Robotics 1

Minimal representations of orientation
(Euler and roll-pitch-yaw angles)
Homogeneous transformations

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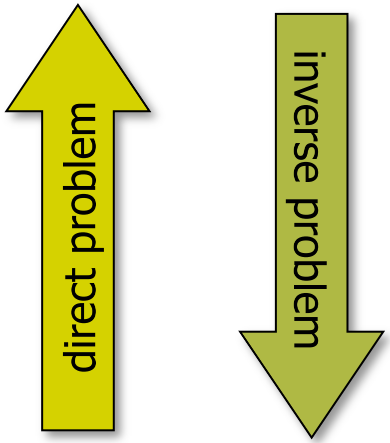
"Minimal" representations

- rotation matrices: $\quad$ 9 elements



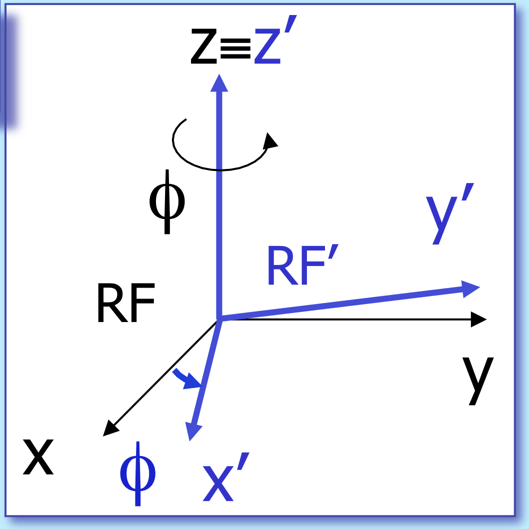
$$\begin{array}{rl} & 9 \text{ elements} \\ - & 3 \text{ orthogonality relationships} \\ - & 3 \text{ unitary relationships} \\ \hline = & 3 \text{ independent variables} \end{array}$$

- sequence of 3 rotations around independent axes
  - fixed (a_i) or moving/current (a'_i) axes
    - generically called Roll-Pitch-Yaw (fixed axes) or Euler (moving axes) angles
  - 12 + 12 possible different sequences (e.g., XYX)
  - actually, only 12 since

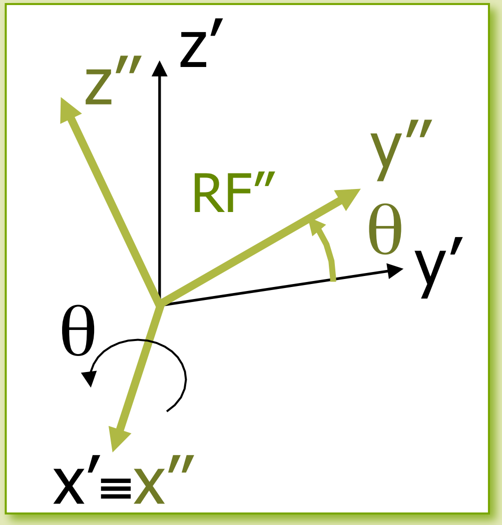
$$\{(a_1\ \alpha_1), (a_2\ \alpha_2), (a_3\ \alpha_3)\} \equiv \{\ (a'_3\ \alpha_3)\ , (a'_2\ \alpha_2), (a'_1\ \alpha_1)\}$$

ZX'Z" Euler angles

1



$$R_z(\phi) = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

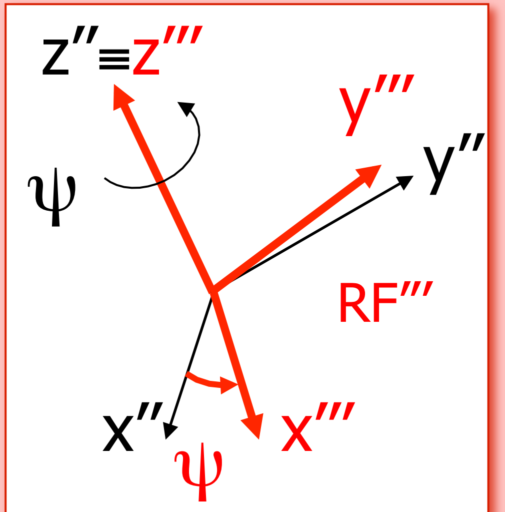


2

$$R_{x'}(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

3

$$R_{z''}(\psi) = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



ZX'Z" Euler angles

- direct problem: given ϕ , θ , ψ ; find R

$$R_{ZX'Z''}(\phi, \theta, \psi) = R_Z(\phi)\, R_{X'}(\theta)\, R_{Z''}(\psi)$$

order of definition in concatenation

$$= \begin{bmatrix} c\phi\, c\psi - s\phi\, c\theta\, s\psi & -c\phi\, s\psi - s\phi\, c\theta\, c\psi & s\phi\, s\theta \\ s\phi\, c\psi + c\phi\, c\theta\, s\psi & -s\phi\, s\psi + c\phi\, c\theta\, c\psi & -c\phi\, s\theta \\ s\theta\, s\psi & s\theta\, c\psi & c\theta \end{bmatrix}$$

- given a vector $v''' = (x''', y''', z''')$ expressed in RF''', its expression in the coordinates of RF is

$$v = R_{ZX'Z''}(\phi, \theta, \psi)\, v'''$$

- the orientation of RF''' is the same that would be obtained with the sequence of rotations:

ψ around z, θ around x (fixed), ϕ around z (fixed)

ZX'Z'' Euler angles

- inverse problem: given $R = \{r_{ij}\}$; find ϕ , θ , ψ

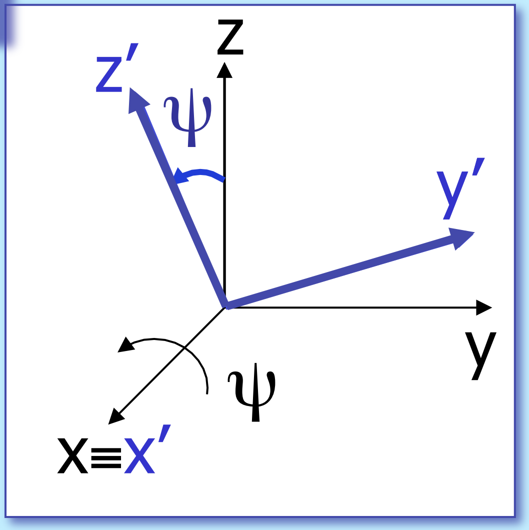
$$\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} c\phi\, c\psi - s\phi\, c\theta\, s\psi & -c\phi\, s\psi - s\phi\, c\theta\, c\psi & s\phi\, s\theta \\ s\phi\, c\psi + c\phi\, c\theta\, s\psi & -s\phi\, s\psi + c\phi\, c\theta\, c\psi & -c\phi\, s\theta \\ s\theta\, s\psi & s\theta\, c\psi & c\theta \end{bmatrix}$$

- $r_{13}^2 + r_{23}^2 = s^2\theta,\ \ r_{33} = c\theta \ \Rightarrow$ $\boxed{\theta = \text{ATAN2}\{\pm\sqrt{r_{13}^2 + r_{23}^2},\, r_{33}\}}$ two values differing just for the sign
- if $r_{13}^2 + r_{23}^2 \neq 0$ (i.e., $s\theta \neq 0$)

  $r_{31}/s\theta = s\psi\ ,\quad r_{32}/s\theta = c\psi \ \Rightarrow$ $\boxed{\psi = \text{ATAN2}\{r_{31}/s\theta,\, r_{32}/s\theta\}}$
- similarly... $\boxed{\phi = \text{ATAN2}\{r_{13}/s\theta,\, -r_{23}/s\theta\}}$
- there is always a pair of solutions
- there are always singularities (here $\theta = 0, \pm\pi$)

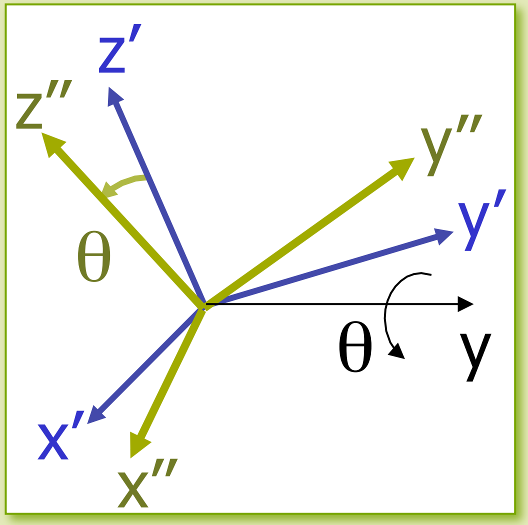
Roll-Pitch-Yaw angles

1 **ROLL**



$$R_X(\psi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & -\sin\psi \\ 0 & \sin\psi & \cos\psi \end{bmatrix}$$

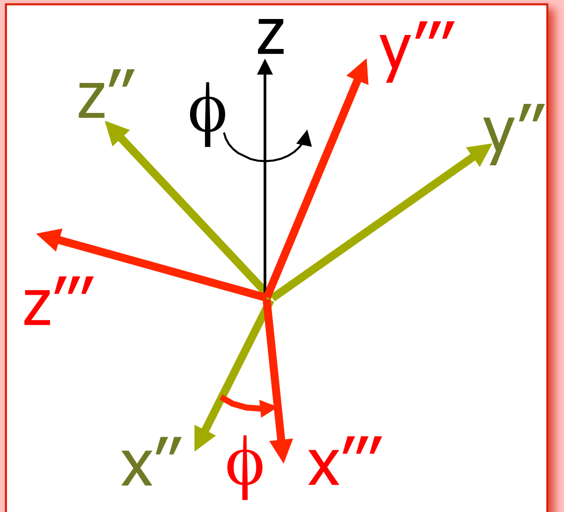
PITCH **2**



$C_1 R_Y(\theta) C_1^T$

with $R_Y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$

3 **YAW**



$C_2 R_Z(\phi) C_2^T$

with $R_Z(\phi) = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Roll-Pitch-Yaw angles (fixed XYZ)

- direct problem: given ψ , θ , ϕ ; find R

$R_{RPY}(\psi, \theta, \phi) = R_Z(\phi)\, R_Y(\theta)\, R_X(\psi)$ $\Leftarrow$ note the order of products!

order of definition

$$= \begin{bmatrix} c\phi\, c\theta & c\phi\, s\theta\, s\psi - s\phi\, c\psi & c\phi\, s\theta\, c\psi + s\phi\, s\psi \\ s\phi\, c\theta & s\phi\, s\theta\, s\psi + c\phi\, c\psi & s\phi\, s\theta\, c\psi - c\phi\, s\psi \\ -s\theta & c\theta\, s\psi & c\theta\, c\psi \end{bmatrix}$$

- inverse problem: given $R = \{r_{ij}\}$; find ψ , θ , ϕ
- $r_{32}^2 + r_{33}^2 = c^2\theta$, $r_{31} = -s\theta$ $\Rightarrow$ $\boxed{\theta = \mathrm{ATAN2}\{-r_{31}, \pm\sqrt{r_{32}^2 + r_{33}^2}\}}$

  for $r_{31}<0$, two symmetric values w.r.t. $\pi/2$
- if $r_{32}^2 + r_{33}^2 \neq 0$ (i.e., $c\theta \neq 0$)

  $r_{32}/c\theta = s\psi$, $r_{33}/c\theta = c\psi$ $\Rightarrow$ $\boxed{\psi = \mathrm{ATAN2}\{r_{32}/c\theta, r_{33}/c\theta\}}$
- similarly ... $\boxed{\phi = \mathrm{ATAN2}\{r_{21}/c\theta, r_{11}/c\theta\}}$
- singularities for $\theta = \pm\pi/2$

…why this order in the product?

$$R_{RPY}(\psi, \theta, \phi) = R_Z(\phi)\, R_Y(\theta)\, R_X(\psi)$$

order of definition

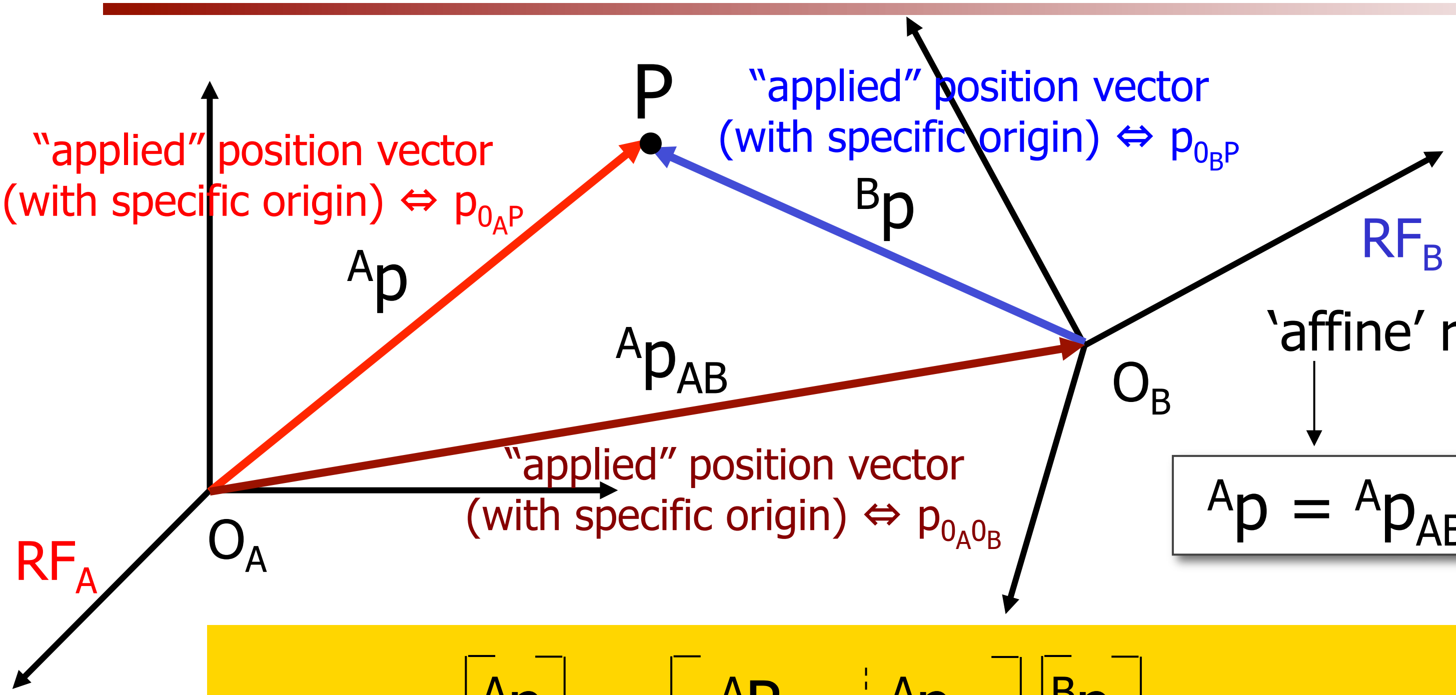
"reverse" order in the product
(pre-multiplication…)

- need to refer each rotation in the sequence to one of the original fixed axes
  - use of the angle/axis technique for each rotation in the sequence: $C\, R(\alpha)\, C^T$, with C being the rotation matrix reverting the previously made rotations (= go back to the original axes)

concatenating three rotations: [ ] [ ] [ ] (post-multiplication…)

$$\begin{aligned} R_{RPY}(\psi, \theta, \phi) &= [R_X(\psi)]\, [R_X^T(\psi)\, R_Y(\theta)\, R_X(\psi)] \\ &\quad [R_X^T(\psi)\, R_Y^T(\theta)\, R_Z(\phi)\, R_Y(\theta)\, R_X(\psi)] \\ &= R_Z(\phi)\, R_Y(\theta)\, R_X(\psi) \end{aligned}$$

Homogeneous transformations



'affine' relationship

$$\boxed{{}^A p = {}^A p_{AB} + {}^A R_B \, {}^B p}$$

$$ {}^A p_{hom} = \begin{bmatrix} {}^A p \\ \hline 1 \end{bmatrix} = \left[\begin{array}{ccc|c} & {}^A R_B & & {}^A p_{AB} \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \begin{bmatrix} {}^B p \\ \hline 1 \end{bmatrix} = {}^A T_B \, {}^B p_{hom} $$

linear relationship

vector in homogeneous coordinates

4x4 matrix of homogeneous transformation

Properties of T matrix

- describes the relation between reference frames (relative pose = position & orientation)
- transforms the representation of a position vector (applied vector starting from the origin of the frame) from a given frame to another frame
- it is a roto-translation operator on vectors in the three-dimensional space
- it is always invertible $({}^{A}T_{B})^{-1} = {}^{B}T_{A}$
- can be composed, i.e., ${}^{A}T_{C} = {}^{A}T_{B}\,{}^{B}T_{C}$ $\leftarrow$ note: it does not commute!

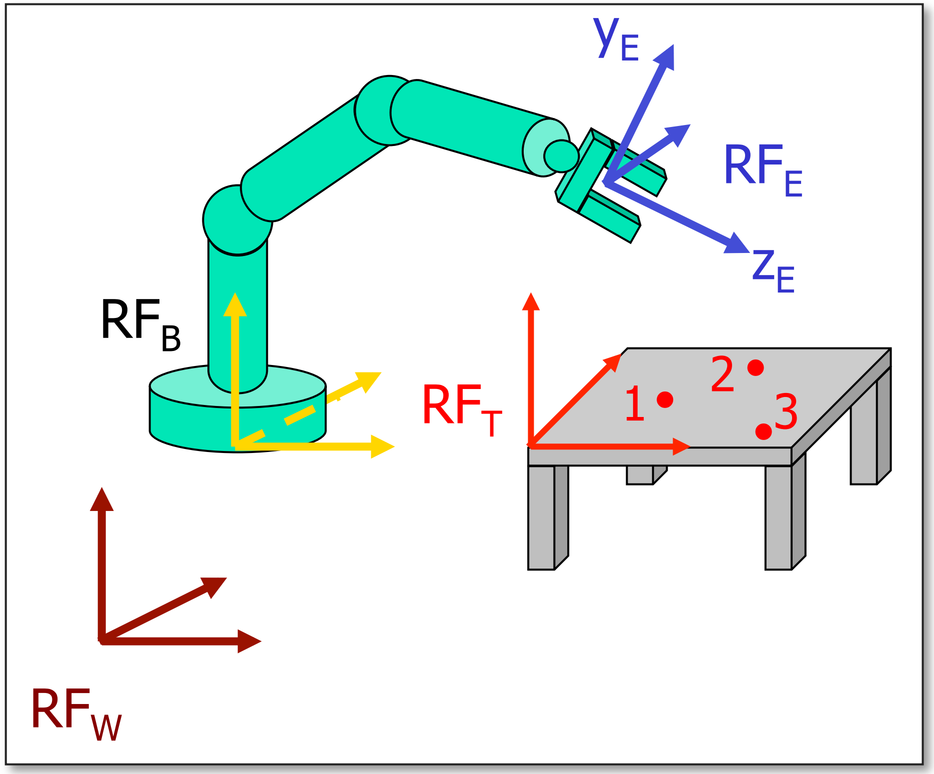
Inverse of a homogeneous transformation

$$ {}^A p = {}^A p_{AB} + {}^A R_B \, {}^B p \qquad {}^B p = {}^B p_{BA} + {}^B R_A \, {}^A p = - {}^A R_B^T \, {}^A p_{AB} + {}^A R_B^T \, {}^A p $$

$$ \left[\begin{array}{ccc|c} & {}^A R_B & & {}^A p_{AB} \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \qquad \left[\begin{array}{ccc|c} & {}^B R_A & & {}^B p_{BA} \\ \hline 0 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{ccc|c} & {}^A R_B^T & & - {}^A R_B^T \, {}^A p_{AB} \\ \hline 0 & 0 & 0 & 1 \end{array}\right] $$

$$ {}^A T_B \qquad {}^B T_A \qquad ({}^A T_B)^{-1} $$

Defining a robot task



absolute definition of task

task definition relative to the robot end-effector

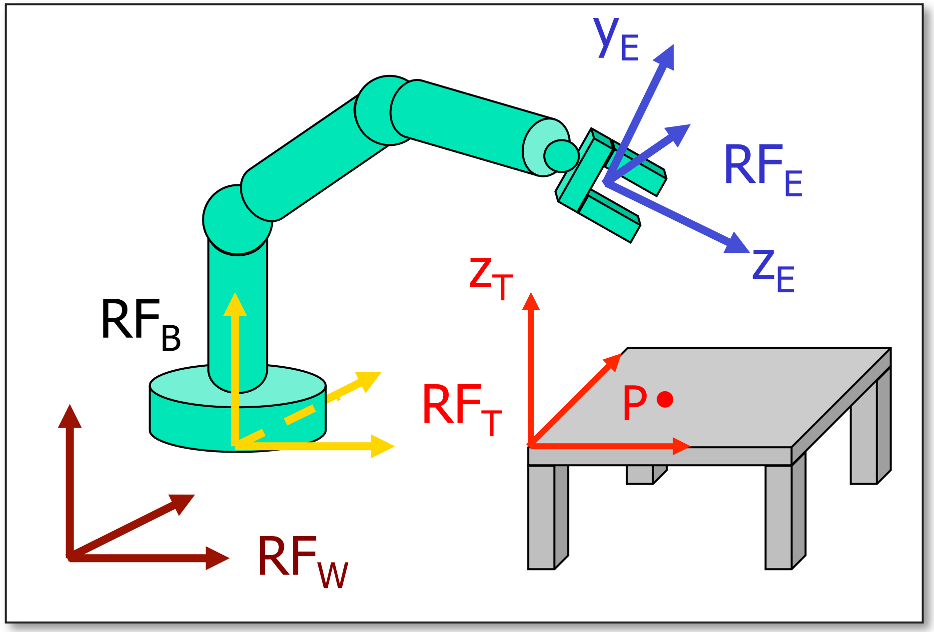
$$ {}^{W}T_T = {}^{W}T_B \, {}^{B}T_E \, {}^{E}T_T $$

known, once the robot is placed

direct kinematics of the robot arm (function of q)

$$ {}^{B}T_E(q) = {}^{W}T_B^{\ -1} \, {}^{W}T_T \, {}^{E}T_T^{\ -1} = \text{constant} $$

Example of task definition



- the robot carries a depth camera (e.g., a Kinect) on the end-effector
- the end-effector should go to a pose above the point P on the table, pointing its approach axis downward and being aligned with the table sides

$$ {}^{E}R_{T} = \begin{pmatrix} {}^{E}x_{T} & {}^{E}y_{T} & {}^{E}z_{T} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} $$

- point P is known in the table frame RF_T

$$ {}^{T}p = \begin{pmatrix} p_x \\ p_y \\ 0 \end{pmatrix} $$

- the depth camera proceeds centering point P in its image until it senses a distance h from the table (in RF_E)

$$ {}^{E}p = \begin{pmatrix} 0 \\ 0 \\ h \end{pmatrix} $$

$$ {}^{E}T_{T}^{-1} = {}^{T}T_{E} = \begin{pmatrix} {}^{T}R_{E} & {}^{T}p_{TE} \\ 0^{T} & 1 \end{pmatrix} $$

with $\quad {}^{T}R_{E} = \left({}^{E}R_{T}\right)^{T} = {}^{E}R_{T}$

$$ {}^{T}p_{TE} = {}^{T}p - {}^{T}R_{E}\,{}^{E}p = \begin{pmatrix} p_x \\ p_y \\ h \end{pmatrix} $$

Final comments on T matrices

- they are the main tool for computing the direct kinematics of robot manipulators
- they are used in many application areas (in robotics and beyond)
  - in positioning/orienting a vision camera (matrix ${}^{b}T_{c}$ with extrinsic parameters of the camera pose)
  - in computer graphics, for the real-time visualization of 3D solid objects when changing the observation point

$$
{}^{A}T_{B} = \left[\begin{array}{ccc|c} & {}^{A}R_{B} & & {}^{A}p_{AB} \\ \hline \alpha_x & \alpha_y & \alpha_z & \sigma \end{array}\right]
$$

all zero in robotics

coefficients of perspective deformation

scaling coefficient

always unitary in robotics