- WMR's → motion control: Regulation
- Regulation:

In regulation we don't analyze the trajectory. Instead of the trajectory, we analyze the fixed target, which is desired configuration $q_d$.

We might solve this regulation by trajectory tracking methods by attaching trajectory for the purpose of having something to track while the robot converges to the goal. However this idea will not work. The 2 methods for trajectory tracking we've seen, do not allow trajectories that decay to a point. It is not working with those methods; because:

→ In state-error feedback method application, the trajectory must be persistent. Because, when cartesian motion stops ($v_d = 0$), then gain blows up ($k_2 \to \infty$) (check previous lecture).

→ In output-error feedback method, the trajectory that we are tracking ($p_d$) is not the contact point of vehicle, but it is the Point B (displaced along the sagittal axis). Thus, when the vehicle will stop we are not going to be sure where the real point that we need to track is.

Thus those 2 methods aren't eligible to be used for REGULATION.

On the other hand Wheeled Mobile Robots (WMR's) have such property that, they don't admit universal controllers (i.e. controllers that can stabilize arbitrary trajectories. It does not matter trajectory is persistent or not.). This property is caused by being nonholonomic.

Note 1: Manipulators admit universal controllers.

If we want to solve regulation or tracking problems, we have to use different (separate) controllers. Because, there isn't such controller that solve both problems at the same time.

Note 2: The main idea of Regulation is converging to the certain point.

- Cartesian regulation.

It is simplified posture regulation. In cartesian regulation we want to drive the unicycle to certain position without considering orientation.

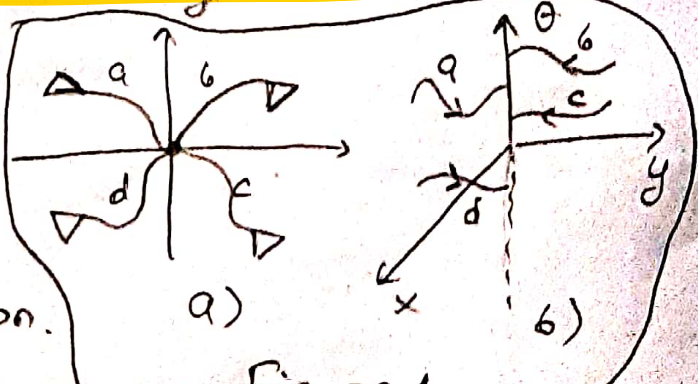



Figure 1.

This idea is demonstrated by the Figure 1.

Figure 1,a show 4 different arbitrary motions. The idea is, from any point in Cartesian space robot starts to move and converges to the origin of the Cartesian space. Notice that, in this case we are not interested in what is the last orientation of the robot. This case can be seen in Configuration space as demonstrated in figure 1,b. All corresponding trajectories are drawn in there, which converge to the $\theta$ axis. In other words, convergence to the origin of Cartesian space is important and the last orientation of robot does not matter for us (for now).

Notice that we analyzed the case by using origin as destination point, however it can be any point.

• Geometry of Cartesian regulation.

We will analyze 2 main approaches in order to analyze geometry of Cartesian regulation, which are DISTANCES and ANGLES.

→ DISTANCES :

We will use figure 2, in order to observe the geometry and will generate required expressions.

According to the Figure 2, we see that destination point is the origin of Configuration Space $O(0,0)$. $x$ and $y$ are corresponding Cartesian coordinates of contact point of Unicycle. Additionally, $\theta$ is the angle of sagittal axis with respect to $x$-axis of Cartesian Space. Then Cartesian error can be computed as below:

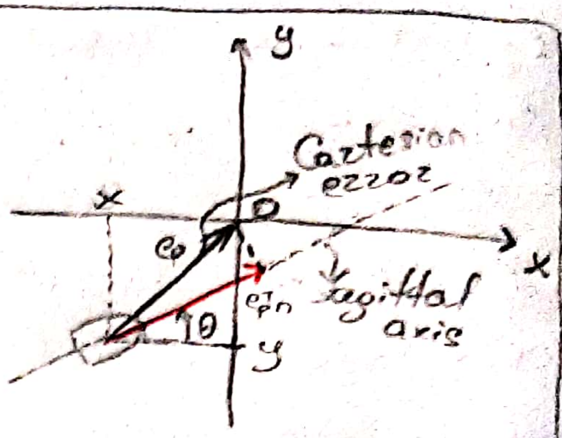



Figure 2.

$$e_p = P_d - P = \begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ -y \end{pmatrix} \quad (1)$$

Unit vector of sagittal axis can be given as below:

$$n = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \quad (2)$$

By using (1) and (2), we can compute the projection of $e_p$ on the sagittal axis (red in Figure 2).

$$e_p^T n = -x\cos\theta - y\sin\theta \quad (3)$$

→ ANGLES:

In order to analyze angles in geometry of Cartesian regulation we will use the Figure 3. We generate required expression according to the Figure 3.

According to the Figure 3, the angle between sagittal axis and x-axis of Cartesian space is $\theta$. On the other hand, the error vector (from contact point to the origin) makes $ATAN2(y,x)+\pi$ angle with x-axis. By using this information, we can compute the angle between error vector and sagittal axis, which is called as pointing error ($\gamma$):

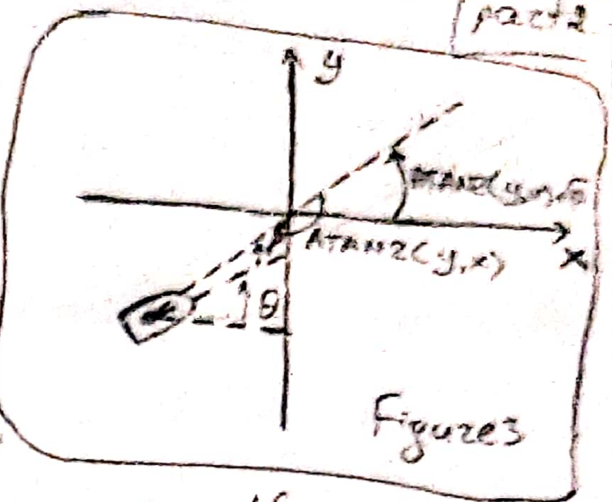



Figure3

$$\gamma = ATAN2(y,x) + \pi - \theta \quad (4)$$

→ At the next step we will consider feedback control law:

→ According to the geometric interpretation, we consider $v$ is proportional to the projection of the cartesian error $e_p$ on the sagittal axis. It can be computed, by considering feedback control law, as below:

$$v = -k_1 \, (x\cos\theta + y\sin\theta) \quad (5)$$

According to (5) we can say that, the driving velocity is reacting to the cartesian error. In particular, it is reacting to the component of cartesian error which is along the sagittal axis.

→ On the other hand, $\omega$ (steering velocity) is proportional to the pointing error (i.e. the difference between the orientation of $e_p$

and that of the unicycle). Thus we can compute $\omega$ in terms of pointing error, by considering feedback control law, as below:

$$\omega = k_2 (ATAN2(y,x) - \theta + \pi) \quad (6)$$

To sum up, the idea is:
Driving velocity try to correct the cartesian position error, which component is aligned with sagittal axis and, meanwhile the Steering velocity is aligning the robot with the cartesian error. The combination of these 2 actions will produce the convergence to the origin.

In order to prove this we will use Lyapunov-like function.

→ Lyapunov-like function.
We will consider Lyapunov-like function as half of squared Cartesian error:

$$V = \frac{1}{2}(x^2+y^2) \quad (7)$$

V is positive semidefinite function, which is never negative.
Additionally we need to find derivative of V, inorder to prove stability. Before finding derivative we need to remember the unicycle equations:

$$\dot{x} = v\cos\theta \quad (8)$$
$$\dot{y} = v\sin\theta \quad (9)$$
$$\dot{\theta} = \omega \quad (10)$$

Now if differentiate V function :

$$\dot{V} = \frac{1}{2}(2x\dot{x} + 2y\dot{y}) = (xv\cos\theta + yv\sin\theta) =$$

$$= v(x\cos\theta + y\sin\theta) \quad (11)$$

If we use (5) in (11) :

$$\dot{V} = -k_1(x\cos\theta + y\sin\theta)(x\cos\theta + y\sin\theta) = -k_1(x\cos\theta + y\sin\theta)^2 \quad (12)$$

$\dot{V}$ is negative semidefine, which is never positive. It means, change of V in time is always to decrease.

Note 2: (12) is called as, computing the derivative of Lyapunov function along the trajectories.

When $\dot{V}$ is zero ?

→ 1st case: If x and y are zero, $\dot{V}$ becomes zero automatically.

→ 2nd case: If cartesian error is orthogonal to the sagittal axis (as demonstrated in Figure 4), $\dot{V}$ becomes zero. In this case, we don't care x and y are zero or not. In this case if we compute $e_p^T n$ it will give us zero :

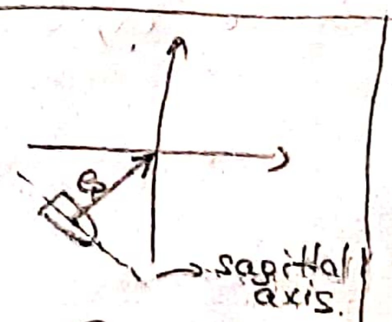


Figure 4

$$e_p^T n = (x \;\; y)\begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix} = (x\cos\theta + y\sin\theta) = 0 \quad (13)$$

We see that, (13) is exactly the part of (12):

$$(12) \Rightarrow \dot{V} = -k_1(x\cos\theta + y\sin\theta)^2 = -k_1(e_p^T n)^2 = 0 \quad (14)$$

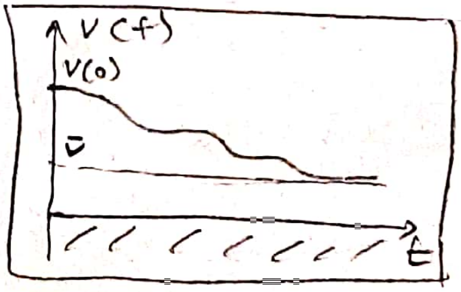
Note 3: We will use this information later.

Because our V is Lyapunov-like function (due to positive semidefinite fact), we can't use La Salle's theorem in order to prove stability. La Salle's theorem can be applied when V is positive definite (Lyapunov function). Additionally, being negative-semidefinite of $\dot{V}$ does not hinder us to use La Salle's theorem. The only issue that we can't apply it, is having positive semidefinite V function.

Why? Because in this case, trajectories that give us V, are not bounded. If we want to use La Salle's theorem our trajectories have to be bounded.

Because of these problems, we will use Barbalat's lemma, instead of La Salle's theorem. Before explaining Barbalat's lemma we will explain why it is important.

Assume we have $V(t)$ function which is positive semi-definite. On the other hand its derivative is negative semidefinite. To sum up, $V(t)$ never becomes negative but it changes constantly or decreasing as demonstrated in figure 5. In this case, we say there is $\bar{V}$ such that, V converges to it, when time increases.




$$\lim_{t \to \infty} V = \bar{V} \quad (15)$$

But does $\dot{V}$ converges to zero? -> NOT NECESSARILY

There are such cases that, the function is positive semidefinite and derivative of it is negative semidefinite. It also converges to some limit. However when we analyze whether its derivative converge or not, we see it diverges. That is why we use Barbalat's lemma.

AMR Lecture 10 / parts

• Barbalat's lemma.

If $V$ is positive semidefinite ($V \geq 0$), $\dot{V}$ is negative semidefinite ($\dot{V} \leq 0$) and $\ddot{V}$ is bounded, then

$$\lim_{t \to \infty} \dot{V}(t) = 0 \quad (16)$$

Without the last condition ($\ddot{V}$ is bounded), we can't compute (16).

In our case it is true (check by computing $\ddot{V}$).

This means (16) is true for our case.

Now we can analyze our system.

1st case: $\dot{V} \to 0$ means $x, y \to 0$. This is okay according to Cartesian regulation.

2nd case: $\dot{V} \to 0$ can happen when $e_p \perp n$

Because $v$ is reacting to the cartesian error, then $v$ will be zero. However is this situation possible steady-state? We know that $\omega$ is reacting to the pointing error which is not zero ($\frac{\pi}{2}$) in this case. Thus, we can say that $\omega \neq 0$.

Having nonzero $\omega$, violates $\dot{V} \to 0$.

That is why, it is not steady-state.

To sum up, the 1st case is only possibility for Cartesian Regulation.

The solution that we have given is for Cartesian Regulation. In other words, we analyzed the regulation of Cartesian point without looking the orientation. What if we want to give a regulation with given orientation?
For these kind of problems' main idea is to bring the vehicle to the origin in terms of x, y and θ. In order to do it, we will use Posture Regulation.

- Posture Regulation.

The main idea is, to bring the vehicle to the origin of Configuration Space → $(0,0,0)^T$.
We will follow, the following steps in order to solve "posture regulation" problem.

→ Convert the model to Polar Coordinates.

It is convenient to write the model of the unicycle in polar coordinates in order to solve this problem (figure 6).

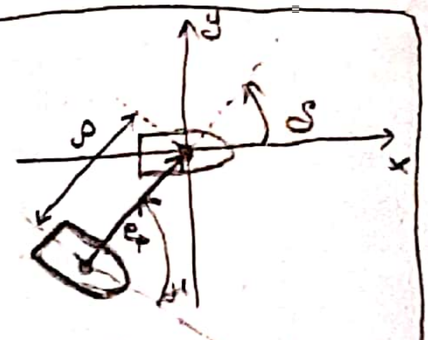



Figure 6

Polar coordinates will be following:

→ ρ is the distance of the unicycle from the origin :

$$\rho = \sqrt{x^2+y^2} \quad (17)$$

→ γ is pointing angle. We already defined this by the equation (4):

$$(4) \Rightarrow \gamma = \mathrm{ATAN2}(y,x) - \theta + \pi$$

$\alpha$ is the angle under which the robot sees the goal.

→ $\delta$ is the orientation of ep which is computed as below:

$$\delta = \alpha + \theta \quad (18)$$

Notice that, by using $\rho$ and $\alpha$ we cannot define pose of the robot. If the robot moves along the circle, then $\rho$ and $\alpha$ will always be same. Thus, these coordinates can't say us where is the robot, exactly. That's why we need to know $\delta$ angle, which gives us orientation of ep.

→ At this step we can define the kinematic model of unicycle. We know that $\rho$, $\alpha$ and $\delta$ define the configuration of unicycle in Polar coordinates. Thus, we can generate kinematic model of unicycle by taking the derivative of them.

Note 2: Derivation of those equations are given by Appendix 10.1.

$$\dot{\rho} = -v \cos \alpha \quad (19)$$

$$\dot{\alpha} = \frac{\sin \alpha}{\rho} v - \omega \quad (20)$$

$$\dot{\delta} = \frac{\sin \alpha}{\rho} v \quad (21)$$

Notice that, at the origin (where $x=0, y=0$) $\alpha$ is not defined. In other words, ep vanishes and the pointing error becomes undefined. That is why, we can say that there is singularity at the origin.

→ Now we can implement the feedback control law.

We will consider the control law as below:

$$v = k_1 \rho \cos \gamma \quad (22)$$

$$\omega = k_2 \gamma + k_1 \frac{\sin \gamma \cos \gamma}{\gamma} (\gamma + k_3 \delta) \quad (23)$$

If we compare (22) with (5), we will see that they give us the same result, but in different coordinate systems. On the other hand, if we compare (23) with (6) we will see there is new term which is:

$$\alpha = k_1 \frac{\sin \gamma \cos \gamma}{\gamma} (\gamma + k_3 \delta) \quad (24)$$

Why $\alpha$ is new term? If $\alpha$ does not exist, we can see that (23) and (6) are same.

On the other hand, $\alpha$ is the term that transforms cartesian regulation controller to a full posture regulation controller. This term allows us to achieve the orientation control as well.

At this point there is a question that raises: How do we come up with this term?

The answer is, it is related with Lyapunov function.

AMR Lecture part 4

Notice that, when we are controlling nonlinear systems Lyapunov function is almost invariably goes to approach.

When we use Lyapunov function, it must be zero at the regulation point. In our case it happens when $\gamma, \rho, \delta$ are zero. Thus we can write Lyapunov function as below:

$$V = \frac{1}{2}(\rho^2 + \gamma^2 + k_3\delta^2) \quad (25)$$

We assume $k_3 = 1$. $V$ is positive definite, because it becomes zero, only when the robot is at the origin. Now we can compute $\dot{V}$:

$$\dot{V} = \frac{1}{2} \cdot 2(\rho\dot{\rho} + \gamma\dot{\gamma} + \delta\dot{\delta}) = \rho\dot{\rho} + \gamma\dot{\gamma} + \delta\dot{\delta} \quad (26)$$

At this step we are going to use the system equations ((19), (20) and (21)) in (26).

$$\dot{V} = \rho\dot{\rho} + \gamma\dot{\gamma} + \delta\dot{\delta} = \rho(-v\cos\gamma) + \gamma\left(\frac{\sin\gamma}{\gamma} v - \omega\right) + \delta\left(\frac{\sin\gamma}{\rho} v\right) =$$

$$= v\left(-\rho\cos\gamma + \gamma\frac{\sin\gamma}{\gamma} + \delta\frac{\sin\gamma}{\rho}\right) - \gamma\omega \quad (27)$$

At this point, we are going to choose some values in order to design our control law.

We choose $v$ as given in the equation (22).

$$\dot{V} = -k_1\rho^2\cos^2\gamma + \frac{v}{\rho}(\delta+\gamma)\sin\gamma - \gamma\omega \quad (28)$$

Choice of $v$ as we did is a good for our Lyapunov function. In order to have negative function we need to have terms with negative signs in Lyapunov function's derivative, we have a term in order to control $\gamma$. Now we need to choose $\omega$ in such way that it must give us negative term and cancel $\frac{v}{\rho}\sin\gamma(\delta+\gamma)$ term from the equation (28). Why we need to cancel it? It has positive sign, which we don't want. Thus, we choose $\omega$ as below:

$$\omega = k_2\gamma + \alpha \quad (29)$$

If we use (29) in (28), we hope that it will cancel the term that we want to cancel and, we know that it will give us negative term which is related with $k_2$.

$$\dot{V} = -k_1\rho^2\cos^2\gamma - k_2\gamma^2 - \alpha\gamma + \frac{v}{\rho}\sin\gamma(\delta+\gamma) \quad (30)$$

In (30), $\alpha\gamma$ will cancel $\frac{v}{\rho}\sin\gamma(\delta+\gamma)$, if:

$$-\alpha\gamma + \frac{v}{\rho}\sin\gamma(\delta+\gamma) = 0 \Rightarrow$$

$$\Rightarrow \alpha = \frac{v}{\rho}\frac{\sin\gamma}{\gamma}(\delta+\gamma) \quad (31)$$

If we use (31) in (29), we can get $\omega$ as below:

$$\omega = k_2\gamma + \frac{v}{\rho}\frac{\sin\gamma}{\gamma}(\delta+\gamma) \quad (32)$$

Notice that, we assumed $k_3 = 1$. If we didn't assume it, it would be:

$$\omega = k_2\gamma + \frac{v}{\rho}\frac{\sin\gamma}{\gamma}(k_3\delta+\gamma) \quad (33)$$

Because of the cancellation process, (30) becomes:

$$\dot{V} = -k_1\rho^2\cos^2\gamma - k_2\gamma^2 \quad (34)$$

According to the (34), $\dot{V}$ is negative semidefinite, because $\delta$ does not appear. In order to make $\dot{V}$ zero, we need to make $\gamma$ and $\rho$ zero at the same time, However, the value of $\delta$ does not matter for $\dot{V}=0$. Thus, we say $\dot{V}$ is negative semidefinite, because it is zero not only at the regulation point, but also at whole $\delta$ axis.

Because, $V$ is positive definite there is possibility to use La Salle's theorem. However, at the origin the model is singular. Thus, we can't use Lasalle's theorem, because the system violates regulatory assumptions and in particular, La Salle's theorem. In this case, we can use Barbalat's lemma, as well.

According to the Barbalat lemma

$$\lim_{t \to \infty} \dot{V} = 0 \qquad (35)$$

Because of (35), $\rho, \gamma, \delta$ go to zero, as well.

Notice that, model is singular at the origin because of the polar coordinate system definition. However, it does not mean that unicycle also singular. We know that the unicycle is perfectly defined at the origin. The singularity of polar model tells us the model is not describing the unicycle correctly anymore.

To sum up, if we define the model in original coordinate system, it will be discontinuous at origin.
Because $\gamma$ is undefined at the origin.

On the other hand, it can be shown that, due to the nonholonomy, all posture stabilizers must be discontinuous w.r.t. the state or time varying (it was proved by Brocket's theorem). In other words, there is no such stabilizers that is smooth.