determinant leads to finding the same singular configurations, which are relative to different values of the third joint variables, though — compare (2.66) and (2.70).

Finally, it is important to remark that, unlike the wrist singularities, the arm singularities are well identified in the operational space, and thus they can be suitably avoided in the end-effector trajectory planning stage.

3.4 Analysis of Redundancy

The concept of *kinematic redundancy* has been introduced in Sect. 2.10.2; redundancy is related to the number n of DOFs of the structure, the number m of operational space variables, and the number r of operational space variables necessary to specify a given task.

In order to perform a systematic analysis of redundancy, it is worth considering differential kinematics in lieu of direct kinematics (2.82). To this end, (3.39) is to be interpreted as the differential kinematics mapping relating the n components of the joint velocity vector to the $r \leq m$ components of the velocity vector $\boldsymbol{v}_e$ of concern for the specific task. To clarify this point, consider the case of a 3-link planar arm; that is not intrinsically redundant ($n = m = 3$) and its Jacobian (3.35) has 3 null rows accordingly. If the task does not specify ω_z ($r = 2$), the arm becomes functionally redundant and the Jacobian to consider for redundancy analysis is the one in (3.36).

A different case is that of the anthropomorphic arm for which only position variables are of concern ($n = m = 3$). The relevant Jacobian is the one in (3.38). The arm is neither intrinsically redundant nor can become functionally redundant if it is assigned a planar task; in that case, indeed, the task would set constraints on the 3 components of end-effector linear velocity.

Therefore, the differential kinematics equation to consider can be formally written as in (3.39), i.e.,

$$\boldsymbol{v}_e = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}}, \tag{3.45}$$

where now $\boldsymbol{v}_e$ is meant to be the $(r \times 1)$ vector of end-effector velocity of concern for the specific task and $\boldsymbol{J}$ is the corresponding $(r \times n)$ Jacobian matrix that can be extracted from the geometric Jacobian; $\dot{\boldsymbol{q}}$ is the $(n \times 1)$ vector of joint velocities. If $r < n$, the manipulator is kinematically redundant and there exist $(n - r)$ *redundant DOFs*.

The Jacobian describes the linear mapping from the joint velocity space to the end-effector velocity space. In general, it is a function of the configuration. In the context of differential kinematics, however, the Jacobian has to be regarded as a constant matrix, since the instantaneous velocity mapping is of interest for a given posture. The mapping is schematically illustrated in Fig. 3.7 with a typical notation from set theory.

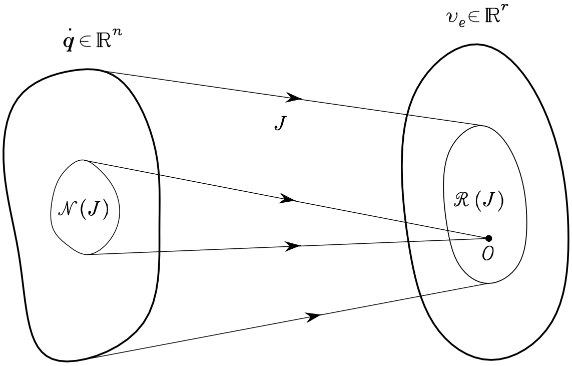

Fig. 3.7. Mapping between the joint velocity space and the end-effector velocity space

The differential kinematics equation in (3.45) can be characterized in terms of the *range* and *null* spaces of the mapping;[2] specifically, one has that:

- The *range* space of $\boldsymbol{J}$ is the subspace $\mathcal{R}(\boldsymbol{J})$ in $\mathbb{R}^r$ of the end-effector velocities that can be generated by the joint velocities, in the given manipulator posture.
- The *null* space of $\boldsymbol{J}$ is the subspace $\mathcal{N}(\boldsymbol{J})$ in $\mathbb{R}^n$ of joint velocities that do not produce any end-effector velocity, in the given manipulator posture.

If the Jacobian has *full rank*, one has

$$\dim\big(\mathcal{R}(\boldsymbol{J})\big) = r \qquad \dim\big(\mathcal{N}(\boldsymbol{J})\big) = n - r$$

and the range of $\boldsymbol{J}$ spans the entire space $\mathbb{R}^r$. Instead, if the Jacobian degenerates at a *singularity*, the dimension of the range space decreases while the dimension of the null space increases, since the following relation holds:

$$\dim\big(\mathcal{R}(\boldsymbol{J})\big) + \dim\big(\mathcal{N}(\boldsymbol{J})\big) = n$$

independently of the rank of the matrix $\boldsymbol{J}$.

The existence of a subspace $\mathcal{N}(\boldsymbol{J}) \neq \emptyset$ for a redundant manipulator allows determination of systematic techniques for handling redundant DOFs. To this end, if $\dot{\boldsymbol{q}}^*$ denotes a solution to (3.45) and $\boldsymbol{P}$ is an $(n \times n)$ matrix so that

$$\mathcal{R}(\boldsymbol{P}) \equiv \mathcal{N}(\boldsymbol{J}),$$

the joint velocity vector

$$\dot{\boldsymbol{q}} = \dot{\boldsymbol{q}}^* + \boldsymbol{P}\dot{\boldsymbol{q}}_0, \tag{3.46}$$

with arbitrary $\dot{\boldsymbol{q}}_0$, is also a solution to (3.45). In fact, premultiplying both sides of (3.46) by $\boldsymbol{J}$ yields

$$\boldsymbol{J}\dot{\boldsymbol{q}} = \boldsymbol{J}\dot{\boldsymbol{q}}^* + \boldsymbol{J}\boldsymbol{P}\dot{\boldsymbol{q}}_0 = \boldsymbol{J}\dot{\boldsymbol{q}}^* = \boldsymbol{v}_e$$

[2] See Sect. A.4 for the linear mappings.

since $\boldsymbol{J}\boldsymbol{P}\dot{\boldsymbol{q}}_0 = \mathbf{0}$ for any $\dot{\boldsymbol{q}}_0$. This result is of fundamental importance for redundancy resolution; a solution of the kind (3.46) points out the possibility of choosing the vector of arbitrary joint velocities $\dot{\boldsymbol{q}}_0$ so as to exploit advantageously the redundant DOFs. In fact, the effect of $\dot{\boldsymbol{q}}_0$ is to generate *internal motions* of the structure that do not change the end-effector position and orientation but may allow, for instance, manipulator reconfiguration into more dexterous postures for execution of a given task.

3.5 Inverse Differential Kinematics

In Sect. 2.12 it was shown how the inverse kinematics problem admits closed-form solutions only for manipulators having a simple kinematic structure. Problems arise whenever the end-effector attains a particular position and/or orientation in the operational space, or the structure is complex and it is not possible to relate the end-effector pose to different sets of joint variables, or else the manipulator is redundant. These limitations are caused by the highly nonlinear relationship between joint space variables and operational space variables.

On the other hand, the differential kinematics equation represents a linear mapping between the joint velocity space and the operational velocity space, although it varies with the current configuration. This fact suggests the possibility to utilize the differential kinematics equation to tackle the inverse kinematics problem.

Suppose that a motion trajectory is assigned to the end-effector in terms of $\boldsymbol{v}_e$ and the initial conditions on position and orientation. The aim is to determine a feasible joint trajectory $(\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t))$ that reproduces the given trajectory.

By considering (3.45) with $n = r$, the joint velocities can be obtained via simple inversion of the Jacobian matrix

$$\dot{\boldsymbol{q}} = \boldsymbol{J}^{-1}(\boldsymbol{q})\boldsymbol{v}_e. \tag{3.47}$$

If the initial manipulator posture $\boldsymbol{q}(0)$ is known, joint positions can be computed by integrating velocities over time, i.e.,

$$\boldsymbol{q}(t) = \int_0^t \dot{\boldsymbol{q}}(\varsigma)d\varsigma + \boldsymbol{q}(0).$$

The integration can be performed in discrete time by resorting to numerical techniques. The simplest technique is based on the Euler integration method; given an integration interval Δt, if the joint positions and velocities at time t_k are known, the joint positions at time $t_{k+1} = t_k + \Delta t$ can be computed as

$$\boldsymbol{q}(t_{k+1}) = \boldsymbol{q}(t_k) + \dot{\boldsymbol{q}}(t_k)\Delta t. \tag{3.48}$$

This technique for inverting kinematics is independent of the solvability of the kinematic structure. Nonetheless, it is necessary that the *Jacobian* be *square* and of *full rank*; this demands further insight into the cases of *redundant* manipulators and kinematic *singularity* occurrence.

3.5.1 Redundant Manipulators

When the manipulator is *redundant* ($r < n$), the Jacobian matrix has more columns than rows and infinite solutions exist to (3.45). A viable solution method is to formulate the problem as a constrained linear optimization problem.

In detail, once the end-effector velocity $\boldsymbol{v}_e$ and Jacobian $\boldsymbol{J}$ are given (for a given configuration $\boldsymbol{q}$), it is desired to find the solutions $\dot{\boldsymbol{q}}$ that satisfy the linear equation in (3.45) and *minimize* the quadratic cost functional of joint velocities[3]

$$g(\dot{\boldsymbol{q}}) = \frac{1}{2}\dot{\boldsymbol{q}}^T \boldsymbol{W} \dot{\boldsymbol{q}}$$

where $\boldsymbol{W}$ is a suitable $(n \times n)$ symmetric positive definite weighting matrix.

This problem can be solved with the *method of Lagrange multipliers*. Consider the modified cost functional

$$g(\dot{\boldsymbol{q}}, \boldsymbol{\lambda}) = \frac{1}{2}\dot{\boldsymbol{q}}^T \boldsymbol{W} \dot{\boldsymbol{q}} + \boldsymbol{\lambda}^T(\boldsymbol{v}_e - \boldsymbol{J}\dot{\boldsymbol{q}}),$$

where $\boldsymbol{\lambda}$ is an $(r \times 1)$ vector of unknown multipliers that allows the incorporation of the constraint (3.45) in the functional to minimize. The requested solution has to satisfy the necessary conditions:

$$\left(\frac{\partial g}{\partial \dot{\boldsymbol{q}}}\right)^T = \boldsymbol{0} \qquad \left(\frac{\partial g}{\partial \boldsymbol{\lambda}}\right)^T = \boldsymbol{0}.$$

From the first one, it is $\boldsymbol{W}\dot{\boldsymbol{q}} - \boldsymbol{J}^T\boldsymbol{\lambda} = \boldsymbol{0}$ and thus

$$\dot{\boldsymbol{q}} = \boldsymbol{W}^{-1}\boldsymbol{J}^T\boldsymbol{\lambda} \tag{3.49}$$

where the inverse of $\boldsymbol{W}$ exists. Notice that the solution (3.49) is a minimum, since $\partial^2 g/\partial \dot{\boldsymbol{q}}^2 = \boldsymbol{W}$ is positive definite. From the second condition above, the constraint

$$\boldsymbol{v}_e = \boldsymbol{J}\dot{\boldsymbol{q}}$$

is recovered. Combining the two conditions gives

$$\boldsymbol{v}_e = \boldsymbol{J}\boldsymbol{W}^{-1}\boldsymbol{J}^T\boldsymbol{\lambda};$$

under the assumption that $\boldsymbol{J}$ has full rank, $\boldsymbol{J}\boldsymbol{W}^{-1}\boldsymbol{J}^T$ is an $(r \times r)$ square matrix of rank r and thus can be inverted. Solving for $\boldsymbol{\lambda}$ yields

$$\boldsymbol{\lambda} = (\boldsymbol{J}\boldsymbol{W}^{-1}\boldsymbol{J}^T)^{-1}\boldsymbol{v}_e$$

[3] Quadratic forms and the relative operations are recalled in Sect. A.6.

which, substituted into (3.49), gives the sought optimal solution

$$\dot{\boldsymbol{q}} = \boldsymbol{W}^{-1}\boldsymbol{J}^T(\boldsymbol{J}\boldsymbol{W}^{-1}\boldsymbol{J}^T)^{-1}\boldsymbol{v}_e. \tag{3.50}$$

Premultiplying both sides of (3.50) by $\boldsymbol{J}$, it is easy to verify that this solution satisfies the differential kinematics equation in (3.45).

A particular case occurs when the weighting matrix $\boldsymbol{W}$ is the identity matrix $\boldsymbol{I}$ and the solution simplifies into

$$\dot{\boldsymbol{q}} = \boldsymbol{J}^{\dagger}\boldsymbol{v}_e; \tag{3.51}$$

the matrix

$$\boldsymbol{J}^{\dagger} = \boldsymbol{J}^T(\boldsymbol{J}\boldsymbol{J}^T)^{-1} \tag{3.52}$$

is the *right pseudo-inverse* of $\boldsymbol{J}$.[4] The obtained solution locally minimizes the norm of joint velocities.

It was pointed out above that if $\dot{\boldsymbol{q}}^*$ is a solution to (3.45), $\dot{\boldsymbol{q}}^* + \boldsymbol{P}\dot{\boldsymbol{q}}_0$ is also a solution, where $\dot{\boldsymbol{q}}_0$ is a vector of arbitrary joint velocities and $\boldsymbol{P}$ is a projector in the null space of $\boldsymbol{J}$. Therefore, in view of the presence of redundant DOFs, the solution (3.51) can be modified by the introduction of another term of the kind $\boldsymbol{P}\dot{\boldsymbol{q}}_0$. In particular, $\dot{\boldsymbol{q}}_0$ can be specified so as to satisfy an additional constraint to the problem.

In that case, it is necessary to consider a new cost functional in the form

$$g'(\dot{\boldsymbol{q}}) = \frac{1}{2}(\dot{\boldsymbol{q}} - \dot{\boldsymbol{q}}_0)^T(\dot{\boldsymbol{q}} - \dot{\boldsymbol{q}}_0);$$

this choice is aimed at minimizing the norm of vector $\dot{\boldsymbol{q}} - \dot{\boldsymbol{q}}_0$; in other words, solutions are sought which satisfy the constraint (3.45) and are as close as possible to $\dot{\boldsymbol{q}}_0$. In this way, the objective specified through $\dot{\boldsymbol{q}}_0$ becomes unavoidably a secondary objective to satisfy with respect to the primary objective specified by the constraint (3.45).

Proceeding in a way similar to the above yields

$$g'(\dot{\boldsymbol{q}}, \boldsymbol{\lambda}) = \frac{1}{2}(\dot{\boldsymbol{q}} - \dot{\boldsymbol{q}}_0)^T(\dot{\boldsymbol{q}} - \dot{\boldsymbol{q}}_0) + \boldsymbol{\lambda}^T(\boldsymbol{v}_e - \boldsymbol{J}\dot{\boldsymbol{q}});$$

from the first necessary condition it is

$$\dot{\boldsymbol{q}} = \boldsymbol{J}^T\boldsymbol{\lambda} + \dot{\boldsymbol{q}}_0 \tag{3.53}$$

which, substituted into (3.45), gives

$$\boldsymbol{\lambda} = (\boldsymbol{J}\boldsymbol{J}^T)^{-1}(\boldsymbol{v}_e - \boldsymbol{J}\dot{\boldsymbol{q}}_0).$$

Finally, substituting $\boldsymbol{\lambda}$ back in (3.53) gives

$$\dot{\boldsymbol{q}} = \boldsymbol{J}^{\dagger}\boldsymbol{v}_e + (\boldsymbol{I}_n - \boldsymbol{J}^{\dagger}\boldsymbol{J})\dot{\boldsymbol{q}}_0. \tag{3.54}$$

[4] See Sect. A.7 for the definition of the pseudo-inverse of a matrix.

As can be easily recognized, the obtained solution is composed of two terms. The first is relative to minimum norm joint velocities. The second, termed *homogeneous solution*, attempts to satisfy the additional constraint to specify via $\dot{\boldsymbol{q}}_0$;[5] the matrix $(\boldsymbol{I} - \boldsymbol{J}^\dagger \boldsymbol{J})$ is one of those matrices $\boldsymbol{P}$ introduced in (3.46) which allows the projection of the vector $\dot{\boldsymbol{q}}_0$ in the null space of $\boldsymbol{J}$, so as not to violate the constraint (3.45). A direct consequence is that, in the case $\boldsymbol{v}_e = \boldsymbol{0}$, is is possible to generate *internal motions* described by $(\boldsymbol{I} - \boldsymbol{J}^\dagger \boldsymbol{J})\dot{\boldsymbol{q}}_0$ that reconfigure the manipulator structure without changing the end-effector position and orientation.

Finally, it is worth discussing the way to specify the vector $\dot{\boldsymbol{q}}_0$ for a convenient utilization of redundant DOFs. A typical choice is

$$\dot{\boldsymbol{q}}_0 = k_0 \left(\frac{\partial w(\boldsymbol{q})}{\partial \boldsymbol{q}} \right)^T \tag{3.55}$$

where $k_0 > 0$ and $w(\boldsymbol{q})$ is a (secondary) objective function of the joint variables. Since the solution moves along the direction of the gradient of the objective function, it attempts to *maximize* it *locally* compatible to the primary objective (kinematic constraint). Typical objective functions are:

- The *manipulability measure*, defined as

$$w(\boldsymbol{q}) = \sqrt{\det\big(\boldsymbol{J}(\boldsymbol{q})\boldsymbol{J}^T(\boldsymbol{q})\big)} \tag{3.56}$$

  which vanishes at a singular configuration; thus, by maximizing this measure, redundancy is exploited to move away from singularities.[6]
- The *distance from mechanical joint limits*, defined as

$$w(\boldsymbol{q}) = -\frac{1}{2n} \sum_{i=1}^{n} \left(\frac{q_i - \bar{q}_i}{q_{iM} - q_{im}} \right)^2 \tag{3.57}$$

  where q_{iM} (q_{im}) denotes the maximum (minimum) joint limit and $\bar{q}_i$ the middle value of the joint range; thus, by maximizing this distance, redundancy is exploited to keep the joint variables as close as possible to the centre of their ranges.
- The *distance from an obstacle*, defined as

$$w(\boldsymbol{q}) = \min_{\boldsymbol{p},\boldsymbol{o}} \|\boldsymbol{p}(\boldsymbol{q}) - \boldsymbol{o}\| \tag{3.58}$$

  where $\boldsymbol{o}$ is the position vector of a suitable point on the obstacle (its centre, for instance, if the obstacle is modelled as a sphere) and $\boldsymbol{p}$ is the

[5] It should be recalled that the additional constraint has secondary priority with respect to the primary kinematic constraint.

[6] The manipulability measure is given by the product of the singular values of the Jacobian (see Problem 3.8).

position vector of a generic point along the structure; thus, by maximizing this distance, redundancy is exploited to avoid collision of the manipulator with an obstacle (see also Problem 3.9).[7]

3.5.2 Kinematic Singularities

Both solutions (3.47) and (3.51) can be computed only when the Jacobian has full rank. Hence, they become meaningless when the manipulator is at a singular configuration; in such a case, the system $\boldsymbol{v}_e = \boldsymbol{J}\dot{\boldsymbol{q}}$ contains linearly dependent equations.

It is possible to find a solution $\dot{\boldsymbol{q}}$ by extracting all the linearly independent equations only if $\boldsymbol{v}_e \in \mathcal{R}(\boldsymbol{J})$. The occurrence of this situation means that the assigned path is physically executable by the manipulator, even though it is at a singular configuration. If instead $\boldsymbol{v}_e \notin \mathcal{R}(\boldsymbol{J})$, the system of equations has no solution; this means that the operational space path cannot be executed by the manipulator at the given posture.

It is important to underline that the inversion of the Jacobian can represent a serious inconvenience not only at a singularity but also in the neighbourhood of a singularity. For instance, for the Jacobian inverse it is well known that its computation requires the computation of the determinant; in the neighbourhood of a singularity, the determinant takes on a relatively small value which can cause large joint velocities (see point **c)** in Sect. 3.3). Consider again the above example of the shoulder singularity for the anthropomorphic arm. If a path is assigned to the end-effector which passes nearby the base rotation axis (geometric locus of singular configurations), the base joint is forced to make a rotation of about π in a relatively short time to allow the end-effector to keep tracking the imposed trajectory.

A more rigorous analysis of the solution features in the neighbourhood of singular configurations can be developed by resorting to the singular value decomposition (SVD) of matrix $\boldsymbol{J}$.[8]

An alternative solution overcoming the problem of inverting differential kinematics in the neighbourhood of a singularity is provided by the so-called *damped least-squares (DLS) inverse*

$$\boldsymbol{J}^{\star} = \boldsymbol{J}^T(\boldsymbol{J}\boldsymbol{J}^T + k^2\boldsymbol{I})^{-1} \tag{3.59}$$

where k is a damping factor that renders the inversion better conditioned from a numerical viewpoint. It can be shown that such a solution can be

[7] If an obstacle occurs along the end-effector path, it is opportune to invert the order of priority between the kinematic constraint and the additional constraint; in this way the obstacle may be avoided, but one gives up tracking the desired path.

[8] See Sect. A.8.

obtained by reformulating the problem in terms of the minimization of the cost functional

$$g''(\dot{\boldsymbol{q}}) = \frac{1}{2}(\boldsymbol{v}_e - \boldsymbol{J}\dot{\boldsymbol{q}})^T(\boldsymbol{v}_e - \boldsymbol{J}\dot{\boldsymbol{q}}) + \frac{1}{2}k^2\dot{\boldsymbol{q}}^T\dot{\boldsymbol{q}},$$

where the introduction of the first term allows a finite inversion error to be tolerated, with the advantage of norm-bounded velocities. The factor k establishes the relative weight between the two objectives, and there exist techniques for selecting optimal values for the damping factor (see Problem 3.10).

3.6 Analytical Jacobian

The above sections have shown the way to compute the end-effector velocity in terms of the velocity of the end-effector frame. The Jacobian is computed according to a *geometric technique* in which the contributions of each joint velocity to the components of end-effector linear and angular velocity are determined.

If the end-effector pose is specified in terms of a minimal number of parameters in the operational space as in (2.80), it is natural to ask whether it is possible to compute the Jacobian via differentiation of the direct kinematics function with respect to the joint variables. To this end, an *analytical technique* is presented below to compute the Jacobian, and the existing relationship between the two Jacobians is found.

The translational velocity of the end-effector frame can be expressed as the time derivative of vector $\boldsymbol{p}_e$, representing the origin of the end-effector frame with respect to the base frame, i.e.,

$$\dot{\boldsymbol{p}}_e = \frac{\partial \boldsymbol{p}_e}{\partial \boldsymbol{q}}\dot{\boldsymbol{q}} = \boldsymbol{J}_P(\boldsymbol{q})\dot{\boldsymbol{q}}. \tag{3.60}$$

For what concerns the rotational velocity of the end-effector frame, the minimal representation of orientation in terms of three variables $\boldsymbol{\phi}_e$ can be considered. Its time derivative $\dot{\boldsymbol{\phi}}_e$ in general differs from the angular velocity vector defined above. In any case, once the function $\boldsymbol{\phi}_e(\boldsymbol{q})$ is known, it is formally correct to consider the Jacobian obtained as

$$\dot{\boldsymbol{\phi}}_e = \frac{\partial \boldsymbol{\phi}_e}{\partial \boldsymbol{q}}\dot{\boldsymbol{q}} = \boldsymbol{J}_\phi(\boldsymbol{q})\dot{\boldsymbol{q}}. \tag{3.61}$$

Computing the Jacobian $\boldsymbol{J}_\phi(\boldsymbol{q})$ as $\partial\boldsymbol{\phi}_e/\partial\boldsymbol{q}$ is not straightforward, since the function $\boldsymbol{\phi}_e(\boldsymbol{q})$ is not usually available in direct form, but requires computation of the elements of the relative rotation matrix.

Upon these premises, the differential kinematics equation can be obtained as the time derivative of the direct kinematics equation in (2.82), i.e.,

$$\dot{\boldsymbol{x}}_e = \begin{bmatrix} \dot{\boldsymbol{p}}_e \\ \dot{\boldsymbol{\phi}}_e \end{bmatrix} = \begin{bmatrix} \boldsymbol{J}_P(\boldsymbol{q}) \\ \boldsymbol{J}_\phi(\boldsymbol{q}) \end{bmatrix} \dot{\boldsymbol{q}} = \boldsymbol{J}_A(\boldsymbol{q})\dot{\boldsymbol{q}} \tag{3.62}$$

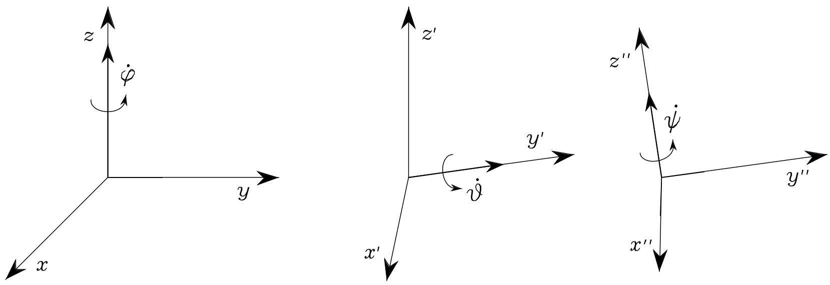

Fig. 3.8. Rotational velocities of Euler angles ZYZ in current frame

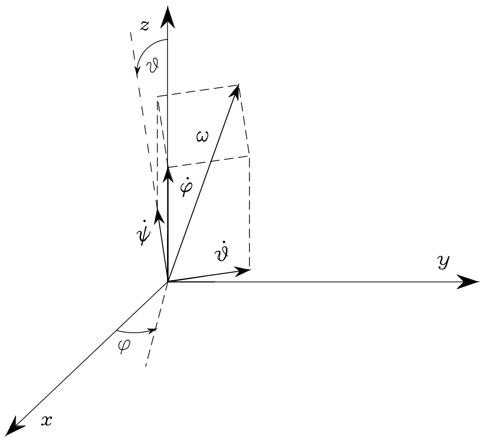

Fig. 3.9. Composition of elementary rotational velocities for computing angular velocity

where the *analytical Jacobian*

$$\boldsymbol{J}_A(\boldsymbol{q}) = \frac{\partial \boldsymbol{k}(\boldsymbol{q})}{\partial \boldsymbol{q}} \tag{3.63}$$

is different from the geometric Jacobian $\boldsymbol{J}$, since the end-effector angular velocity $\boldsymbol{\omega}_e$ with respect to the base frame is not given by $\dot{\boldsymbol{\phi}}_e$.

It is possible to find the relationship between the angular velocity $\boldsymbol{\omega}_e$ and the rotational velocity $\dot{\boldsymbol{\phi}}_e$ for a given set of orientation angles. For instance, consider the Euler angles ZYZ defined in Sect. 2.4.1; in Fig. 3.8, the vectors corresponding to the rotational velocities $\dot{\varphi}$, $\dot{\vartheta}$, $\dot{\psi}$ have been represented with reference to the current frame. Figure 3.9 illustrates how to compute the contributions of each rotational velocity to the components of angular velocity about the axes of the reference frame:

- as a result of $\dot{\varphi}$: $[\,\omega_x \quad \omega_y \quad \omega_z\,]^T = \dot{\varphi}\,[\,0 \quad 0 \quad 1\,]^T$
- as a result of $\dot{\vartheta}$: $[\,\omega_x \quad \omega_y \quad \omega_z\,]^T = \dot{\vartheta}\,[\,-s_{\varphi} \quad c_{\varphi} \quad 0\,]^T$

- as a result of $\dot{\psi}$: $\quad [\,\omega_x \quad \omega_y \quad \omega_z\,]^T = \dot{\psi}\,[\,c_\varphi s_\vartheta \quad s_\varphi s_\vartheta \quad c_\vartheta\,]^T$,

and then the equation relating the angular velocity $\boldsymbol{\omega}_e$ to the time derivative of the Euler angles $\dot{\boldsymbol{\phi}}_e$ is[9]

$$\boldsymbol{\omega}_e = \boldsymbol{T}(\boldsymbol{\phi}_e)\dot{\boldsymbol{\phi}}_e, \tag{3.64}$$

where, in this case,

$$\boldsymbol{T} = \begin{bmatrix} 0 & -s_\varphi & c_\varphi s_\vartheta \\ 0 & c_\varphi & s_\varphi s_\vartheta \\ 1 & 0 & c_\vartheta \end{bmatrix}.$$

The determinant of matrix $\boldsymbol{T}$ is $-s_\vartheta$, which implies that the relationship cannot be inverted for $\vartheta = 0, \pi$. This means that, even though all rotational velocities of the end-effector frame can be expressed by means of a suitable angular velocity vector $\boldsymbol{\omega}_e$, there exist angular velocities which cannot be expressed by means of $\dot{\boldsymbol{\phi}}_e$ when the orientation of the end-effector frame causes $s_\vartheta = 0$.[10] In fact, in this situation, the angular velocities that can be described by $\dot{\boldsymbol{\phi}}_e$ should have linearly dependent components in the directions orthogonal to axis z $(\omega_x^2 + \omega_y^2 = \dot{\vartheta}^2)$. An orientation for which the determinant of the transformation matrix vanishes is termed *representation singularity* of $\boldsymbol{\phi}_e$.

From a physical viewpoint, the meaning of $\boldsymbol{\omega}_e$ is more intuitive than that of $\dot{\boldsymbol{\phi}}_e$. The three components of $\boldsymbol{\omega}_e$ represent the components of angular velocity with respect to the base frame. Instead, the three elements of $\dot{\boldsymbol{\phi}}_e$ represent nonorthogonal components of angular velocity defined with respect to the axes of a frame that varies as the end-effector orientation varies. On the other hand, while the integral of $\dot{\boldsymbol{\phi}}_e$ over time gives $\boldsymbol{\phi}_e$, the integral of $\boldsymbol{\omega}_e$ does not admit a clear physical interpretation, as can be seen in the following example.

Example 3.3

Consider an object whose orientation with respect to a reference frame is known at time $t = 0$. Assign the following time profiles to $\boldsymbol{\omega}$:

- $\boldsymbol{\omega} = [\,\pi/2 \quad 0 \quad 0\,]^T \quad 0 \le t \le 1 \qquad \boldsymbol{\omega} = [\,0 \quad \pi/2 \quad 0\,]^T \quad 1 < t \le 2$,
- $\boldsymbol{\omega} = [\,0 \quad \pi/2 \quad 0\,]^T \quad 0 \le t \le 1 \qquad \boldsymbol{\omega} = [\,\pi/2 \quad 0 \quad 0\,]^T \quad 1 < t \le 2$.

The integral of $\boldsymbol{\omega}$ gives the same result in the two cases

$$\int_0^2 \boldsymbol{\omega}\,dt = [\,\pi/2 \quad \pi/2 \quad 0\,]^T$$

but the final object orientation corresponding to the second timing law is clearly different from the one obtained with the first timing law (Fig. 3.10).

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[9] This relation can also be obtained from the rotation matrix associated with the three angles (see Problem 3.11).

[10] In Sect. 2.4.1, it was shown that for this orientation the inverse solution of the Euler angles degenerates.

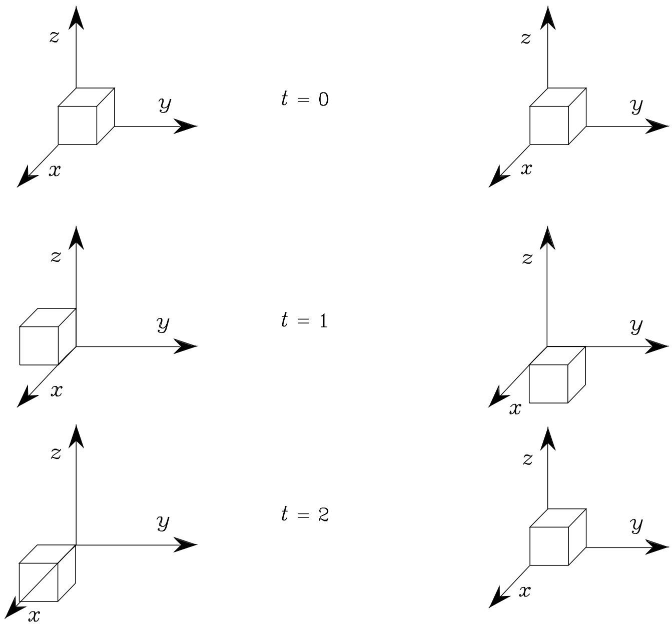

Fig. 3.10. Nonuniqueness of orientation computed as the integral of angular velocity

Once the transformation $\boldsymbol{T}$ between $\boldsymbol{\omega}_e$ and $\dot{\boldsymbol{\phi}}_e$ is given, the analytical Jacobian can be related to the geometric Jacobian as

$$\boldsymbol{v}_e = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{T}(\boldsymbol{\phi}_e) \end{bmatrix} \dot{\boldsymbol{x}}_e = \boldsymbol{T}_A(\boldsymbol{\phi}_e)\dot{\boldsymbol{x}}_e \tag{3.65}$$

which, in view of (3.4), (3.62), yields

$$\boldsymbol{J} = \boldsymbol{T}_A(\boldsymbol{\phi})\boldsymbol{J}_A. \tag{3.66}$$

This relationship shows that $\boldsymbol{J}$ and $\boldsymbol{J}_A$, in general, differ. Regarding the use of either one or the other in all those problems where the influence of the Jacobian matters, it is anticipated that the geometric Jacobian will be adopted whenever it is necessary to refer to quantities of clear physical meaning, while the analytical Jacobian will be adopted whenever it is necessary to refer to differential quantities of variables defined in the operational space.

For certain manipulator geometries, it is possible to establish a substantial equivalence between $\boldsymbol{J}$ and $\boldsymbol{J}_A$. In fact, when the DOFs cause rotations of the end-effector all about the same fixed axis in space, the two Jacobians are essentially the same. This is the case of the above three-link planar arm. Its geometric Jacobian (3.35) reveals that only rotations about axis z_0 are permitted. The (3×3) analytical Jacobian that can be derived by considering the end-effector position components in the plane of the structure and defining the end-effector orientation as $\phi = \vartheta_1 + \vartheta_2 + \vartheta_3$ coincides with the matrix that is obtained by eliminating the three null rows of the geometric Jacobian.

3.7 Inverse Kinematics Algorithms

In Sect. 3.5 it was shown how to invert kinematics by using the differential kinematics equation. In the numerical implementation of (3.48), computation of joint velocities is obtained by using the inverse of the Jacobian evaluated with the joint variables at the previous instant of time

$$\boldsymbol{q}(t_{k+1}) = \boldsymbol{q}(t_k) + \boldsymbol{J}^{-1}(\boldsymbol{q}(t_k))\boldsymbol{v}_e(t_k)\Delta t.$$

It follows that the computed joint velocities $\dot{\boldsymbol{q}}$ do not coincide with those satisfying (3.47) in the continuous time. Therefore, reconstruction of joint variables $\boldsymbol{q}$ is entrusted to a numerical integration which involves *drift* phenomena of the solution; as a consequence, the end-effector pose corresponding to the computed joint variables differs from the desired one.

This inconvenience can be overcome by resorting to a solution scheme that accounts for the *operational space error* between the desired and the actual end-effector position and orientation. Let

$$\boldsymbol{e} = \boldsymbol{x}_d - \boldsymbol{x}_e \tag{3.67}$$

be the expression of such error.

Consider the time derivative of (3.67), i.e.,

$$\dot{\boldsymbol{e}} = \dot{\boldsymbol{x}}_d - \dot{\boldsymbol{x}}_e \tag{3.68}$$

which, according to differential kinematics (3.62), can be written as

$$\dot{\boldsymbol{e}} = \dot{\boldsymbol{x}}_d - \boldsymbol{J}_A(\boldsymbol{q})\dot{\boldsymbol{q}}. \tag{3.69}$$

Notice in (3.69) that the use of operational space quantities has naturally lead to using the analytical Jacobian in lieu of the geometric Jacobian. For this equation to lead to an *inverse kinematics algorithm*, it is worth relating the computed joint velocity vector $\dot{\boldsymbol{q}}$ to the error $\boldsymbol{e}$ so that (3.69) gives a differential equation describing error evolution over time. Nonetheless, it is necessary to choose a relationship between $\dot{\boldsymbol{q}}$ and $\boldsymbol{e}$ that ensures convergence of the error to zero.

Having formulated inverse kinematics in algorithmic terms implies that the joint variables $\boldsymbol{q}$ corresponding to a given end-effector pose $\boldsymbol{x}_d$ are accurately computed only when the error $\boldsymbol{x}_d - \boldsymbol{k}(\boldsymbol{q})$ is reduced within a given threshold; such settling time depends on the dynamic characteristics of the error differential equation. The choice of $\dot{\boldsymbol{q}}$ as a function of $\boldsymbol{e}$ permits finding inverse kinematics algorithms with different features.