begin{pmatrix} ^{A}\boldsymbol{R}_B & ^{A}\boldsymbol{p}_{\boldsymbol{O}_A\boldsymbol{O}_B} \\ \boldsymbol{0}^T & 1 \end{pmatrix} = \begin{pmatrix} 0 & -\sqrt{2}/2 & -\sqrt{2}/2 & 3 \\ 0 & \sqrt{2}/2 & -\sqrt{2}/2 & 7 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Finally

$$^{A}\boldsymbol{p}_{\boldsymbol{O}_A P,h} = {}^{A}\boldsymbol{T}_B\, {}^{B}\boldsymbol{p}_{\boldsymbol{O}_B P,h} = {}^{A}\boldsymbol{T}_B \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 - \frac{\sqrt{2}}{2} \\ 7 + \frac{\sqrt{2}}{2} \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2.2929 \\ 7.7071 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} ^{A}\boldsymbol{p}_{\boldsymbol{O}_A P} \\ 1 \end{pmatrix}.$$

Exercise 3

An assignment of frames and the associated table of Denavit-Hartenberg are given in Fig. 4 and Tab. 2, respectively. The origin of frame RF_2 is conveniently placed at point P.

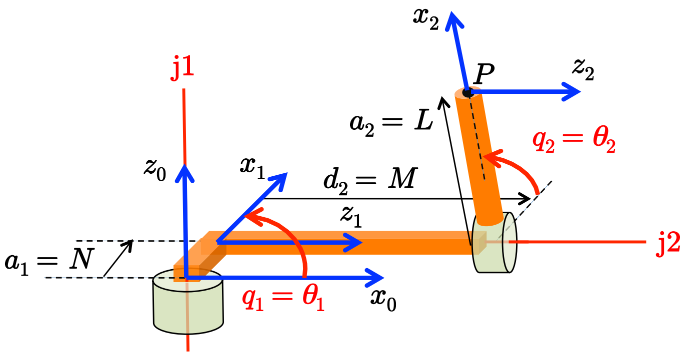

Figure 4: A possible assignment of DH frames for the 2R robot of Fig. 2.

i	α_i	a_i	d_i	θ_i
1	$\pi/2$	N	0	q_1
2	0	L	M	q_2

Table 2: Parameters associated to the DH frames in Fig. 4.

From this, the two homogeneous transformation matrices are computed

$$^{0}\boldsymbol{A}_1(q_1) = \begin{pmatrix} \cos q_1 & 0 & \sin q_1 & N\cos q_1 \\ \sin q_1 & 0 & -\cos q_1 & N\sin q_1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad ^{1}\boldsymbol{A}_2(q_2) = \begin{pmatrix} \cos q_2 & -\sin q_2 & 0 & L\cos q_2 \\ \sin q_2 & \cos q_1 & 0 & L\sin q_2 \\ 0 & 0 & 1 & M \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Thus, the symbolic expression in frame RF_0 of the position vector associated to point P (in homogeneous coordinates) is

$$
{}^0\boldsymbol{p}_{OP,h}(\boldsymbol{q}) = {}^0\boldsymbol{A}_1(q_1)\,{}^1\boldsymbol{A}_2(q_2)\begin{pmatrix}0\\0\\0\\1\end{pmatrix} = {}^0\boldsymbol{A}_1(q_1)\begin{pmatrix}L\cos q_2\\L\sin q_2\\M\\1\end{pmatrix} = \begin{pmatrix}L\cos q_1\cos q_2 + M\sin q_1 + N\cos q_1\\L\sin q_1\cos q_2 - M\cos q_1 + N\sin q_1\\L\sin q_2\\1\end{pmatrix}
$$

being ${}^0\boldsymbol{p}^T_{OP,h}(\boldsymbol{q}) = \begin{pmatrix} {}^0\boldsymbol{p}^T_{OP}(\boldsymbol{q}) & 1 \end{pmatrix}$.

The numerical value of ${}^0\boldsymbol{p}_{OP}(\boldsymbol{q})$ with the data $L = 1$, $M = 2$, $N = 0.3$ [m] and at the requested robot configuration $\boldsymbol{q} = \begin{pmatrix} \pi/2 & -\pi/4 \end{pmatrix}^T$ [rad] is

$$
{}^0\boldsymbol{p}_{OP} = \begin{pmatrix} 2 & 0.3 + \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{pmatrix}^T = \begin{pmatrix} 2 & 1.0071 & -0.7071 \end{pmatrix}^T.
$$

Exercise 4

One needs to verify first the existence of a scalar $a > 0$ such that $\boldsymbol{A}$ is a rotation matrix (i.e., an orthonormal matrix with determinant $= +1$). The orthogonality among the three columns is already in place (and for any value of a). Imposing a unit norm to the first to columns leads to $a = \pm\sqrt{3}/2$, and the $+$ sign is taken to obtain $\det \boldsymbol{A} = +1$. The matrix equation

$$
\boldsymbol{R}(\boldsymbol{r},\theta) = \boldsymbol{A} = \begin{pmatrix} -0.5 & -\sqrt{3}/2 & 0\\ 0 & 0 & -1\\ \sqrt{3}/2 & -0.5 & 0 \end{pmatrix}
$$

is solved for $\boldsymbol{r}$ and θ, using the inverse mapping of the axis-angle representation. Denoting by A_{ij} the elements of $\boldsymbol{A}$, we find that the problem at hand is a regular one since

$$
\sin\theta = \pm\frac{1}{2}\sqrt{(A_{12}-A_{21})^2 + (A_{13}-A_{31})^2 + (A_{23}-A_{32})^2} = \pm\,0.6614 \neq 0. \tag{1}
$$

Therefore, from

$$
\cos\theta = \frac{1}{2}\left(A_{11} + A_{22} + A_{33} - 1\right) = -0.75,
$$

taking the $+$ sign in (1) we obtain

$$
\theta^{\{1\}} = \text{ATAN2}\left\{0.6614, -0.75\right\} = 2.4189 \text{ [rad]} = 138.59^\circ
$$

and then

$$
\boldsymbol{r}^{\{1\}} = \frac{1}{2\sin\theta^{\{1\}}}\begin{pmatrix} A_{32}-A_{23}\\ A_{13}-A_{31}\\ A_{21}-A_{12} \end{pmatrix} = \begin{pmatrix} 0.3780\\ -0.6547\\ 0.6547 \end{pmatrix}.
$$

The second solution is simply given by $\theta^{\{2\}} = -\theta^{\{1\}}$, $\boldsymbol{r}^{\{2\}} = -\boldsymbol{r}^{\{1\}}$. Indeed, one can check, e.g., that

$$
\boldsymbol{R}(\boldsymbol{r}^{\{2\}},\theta^{\{2\}}) = \boldsymbol{r}^{\{2\}}\boldsymbol{r}^{\{2\}T} + \left(\boldsymbol{I} - \boldsymbol{r}^{\{2\}}\boldsymbol{r}^{\{2\}T}\right)\cos\theta^{\{2\}} + \boldsymbol{S}(\boldsymbol{r}^{\{2\}})\sin\theta^{\{2\}} = \boldsymbol{A}.
$$

* * * * *