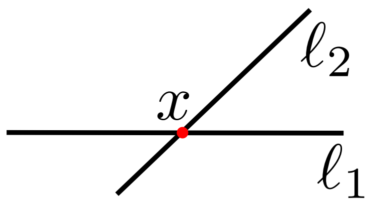
Homework assignment
Vision and Perception

Student: Artur Back de Luca
Professor: Fiora Pirri

April 19, 2020

Exercise 1. Compute the degenerate conic M formed by two lines and then its null space and the cross product of the null space

Consider the following two lines taken from one of the lecture's example:



$$\ell_1 = (0.2500, 3.2000, 1.0000)^\top$$

$$\ell_2 = (-2.0000, 0.5000, 1.0000)^\top$$

$$\mathrm{x} = \ell_1 \times \ell_2 = (-0.4138, 0.3448, 1.0000)^\top$$

$$M_1 = \ell_1 \ell_1^\intercal = \begin{bmatrix} 0.0625 & 0.8000 & 0.2500 \\ 0.8000 & 10.2400 & 3.2000 \\ 0.2500 & 3.2000 & 1.0000 \end{bmatrix}$$

Computing the degenerate conic for ℓ_2, we get:

$$M_2 = \ell_2 \ell_2^\intercal = \begin{bmatrix} 4.0000 & -1.0000 & -2.0000 \\ -1.0000 & 0.2500 & 0.5000 \\ -2.0000 & 0.5000 & 1.0000 \end{bmatrix}.$$

The null space of M_1 and M_2 are the following:

$$\mathrm{Null}\left(M_1\right) = \mathrm{Span}\left\{ \begin{pmatrix} 0.2210 \\ 0.2751 \\ -0.9357 \end{pmatrix}, \begin{pmatrix} -0.9724 \\ 0.1353 \\ -0.1899 \end{pmatrix} \right\} \quad \text{and} \quad \mathrm{Null}\left(M_2\right) = \mathrm{Span}\left\{ \begin{pmatrix} 0 \\ 0.8944 \\ -0.4472 \end{pmatrix}, \begin{pmatrix} -0.4880 \\ -0.3904 \\ -0.7807 \end{pmatrix} \right\}$$

Calculating the cross-product between the null-space components of M_1:

$$(0.2210, 0.2751, -0.9357) \times (-0.9724, 0.1353, -0.1899) = (0.0744, 0.9518, 0.2974)^\intercal$$

and for M_2 :

$$(0, 0.8944 - 0.4472) \times (-0.4880, -0.3904, -0.7807) = (-0.8728, 0.2182, 0.4365)^\intercal$$

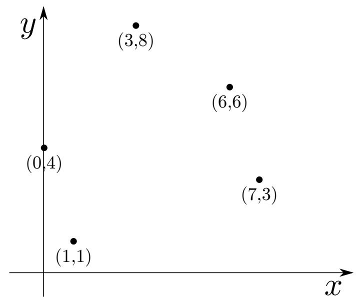
These two results are no more than the original lines, ℓ_1 and ℓ_2 with unitary norm. If we mutliply these by the respective original norms, i.e. $\|\ell_1\| = 3.3619$ and $\|\ell_2\| = 2.2913$, we retrieve their original values.

Exercise 2. Compute a conic based on the points given

The equation of a conic is defined as follows:

$$a{x_i}^2 + bx_iy_i + c{y_i}^2 + dx_i + ey_i + f = 0,$$

thus to determine the parameters of the equation, we need at least five points. Consider the following:



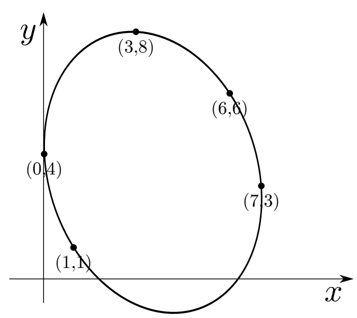
From this collection of points, we have the following set of equations:

$$\begin{bmatrix} 0 & 0 & 16 & 0 & 4 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 9 & 24 & 64 & 3 & 8 & 1 \\ 36 & 36 & 36 & 6 & 6 & 1 \\ 49 & 21 & 9 & 7 & 3 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \end{bmatrix} = 0$$

from which we obtain the following values:

$$a = 0.0876, \quad b = 0.0258, \quad c = 0.0532, \quad d = -0.7038, \quad e = -0.4629, \text{ and } f = 1,$$

thus giving us the following result:



Exercise 3. Compute the DLT (Direct Linear Transformation) algorithm on a distorted shape

Consider the following picture:

Our goal is to apply a homography so that the outline shape of bridge forms a rectangle. The homogeneous coordinates for the shape are:

$$A = \begin{bmatrix} 135.0 & 548.0 & 1.0 \\ 121.0 & 767.0 & 1.0 \\ 1580.0 & 712.0 & 1.0 \\ 1560.0 & 408.0 & 1.0 \end{bmatrix}$$

and our target shape has the coordinates:

$$B = \begin{bmatrix} 103.0 & 408.0 & 1.0 \\ 103.0 & 712.0 & 1.0 \\ 1580.0 & 712.0 & 1.0 \\ 1580.0 & 408.0 & 1.0 \end{bmatrix}$$

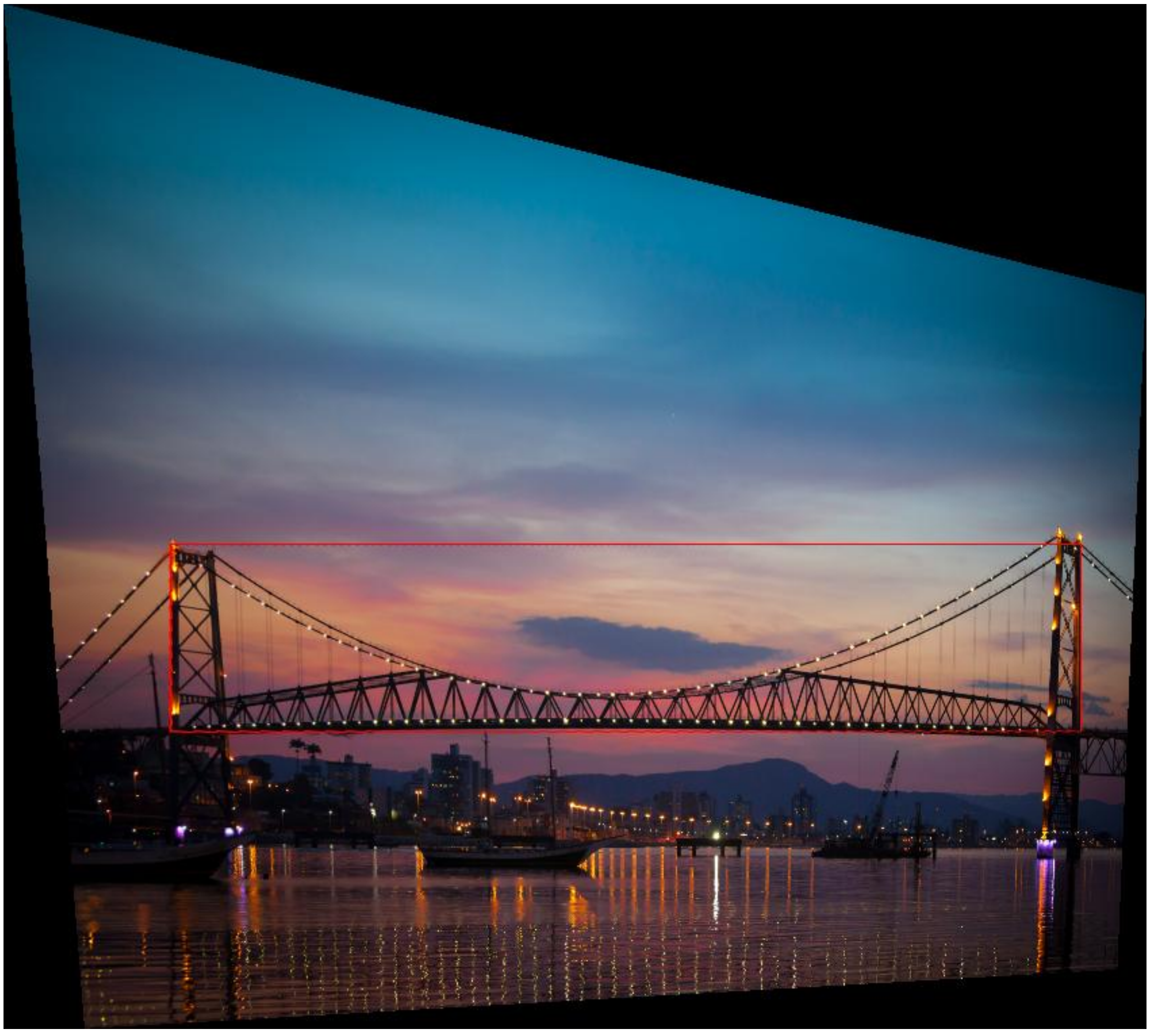
thus yielding the following equation:

$$B = AH$$

in which we ought to find the homography that produces such transformation. The matrix A is non-square, thus it cannot be inverted, we have to resort to the Direct Linear Transformation algorithm, in which at least three non-colinear points are used to find the 8 independent terms of the transformation matrix. Since our settings have four points, our system is overdetermined, from which we derive the following transformation matrix:

$$H = \begin{bmatrix} -0.667 & -0.0463 & 68.9 \\ -0.115 & -0.659 & 192.0 \\ 0 & 0 & -0.41 \end{bmatrix}$$

from which we finally obtain the following transformation:



Exercise 3b. Show that an affine transformation preserves parallel lines and areas.

Demonstration 1: parallel lines

Given the example of the lecture, we have:

$$\ell_1 = (0, 1, 1)^\top \quad \text{and} \quad \ell_2 = (0, 2.67, 1)^\top$$

These lines are parallel as they cross-product give us the following:

$$\ell_1 \times \ell_2 = (-1.67, 0, 0)^\top$$

Points of the two lines can be obtained as follows:

$$p_1, p_2 = \text{Null}(\ell_1)\,,\ p_1 = (-0.707, 0.500, -0.500)^\top \ \text{ and } p_2 = (-0.707, -0.500, 0.500)^\top$$

Conversely

$$q_1, q_2 = \text{Null}(\ell_2)\,,\ q_1 = (-0.936, 0.123, -0.328)^\top \ \text{ and } q_2 = (-0.351, -0.328, 0.877)^\top$$

Given the following transformation parameters: $\alpha = 0.7, s_1 = 1.4, s_2 = 0.9, t_x = t_y = 1.2$, we obtain the following affine transformation matrix:

$$H = \begin{bmatrix} 1.07080 & 0.0618 & 1.2000 \\ 0.9019 & 1.8825 & 1.20000 \\ 0 & 0 & 1.0000 \end{bmatrix}.$$

Applying the affine transformation onto the lines, we obtain:

$$\ell_1^{'} = \ell_1 H = (0.902, 1.88, 2.20) \ \text{ and } \ \ell_2^{'} = \ell_2 H = (2.40, 5.02, 4.20)\,,$$

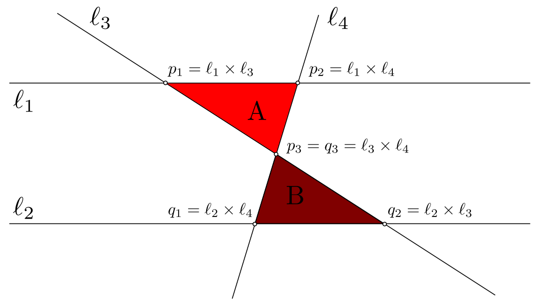
and so, we verify:

$$\ell_1^{'} \times \ell_2^{'} = (-3.14, 1.51, 0)^{\top}\,.$$

Given the last coordinate in 0, we can ascertain that the lines remain parallel after the transformation.

Demonstration 2: conservation of the ratio of areas

Given the the two triangles formed by an arbitrary collection of lines, ℓ_1,ℓ_2,ℓ_3 and ℓ_4:



where $\ell_1$and ℓ_2 are parallel. We can calculate the area S of the triangles A and B, using Heron's relation:

$$S = \sqrt{s(s-a)(s-b)(s-c)},$$

where s is the semi-perimeter of the triangle:

$$s = \frac{a+b+c}{2}.$$

For triangle A, the relation is as follows:

$$S_A = \sqrt{s_A(s_A - \overline{p_1p_3})(s_A - \overline{p_1p_2})(s_A - \overline{p_2p_3})},$$

where:

$$s_A = \frac{\overline{p_1p_3} + \overline{p_1p_2} + \overline{p_2p_3}}{2}.$$

As for triangle B, we have:

$$S_B = \sqrt{s_B(s_B - \overline{q_1q_3})(s_B - \overline{q_1q_2})(s_B - \overline{q_2q_3})},$$

and, conversely:

$$s_B = \frac{\overline{q_1q_3} + \overline{q_1q_2} + \overline{q_2q_3}}{2}.$$

Hence, for both triangles, their respective areas can be simply described by the distance of the intersecting points. Given an arbitrarily affine transformation H, that can be described as:

$$H = \begin{bmatrix} A & t \\ \vec{0} & 1 \end{bmatrix}$$

Taking a line segment of each triangles, e.g. $\overline{p_1p_3}$ and $\overline{q_2q_3}$, we can devise the ratio $\frac{\overline{p_1p_3}}{\overline{q_2q_3}}$:

$$\frac{\overline{p_1p_3}}{\overline{q_2q_3}} = \frac{\sqrt{(\ell_3 \times \ell_4 - \ell_1 \times \ell_3)\,(\ell_3 \times \ell_4 - \ell_1 \times \ell_3)^{\top}}}{\sqrt{(\ell_3 \times \ell_4 - \ell_2 \times \ell_3)\,(\ell_3 \times \ell_4 - \ell_2 \times \ell_3)^{\top}}}$$

Now we analyze the ratio's behavior under the transformation H:

$$\frac{\overline{p_1p_3}'}{\overline{q_2q_3}'} = \frac{\sqrt{(H\ell_3 \times H\ell_4 - H\ell_1 \times H\ell_3)\,(H\ell_3 \times H\ell_4 - H\ell_1 \times H\ell_3)^{\top}}}{\sqrt{(H\ell_3 \times H\ell_4 - H\ell_2 \times H\ell_3)\,(H\ell_3 \times H\ell_4 - H\ell_2 \times H\ell_3)^{\top}}}$$

Using the distributive property of the cross product, we have:

$$\frac{\overline{p_1p_3}'}{\overline{q_2q_3}'} = \frac{\cancel{\det(H)\left(H^{-1}\right)^\top}\sqrt{\left(\ell_3 \times \ell_4 - \ell_1 \times \ell_3\right)\left(\ell_3 \times \ell_4 - \ell_1 \times \ell_3\right)^\top}}{\cancel{\det(H)\left(H^{-1}\right)^\top}\sqrt{\left(\ell_3 \times \ell_4 - \ell_2 \times \ell_3\right)\left(\ell_3 \times \ell_4 - \ell_2 \times \ell_3\right)^\top}}$$

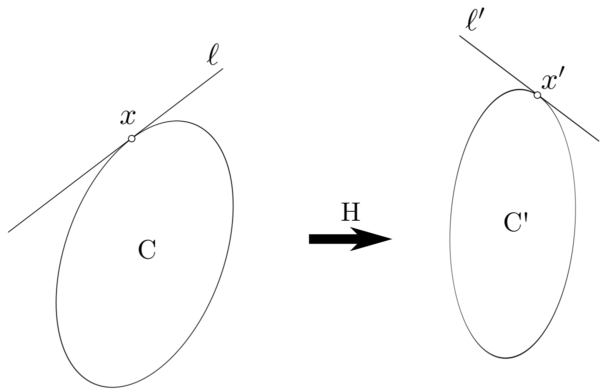
where the left-hand side of the equation give us:

$$\frac{\overline{p_1p_3}'}{\overline{q_2q_3}'} = \frac{\overline{p_1p_3}}{\overline{q_2q_3}}.$$

This relation can be naturally extended to any two line segments, thus the ratio between the areas of the triangle are also preserved.

Exercise 3c. Show homography keeps lines tangent to conics

Consider the following line ℓ, tangent to the conic C at the point x



From this property, we know that $\ell = Cx$ and $x^\top \ell = x^\top Cx = 0$. Using the homography H on x, we get $x' = Hx$, and to reverse this transformation, we can use the inverse homography H^{-1}. With the aforementioned properties, we can retwrite the equations like so:

$$x^\top Cx = \left(H^{-1}x'\right)^\top C\left(H^{-1}x'\right) = 0$$

this, in turn, is equilavent to:

$$x^\top Cx = x'\underbrace{\left(H^{-1}\right)^\top CH^{-1}}_{C'}x' = x'C'x' = 0.$$

From the relation $\ell = Cx$, we have:

$$\left(H^{-1}\right)^\top \ell = \left(H^{-1}\right)^\top Cx,$$

which can be rewritten to:

$$\left(H^{-1}\right)^\top \ell = \left(H^{-1}\right)^\top C\left(H^{-1}H\right)x.$$

As we have previously seen: $C' = \left(H^{-1}\right)^\top CH^{-1}$, and so the previous equation becomes:

$$\left(H^{-1}\right)^\top \ell = C'Hx,$$

thus finally arriving at:

$$\ell' = C'x'$$

which implies that ℓ' is also tangent to C'.

Exercise 4. Compute the Singular Value Decomposion for the given matrix

The matrix A which we ought to decompose is the following:

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 4 & -2 \end{bmatrix}$$

The singular value decomposition property proposes that any given matrix can be decomposed in three matrices, that is:

$$A = U\Sigma V^\top,$$

where Σ is a rectangular diagonal matrix and, U and $V^\top$ are orthonormal matrices. To obtain these matrices, we first compute the product $A^\top A$:

$$A^\top A = \left(U\Sigma V^\top\right)^\top U\Sigma V^\top = V\left(\Sigma^\top\Sigma\right)V^\top = \begin{bmatrix} 5 & 10 & 0 \\ 10 & 25 & -5 \\ 0 & -5 & 5 \end{bmatrix}$$

from this relation, we extract the eigenvalues of $A^\top A$ as follows:

$$\left(A^\top A - I\lambda\right).x = 0,$$

in which, in order to obtain a non-trivial solution, the following property must hold:

$$det\left(A^\top A - I\lambda\right) = 0,$$

which gives the following equation:

$$(5-\lambda)(25-\lambda)(5-\lambda) - 25(5-\lambda) - 100(5-\lambda) = 0.$$

From this equation, we arrive at the following solutions:

$$\lambda_1 = 30, \quad \lambda_2 = 5 \quad \text{and} \lambda_3 = 0.$$

To obtain the eingenvectors, we must find the vectors that lie in the null-space of the resulting matrices once each eigenvalue is substituted:
For $\lambda = 30$:

$$\begin{bmatrix} -25 & 10 & 0 \\ 10 & -5 & -5 \\ 0 & -5 & -25 \end{bmatrix}\vec{x} = 0,$$

Giving us the following expressions:

$$\begin{aligned} -25x_1 + 10x_2 &= 0 \\ 10x_1 - 5x_2 - 5x_3 &= 0 \\ -5x_2 - 25x_3 &= 0 \end{aligned}$$

from this linear set of equations, we arrive at the following solution:

$$\vec{x} = [-1, -2.5, 0.5]^\top,$$

which is then normalized:

$$\vec{x} = [-0.3651, -0.9129, 0.1826]^\top.$$

Repeating the same procedure for the other eigenvalues, we arrive at:

$$\vec{x} = [-0.4472, 0, -0.8944]^\top \quad \text{and} \quad \vec{x}{=}[0.8165, -0.4082, -0.4082]^\top.$$

Like so, we obtain the matrix V, compose of the aforementioned column vectors:

$$V = \begin{bmatrix} -0.3651 & -0.4472 & 0.8165 \\ -0.9129 & 0 & -0.4082 \\ 0.1826 & -0.8944 & -0.4082 \end{bmatrix},$$

furthermore, we also obtain the matrix Σ, which is the square root of the diagonal matrix composed of the calculated eigenvalues:

$$\Sigma = \begin{bmatrix} \sqrt{30} & 0 & 0 \\ 0 & \sqrt{5} & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

since the last row is a product of any above row, we can write:

$$\Sigma = \begin{bmatrix} \sqrt{30} & 0 & 0 \\ 0 & \sqrt{5} & 0 \end{bmatrix}.$$

The last step is to determine the orthonormal matrix U. This can be obtained by the following product:

$$AV = U\Sigma \cancel{V^\top V}.$$

Hence:

$$\begin{bmatrix} 2 & 3 & 1 \\ 1 & 4 & -2 \end{bmatrix} \begin{bmatrix} -0.3651 & -0.4472 & 0.8165 \\ -0.9129 & 0 & -0.4082 \\ 0.1826 & -0.8944 & -0.4082 \end{bmatrix} = U \begin{bmatrix} \sqrt{30} & 0 & 0 \\ 0 & \sqrt{5} & 0 \end{bmatrix}$$
$$\begin{bmatrix} -3.2863 & -1.7889 & 0 \\ -4.3818 & 1.3416 & 0 \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} \begin{bmatrix} \sqrt{30} & 0 & 0 \\ 0 & \sqrt{5} & 0 \end{bmatrix},$$

which give us the following matrix U:

$$U = \begin{bmatrix} -0.6 & -0.8 \\ -0.8 & 0.6 \end{bmatrix}$$

Exercise 6. Given the equation of a quadric find the projective transformation H mapping it into a diagonal form and determine the corresponding type of quadric.

The equation of the quadric is as follows:

$$4x_1^2 + 4x_1x_2 - 2x_1x_3 + 2x_1x_4 + 5{x_2}^2 - 2x_2x_4 + 2{x^2}_3 + 2x_3x_4 + 2x_4^2$$

The equation can be rewritten as:

$$X^T AX, \text{ where } X = (x_1, x_2, x_3, x_4)^\top,$$

where:

$$A = \begin{bmatrix} 4 & 2 & -1 & 1 \\ 2 & 5 & 0 & -1 \\ -1 & 0 & 2 & 1 \\ 1 & -1 & 1 & 2 \end{bmatrix}.$$

We now introduce a transformation H with has the following property:

$$A' = \left(H^{-1}\right)^\top AH^{-1} = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix},$$

Since we have the following property:

$$A = U^T DU,$$

in which U is an orthonormal matrix and D is a diagonal matrix containing the eigenvalues of A, we can conclude that the matrix H is equal to U^{-1}, so that the same properties can be obtained. The eigenvalues of matrix A are the following:

$$\lambda_1 = -0.0152 \qquad \lambda_2 = 2.7154 \qquad \lambda_3 = 3.6361 \qquad \lambda_4 = 6.6637.$$

From these, we obtain the following column vectors:

$$U = \begin{bmatrix} 0.4475 & 0.2008 & -0.6208 & -0.6115 \\ -0.3066 & -0.3472 & 0.4300 & -0.7749 \\ 0.5410 & -0.8281 & -0.0210 & 0.1453 \\ -0.6427 & -0.3917 & -0.6551 & 0.0662 \end{bmatrix},$$

finally, we conclude that matrix H is:

$$H = U^{-1} = \begin{bmatrix} 0.4475 & -0.3066 & 0.5410 & -0.6427 \\ 0.2008 & -0.3472 & -0.8281 & -0.3917 \\ -0.6208 & 0.4300 & -0.0210 & -0.6551 \\ -0.6115 & -0.7749 & 0.1453 & 0.0662 \end{bmatrix}$$

For the sake of confirmation, we get:

$$\left(H^{-1}\right)^\top AH^{-1} = \begin{bmatrix} -0.0152 & 0 & 0 & 0 \\ 0 & 2.7154 & 0 & 0 \\ 0 & 0 & 3.6361 & 0 \\ 0 & 0 & 0 & 6.6637 \end{bmatrix}.$$

The resulting quadric is a circular hyperboloid.

Exercise 6. Given the homogeneous coordinate system X_1,X_2,X_3, and X_4 define the pairs of point equations for directions of the axis and pairs of plane equations for the coordinate planes. Define the equation of a coninc resulting from the intersection of a plane with a quadric in $\mathbb{P}^3$.

For a given homongeous coordinate system in $\mathbb{P}^3$,we have the following set of point equations that define the axes:

$$x = (1,0,0,1)^\top \qquad y = (0,1,0,1)^\top \qquad z = (0,0,1,1)^\top$$

and the coordinate planes:

$$x\text{-}y = (0,0,1,1)^\top \qquad y - z = (1,0,0,1)^\top \qquad x\text{-}z = (0,1,0,1)^\top$$

Given the equation of the quadric used in the previous exercise:

$$4x_1^2 + 4x_1x_2 - 2x_1x_3 + 2x_1x_4 + 5{x_2}^2 - 2x_2x_4 + 2{x^2}_3 + 2x_3x_4 + 2x_4^2,$$

which can be rewritten as:

$$Q = \begin{bmatrix} 4 & 2 & -1 & 1 \\ 2 & 5 & 0 & -1 \\ -1 & 0 & 2 & 1 \\ 1 & -1 & 1 & 2 \end{bmatrix}$$

as well as the following coordinates of a plane:

$$\pi = (2,3,1,1)^\top$$

thus giving us the following null-space:

$$M_\pi = \begin{bmatrix} -0.775 & -0.258 & -0.258 \\ 0.604 & -0.132 & -0.132 \\ -0.132 & 0.956 & -0.044 \\ -0.132 & -0.044 & 0.956 \end{bmatrix}.$$

From the relation $C = M_\pi^\top Q M_\pi$, we obtain the relation for the conic:

$$C = \begin{bmatrix} 2.617 & 0.717 & -1.437 \\ 0.717 & 2.742 & 1.358 \\ -1.437 & 1.358 & 1.973 \end{bmatrix},$$

thus, the conic is represented as following:

$$C = 2.617x_1^2 + 0.358x_1x_2 + 2.742x_2^2 - 0.718x_1x_3 + 0.679x_2x_3 + 1.973x_3^2.$$