- Comments on Bicycle

Figure 1 demonstrates the bicycle.

$$q = \begin{pmatrix} x \\ y \\ \theta \\ \phi \end{pmatrix} \quad (1)$$

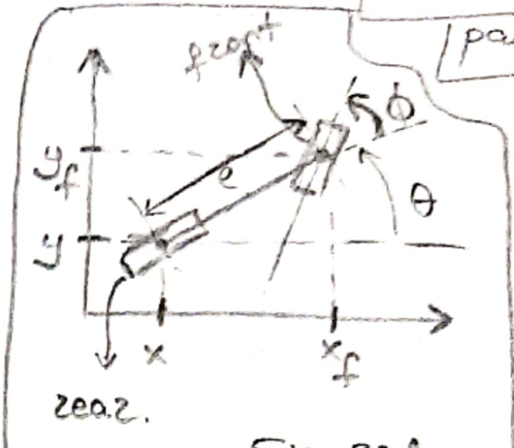

Figure 1

Forward Wheel Drive (FWD):

$$\dot{q} = \begin{pmatrix} \cos\theta \cos\phi \\ \sin\theta \cos\phi \\ (\sin\phi)/e \\ 0 \end{pmatrix} \underset{\substack{\downarrow \\ v}}{u_1} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \underset{\substack{\downarrow \\ \omega}}{u_2} \quad (2)$$

Rear Wheel Drive (RWD):

$$\dot{q} = \begin{pmatrix} \cos\theta \\ \sin\theta \\ (\tan\phi)/e \\ 0 \end{pmatrix} \underset{\substack{\downarrow \\ v}}{u_1} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \underset{\substack{\downarrow \\ \omega}}{u_2} \quad (3)$$

1^{st} comment:

What does happen when $\phi = \frac{\pi}{2}$? (Figure 2)

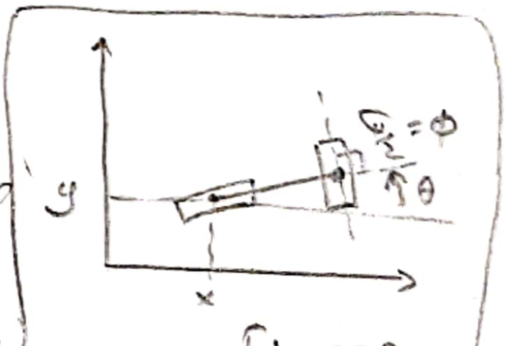

Figure 2.

In Kinematic Model of RWD **$g_1(q)$ diverges**. Why? If we check the equation (3), we will see that the 3^{rd} element of $g_1(q)$ is $(\tan\phi)/e$ which is not defined for $\phi = \frac{\pi}{2}$. This situation leads us to have **singularity** in that position. Additionally, we can say that is related to **MECHANICAL JAM**.

• 2 zero motion lines

For each scenario (RWD/FWD), rolling without slipping (RWS) constraint defines zero motion lines. For each scenario there are 2 wheels → 2 constraints → 2 zero motion lines. We will analyze it on Figure 3.

In figure 3 :

Red → Zero motion line associated with rear wheel

Blue → Zero motion line associated with front wheel

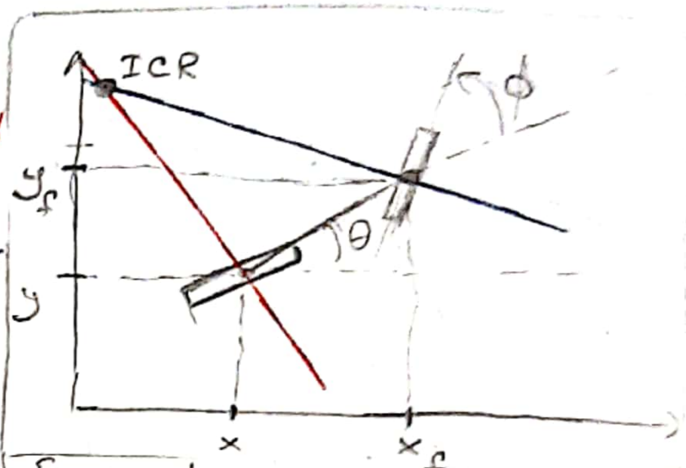

Figure 3

These 2 ZML's meet at the ICR (Instantaneous Center of Rotation). If ZML's are parallel, ICR is at infinity. In that case not only wheels but also any point of bicycle are instantaneously rotating around the ICR point. If we put some frame to bicycle any point on that frame will follow the fixed wheel (rear). That will lead the frame to rotate instantaneously around ICR point.

Is the bicycle controllable?

In linear systems controllability can be checked by using Linear Algebra methods. Can we do it for nonlinear systems?

• Controllability of nonlinear driftless systems

Note: Its special part is → driftless nonlinear system. We'll use this as an advantage.

General Nonlinear Driftless system can be written as below:

$$\dot{x} = \sum_{i=1}^{m} g_i(x)\, u_i \quad (4)$$

In the equation (4): $x \to$ state $\in \mathbb{R}^n$,
$u \to$ input $\in \mathbb{R}^m$

The definition of controllability:

(4) is controllable if for any $x_i, x_f \in \mathbb{R}^n$ there exists a time T and any input $u|_{[0,T]}$ such that moving from x_i:

$$x(T) = x_f$$

Note: x_i and x_f are initial and final points, respectively.

Before going on we have to know some preliminary concepts.

→ an operation between 2 vector fields.

Lie BRACKET of g_1, g_2:

$$[g_1(x), g_2(x)] = \frac{\partial g_2}{\partial x} g_1 - \frac{\partial g_1}{\partial x} g_2 \quad (5)$$

$\frac{\partial g_2}{\partial x} \to n \times n$; $\frac{\partial g_1}{\partial x} \to n \times n$; $g_1 \to n \times 1$; $g_2 \to n \times 1$ → Result of (5) is $n \times 1$ vector field.

It gives us new vector field.

In general (it may, it may not) it is not a linear combination of g_1, g_2.

• Interpretation of $[g_1, g_2]$

$$\dot{x} = g_1(x) u_1 + g_2(x) u_2 \quad (6)$$

We consider (6) is a particular control sequence

$$u = \begin{cases} u_1(t) = 1 \; ; \; u_2(t) = 0 & t \in [0, \varepsilon) \\ u_1(t) = 0 \; ; \; u_2(t) = 1 & t \in [\varepsilon, 2\varepsilon) \\ u_1(t) = -1 \; ; \; u_2(t) = 0 & t \in [2\varepsilon, 3\varepsilon) \\ u_1(t) = 0 \; ; \; u_2(t) = -1 & t \in [3\varepsilon, 4\varepsilon) \end{cases} \quad (7)$$

We can show (7) as in figure 4.

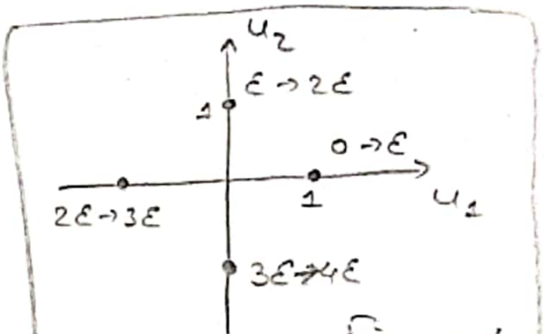

Figure 4

It can be proven that:

$$x(4\varepsilon) = x_0 + \varepsilon^2 [g_1(x), g_2(x)]\Big|_{x_0} + o(\varepsilon^3) \quad (8)$$

(8) says that, final state can be defined by initial state and some displacement.

$$\text{displacement} = \varepsilon^2 [g_1(x), g_2(x)]\Big|_{x_0} + o(\varepsilon^3) \quad (9)$$

for $\varepsilon \to 0$, the DISPLACEMENT is eventually in the direction of $[g_1(x), g_2(x)]$. Because $o(\varepsilon^3)$ will converge zero faster.

Note: $\varepsilon \to 0$ means ε is infinitesimal value (so small)

We can analyze those by using the figure (5) and the equation (7) and (6).

We started from x_0. From (7):

- (6) $\rightarrow$ $\dot{x} = g_1(x)$ (a); $t \in [0, \varepsilon)$

According to the a, we'll have such path that $g_1(x)$ must be tangent of it. Then we got to the state: $x(\varepsilon)$ -> Point $\vec{B}$.

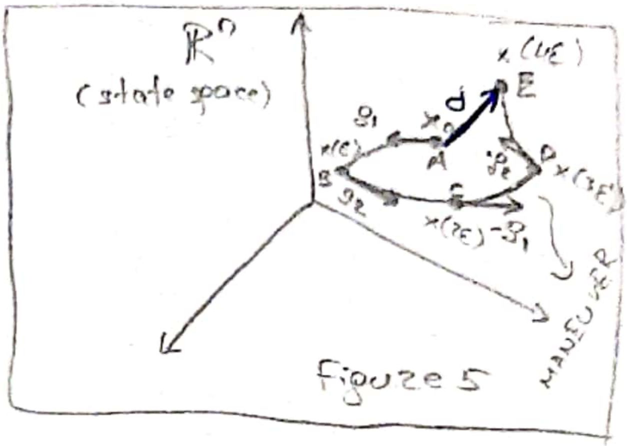

Figure 5

- (6) $\rightarrow$ $\dot{x} = g_2(x)$ (b); $t \in [\varepsilon, 2\varepsilon)$

By using the same methodology, we got to the Point B which is $x(2\varepsilon)$.

- (6) $\rightarrow$ $\dot{x} = -g_1(x)$ (c); $t \in [2\varepsilon, 3\varepsilon)$ => The point we reach is D.

- (6) $\rightarrow$ $\dot{x} = -g_2(x)$ (d) $t \in [3\varepsilon, 4\varepsilon)$ => The point is E.

We reach the states by using the equations (a), (b), (c) and (d). Finally, we see from the Figure 5 that, there is blue vector which is displacement (starts from x_0 ends at $x(4\varepsilon)$).

$$d = \varepsilon^2 [g_1(x), g_2(x)] + o(\varepsilon^3) \quad (10)$$

higher order terms

This line (Fig 5) we follow is a MANEUVER (result of a control sequence

In order to explain next concept we will recall the equation (4) :

$$(4) \Rightarrow \quad \dot{x} = \sum_{i=1}^{m} g_i(x)\, u_i$$

In expansion of (4) we'll have collection of g_i's.

$$\{g_1(x), \ldots, g_m(x)\} \rightarrow \text{input vf.}$$

These are called input vector fields, because they're multiplied by inputs.

At every change of x, the linear space which is generated by $g_i(x)$ vector fields also will change. This association of the change of linear space is called as DISTRIBUTION.

$$\Delta = \text{span}\{g_1(x), \ldots, g_m(x)\} \quad (11)$$

DISTRIBUTION $\rightarrow$ associates to each $x \in \mathbb{R}^n$;
↳ a linear space, the linear combinations of $g_1(x), g_2(x), \ldots, g_m(x)$.

Figure 6 demonstrates DISTRIBUTION.

Point B and Point A correspond different planes and this change is obtained by DISTRIBUTION

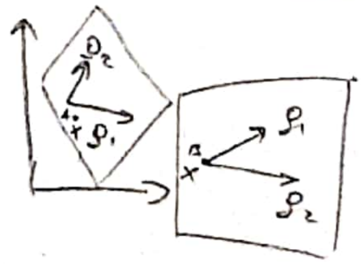

Figure6.

There is also special distribution :

$$\Delta_A : \text{span}\{g_1(x), \dots g_m(x), \underbrace{[g_1(x), g_2(x)], \dots}_{\text{all 1st order Lie Brackets}}, \underbrace{[g_1(x), [g_2(x), g_3(x)]] \dots}_{\text{all 2nd order Lie Brackets}}\} \quad (12)$$

$\Delta_A \rightarrow$ is accessibility DISTRIBUTION.

Then we can speak about controllability.

CHOW'S THEOREM

(4) is controllable if and only if

$$\dim \Delta_A = n \quad (13)$$

In practice we try to find this dimension by computing the rank.

$$\text{rank}\begin{pmatrix} g_1 \dots g_m & \dots [g_1, g_2] & \dots [g_1, [g_2, g_3]] \end{pmatrix} = n \quad (14)$$

If (14) is true, then we can say model is controllable.

By checking this, we actually investigating our possible vector fields that we can have (or maneuvers)

Controllability check

→ Unicycle.

The model of the unicycle:

$$q = \begin{pmatrix} x \\ y \\ \theta \end{pmatrix} \; ; \quad \dot{q} = \underbrace{\begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}}_{\text{drive}} v + \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{\text{steer}} \omega \qquad (15).$$

In order to check controllability, we have to get Lie-Bracket:

$$[g_1, g_2] = \frac{\partial g_2}{\partial q} \cdot g_1 - \frac{\partial g_1}{\partial q} \cdot g_2 =$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & -\sin\theta \\ 0 & 0 & \cos\theta \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{bmatrix} \quad (16)$$

We can apply CHOW's theorem, because we don't have second order brackets.

$$\text{rank} \begin{bmatrix} \cos\theta & 0 & +\sin\theta \\ \sin\theta & 0 & -\cos\theta \\ 0 & 1 & 0 \end{bmatrix} = 3 = n. \quad (17).$$

According to (17), unicycle is not controllable

Geometrical interpretation:

Note: The Figure 7 is not an instantaneous rotation's result. It's the result of the Lie-Bracketmaneuver.

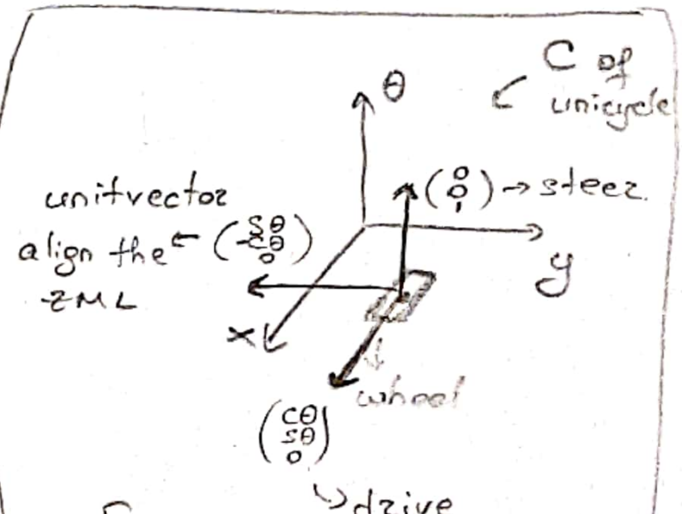

Figure 7

The LB maneuver (Lie-Bracket)

Figure 8 demonstrates it.

→ Note: Our velocity is 1.

We'll use equations (6) & (7) for that.

→ $t \in [0, \varepsilon) \rightarrow q_0 \xrightarrow{to} q(\varepsilon)$

→ $t \in [\varepsilon, 2\varepsilon) \rightarrow q(\varepsilon) \xrightarrow{to} q(2\varepsilon)$

→ $t \in [2\varepsilon, 3\varepsilon) \rightarrow q(2\varepsilon) \xrightarrow{to} q(3\varepsilon)$

→ $t \in [3\varepsilon, 4\varepsilon) \rightarrow q(3\varepsilon) \xrightarrow{to} q(4\varepsilon)$

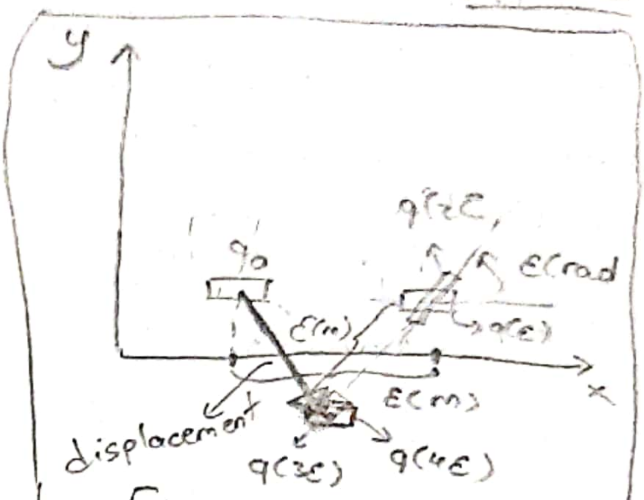

Figure 8

Note, velocity is 1, thus distance is ε(m), angles are ε(rad)

If we analyze the displacement vector, we'll see that is not directly along the ZML. Why?

Answer: Because, ε is not infinitesimal, thus there are also high order brackets as well.

As $\varepsilon \rightarrow 0$, displacement is alligned with ZML ($[g_1, g_2]$).

To sum up, according to the equation (17) unicycle is controllable. Additionally, under the RWS constraint unicycle is Non-Holonomic.

Bicycle

RWD model of Bicycle

$$\dot{q} = \underbrace{\begin{pmatrix} \cos\theta \\ \sin\theta \\ (\tan\phi)/l \\ 0 \end{pmatrix}}_{g_1} v + \underbrace{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}}_{g_2} \omega \qquad (18)$$

We'll compute Lie-Bracket. Because $n=4$ we'll need at least 2 more vector-fields in order to apply Chow's theorem.

$$[g_1, g_2] \triangleq g_3 = -\begin{pmatrix} 0 & 0 & -\sin\theta & 0 \\ 0 & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = -\begin{pmatrix} 0 \\ 0 \\ 1/l\cos^2\phi \\ 0 \end{pmatrix} \qquad (19)$$

In the equation (19), $*$ is $\dfrac{1}{l\cos^2\phi}$.

If we analyze (19), we can see that this vector field is related with a motion along θ. In other words, it doesn't change x, y position of rear wheel and the rotation of the front wheel (Figure 9).

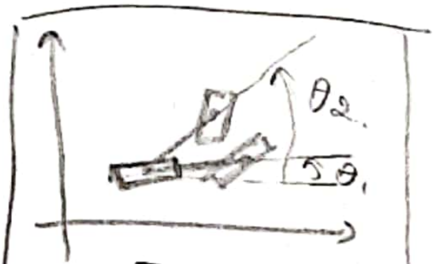

Figure 9

This kind of motion is called as wriggling. Thus we call this vector field (g_3) as "wriggle". g_3 is good vector field for computing the controllability. Why? It's linear independent on g_1 and g_2. Now we need another one.

We can do either $[g_1, [g_1, g_2]]$ or $[g_2, [g_1, g_2]]$. If we choose the first one we'll see the following computations.

$$g_4 \triangleq [g_1,[g_1,g_2]] = [g_1,g_3] = \frac{\partial g_3}{\partial q} g_1 - \frac{\partial g_1}{\partial q} g_3 =$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \sin\theta \\ \cos\theta \\ (\tan\phi)/\ell \\ 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & -\sin\theta & 0 \\ 0 & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1/\ell\cos^2\phi \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1/\ell\cos^2\phi \\ 0 \end{bmatrix} =$$

$$= \begin{pmatrix} -\frac{\sin\theta}{L\cos^2\phi} \\ \frac{\cos\theta}{L\cos^2\phi} \\ 0 \\ 0 \end{pmatrix} \quad (20)$$

(20) is slide vector field. Now we can apply Chow's theorem.

$$\Delta_A = \text{span}\{g_1, g_2, g_3, g_4, \underbrace{\dots}\} \quad (21)$$

↳ do not need them, because these 4 g. vectors will give us 4 dimension.

$$(21) \Rightarrow \text{span}\left\{ \begin{pmatrix} \cos\theta \\ \sin\theta \\ (\tan\theta)/\ell \\ 0 \end{pmatrix} \begin{pmatrix} -\sin\theta/\ell\cos^2\phi \\ \cos\theta/\ell\cos^2\phi \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ -\frac{1}{L\cos^2\phi} \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\} \quad (22)$$

Rank of (22) is (4) and because of that bicycle is controllable.

FWP → do it as exercise.

Balance (mechanical)

Mechanically balanced versions of unicycle are:

unicycle { differential-drive robots
synchro-drive robots

↑ equivalent to unicycle but statically balanced (3 wheels)

Differential drive robots have 3 wheels as in figure 10.

Note: 2 fixed wheels are independent:

→ they have independent motors

→ But they're mounted parallel

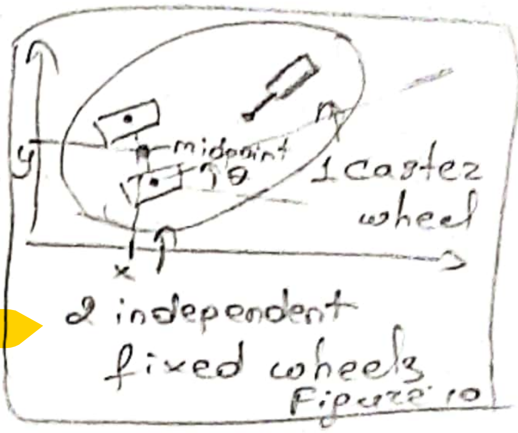

Figure 10

q can be given as general:

$$q = \begin{pmatrix} x \\ y \\ \theta \end{pmatrix} \quad (23)$$

Kinematic model can be given as below:

$$\dot{q} = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix} v + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \omega \quad (24)$$

However we've said that these 2 wheels are independent (independent motors).

ω_L and ω_R are rotational speed of left and right wheels, respectively

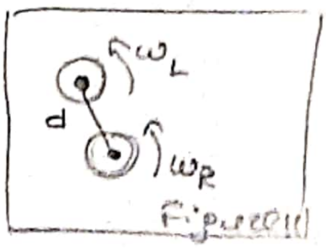

Figure 11

$$v = \frac{\omega_L + \omega_R}{2} \cdot r \quad (25)$$

(25) is driving velocity of model.

$$\omega = \frac{\omega_R - \omega_L}{d} \quad (26)$$

(26) is steering velocity of model where d is the distance between wheels.

Synchro drive model which is demonstrated in Figure 12, has 3 wheels as well.

All 3 wheels are synchronously actuated by 2 actuators.

All wheels have same velocity and rotation and position is determined by common point of 3 wheels.

Figure 12

The kinematic model of robot is again given by the equation (24).

In this case, because they're synchronously actuated, 3 wheels' driving and steering velocities are the same:

$$(24) \Rightarrow \dot{q} = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix} v + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \omega$$

$v \rightarrow$ driving velocity of 3 wheels

$\omega \rightarrow$ steering velocity of 3 wheels