Vision and Perception

## Nijat Mursali

May 16, 2020

Abstract

This document is used to illustrate all the homework exercises that has been given during Vision and Perception course in 2020/21. There are overall 14 exercises that needed to be done during this time.

1 Homework 1 - Degenerate Conic

As mentioned during the video of Geometric Parameters, we needed to compute the M_2 as we did for M_1 and then computing the null space of M_1 and verifying that is 2, then computing the cross product of null space vectors and after normalizing to obtain the $x_3 = 1$, we needed to think about the result.

The idea is here space of M_1 and verifying that is 2, then computing the cross product of null space vectors and after normalizing to obtain the $x_3 = 1$, we needed to think about the result.
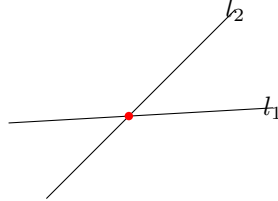
Thus, we need to consider the two intersecting lines that we have seen in our lecture:

$$l_1 = (0.2500, 3.200, 1.0000)^T \quad l_2 = (-2.0000, 0.5000, 1.0000)^T$$
$$x = l_1 \times l_2 = (-0.4138, 0.3448, 1.0000)^T$$

$$M_1 = l_1 l_1^T = \begin{pmatrix} 0.0625 & 0.8000 & 0.2500 \\ 0.8000 & 10.2400 & 3.2000 \\ 0.2500 & 3.2000 & 1.0000 \end{pmatrix}$$

Thus, by computing the degenerate conic for l_2, we could get something following:

$$M_2 = l_2 l_2^T = \begin{pmatrix} 4.0000 & -1.0000 & -2.0000 \\ -1.0000 & 0.2500 & 0.5000 \\ -2.0000 & 0.5000 & 1.0000 \end{pmatrix}$$

As we see from the book, the null space can be computed by span as following:

$$Null(M_1) = Span\{ \begin{pmatrix} 0.2210 \\ 0.2751 \\ -0.9357 \end{pmatrix}, \begin{pmatrix} -0.9724 \\ 0.1353 \\ -0.1899 \end{pmatrix} \}$$

and

$$Null(M_2) = Span\{ \begin{pmatrix} 0 \\ 0.8944 \\ -0.4472 \end{pmatrix}, \begin{pmatrix} -0.4880 \\ -0.3904 \\ -0.7807 \end{pmatrix} \}$$

Finally, we need to calculate the cross-product of the null-space components of M_1 as:

$M_1 = (0.2210, 0.2751, -0.9357) \times (-0.9724, 0.1353, -0.18, 99) = (0.0744, 0.9518, 0.2974)^T$

then, for M_2 it will be like:

$M_2 = (0, 0.8944 - 0.4472) \times (-0.4880, -0.3904, -0.7807) = (-.8728, 0.2182, 0.4365)^T$

We come up to the result that these two results are no more than the original lines, l_1 and l_2 with unitary norm.

2   Homework 2 - Five Points Define a Conic

As we know for each point the conic passes through
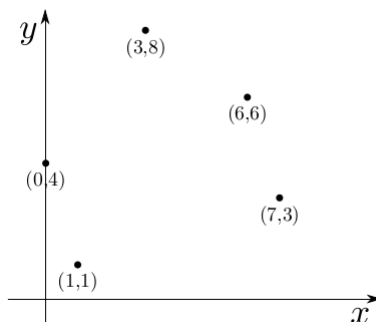$$ax_i^2 + bx_iy_i + cy_i^2 + dx_i + ey_i + f = 0$$

or
$$(x_i^2, x_iy_i, y_i^2, x_i, y_i, f)c = 0, \text{ where } c = (a, b, c, d, e, f)$$

stacking constraints yields
$$\begin{bmatrix} x_1^2 & x_1y_1 & y_1^2 & x_1 & y_1 & 1 \\ x_2^2 & x_2y_2 & y_2^2 & x_2 & y_2 & 1 \\ x_3^2 & x_3y_3 & y_3^2 & x_3 & y_3 & 1 \\ x_4^2 & x_4y_4 & y_4^2 & x_4 & y_4 & 1 \\ x_5^2 & x_5y_5 & y_5^2 & x_5 & y_5 & 1 \end{bmatrix} c = 0$$

thus to determine the parameters of the equation, we need at least five points. Consider the following:


As we see from the equation, we need to determine the parameters of the equation with at least five points. So, for this homework we just chose 4 points

in xy plane with different values, thus we have the following set of equations:

$$
\begin{bmatrix}
0 & 0 & 16 & 0 & 4 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
9 & 24 & 64 & 3 & 8 & 1 \\
36 & 36 & 36 & 6 & 6 & 1 \\
49 & 21 & 9 & 7 & 3 & 1
\end{bmatrix}
\begin{bmatrix}
a \\ b \\ c \\ d \\ e \\ f
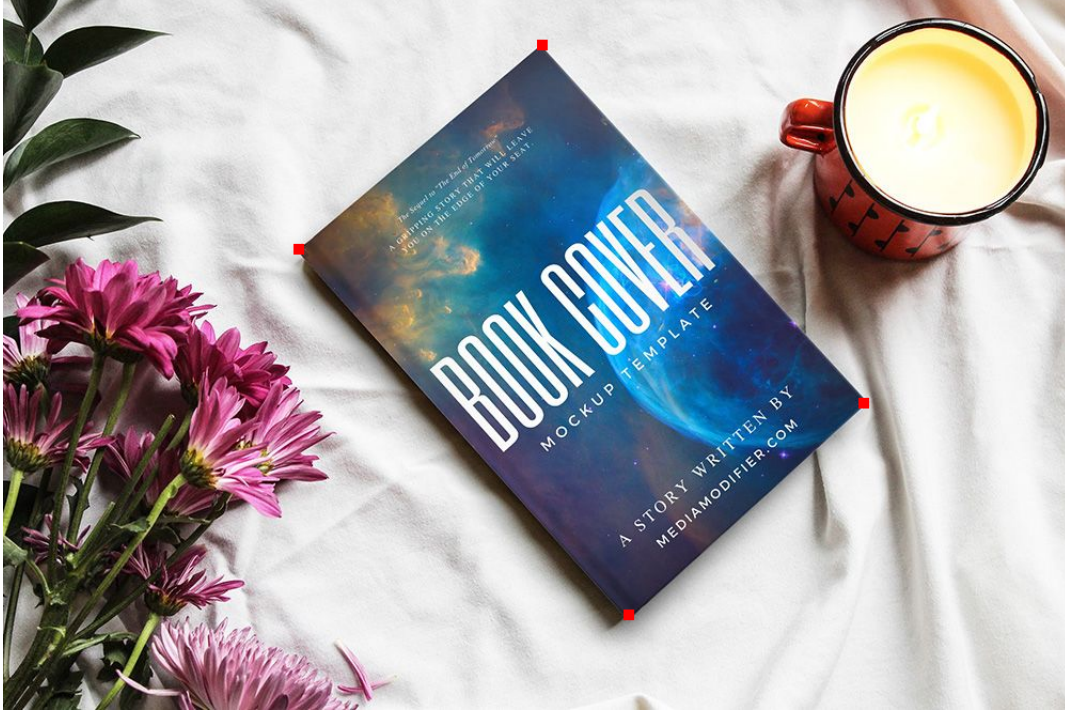\end{bmatrix} = 0
$$

from which we obtain the following values:

$a = 0.0876$, $b = 0.0258$, $c = 0.0532$, $d = -0.7038$, $e = -0.4629$, $f = 1$,

thus giving us following result:
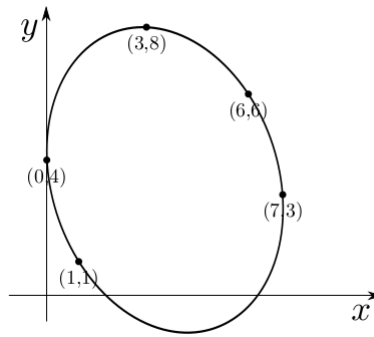
3 Homework 3 - Compute the DLT algorithm

In order to solve this problem, we needed to take one picture and apply the algorithm to check the coordinates of the shape. The idea is to pick 4 points on two different images by hand and apply the DLT algorithm showing the calculations. Additionally, the point selection is arbitrary.

Thus, we consider the following picture where we selected the four points as DLT algorithm stated in order to get 2D point correspondences $x_i \leftrightarrow x_i'$.

From the 2D to 2D point correspondences we need to determine 2D homography matrix H such that $x_i' = Hx_i$.

The fundamental goal of ours is to apply homography so that the outline shape of object forms a rectangle. The homogeneous coordinates for the shape are:

$$A = \begin{bmatrix} 281.0 & 234.0 & 1.0 \\ 587.0 & 580.0 & 1.0 \\ 808.0 & 375.0 & 1.0 \\ 506.0 & 45.0 & 1.0 \end{bmatrix}$$

and our target shape has the coordinates:

$$B = \begin{bmatrix} 172.0 & 347.0 & 1.0 \\ 566.0 & 580.0 & 1.0 \\ 763.0 & 340.0 & 1.0 \\ 336.0 & 166.0 & 1.0 \end{bmatrix}$$

thus yielding the following equation:

$$B = AH$$

in which we ought to find the homography that produces such transformation. The matrix A is non-square, thus it cannot be inverted, we have to resort to the Direct Linear Transformation algorithm, in which at least three non-co-linear points are used to find the 8 independent terms of the transformation matrix. Since our settings have four points, our system is over determined, from which we derive the following transformation matrix:

$$H = \begin{bmatrix} 9.7955 & 2.1074 & -1.4142 \\ -3.4637 & 7.1861 & 2.9439 \\ -4.2166 & -1.3235 & 1.000 \end{bmatrix}$$

from which we finally obtain the following transformation:
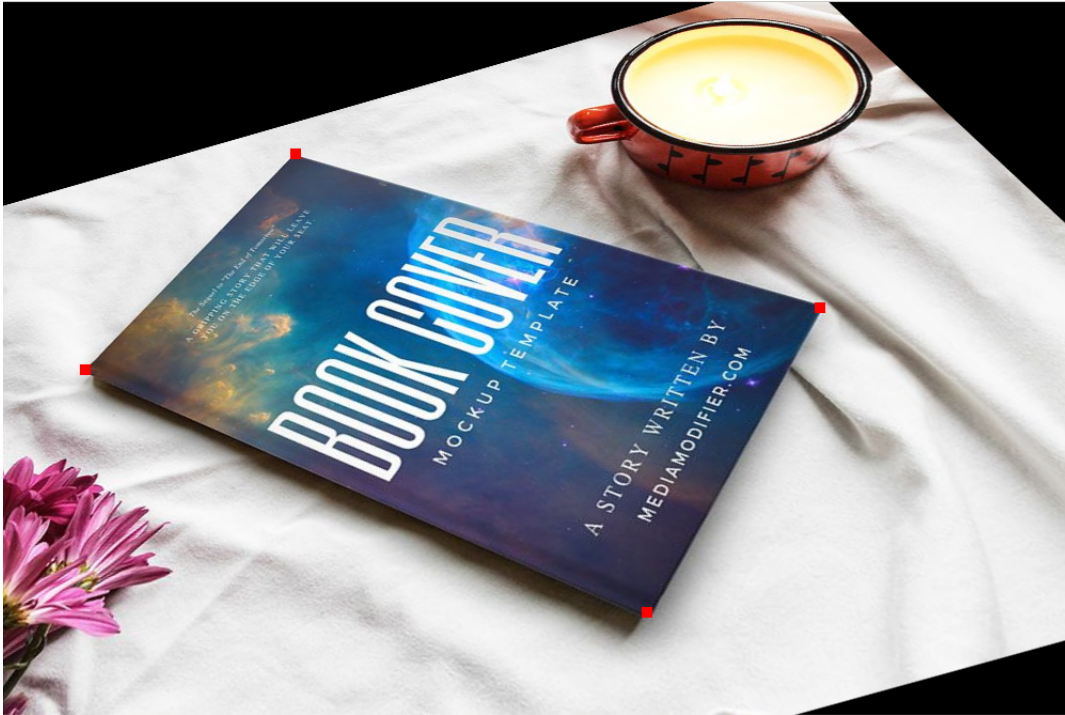
4 Homework 4 - Affine Transformations

The task for this exercise is to show that an affine transformation preserves both parallel lines and area. First of all let's describe what exactly affine transformation is. Affine transformation is a function mapping an affine space onto itself that preserves the dimension of any affine subspaces and also preserves the ratio of the lengths of parallel line segments. Additionally, those sets of parallel affine subspaces still remain the same parallel after an affine transformation and and in this exercise we needed to show that.

4.1 Affine transformation preserves parallel lines

As we have talked in the lecture, we have:

$$l_1 = (0, 1, 1)^T \text{ and } l_2 = (0, 2.45, 1)^T$$

These lines are parallel as they cross-product give us the following:

$$l_1 \times l_2 = (-1.45, 0, 0)^T$$

Points of the two lines can be obtained as follows:

$$p_1,\ p_2 = Null(l_1),$$

$$p_1 = (-0.707, 0.500, -0.500)^T \text{ and } p_2 = (-0.707, -0.500, 0.500)$$

Conversely,

$$q_1,\ q_2 = Null(l_2),$$

$$q_1 = (-0.936, 0.123, -0.328)^T \text{ and } q_2 = (-0.351, -0.328, 0.877)$$

Given the following transformation parameters: $\alpha = 0.7$, $s_1 = 1.4$, $s_2 = 0.9$, $t_x = t_y = 1.2$, we obtain the following affine transformation matrix:

$$H = \begin{bmatrix} 1.07080 & 0.0618 & 1.2000 \\ 0.9019 & 1.8825 & 1.2000 \\ 0 & 0 & 1.0000 \end{bmatrix}$$

Applying the affine transformation onto the lines, we obtain:

$$l_1' = l_1 H = (0.902, 1.88, 2.20) \text{ and } l_2' = l_2 H = (2.40, 5.02, 4.20)$$
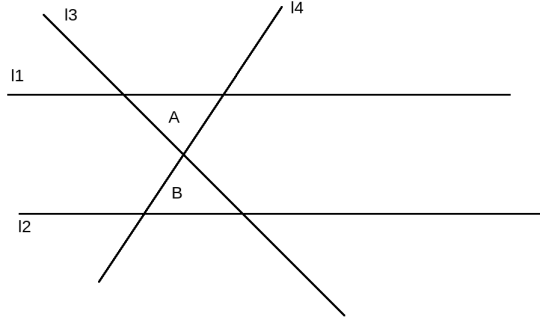
and so, we verify:

$$l_1' \times l_2' = (-3.14, 1.51, 0)^T$$

Given the last coordinate in 0, we can ascertain that the lines remain parallel after the transformation.

4.2 Affine transformation preserves area

Given the two triangles formed by an arbitrary collection of lines, l_1, l_2, l_3 and l_4:

where l_1 and l_2 are parallel. We also know from the first example that, we can calculate the p_i as $p_1 = l_1 \times l_3$, $p_2 = l_1 \times l_4$, $p_3 = l_3 \times l_4$ and also finding q_i will be as $q_1 = l_2 \times l_4$, $q_2 = l_2 \times l_3$. We can calculate the area S of the triangles A and B, using Heron's relation:

$$S_A = \sqrt{s(s-a)(s-b)(s-c)}$$

where s is the semi-perimeter of the triangle S.

For triangle A, the relation is as follows:

$$S_A = \sqrt{s_A(s_A - \overline{p_1 p_3})(s_A - \overline{p_1 p_2})(s_A - \overline{p_2 p_3})}$$

6

As for triangle B, we have:
$$S_B = \sqrt{s_B(s_B - \overline{p_1 p_3})(s_B - \overline{p_1 p_2})(s_B - \overline{p_2 p_3})}$$

Hence, for both triangles, their respective areas can be simply described by the distance of the intersecting points. Given an arbitrarily affine transformation H, that can be described as:
$$H = \left[\begin{array}{cc} A & t \\ \overrightarrow{0} & 1 \end{array} \right]$$

Taking a line segment of each triangles, we can get the ratio as:
$$\frac{\overline{p_1 p_3}}{\overline{q_2 q_3}} = \frac{\sqrt{(l_3 \times l_4 - l_1 \times l_3)(l_3 \times l_4 - l_1 \times l_3)}}{\sqrt{(l_3 \times l_4 - l_2 \times l_3)(l_3 \times l_4 - l_2 \times l_3)}}$$

If we analyze the ratio's behavior under the transformation H by simply multiplying them with H, we get the following equation:
$$\frac{\overline{p_1 p_3}}{\overline{q_2 q_3}}' = \frac{\overline{p_1 p_3}}{\overline{q_2 q_3}}$$

This relation can be naturally extended to any two line segments, thus the ratio between the areas of the triangle are also preserved.

5 Homework 5 - Homography keeps lines tangent to conics

Consider the following line l, tanget to the conic C at the point x,
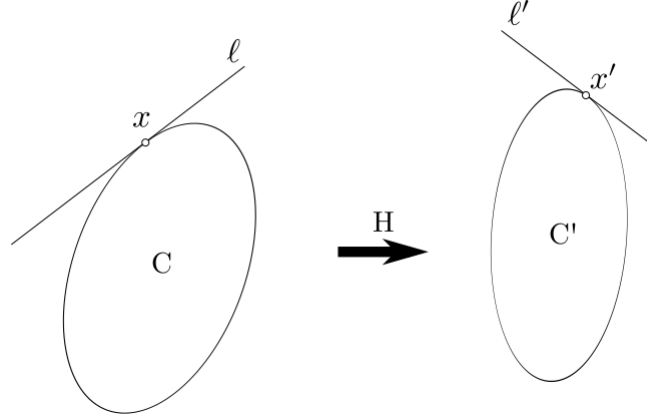
As we have seen in our class slides, we know that $l = Cx$ and $x^T l = x^T C x = 0$. For this exercise, we need to show that $X^T C X = 0$ that $l = Cx$. Additionally it was required to show the configuration $l^T x = 0$. Using the homography H on x, we get $x' = Hx$, and to reverse this transformation, we can use the inverse homography H^{-1}. With the aformentioned properties, we can rewrite the equations like so:
$$x^T C x = (H^{-1} x')^T C (H^{-1} x') = 0$$

this, in turn, is equivalent to :
$$x^T C x = x'(H^{-1})^T C H^{-1} x' = x' C' x' = 0.$$

From the relation $l = Cx$, we have:

$$(H^{-1})^T l = (H^{-1})^T Cx,$$

which can be rewritten to:

$$(H^{-1})^T l = (H^{-1})^T C(H^{-1}H)x.$$

As we have previously seen: $C' = (H^{-1})^T C H^{-1}$, and so the previous equation becomes:

$$(H^{-1})^T l = C' H x$$

which finally gives us:

$$l' = C'x'$$

which it shows from the equation that l' is also tangent to C'.

6 Homework 6 - SVD for given matrix

As we have seen from the example, the matrix A which we out to decompose is the following:

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 4 & -2 \end{bmatrix}$$

The singular value decomposition (SVD) property proposes that any given matrix can be decomposed in three matrices that is:

$$A = U\Sigma V^T$$

where Σ is a rectangular diagonal matrix and, U and V^T are orthogonal matrices. To obtain these matrices, we first compute the product $A^T A$ as following:

$$A^T A = (U\Sigma V^T)^T U\Sigma V^T = V(\Sigma^T \Sigma)V^T$$

which will be equal to

$$A^T A = \begin{bmatrix} 5 & 10 & 0 \\ 10 & 25 & -5 \\ 0 & -5 & 5 \end{bmatrix}$$

from this relation, we extract the eigenvalues of $A^T A$ as follows:

$$(A^T A - I\lambda)x = 0$$

in which, in order to obtain a non-trivial solution, the following property must hold:

$$det(A^T A - I\lambda) = 0$$

which gives the following equation:

$$(5 - \lambda)(25 - \lambda)(5 - \lambda) - 25(5 - \lambda) - 100(5 - \lambda) = 0$$

From this equation, we arrive at the following solutions:

$$\lambda_1 = 30, \lambda_2 = 5, \lambda_3 = 0$$

8

To obtain the eingenvectors, we must find the vectors that lie in the null-space of the resulting matrices once each eigenvalue is substituted:

For $\lambda = 30$:

$$\begin{bmatrix} -25 & 10 & 0 \\ 10 & -5 & -5 \\ 0 & -5 & -25 \end{bmatrix} x = 0,$$

Giving us the following expressions:

$$-25x_1 + 10x_2 = 0,$$
$$10x_1 - 5x_2 - 5x_3 = 0,$$
$$-5x_2 - 25x_3 = 0$$

from this linear set of equations, we arrive at the following solution:
$$x = [-1, -2.5, 0.5]^T,$$

which is then normalized:

$$x = [-0.3651, -0.9129, 0.1826]^T.$$

Repeating the same procedure for the other eigenvalues, we arrive at:

$$x = [-0.4472, 0, -0.8944]^T \text{ and } x = [0.8165, -0.4082, -0.4082]^T$$

Like so, we obtain the matrix V, compose of the column vectors:

$$V = \begin{bmatrix} -0.3651 & -0.4472 & 0.8165 \\ -0.9129 & 0 & -0.4082 \\ 0.1826 & -0.8944 & -0.4082 \end{bmatrix}$$

furthemore, we also obtain the matrix λ, which is the square root of the diagonal matrix composed of the calculated eigenvalues:

$$\Sigma = \begin{bmatrix} \sqrt{30} & 0 & 0 \\ 0 & \sqrt{5} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

since the last row is a product of any above row, we can write:

$$\Sigma = \begin{bmatrix} \sqrt{30} & 0 & 0 \\ 0 & \sqrt{5} & 0 \end{bmatrix}$$

The last step is to determine the orthonormal matrix U. This can be obtained by the following product:

$$AV = U\Sigma V^T V$$

where we cross out the $\Sigma V^T V$ that gives us only $AV = U$. Hence:

$$\begin{bmatrix} 2 & 3 & 1 \\ 1 & 4 & -2 \end{bmatrix} \begin{bmatrix} -0.3651 & -0.4472 & 0.8165 \\ -0.9129 & 0 & -0.4082 \\ 0.1826 & -0.8944 & -0.4082 \end{bmatrix} = U \begin{bmatrix} \sqrt{30} & 0 & 0 \\ 0 & \sqrt{5} & 0 \end{bmatrix}$$

which gives us the following matrix U:

$$U = \begin{bmatrix} -0.6 & -0.8 \\ -0.8 & 0.6 \end{bmatrix}$$

7 Homework 7 - Projective Transformation

The task here is to find the projective transformation H and define the type of quadric from the quadric equation.

Firstly, let's clarify some points we have learned in our lectures.

3D point in R^3 which is $X = (X, Y, Z)^T$ in $P^3 : (x_1, x_2, x_3, x_4)^T$,

Euclidean frame $\pi : ax + by + cz + d = 0$, where $\pi^T X = 0 => (abcd)^T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$

Thus,

$$X = (X_1, X_2, X_3, X_4) \text{ derives,}$$

$$X^T A X = 4x_1^2 + 4x_1x_2 - 2x_1x_3 + 2x_1x_4 + 5x_2^2 - 2x_2x_4 + 2x_3^2 + 2x_3x_4 + 2x_4^2 = 0$$

$$\textstyle\sum_{1 \leq i \leq j \leq 4}^{4} a_{ij}x_i y_i = 4 * x_1^2 + 2 * 2x_1x_2 +$$

$$+2 * (-1)x_1x_3 + 2 * (1)x_1x_4 + 5 * x_2^2 + 2(0)x_2x_3 + 2(-1)x_2x_4 +$$

$$+2 * x_3^2 + 2 * (-1)x_2x_4 + 2 * x_3^2 + 2 * (1)x_3x_4 + 2 * x_4 = 0$$

Then, we get the following A matrix,

$$A = a_i a_j = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{31} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} = \begin{bmatrix} 4 & 2 & -1 & 1 \\ 2 & 5 & 0 & -1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix} = 0$$

We also need to remember $Q = U^T D U$ which is equal to $H^T D H$ where U is orthogonal matrix and D is diagonal matrix.

Then from the following equation we are getting the λ values.

$$|\lambda I - A| = \begin{bmatrix} \lambda - 4 & -2 & 1 & -1 \\ -2 & \lambda - 5 & 0 & 1 \\ 1 & 0 & \lambda - 2 & -1 \\ -1 & 1 & -1 & \lambda - 2 \end{bmatrix} = 0$$

$$\lambda^4 - 13\lambda^3 + 52\lambda^2 - 81\lambda + 25 = 0,$$

$$\lambda_1 = 6.6637, \lambda_2 = 3.6360, \lambda_3 = 2.7153, \lambda_4 = -0.0151$$

Then, from the matrix we get the V values,

$$V_1 = \begin{bmatrix} -9.2387 \\ -11.7071 \\ 2.1953 \\ 1 \end{bmatrix}, V_2 = \begin{bmatrix} 0.9476 \\ -0.6564 \\ 0.0319 \\ 1 \end{bmatrix}, V_3 = \begin{bmatrix} -0.5125 \\ 0.9964 \\ 2.1143 \\ 1 \end{bmatrix}, V_4 = \begin{bmatrix} -0.6963 \\ 0.4770 \\ -0.8417 \\ 1 \end{bmatrix},$$

Then, from the formula of $U_i = \frac{AV_i}{\sigma_i}$, we tried to find the U value and got the following values:

$$U_1 = \frac{AV_i}{\sigma_i} = \begin{bmatrix} -23.8491 \\ -30.312 \\ 5.6681 \\ 2.5814 \end{bmatrix}, U_2 = \frac{AV_i}{\sigma_i} = \begin{bmatrix} 1.6564 \\ -1.2517 \\ 0.3620 \\ 2.0575 \end{bmatrix}$$

$$U_3 = \frac{AV_i}{\sigma_i} = \begin{bmatrix} -0.3444 \\ 1.4607 \\ 3.4840 \\ 1.6768 \end{bmatrix}, U_4 = \frac{AV_i}{\sigma_i} = \begin{bmatrix} -0.0856 \\ 0.0418 \\ -0.1043 \\ 0.1220 \end{bmatrix}$$

Thus, the final U will be:

$$U = \frac{AV_i}{\sigma_i} = \begin{bmatrix} 0.4475 & 0.2008 & 0.6208 & 0.6115 \\ 0.3066 & 0.3472 & 0.4300 & 0.7749 \\ 0.5410 & 0.8281 & 0.0210 & 0.1453 \\ 0.6427 & 0.3917 & 0.6551 & 0.0662 \end{bmatrix}$$

As we have found the U matrix, now we could get the homography H by finding the inverse of U which gives us:

$$H = U^{-1} = \frac{AV_i}{\sigma_i} = \begin{bmatrix} 0.4475 & 0.3066 & 0.5410 & 0.6427 \\ 0.2008 & 0.3472 & 0.8281 & 0.3917 \\ 0.6208 & 0.4300 & 0.0210 & 0.6551 \\ 0.6115 & 0.7749 & 0.1453 & 0.0662 \end{bmatrix}$$

Finally, we could find the class by calculating $(H^{-1})^T A H^{-1}$ which gives the following matrix:

$$(H^{-1})^T A H^{-1} = \begin{bmatrix} -0.0152 & 0 & 0 & 0 \\ 0 & 2.7154 & 0 & 0 \\ 0 & 0 & 3.6361 & 0 \\ 0 & 0 & 0 & 6.6637 \end{bmatrix}$$

As we know from our lectures, the result is circular hyperboloid.

8 Homework 8 - Point Equations

The task here is to define the pairs of point equations for the direction and pair of plane equations for coordinate planes. As we know that, in $P2$ the lines and points are dual, but in $P3$ two points have 6 degree of freedom which is represented by two 4-vectors. Thus, for a given homogeneous coordinate system in $P3$, we have the following set of point equations that define the axes:

As we have learned from the lecture

$$ax_1 + bx_2 + cx_3 + dx_4 = 0 \text{ gives } (a : b : c : d)$$

where

$$\pi^T X = 0 \text{ gives } (a, b, c, d)^T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$$

11

$$x = (1, 0, 0, 1)^T, \ y = (0, 1, 0, 1)^T \text{ and } z = (0, 0, 1, 1)^T$$

and the coordinate planes:

$$x - y = (0, 0, 1, 1)^T, \ y - z = (1, 0, 0, 1)^T \text{ and } x - z = (0, 1, 0, 1)^T$$

Given the equation of the quadric used in the previous exercise:

$$X^T A X = 4x_1^2 + 4x_1 x_2 - 2x_1 x_3 + 2x_1 x_4 + 5x_2^2 - 2x_2 x_4 + 2x_3^2 + 2x_3 x_4 + 2x_4^2$$

which can be rewritten as:

$$Q = \begin{bmatrix} 4 & 2 & -1 & 1 \\ 2 & 5 & 0 & -1 \\ -1 & 0 & 2 & 1 \\ 1 & -1 & 1 & 2 \end{bmatrix},$$

as well as the following coordinates of a plane:

$$\pi = (2, 3, 1, 1)^T$$

thus giving us the following null space:

$$M_\pi = \begin{bmatrix} 0.775 & 0.258 & 0.258 \\ 0.604 & -0.132 & -0.132 \\ -0.132 & 0.956 & -0.044 \\ -0.132 & -0.044 & 0.956 \end{bmatrix},$$

From the relation of $C = M_\pi^T Q M_\pi$, we obtain the relation for the conic:

$$C = \begin{bmatrix} 2.617 & 0.717 & 1.437 \\ 0.717 & 2.742 & 1.358 \\ -1.437 & 1.358 & 1.973 \end{bmatrix},$$

thus, the conic is represented as following:

$$C = 2.617x_1^2 + 0.358x_1 x_2 + 2.742x_2^2 - 0.718x_1 x_3 + 0.679x_2 x_3 + 1.973x_3^2$$

9 Homework 9 - Equation of Conic

First of all, as we know a quadric is a surface in $P3$ defined by the equation

$$X^T Q X = 0$$

where Q is a symmetric 4×4 matrix.
It's also clear that plane π can be found as following:

$$\pi : ax + by + cz + d = 0 \text{ and } n(x, y, z)^T + d = 0$$

where n is the normal to the plane computed by $n = (a, b, c)^T$ and a, b, c, d are the points.

The intersection of a plane π with a quadric Q is a conic. Computing the conic can be tricky because it requires a coordinate system for the plane. As we know, a coordinate system for the plane can be defined by complement space to π as

$$X = Mx.$$

Points on π are on Q if

$$X^T Q X = x^T M^T Q M x = 0$$

These points lie on a conic C, since $x^T C x = 0$ with $C = M^T Q M$.

10 Homework 10 - Proof for $cos(\theta)$

Points on the plane at infinity (π_∞), which may be written as $X_\infty = (d^T, 0)^T$ are mapped to the image plane by a general camera $P = CR[I|t]$ as

$$x = PX_\infty = CR[I|t](d^T, 0)^T = CRd$$

Thus, in this case $H = CR$ is the planar homography between (π_∞) and the image plane. Since the absolute conic (Ω_∞) is on (π_∞), we can compute its image as

$$\omega = (CC^T)^{-1} = C^{-T}C^1$$

We can simply prove this by $\omega = (CR)^{-T}I(CR)^{-1}$ which is lastly equal to $C^{-T}C^{-1}$.

Like (Ω_∞), ω is an imaginary point conic with no real points. It cannot really be observed in an image. ω is dependent only on the internal parameters of the camera and is independent of the camera's position or orientation. Thus, it follows from above that the angle between two rays is given by the simple equation:

$$cos\theta = \frac{x_1^T(C^{-T}C^{-1})x_2}{\sqrt{x_1^T(C^{-T}C^{-1})x_x}\sqrt{x_2^T(C^{-T}C^{-1})x_2}} = \frac{x_1^T\omega x_1}{\sqrt{x_1^T\omega x_1}\sqrt{x_2^T\omega x_2}}$$

The above expression is independent of the choice of the projective coordinate on the image. To see this consider any 2D projective transformation H. The points x_i are transformed to Hx_i, and ω transforms to $H^{-T}\omega H^{-1}$. Hence, the expression for $cos(\theta)$ is unchanged. Thus, it will still be valid for any projective frame.

11 Homework 11 - Euclidean Rotation

The idea here is that dual conic here the duality is between planes and points. Thus, Q_∞^* is made by planes tangent to (Ω_∞), therefore any plane that belongs to Q_∞^* envelope is tangent to Ω.

$$\pi = \Omega_\infty X = \pi^T Q_\infty^* = 0$$

$$Q_\infty^* = \begin{bmatrix} I & 0 \\ 0^T & 0 \end{bmatrix},$$

When we consider the Euclidean transformation represented by the matrix

$$H_E = \begin{bmatrix} R & 0 \\ 0^T & 1 \end{bmatrix} = \begin{bmatrix} cos\theta & -sin\theta & 0 & 0 \\ sin\theta & cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

This is a rotation by θ about the z-axis with a zero translation. Geometrically it is evident that the family of XY-planes orthogonal to the rotation axis are simply rotated about the Z-axis by this transformation.

This means that there is a pencil of fixed planes orthogonal to the Z-axis. The planes are fixed as sets, but not point-wise as any (finite) point (not on the axis) is rotated in horizontal circles by this Euclidean action. Algebraically, the fixed planes of H are the eigenvectors of H^T. We also need to mention that fixed planes are eigen-vectors of H_E^T which we need to find it in order to define the planes.

For defining planes, we have to find the eigenvectors of H_E^T by:

$$[H_E^T - \lambda I]v = 0 \text{ which gives } \begin{bmatrix} cos\theta - \lambda & sin\theta & 0 & 0 \\ -sin\theta & cos\theta - \lambda & 0 & 0 \\ 0 & 0 & 1 - \lambda & 0 \\ 0 & 0 & 0 & 1 - \lambda \end{bmatrix}$$

Thus, we have to find such λ that satisfies $det(H_E^T - \lambda I) = 0$

Corresponding eigenvectors of H_E^T are

$$E_1 = \begin{pmatrix} 1 \\ i \\ 0 \\ 0 \end{pmatrix}, E_2 = \begin{pmatrix} 1 \\ -i \\ 0 \\ 0 \end{pmatrix}, E_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, E_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

The eigenvectors E_3 and E_4 are degenerate. However, E_1 and E_2 doesn't correspond to real planes. Thus there is a pencil of fixed planes which is spanned by these eigenvectors (E_3 and E_4). The axis of this pencil is the line of intersection of the the planes (perpendicular to the Z-axis) with π_∞, and the pencil includes π_∞.

The example also illustrates the connection between the geometry of the projective plane, P_2, and projective 3-space, P_3. A plane π intersects π_∞ in a line which is the line at infinity, l_∞, of the plane π. A projective transformation of $P3$ induces a subordinate plane projective transformation on π.

To conclude the ideas, those planes are span of eigen-vectors and they will not be affected because of transformation (not translation, but rotation).

12 Homework 12 - Camera Centre

The camera centre C is the point for which $PC = 0$. Numerically, this right null-vector may be obtained from the SVD of R. Algebraically, the centre $C = (X, Y, Z, T)^T$, where

$$X = det([p_2, p_3, p_4]), \ Y = -det([p_1, p_3, p_4]),$$
$$Z = det([p_1, p_2, p_4]), \ T = -det([p_1, p_2, p_3])$$

and the P is
$$P = [M| - MC] = K[R| - RC]$$
We can easily find K nad R by decomposing M as $M = KR$ using the RQ decomposition. The matrix R gives the orientation of camera, whereas K is calibration matrix.
$$K = \begin{bmatrix} \alpha_x & s & x_0 \\ 0 & \alpha_y & y_0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } R|t = \begin{bmatrix} r_{11} & r_{12} & r_{13} & t_1 \\ r_{21} & r_{22} & r_{23} & t_2 \\ r_{31} & r_{32} & r_{33} & t_3 \end{bmatrix}$$
For our problem we have P as:
$$P =$$
Thus, when we decompose the matrix M as KR using QR-decomposition
$$P = [M| - MC] \text{ the centre } C = -R^T t$$
For our problem, M and MC equals
$$M = \begin{bmatrix} 0.6773 & 8.6606 & 7.9236 \\ -4.2765 & 5.4432 & 7.8011 \\ -0.91189 & 0.4426 & 10.4433 \end{bmatrix}, \ MC = \begin{bmatrix} -2000000.61370 \\ 1000 \\ 30.0518 \end{bmatrix}$$
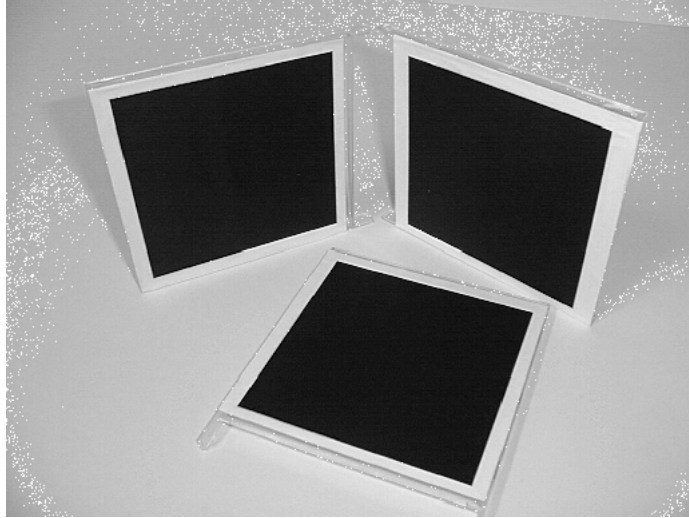Then, we can compute C by
We get C
$$C = \begin{bmatrix} 27964.6 & 19.410.2 & 1635.09 \end{bmatrix}^T$$
b) For this exercise, we needed to find the translation vector which can be calculated as
$$t = -RC, \text{ thus } t = \begin{bmatrix} 31751.7 \\ 19688.7 \\ -15127.2 \end{bmatrix}$$

13   Homework 13 - Calibration Device

13.1   Implement a example of a simple calibration device

The idea here is to take the image of three squares and compute the K. For this, we need to take several steps. As we know, for implementing the simple calibration device, we used to follow the steps that showed in the book. Firstly, we need to find the homography H for each square.

Then, we need to compute the imaged circular points for the plane of that square as $H(1, \pm i, 0)^T$.

Before the last step, we need to fit a conic ω to the six imaged circular points. The constraint that the imaged circular points lie on ω may be rewritten as two real contrains. If $h_1 \pm ih_2$ lies on ω then $(h_1 \pm ih_2)^T \omega() h_1 \pm ih_2) = 0$, and the imaginary and real parts give respectively:

$$h_1^T \omega h_2 = 0 \text{ and } h_1^T \omega h_1 = h_2^T \omega h_2$$

which are equations linear in ω.

Finally, we need to compute the calibration K from $\omega = (KK^T) - 1$ using the Cholesky factorization.

Thus, for the first step, we used OpenCV library in order to get the corners of the squares. The points are as following:

$$S_1 = \begin{bmatrix} 77.7704 & 75.5438 & 1.0 \\ 111.0461 & 208.2040 & 1.0 \\ 246.7895 & 165.1987 & 1.0 \\ 243.4158 & 38.8666 & 1.0 \end{bmatrix}, S_2 = \begin{bmatrix} 302.4889 & 42.3844 & 1.0 \\ 301.1318 & 164.1352 & 1.0 \\ 423.8295 & 229.9800 & 1.0 \\ 452.5454 & 100.7878 & 1.0 \end{bmatrix}$$

and the last square's coordinates are:

$$S_3 = \begin{bmatrix} 250.2883 & 197.7703 & 1.0 \\ 174.3199 & 164.1352 & 1.0 \\ 423.8295 & 229.9800 & 1.0 \\ 348.1797 & 362.9188 & 1.0 \end{bmatrix}$$

Then, we needed to find the homographies, and we have done it also using the library which gave overall three homographies.

$$H_1 = \begin{bmatrix} 1.09317181e - 03 & 4.96735473e - 03 & -4.61466910e - 01 \\ 4.28243203e - 03 & -1.13379462e - 03 & -2.42372406e - 01 \\ -8.34657587e - 04 & -1.04738101e - 03 & 1.00000000e + 00 \end{bmatrix}$$

$$H_2 = \begin{bmatrix} -4.38946955e - 03 & 1.28792782e - 02 & 7.44873229e - 01 \\ 1.49341334e - 02 & -1.30224096e - 04 & -4.48894349e + 00 \\ 3.27712794e - 03 & -2.49170706e - 03 & 1.00000000e + 00 \end{bmatrix}$$

$$H_3 = \begin{bmatrix} -4.73274525e - 03 & 1.87530587e - 02 & -2.50705370e + 00 \\ 1.22991572e - 02 & 8.54371755e - 03 & -4.48894349e + 00 \\ 2.36397229e - 04 & 4.24985472e - 03 & 1.00000000e + 00 \end{bmatrix}$$

Then, we had to compute the imaged circular points for the plane of that square as we mentioned above. Writing $H = [h_1, h_2, h_3]$, the imaged circular points are $h_1 + ih_2$ and $h_1 - ih_2$. From those equations, we have got the P matrices.

As the following step, we needed to fit a conic ω to the six imaged circular

points. The constraint that the imaged circular points lie on ω may be written as two real constraints. Then, we had to find the ω which we found by A matrix which is found by the columns of H matrix. We have found the ω as:

$$\omega = \begin{bmatrix} 0.00000000e + 00 \\ 7.07106781e - 01 \\ 1.11022302e - 16 \\ 1.00613962e - 16 \\ -7.07106781e - 01 \\ -5.32072216e - 16 \end{bmatrix}$$

Then, we have divided the ω with Numpy array into three conics and stored it into the array to find the Cholensky factorization. We have got the new matrix as following:

$$W = \begin{bmatrix} 5.15012402e - 01 & 7.07106781e - 01 & 1.00613962e - 16 \\ 7.07106781e - 01 & 1.11022302e - 16 & 2.55616781e - 01 \\ 1.00613962e - 16 & 5.42056781e - 05 & 3.45077216e - 16 \end{bmatrix}$$

Finally, using the Cholensky factorization (with *np.linalg*), we have computed the K as following:

$$K = \begin{bmatrix} 1046.5 & -9.5 & 505.2 \\ 0 & 1064.2 & 374.4 \\ 0 & 0 & 1 \end{bmatrix}$$

13.2   Implement Vanishing Points

Firstly, vanishing point is the image of a point at infinity. Since parallel lines intersect at infinity, the intersection of parallel lines in the image is the vanishing point. As a result, we can two parallel lines, and find the intersection in the image, and the intersection is one vanishing points. Repeat this process, we can find a second and third vanishing point.

As we have discussed in class, there can be several vanishing in an image, depending on where the images were taken. It's also dependent on how many objects you have in the image. In order to find the vanishing points, you need to have several steps in mind:

1. remember that all parallel lines will meet in a vanishing point
2. know which lines in the image are really parallel

For example, in the following picture we have found three vanishing points.
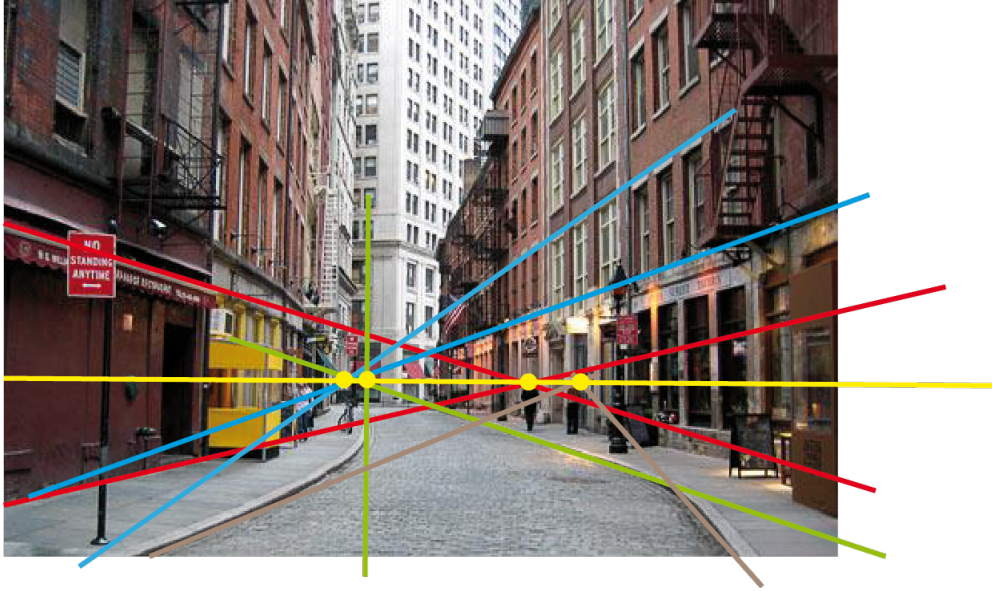
14 Homework 14 - Computing the DLT Algorithm

The task here was to build a reference shape for calibration as a check-board typically checker-board must not be square. One side must contain an even number of squares and the other side must contain an odd number of squares. We need to measure the dimension of the check-board square.

Using the harry corner detector, we first found the corners and added them into our code for further computation. For our example, we have taken overall 8 points as shown in the following matrix where first row shows the x axis and second row shows the y axis. As the checker-board is in 2D we don't care about z-axis. At this stage, we get the matrix as following:

$$
x = \begin{bmatrix} 454 & 663 & 238 & 658 & 237 & 25 & 442 & 445 \\ 236 & 442 & 1382 & 623 & 814 & 442 & 1317 & 412 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}
$$

When we normalize the matrices (by checking them with the ruler as mentioned in class), we get the following results:

$$
X = \begin{bmatrix} 6 & 9 & 3 & 9 & 3 & 0 & 6 & 6 \\ 3 & 6 & 18 & 9 & 12 & 6 & 18 & 6 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}
$$

The next step was to compute the A matrix using the points we have got, but before that we needed to get the correspondences of the coordination points. When, we computed them we also computed the A matrix which had overall 12

18

columns.

Then, we have tried to compute H by $H = AA^T$, as our A matrix had 12 columns, when we multiply it with it's transpose it will also give the same columns as A. After computing the H matrix, we could get the V by using eigen-vectors and after that we could get the P which is the calibration matrix. The following matrix is calibration matrix which is 3×4:

$$P = \begin{bmatrix} 65.4500 & 7.1226 & -14.6073 & 283.6928 \\ -15.0209 & 39.2664 & 64.1435 & -1175.6065 \\ 0.0094 & 0.0249 & -0.0506 & 1.7929 \end{bmatrix}$$

As we know from the $Exercise$ 12, we could find M (which is the first 3×3 matrix in P) and then we could get the R and K from the QR-decomposition of M. The following matrices are M, R accordingly.

$$M = \begin{bmatrix} 65.4500 & 7.1226 & -14.6073 \\ -15.0209 & 39.2664 & 64.1435 \\ 0.0094 & 0.0249 & -0.0506 \end{bmatrix}, R = \begin{bmatrix} -0.9746 & -0.2236 & -0.0002 \\ 0.2236 & -0.9746 & -0.0005 \\ -0.0001 & -0.0006 & 0.9999 \end{bmatrix}$$

Finally, we could find the K which is the following matrix:

$$K = \begin{bmatrix} -67.1516 & 1.84118 & 28.5852 \\ 0.00000 & -39.8647 & -59.2506 \\ 0.00000 & 0.00000 & -0.0839 \end{bmatrix},$$

Finally, it was asked to compute the algebraic and geometric errors and then comparing the results obtained by the DLT with the results obtained with the calibration using the absolute conic. In our case error decreased after every iteration and the table is as follows:

$iter$	$error$
1	1.53
2	1.10
...	...
8	0.35