Robotics 1

Differential kinematics

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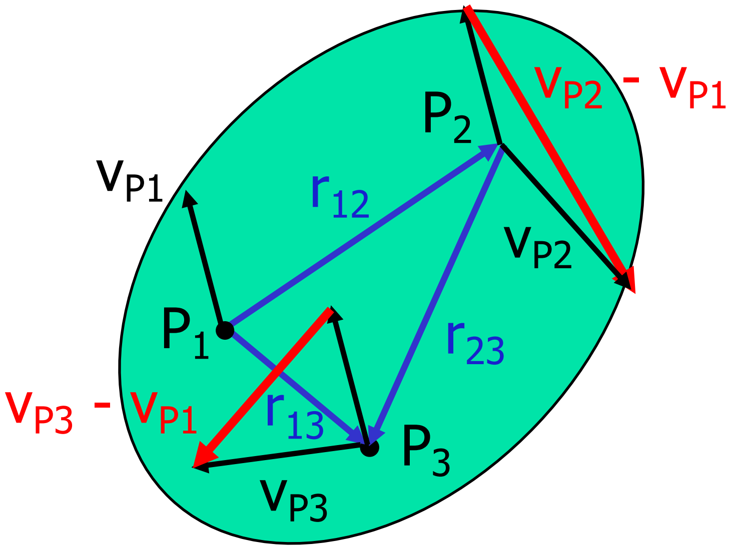
Differential kinematics

- "relations between motion (velocity) in joint space and motion (linear/angular velocity) in task space (e.g., Cartesian space)"
- instantaneous velocity mappings can be obtained through time differentiation of the direct kinematics or in a geometric way, directly at the differential level
  - different treatments arise for rotational quantities
  - establish the link between angular velocity and
    - time derivative of a rotation matrix
    - time derivative of the angles in a minimal representation of orientation

Angular velocity of a rigid body



"rigidity" constraint on distances among points:

$$\|r_{ij}\| = \text{constant}$$

$\Rightarrow$ $v_{Pi} - v_{Pj}$ orthogonal to r_{ij}

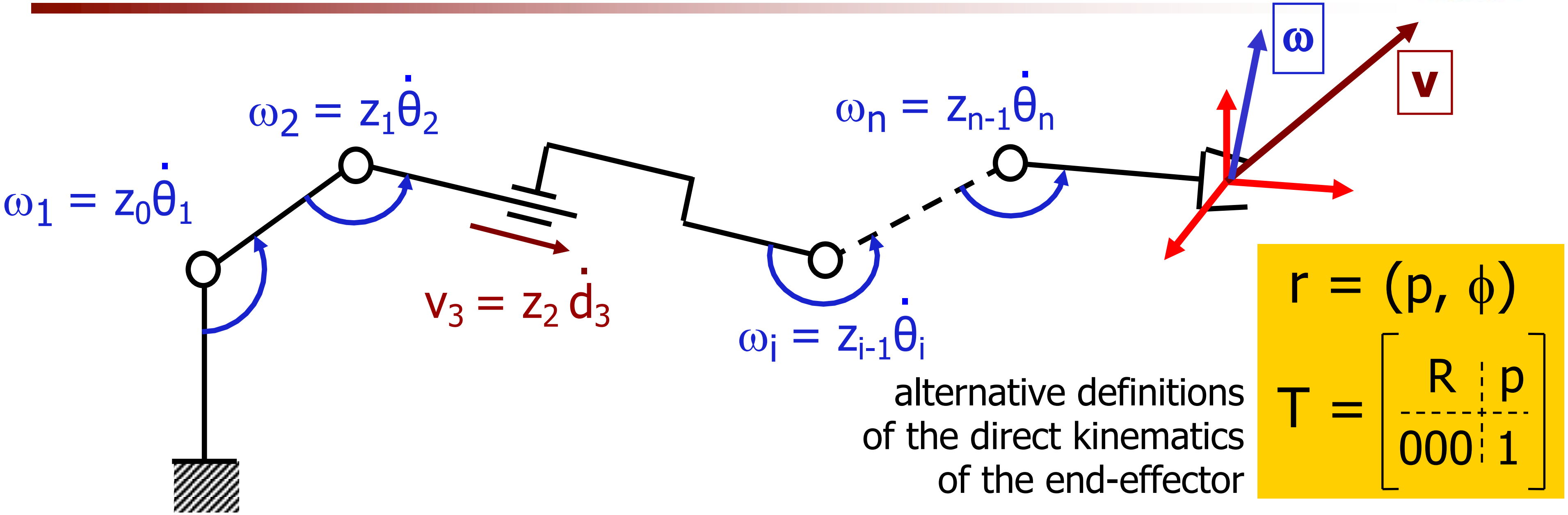
1	$v_{P2} - v_{P1} = \omega_1 \times r_{12}$
2	$v_{P3} - v_{P1} = \omega_1 \times r_{13}$
3	$v_{P3} - v_{P2} = \omega_2 \times r_{23}$

$\forall\ P_1, P_2, P_3$ $\quad$ 2 - 1 = 3 $\Rightarrow$ $\omega_1 = \omega_2 = \omega$

$$\boxed{v_{Pj} = v_{Pi} + \omega \times r_{ij} = v_{Pi} + S(\omega)\, r_{ij}} \Leftrightarrow \boxed{\dot{r}_{ij} = \omega \times r_{ij}}$$

- the angular velocity $\boldsymbol{\omega}$ is associated to the whole body (**not** to a point)
- if $\exists\ P_1, P_2$ with $v_{P1}=v_{P2}=0$: pure rotation (circular motion of all $P_j \notin$ line P_1P_2)
- $\omega=0$: pure translation (**all** points have the same velocity v_P)

Linear and angular velocity of the robot end-effector

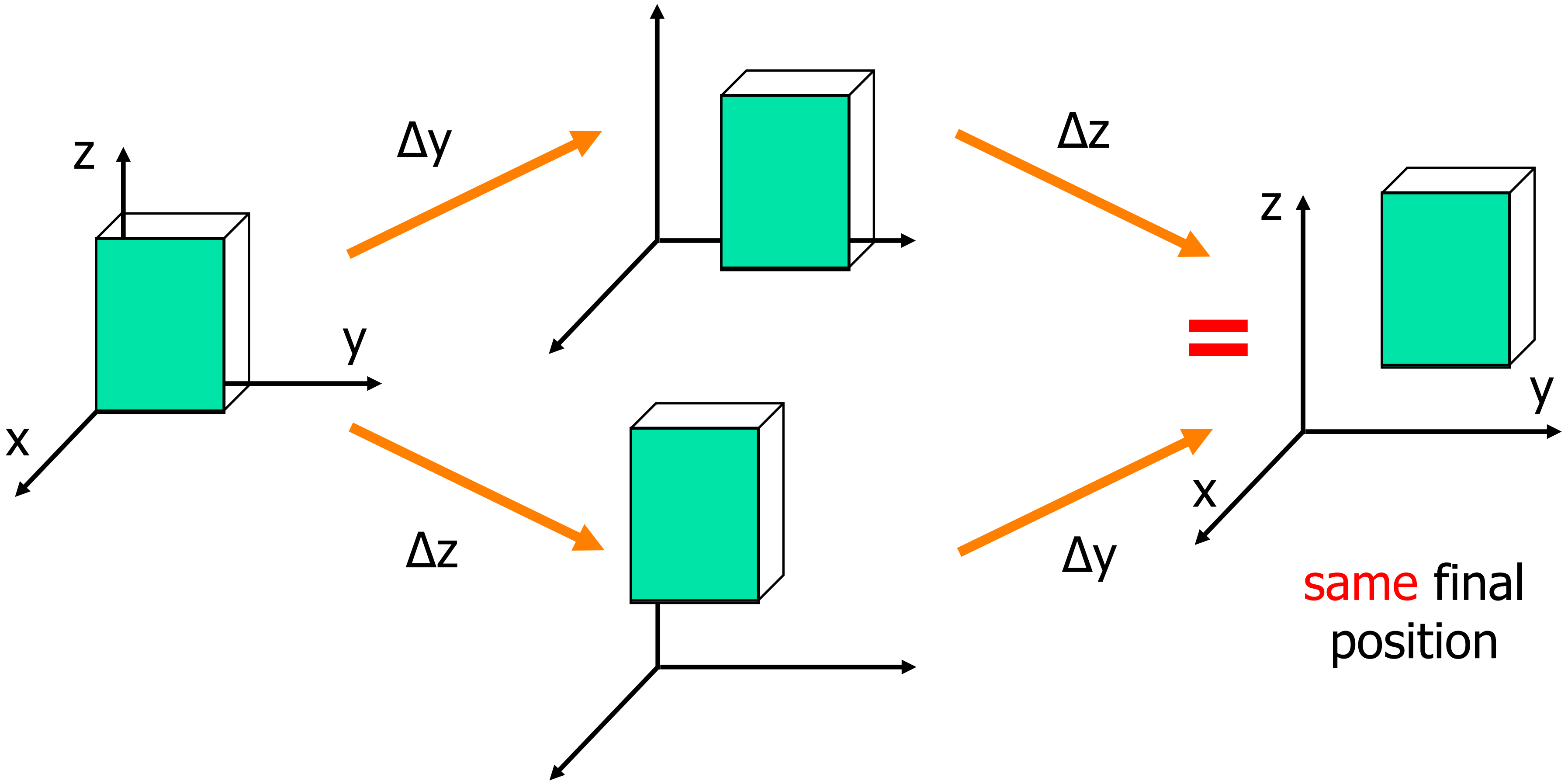


- v and ω are "vectors", namely are elements of vector spaces
  - they can be obtained as the sum of single contributions (in any order)
  - such contributions will be given by the single joint velocities
- on the other hand, ϕ (and $\dot{\phi}$) is not an element of a vector space
  - a minimal representation of a sequence of two rotations is not obtained summing the corresponding minimal representations (accordingly, for their time derivatives)

in general, $\omega \neq \dot{\phi}$

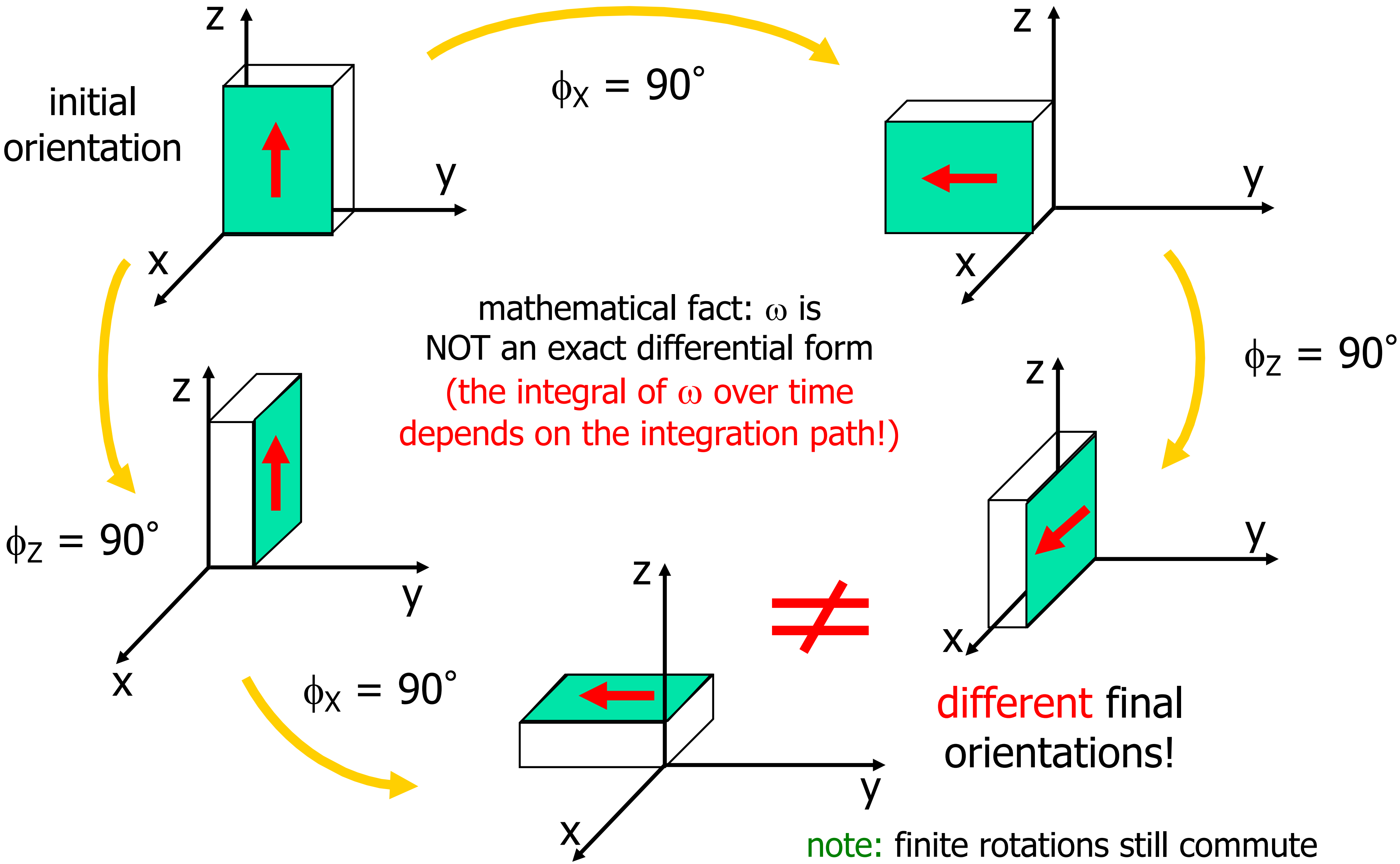
Finite and infinitesimal translations

- finite $\Delta x, \Delta y, \Delta z$ or infinitesimal dx, dy, dz translations (linear displacements) always commute



Finite rotations do not commute

example



mathematical fact: ω is NOT an exact differential form (the integral of ω over time depends on the integration path!)

different final orientations!

note: finite rotations still commute when made around the same fixed axis

Infinitesimal rotations commute!

- infinitesimal rotations $d\phi_X, d\phi_Y, d\phi_Z$ around x, y, z axes

$$R_X(\phi_X) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi_X & -\sin\phi_X \\ 0 & \sin\phi_X & \cos\phi_X \end{bmatrix} \Rightarrow R_X(d\phi_X) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -d\phi_X \\ 0 & d\phi_X & 1 \end{bmatrix}$$

$$R_Y(\phi_Y) = \begin{bmatrix} \cos\phi_Y & 0 & \sin\phi_Y \\ 0 & 1 & 0 \\ -\sin\phi_Y & 0 & \cos\phi_Y \end{bmatrix} \Rightarrow R_Y(d\phi_Y) = \begin{bmatrix} 1 & 0 & d\phi_Y \\ 0 & 1 & 0 \\ -d\phi_Y & 0 & 1 \end{bmatrix}$$

$$R_Z(\phi_Z) = \begin{bmatrix} \cos\phi_Z & -\sin\phi_Z & 0 \\ \sin\phi_Z & \cos\phi_Z & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow R_Z(d\phi_Z) = \begin{bmatrix} 1 & -d\phi_Z & 0 \\ d\phi_Z & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- $$R(d\phi) = R(d\phi_X, d\phi_Y, d\phi_Z) = \begin{bmatrix} 1 & -d\phi_Z & d\phi_Y \\ d\phi_Z & 1 & -d\phi_X \\ -d\phi_Y & d\phi_X & 1 \end{bmatrix} \leftarrow$$

neglecting second- and third-order (infinitesimal) terms

in any order

$$= I + S(d\phi)$$

Time derivative of a rotation matrix

- let $R = R(t)$ be a rotation matrix, given as a function of time
- since $I = R(t)R^T(t)$, taking the time derivative of both sides yields

$$0 = d[R(t)R^T(t)]/dt = dR(t)/dt\, R^T(t) + R(t)\, dR^T(t)/dt$$

$$= dR(t)/dt\, R^T(t) + [dR(t)/dt\, R^T(t)]^T$$

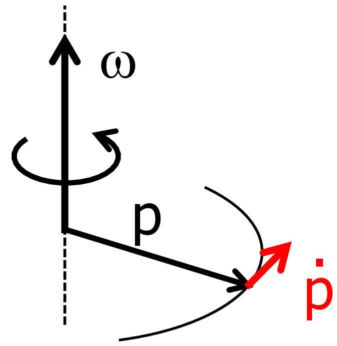
thus $dR(t)/dt\, R^T(t) = S(t)$ is a skew-symmetric matrix

- let $p(t) = R(t)p'$ a vector (with constant norm) rotated over time
- comparing

$$dp(t)/dt = dR(t)/dt\, p' = S(t)R(t)p' = S(t)p(t)$$

$$dp(t)/dt = \omega(t) \times p(t) = S(\omega(t))p(t)$$

we get $S = S(\omega)$



$$\boxed{\dot{R} = S(\omega)\, R} \iff \boxed{S(\omega) = \dot{R}\, R^T}$$

Example

Time derivative of an elementary rotation matrix

$$R_X(\phi(t)) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi(t) & -\sin\phi(t) \\ 0 & \sin\phi(t) & \cos\phi(t) \end{bmatrix}$$

$$\dot{R}_X(\phi)\, R^T_X(\phi) = \dot{\phi} \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\sin\phi & -\cos\phi \\ 0 & \cos\phi & -\sin\phi \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\dot{\phi} \\ 0 & \dot{\phi} & 0 \end{bmatrix} = S(\omega)$$

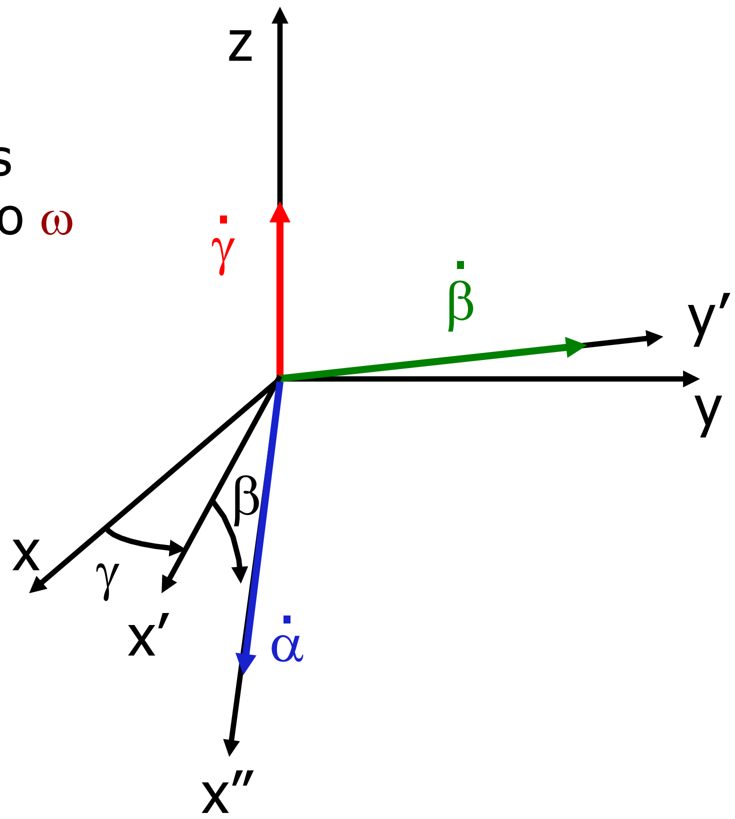
$$\Downarrow$$

$$\omega = \begin{bmatrix} \dot{\phi} \\ 0 \\ 0 \end{bmatrix}$$

Time derivative of RPY angles and ω

$$R_{RPY}(\alpha_x, \beta_y, \gamma_z) = R_{ZY'X''}(\gamma_z, \beta_y, \alpha_x)$$

the three contributions $\dot{\gamma}z$, $\dot{\beta}y'$, $\dot{\alpha}x''$ to ω are simply summed as vectors



$$\omega = \underbrace{\begin{bmatrix} c\beta\, c\gamma & -s\gamma & 0 \\ c\beta\, s\gamma & c\gamma & 0 \\ -s\beta & 0 & 1 \end{bmatrix}}_{T_{RPY}(\beta,\gamma)} \begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \end{bmatrix}$$

$$\begin{array}{ccc} x'' & y' & z \\ \uparrow & \uparrow & \\ \text{1st col in } R_Z(\gamma_z)R_{Y'}(\beta_y) & \text{2nd col in } R_Z(\gamma_z) & \end{array}$$

$\det T_{RPY}(\beta,\gamma) = c\beta = 0$
for $\beta = \pm\pi/2$
(singularity of the RPY representation)

similar treatment for the other 11 minimal representations...

Robot Jacobian matrices

- analytical Jacobian (obtained by time differentiation)

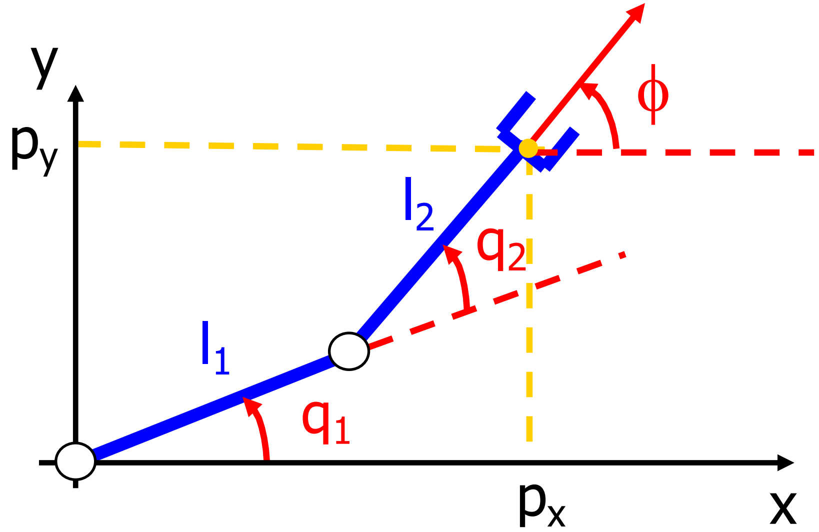
$$r = \begin{pmatrix} p \\ \phi \end{pmatrix} = f_r(q) \quad \Rightarrow \quad \dot{r} = \begin{pmatrix} \dot{p} \\ \dot{\phi} \end{pmatrix} = \frac{\partial f_r(q)}{\partial q}\,\dot{q} = J_r(q)\,\dot{q}$$

- geometric Jacobian (no derivatives)

$$\begin{pmatrix} v \\ \omega \end{pmatrix} = \begin{pmatrix} \dot{p} \\ \omega \end{pmatrix} = \begin{pmatrix} J_L(q) \\ J_A(q) \end{pmatrix} \dot{q} = J(q)\,\dot{q}$$

- in both cases, the Jacobian matrix depends on the (current) configuration of the robot

Analytical Jacobian of planar 2R arm



direct kinematics

$$
r \left\{
\begin{aligned}
p_x &= l_1\, c_1 + l_2\, c_{12} \\
p_y &= l_1\, s_1 + l_2\, s_{12} \\
\phi &= q_1 + q_2
\end{aligned}
\right.
$$

$$\dot{p}_x = -\, l_1\, s_1\, \dot{q}_1 - l_2\, s_{12}\, (\dot{q}_1 + \dot{q}_2)$$

$$\dot{p}_y = l_1\, c_1\, \dot{q}_1 + l_2\, c_{12}\, (\dot{q}_1 + \dot{q}_2)$$

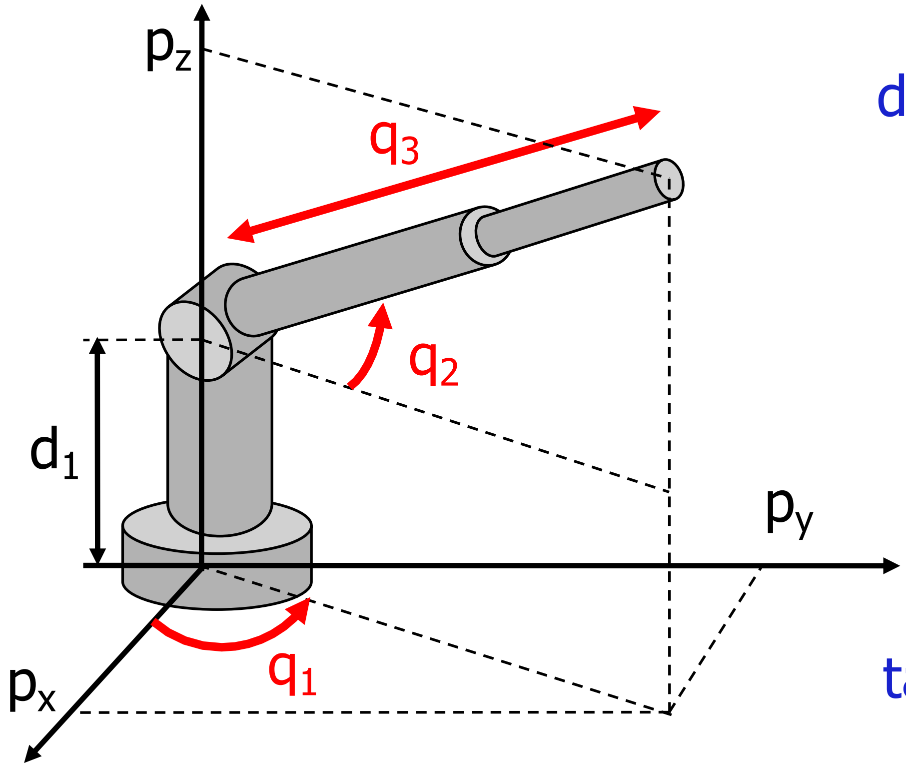
$$\dot{\phi} = \omega_z = \dot{q}_1 + \dot{q}_2$$

$$\Rightarrow \quad J_r(q) = \begin{pmatrix} -\, l_1\, s_1 - l_2\, s_{12} & -\, l_2\, s_{12} \\ l_1\, c_1 + l_2\, c_{12} & l_2\, c_{12} \\ 1 & 1 \end{pmatrix}$$

here, all rotations occur around the same fixed axis z (normal to the plane of motion)

given r, this is a 3 x 2 matrix

Analytical Jacobian of polar robot



direct kinematics (here, $r = p$)

$$\left.\begin{aligned} p_x &= q_3\, c_2\, c_1 \\ p_y &= q_3\, c_2\, s_1 \\ p_z &= d_1 + q_3\, s_2 \end{aligned}\right\} f_r(q)$$

taking the time derivative

$$v = \dot{p} = \underbrace{\begin{pmatrix} -q_3c_2s_1 & -q_3s_2c_1 & c_2c_1 \\ q_3c_2c_1 & -q_3s_2s_1 & c_2s_1 \\ 0 & q_3c_2 & s_2 \end{pmatrix}}_{\dfrac{\partial f_r(q)}{\partial q}} \dot{q} = J_r(q)\,\dot{q}$$

Geometric Jacobian

always a 6 x n matrix

end-effector instantaneous velocity $\begin{pmatrix} v_E \\ \omega_E \end{pmatrix} = \begin{pmatrix} J_L(q) \\ J_A(q) \end{pmatrix} \dot{q} = \begin{pmatrix} J_{L1}(q) & \dots & J_{Ln}(q) \\ J_{A1}(q) & \dots & J_{An}(q) \end{pmatrix} \begin{pmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{pmatrix}$

superposition of effects

$v_E = \underbrace{J_{L1}(q)\,\dot{q}_1}\ + \dots + J_{Ln}(q)\,\dot{q}_n \qquad \omega_E = \underbrace{J_{A1}(q)\,\dot{q}_1}\ + \dots + J_{An}(q)\,\dot{q}_n$

contribution to the linear e-e velocity due to $\dot{q}_1$

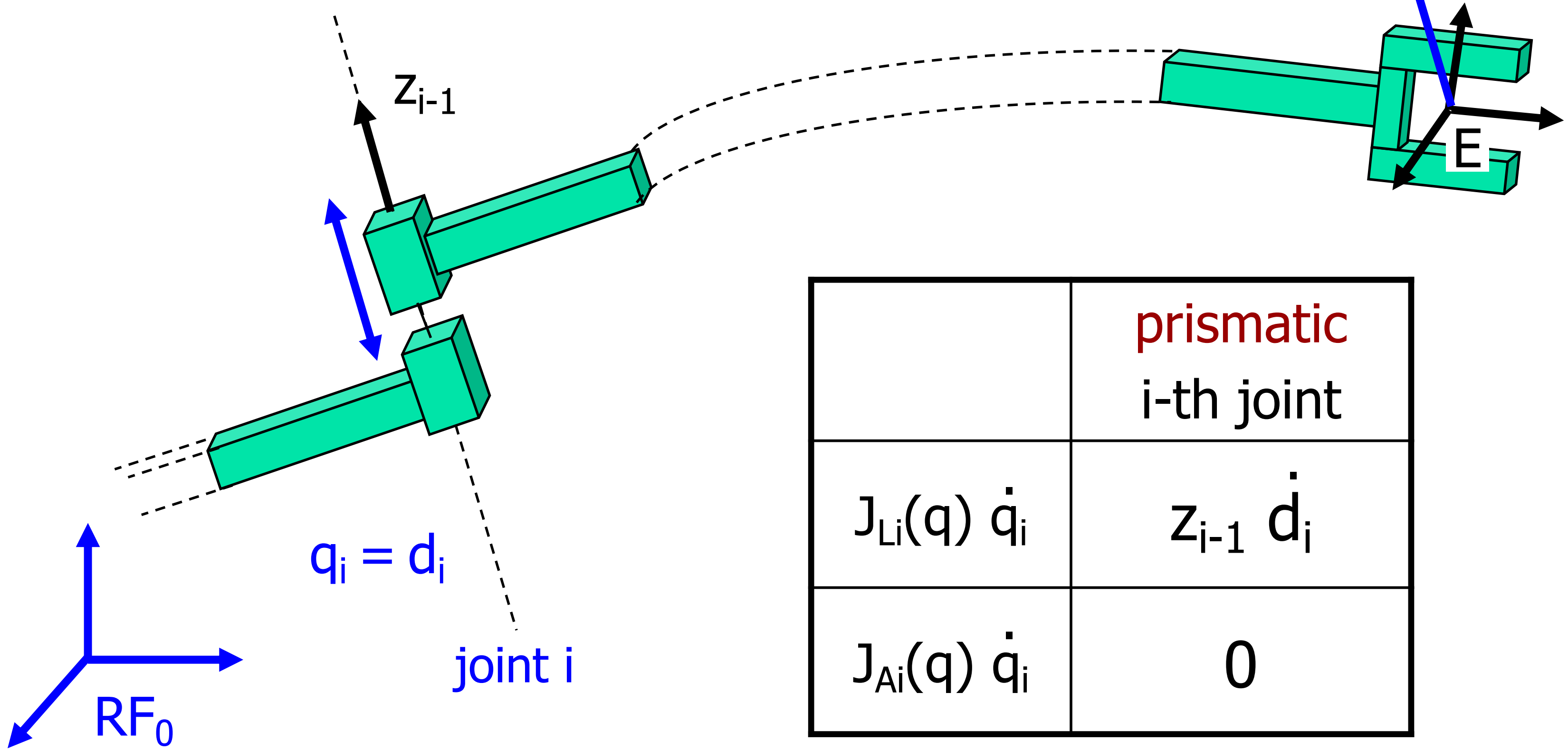
contribution to the angular e-e velocity due to $\dot{q}_1$

linear and angular velocity belong to (linear) vector spaces in R^3

Contribution of a prismatic joint

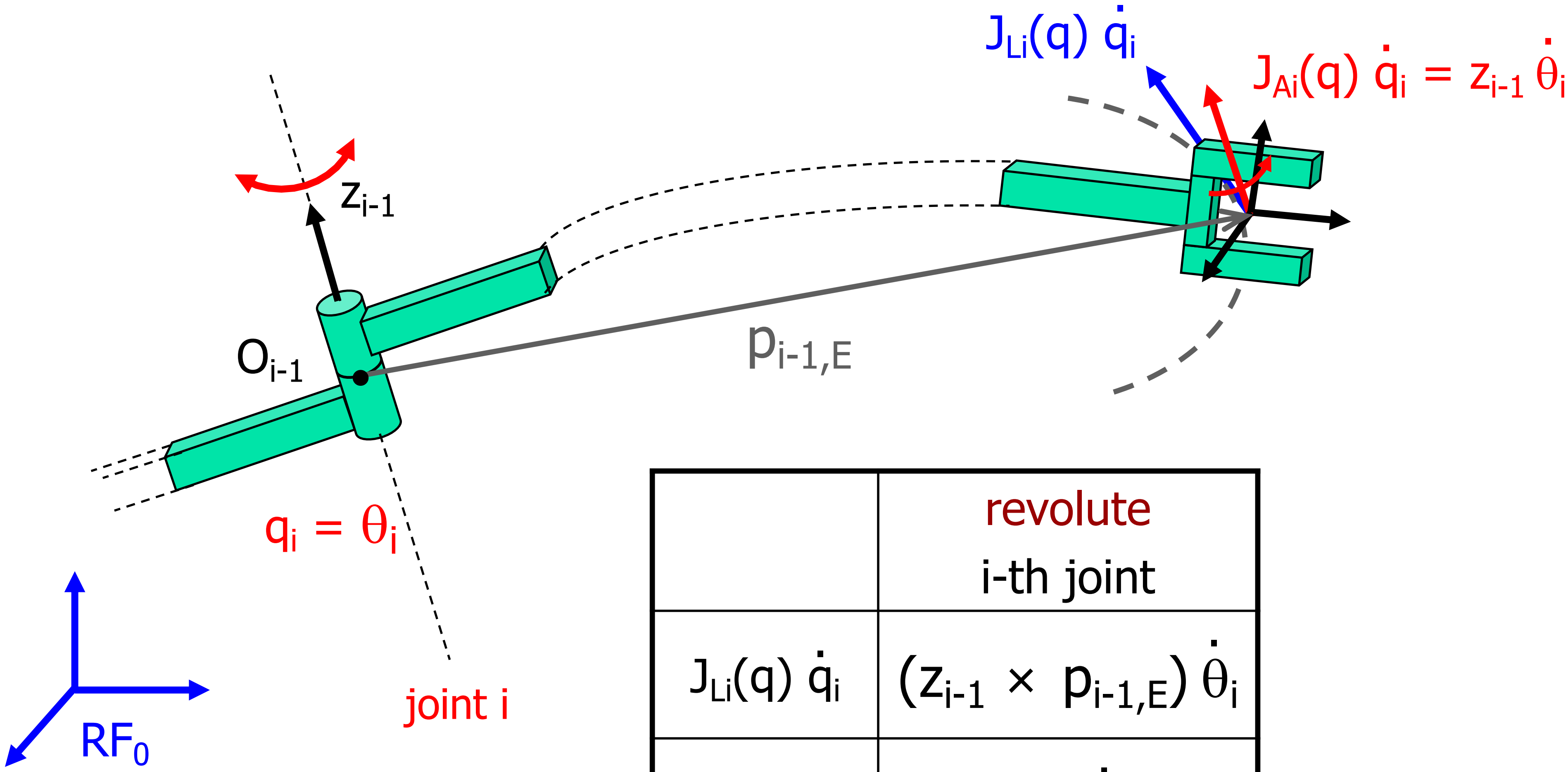
note: joints beyond the i-th one are considered to be "frozen", so that the distal part of the robot is a single rigid body

$J_{Li}(q)\,\dot{q}_i = z_{i-1}\,\dot{d}_i$



	prismatic i-th joint
$J_{Li}(q)\,\dot{q}_i$	$z_{i-1}\,\dot{d}_i$
$J_{Ai}(q)\,\dot{q}_i$	0

Contribution of a revolute joint



	revolute i-th joint
$J_{Li}(q)\,\dot{q}_i$	$(z_{i-1} \times p_{i-1,E})\,\dot{\theta}_i$
$J_{Ai}(q)\,\dot{q}_i$	$z_{i-1}\,\dot{\theta}_i$

Expression of geometric Jacobian

$$\left(\begin{pmatrix} \dot{p}_{0,E} \\ \omega_E \end{pmatrix} =\right) \begin{pmatrix} v_E \\ \omega_E \end{pmatrix} = \begin{pmatrix} J_L(q) \\ J_A(q) \end{pmatrix} \dot{q} = \begin{pmatrix} J_{L1}(q) & \dots & J_{Ln}(q) \\ J_{A1}(q) & \dots & J_{An}(q) \end{pmatrix} \begin{pmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_n \end{pmatrix}$$

	prismatic i-th joint	revolute i-th joint
$J_{Li}(q)$	z_{i-1}	$z_{i-1} \times p_{i-1,E}$
$J_{Ai}(q)$	0	z_{i-1}

this can be also computed as

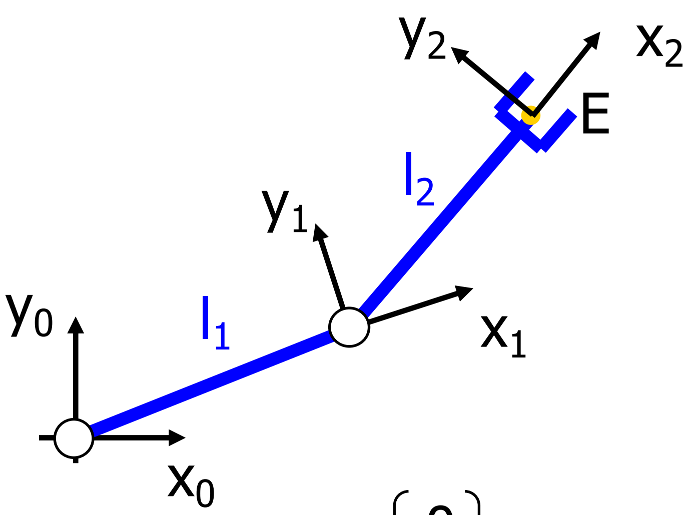
$$= \frac{\partial p_{0,E}}{\partial q_i}$$

$$z_{i-1} = {}^0R_1(q_1)\dots{}^{i-2}R_{i-1}(q_{i-1}) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$p_{i-1,E} = p_{0,E}(q_1,\dots,q_n) - p_{0,i-1}(q_1,\dots,q_{i-1})$$

all vectors should be expressed in the same reference frame (here, the base frame RF_0)

Example: planar 2R arm



DENAVIT-HARTENBERG table

joint	α_i	d_i	a_i	θ_i
1	0	0	l_1	q_1
2	0	0	l_2	q_2

$$z_0 = z_1 = z_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

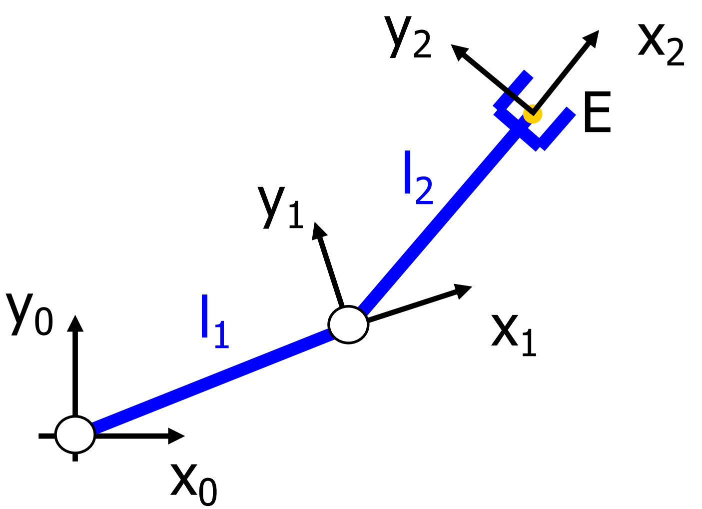
$$^0A_1 = \begin{pmatrix} c_1 & -s_1 & 0 & l_1c_1 \\ s_1 & c_1 & 0 & l_1s_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \leftarrow p_{0,1}$$

$$p_{1,E} = p_{0,E} - p_{0,1}$$

$$J = \begin{pmatrix} z_0 \times p_{0,E} & z_1 \times p_{1,E} \\ z_0 & z_1 \end{pmatrix}$$

$$^0A_2 = \begin{pmatrix} c_{12} & -s_{12} & 0 & l_1c_1 + l_2c_{12} \\ s_{12} & c_{12} & 0 & l_1s_1 + l_2s_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \leftarrow p_{0,E}$$

Geometric Jacobian of planar 2R arm



note: the Jacobian is here a 6×2 matrix, thus its maximum rank is 2

at most 2 components of the linear/angular end-effector velocity can be independently assigned

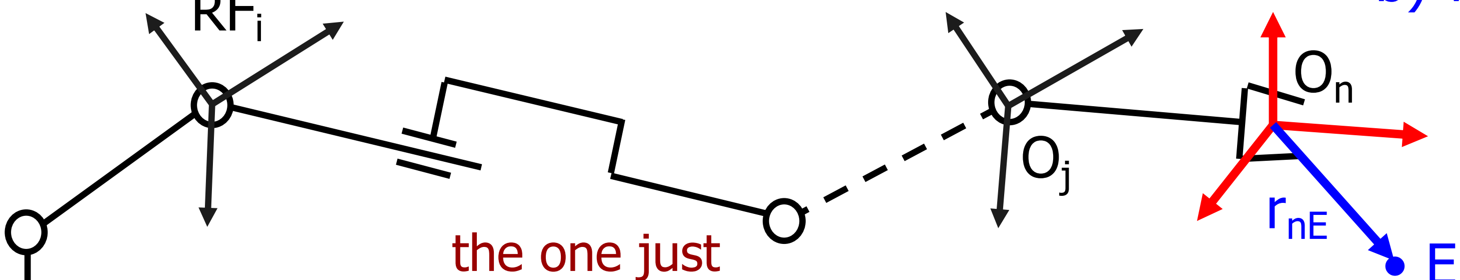
$$J = \begin{bmatrix} z_0 \times p_{0,E} & z_1 \times p_{1,E} \\ z_0 & z_1 \end{bmatrix}$$

$$= \begin{bmatrix} -l_1 s_1 - l_2 s_{12} & -l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$$

compare rows 1, 2, and 6 with the analytical Jacobian in slide #12!

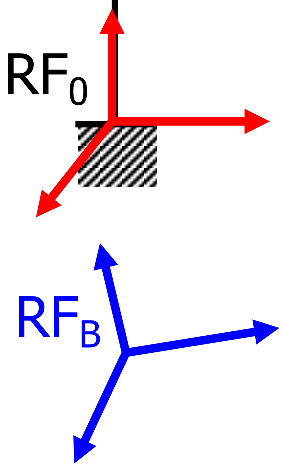
Transformations of the Jacobian matrix

b) we may choose
$E \Rightarrow O_j(q)$



the one just computed …

$$\begin{bmatrix} {}^0v_n \\ {}^0\omega \end{bmatrix} = {}^0J_n(q)\,\dot{q}$$



$$v_E = v_n + \omega \times r_{nE} \\ = v_n + S(r_{En})\,\omega$$

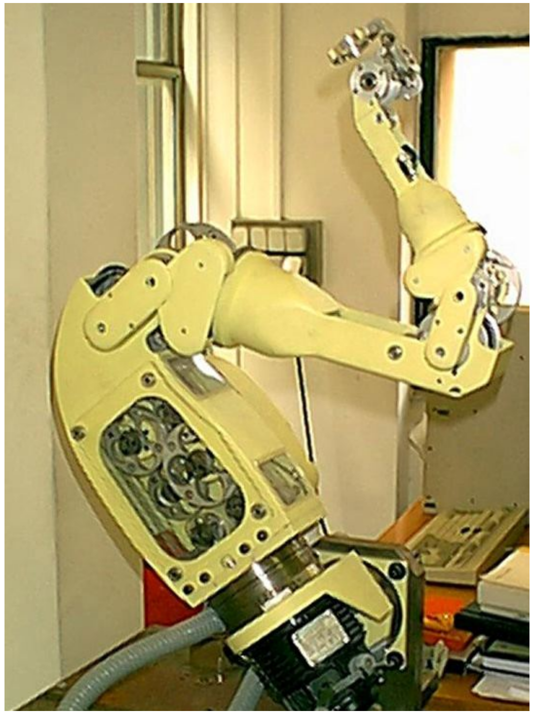
$$\begin{bmatrix} {}^Bv_E \\ {}^B\omega \end{bmatrix} = \begin{bmatrix} {}^BR_0 & 0 \\ 0 & {}^BR_0 \end{bmatrix} \begin{bmatrix} I & S({}^0r_{En}) \\ 0 & I \end{bmatrix} \begin{bmatrix} {}^0v_n \\ {}^0\omega \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} {}^BR_0(q) & 0 \\ 0 & {}^BR_0(q) \end{bmatrix} \begin{bmatrix} I & S({}^0r_{En}(q)) \\ 0 & I \end{bmatrix}}_{\text{never singular!}} {}^0J_n(q)\,\dot{q} = {}^BJ_E(q)\,\dot{q}$$

a) we may choose
$RF_B \Rightarrow RF_i(q)$

Example: Dexter robot

- 8R robot manipulator with transmissions by pulleys and steel cables (joints 3 to 8)
  - lightweight: only 15 kg in motion
  - motors located in second link
  - incremental encoders (homing)
  - redundancy degree for e-e pose task: n-m=2
  - compliant in the interaction with environment



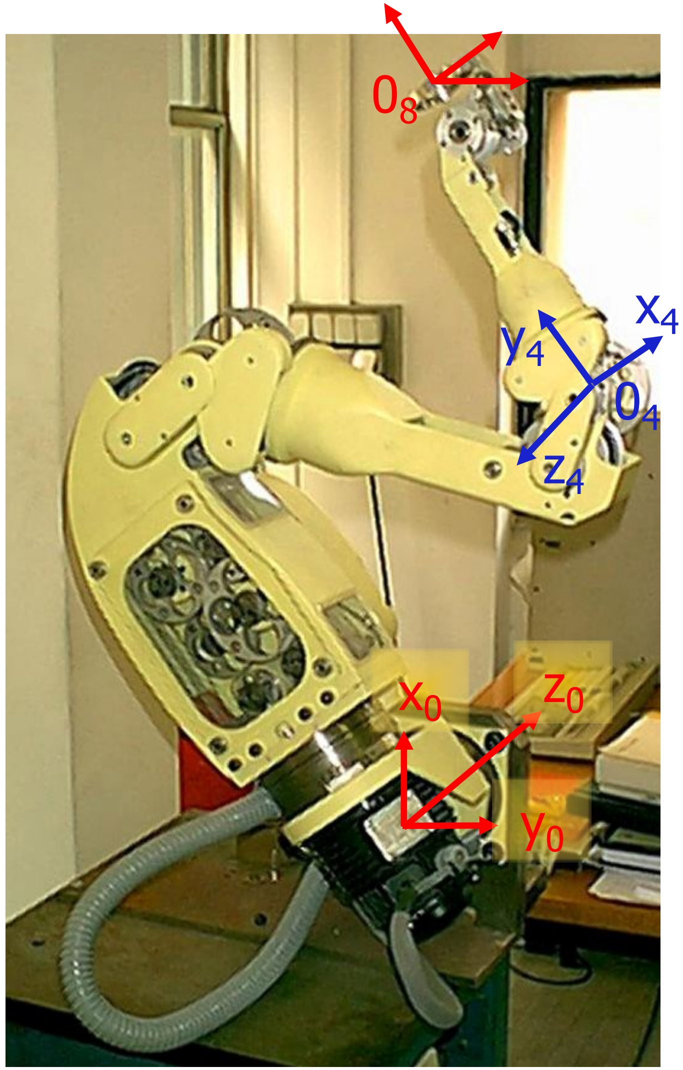
i	a (mm)	d (mm)	α (rad)	range θ (deg)
0	0	0	$-\pi/2$	[-12.56, 179.89]
1	144	450	$-\pi/2$	[-83, 84]
2	0	0	$\pi/2$	[7, 173]
3	100	350	$\pi/2$	[65, 295]
4	0	0	$-\pi/2$	[-174, -3]
5	24	250	$-\pi/2$	[57, 265]
6	0	0	$-\pi/2$	[-129.99, -45]
7	100	0	π	[-55.05, 30]

Mid-frame Jacobian of Dexter robot

- geometric Jacobian ${}^0J_8(q)$ is very complex
- "mid-frame" Jacobian ${}^4J_4(q)$ is relatively simple!

$$
{}^4\hat{J}_4 = \begin{bmatrix}
d_1s_1s_3+d_3s_3c_2s_1-a_1c_3c_1s_2-d_1c_3c_1c_2-d_3c_1c_3 & a_1s_3+d_3s_3s_2 & d_3c_3 & 0 & 0 & 0 & -a_5s_4-d_5c_5c_4 & -a_5s_5c_4c_6+d_5s_5s_6c_4 \\
-a_3s_3c_2s_1+a_3c_3c_1+a_1c_1c_2-d_1c_1s_2 & -a_3s_3s_2 & -a_3c_3 & 0 & 0 & 0 & -d_5c_5s_4+a_5c_4 & d_5s_5s_6s_4-a_5s_5s_4c_6 \\
-d_3c_3c_2s_1-a_1s_3c_1s_2-d_1s_3c_1c_2-d_3s_3c_1-d_1s_1c_3+a_3s_2s_1 & -a_1c_3-d_3c_3s_2-a_3c_2 & d_3s_3 & -a_3 & 0 & 0 & d_5s_5 & -a_5c_6c_5+d_5c_5s_6 \\
-c_3c_2s_1-s_3c_1 & -c_3s_2 & s_3 & 0 & 0 & -s_4 & -c_4s_5 & -c_4c_5s_6+s_4c_6 \\
-s_2s_1 & c_2 & 0 & 1 & 0 & c_4 & -s_4s_5 & -s_4c_5s_6-c_4c_6 \\
-s_3c_2s_1+c_3c_1 & -s_3s_2 & -c_3 & 0 & 1 & 0 & -c_5 & s_5s_6
\end{bmatrix}
$$

6 rows,
8 columns



Summary of differential relations

$\dot{p} \rightleftarrows v$ $\quad \dot{p} = v$

$\dot{R} \rightleftarrows \omega$ $\quad \dot{R} = S(\omega)\, R \quad \Longleftrightarrow$ for each column r_i of R (unit vector of a frame), it is

$$\dot{r}_i = \omega \times r_i$$

$\dot{\phi} \rightleftarrows \omega$

$$\omega = \omega_{\dot{\phi}_1} + \omega_{\dot{\phi}_2} + \omega_{\dot{\phi}_3} = a_1\, \dot{\phi}_1 + a_2(\phi_1)\, \dot{\phi}_2 + a_3(\phi_1, \phi_2)\, \dot{\phi}_3 = T(\phi)\dot{\phi}$$

(moving) axes of definition for the sequence of rotations ϕ_i

$$r = \begin{pmatrix} p \\ \phi \end{pmatrix} \;\Rightarrow\; J(q) = \begin{pmatrix} I & 0 \\ 0 & T(\phi) \end{pmatrix} J_r(q) \;\Longleftrightarrow\; J_r(q) = \begin{pmatrix} I & 0 \\ 0 & T^{-1}(\phi) \end{pmatrix} J(q)$$

$T(\phi)$ has always a singularity $\Leftrightarrow$ singularity of the specific minimal representation of orientation

Acceleration relations (and beyond…)

Higher-order differential kinematics

- differential relations between motion in the joint space and motion in the task space can be established at the second order, third order, ...
- the analytical Jacobian always "weights" the highest-order derivative

velocity $\quad \dot{r} = J_r(q)\,\dot{q}$

matrix function $N_2(q,\dot{q})$

acceleration $\quad \ddot{r} = J_r(q)\,\ddot{q} + \dot{J}_r(q)\,\dot{q}$

jerk $\quad \dddot{r} = J_r(q)\,\dddot{q} + 2\,\dot{J}_r(q)\,\ddot{q} + \ddot{J}_r(q)\,\dot{q}$

snap $\quad \ddddot{r} = J_r(q)\,\ddddot{q} + \ldots$

matrix function $N_3(q,\dot{q},\ddot{q})$

- the same holds true also for the geometric Jacobian $J(q)$

Primer on linear algebra

given a matrix J: $m \times n$ (m rows, n columns)

- rank $\rho(J)$ = max # of rows or columns that are linearly independent
  - $\rho(J) \leq \min(m,n)$ (if equality holds, J has "full rank")
  - if $m = n$ and J has full rank, J is "non singular" and the inverse J^{-1} exists
  - $\rho(J)$ = dimension of the largest non singular square submatrix of J
- range $\Re(J)$ = vector subspace generated by all possible linear combinations of the columns of J

  also called "image" of J

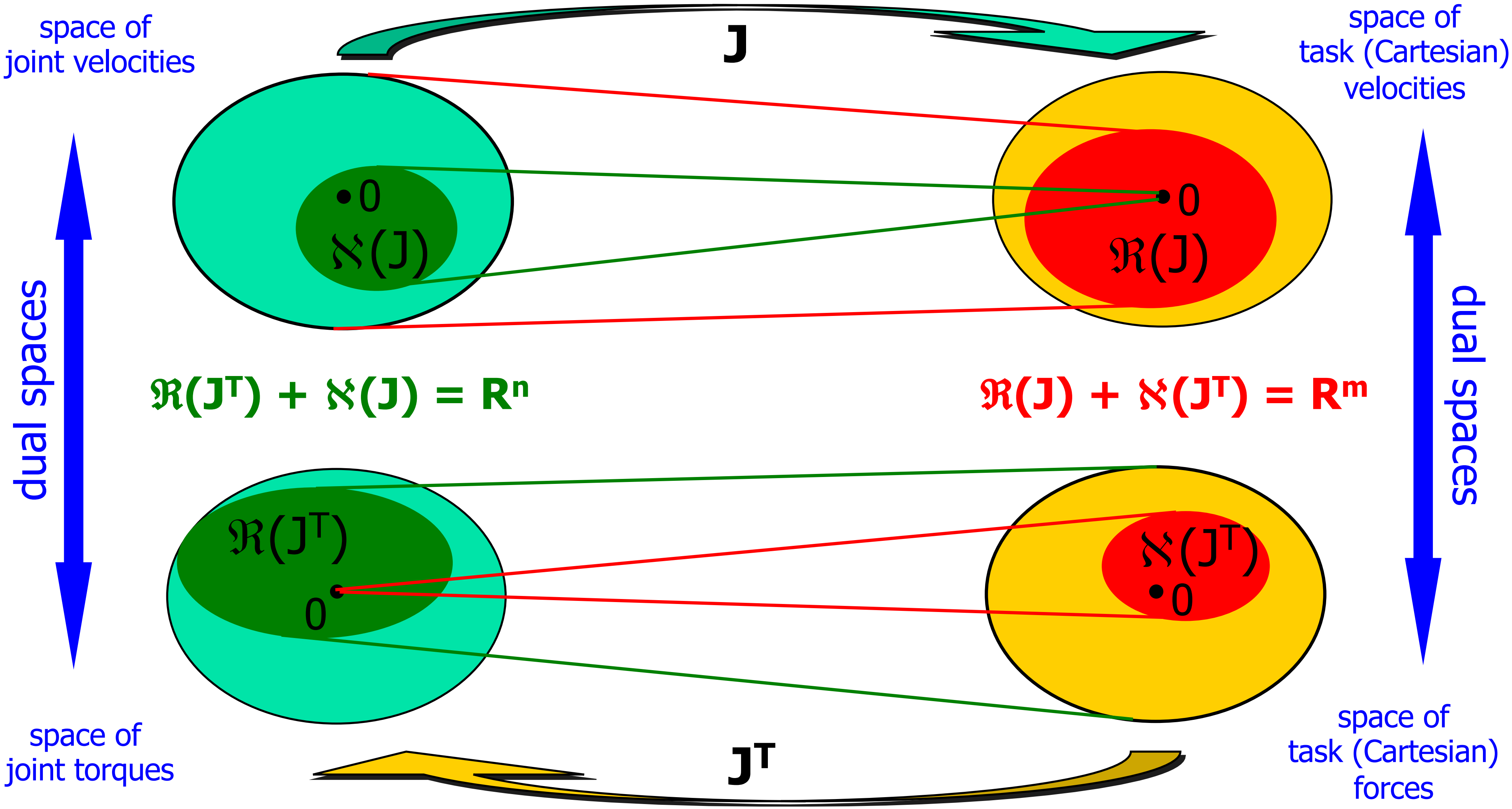
  $$\Re(J)=\{v \in R^m : \exists\, \xi \in R^n,\ v = J\,\xi\}$$

  - $\dim(\Re(J)) = \rho(J)$
- kernel $\aleph(J)$ = vector subspace of all vectors $\xi \in R^n$ such that $J \cdot \xi = 0$
  - $\dim(\aleph(J)) = n - \rho(J)$

  also called "null space" of J
- $\boxed{\Re(J) + \aleph(J^T) = R^m}$ e $\boxed{\Re(J^T) + \aleph(J) = R^n}$
  - sum of vector subspaces $V_1 + V_2$ = vector space where any element v can be written as $v = v_1 + v_2$, with $v_1 \in V_1$, $v_2 \in V_2$
- all the above quantities/subspaces can be computed using, e.g., Matlab

Robot Jacobian

decomposition in linear subspaces and duality



(in a given configuration q)

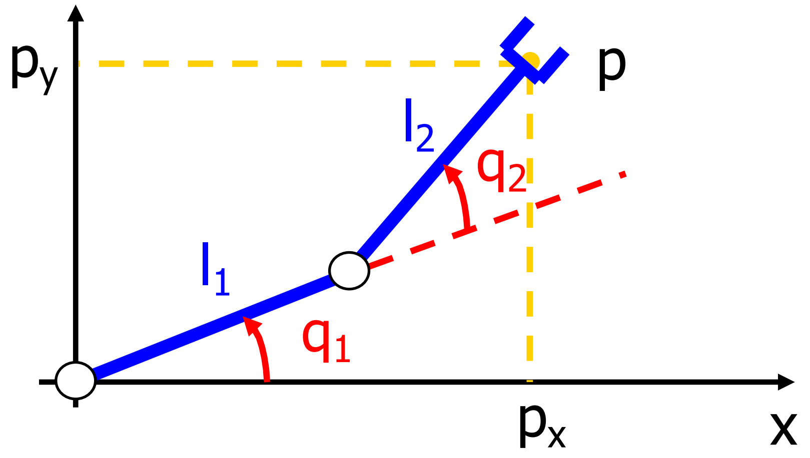
Mobility analysis

- $\rho(J) = \rho(J(q))$, $\Re(J) = \Re(J(q))$, $\aleph(J^T) = \aleph(J^T(q))$ are locally defined, i.e., they depend on the current configuration q
- $\Re(J(q))$ = subspace of all "generalized" velocities (with linear and/or angular components) that can be instantaneously realized by the robot end-effector when varying the joint velocities at the configuration q
- if J(q) has max rank (typically = m) in the configuration q, the robot end-effector can be moved in any direction of the task space R^m
- if $\rho(J(q)) < m$, there exist directions in R^m along which the robot end-effector cannot move (instantaneously!)
  - these directions lie in $\aleph(J^T(q))$, namely the complement of $\Re(J(q))$ to the task space R^m, which is of dimension $m - \rho(J(q))$
- when $\aleph(J(q)) \neq \{0\}$, there exist non-zero joint velocities that produce zero end-effector velocity ("self motions")
  - this always happens for $m<n$, i.e., when the robot is redundant for the task

Kinematic singularities

- **configurations where the Jacobian loses rank**
  $\Leftrightarrow$ loss of instantaneous mobility of the robot end-effector
- for $m = n$, they correspond to Cartesian poses at which the number of solutions of the inverse kinematics problem differs from the "generic" case
- "in" a singular configuration, we cannot find a joint velocity that realizes a desired end-effector velocity in an arbitrary direction of the task space
- "close" to a singularity, large joint velocities may be needed to realize some (even small) velocity of the end-effector
- finding and analyzing in advance all singularities of a robot helps in avoiding them during trajectory planning and motion control
  - when $m = n$: find the configurations q such that $\det J(q) = 0$
  - when $m < n$: find the configurations q such that all $m \times m$ minors of J are singular (or, equivalently, such that $\det [J(q)\, J^T(q)] = 0$)
- finding all singular configurations of a robot with a large number of joints, or the actual "distance" from a singularity, is a hard computational task

Singularities of planar 2R arm



direct kinematics

$$p_x = l_1\, c_1 + l_2\, c_{12}$$

$$p_y = l_1\, s_1 + l_2\, s_{12}$$

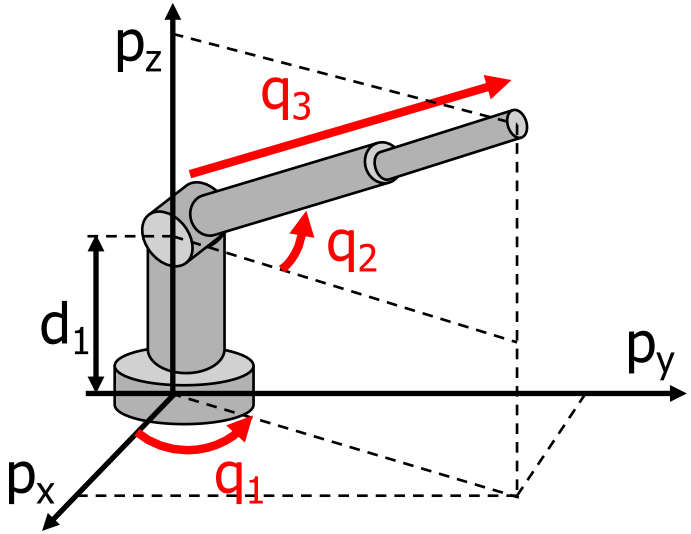
analytical Jacobian

$$\dot{p} = \begin{pmatrix} -l_1 s_1 - l_2 s_{12} & -l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} \end{pmatrix} \dot{q} = J(q)\,\dot{q}$$

$$\boxed{\det J(q) = l_1 l_2 s_2}$$

- singularities: arm is stretched ($q_2 = 0$) or folded ($q_2 = \pi$)
- singular configurations correspond here to Cartesian points on the boundary of the workspace
- in many cases, these singularities separate regions in the joint space with distinct inverse kinematic solutions (e.g., "elbow up" or "down")

Singularities of polar (RRP) arm



$$\boxed{\det J(q) = q_3^2\, c_2}$$

direct kinematics

$$p_x = q_3\, c_2\, c_1$$

$$p_y = q_3\, c_2\, s_1$$

$$p_z = d_1 + q_3\, s_2$$

analytical Jacobian

$$\dot{p} = \begin{pmatrix} -q_3 s_1 c_2 & -q_3 c_1 s_2 & c_1 c_2 \\ q_3 c_1 c_2 & -q_3 s_1 s_2 & s_1 c_2 \\ 0 & q_3 c_2 & s_2 \end{pmatrix} \dot{q} = J(q)\, \dot{q}$$

- singularities
  - E-E is along the z axis ($q_2 = \pm\pi/2$): simple singularity $\Rightarrow$ rank $J = 2$
  - third link is fully retracted ($q_3 = 0$): double singularity $\Rightarrow$ rank J drops to 1
- all singular configurations correspond here to Cartesian points internal to the workspace (supposing no limits for the prismatic joint)

Singularities of robots with spherical wrist

- $n = 6$, last three joints are revolute and their axes intersect at a point
- without loss of generality, we set $O_6 = W$ = center of spherical wrist (i.e., choose $d_6 = 0$ in the DH table)

$$J(q) = \begin{pmatrix} J_{11} & 0 \\ J_{21} & J_{22} \end{pmatrix}$$

- since $\det J(q_1,\ldots,q_5) = \det J_{11} \cdot \det J_{22}$, there is a decoupling property
  - $\det J_{11}(q_1,\ldots,q_3) = 0$ provides the arm singularities
  - $\det J_{22}(q_4, q_5) = 0$ provides the wrist singularities
- being $J_{22} = [z_3\ z_4\ z_5]$ (in the geometric Jacobian), wrist singularities correspond to when z_3, z_4 and z_5 become linearly dependent vectors

  $\Rightarrow$ when either $q_5 = 0$ or $q_5 = \pm\pi/2$
- inversion of J is simpler (block triangular structure)
- the determinant of J will never depend on q_1: why?