Autonomous and Mobile Robotics

Prof. Giuseppe Oriolo

Humanoid Robots 2:

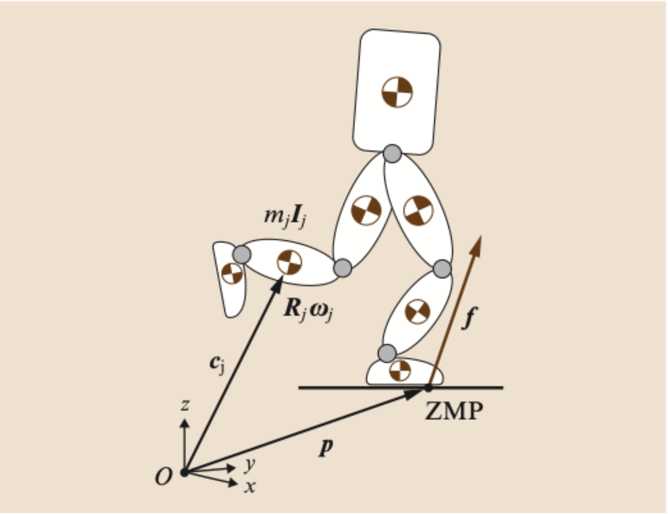
Dynamic Modeling

Dipartimento di Ingegneria Informatica
Automatica e Gestionale Antonio Ruberti

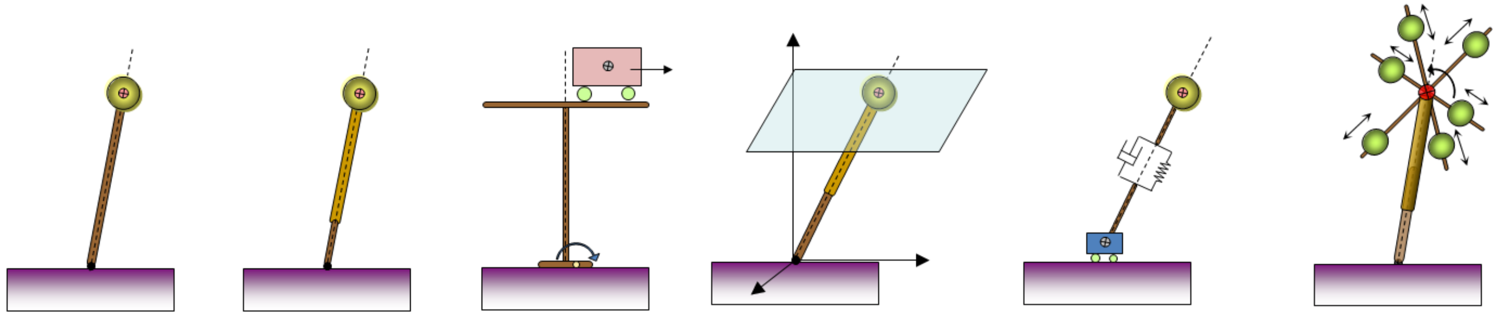
Sapienza
Università di Roma

modeling

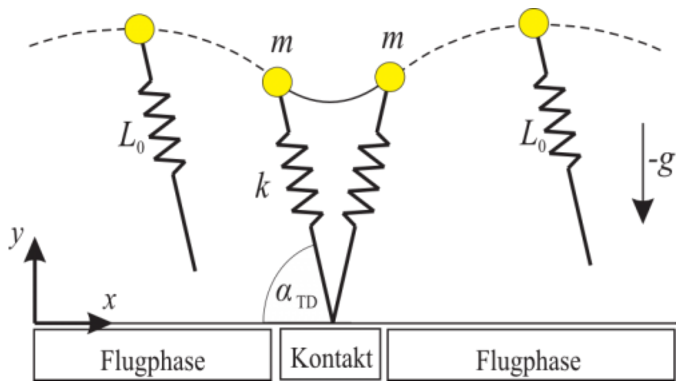
- multi-body free floating complete model



- conceptual models



rigid inverted pendulum | telescopic inverted pendulum | cart-table model | linear inverted pendulum model (LIPM) | variable impedance inverted pendulum | reaction mass pendulum (RMP)



for walking/balancing

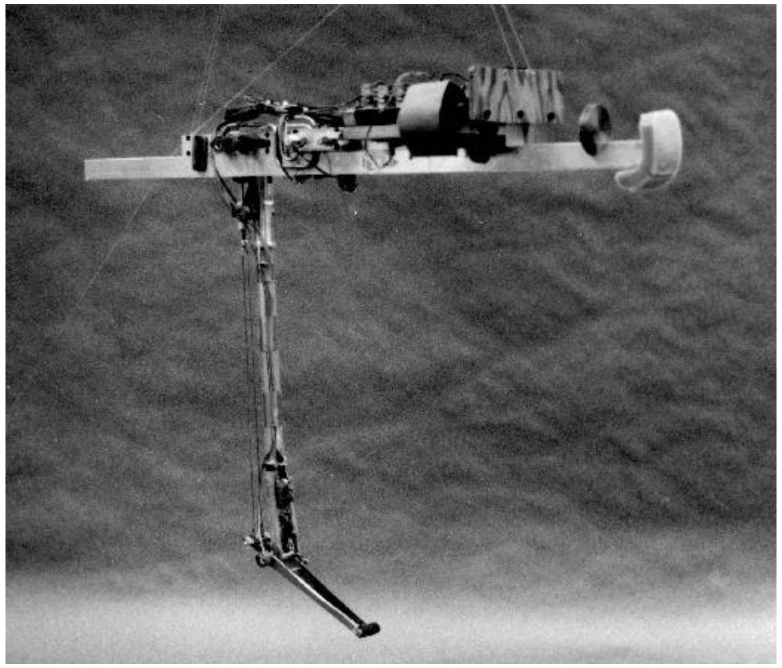
for running

like a manipulator?

can we consider this as a part (leg) of a legged robot?

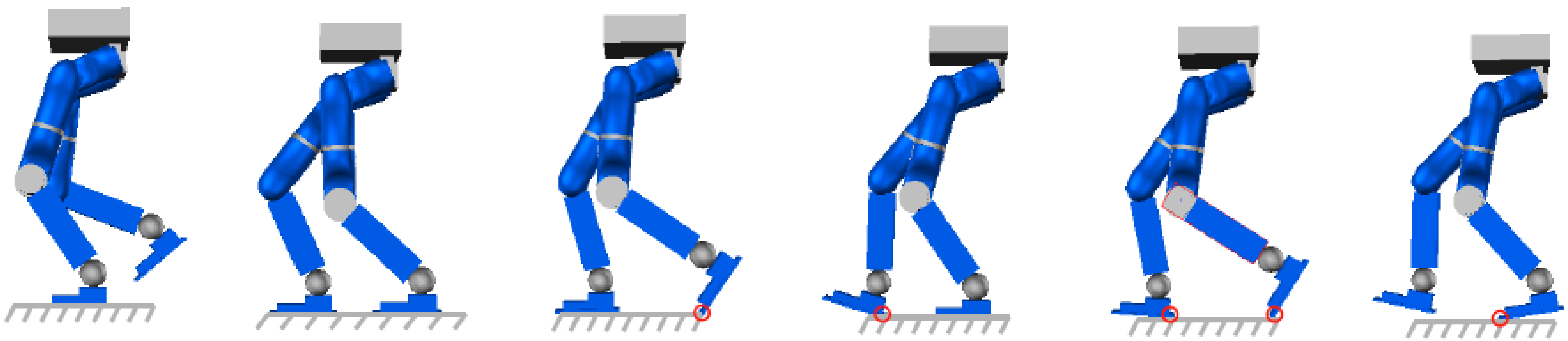
NO: this manipulator cannot fall because its base is clamped to the ground

this is a one-legged robot:
Monopod from MIT



floating-base model

the difference lies in the contact forces

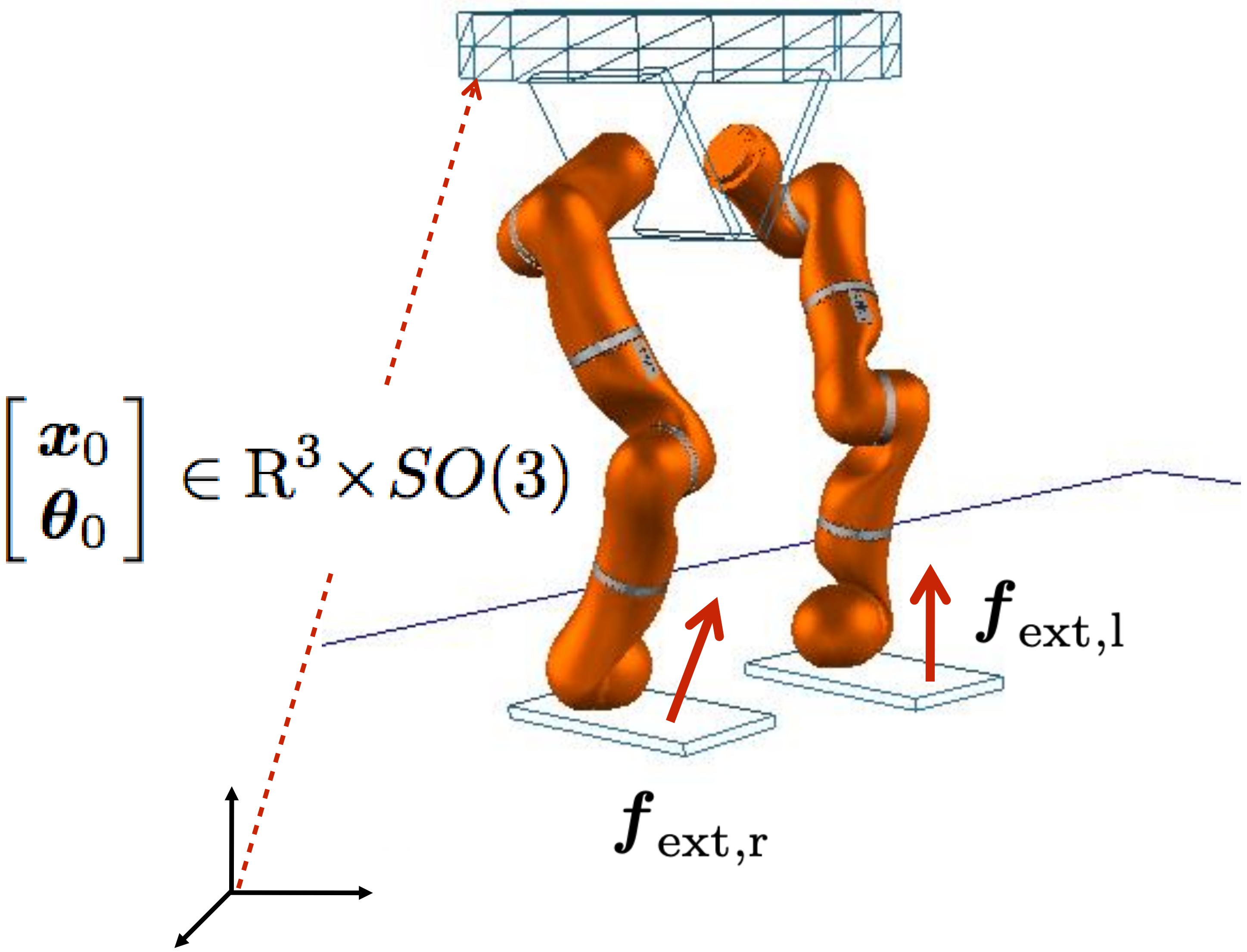


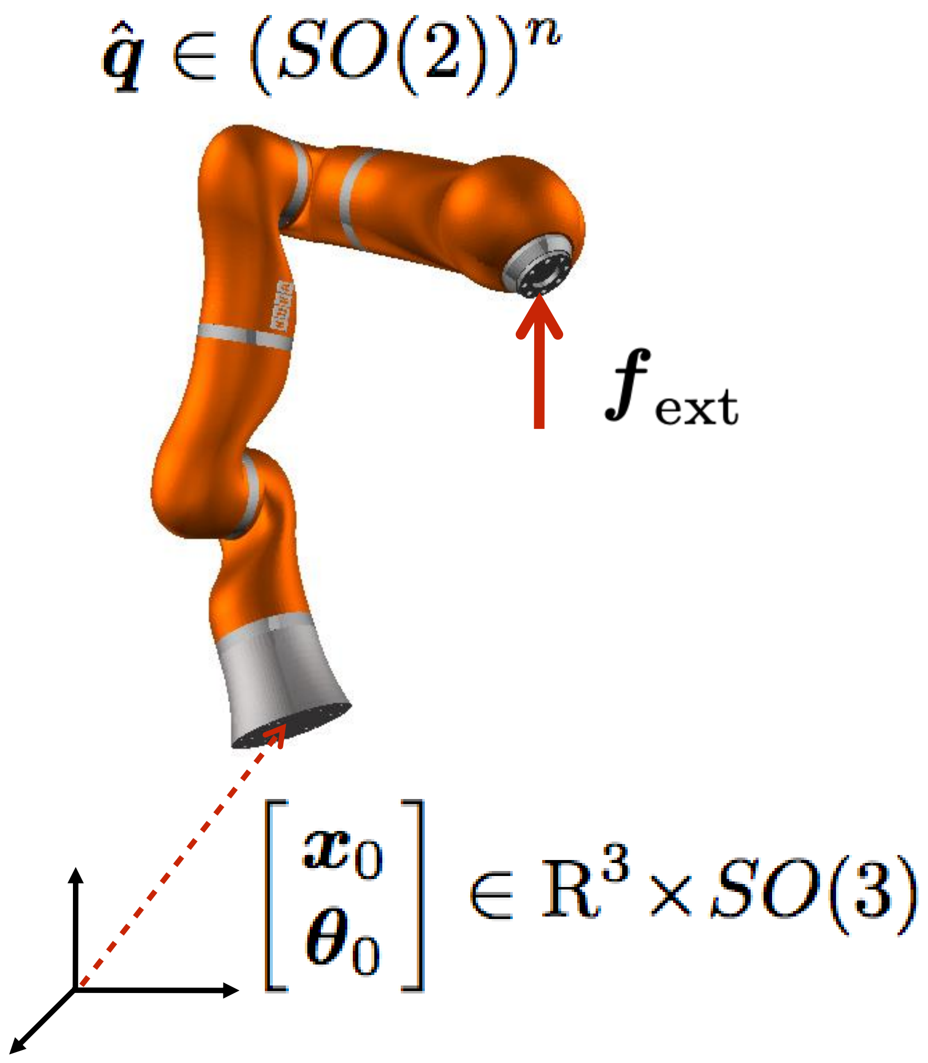
one may look at these contact configurations as different fixed-base robots, each with a specific kinematic and dynamic model

$\longrightarrow$ or consider a single floating-base system with limbs that may establish contacts

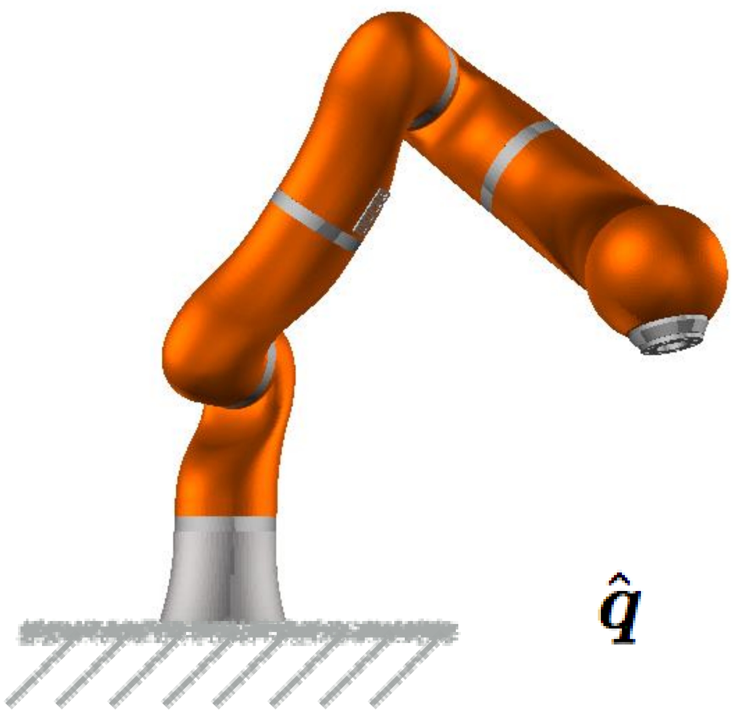
floating-base model

the general model is that of a floating-base multi-body



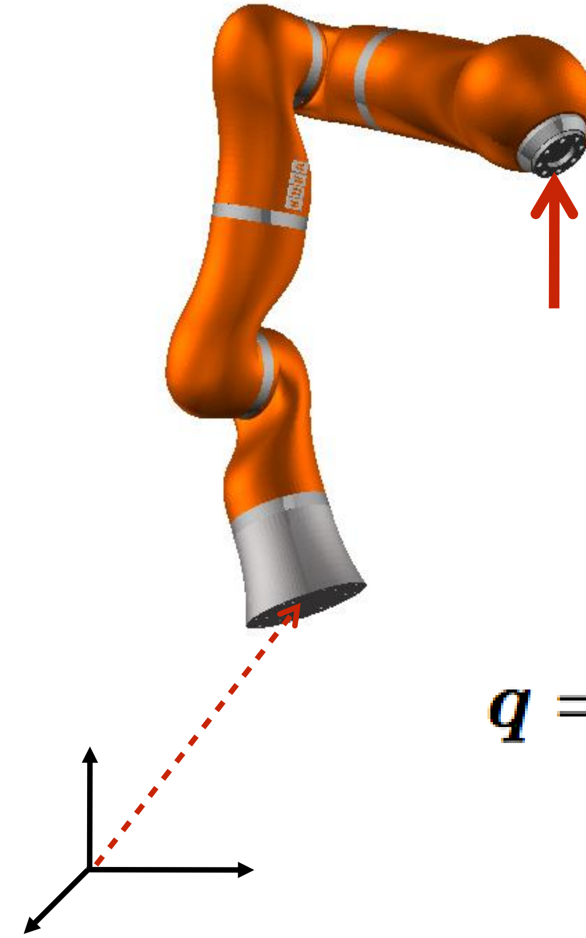


configuration



$\hat{\boldsymbol{q}}$

vs.



$$\boldsymbol{q} = \begin{bmatrix} \hat{\boldsymbol{q}} \\ \boldsymbol{x}_0 \\ \boldsymbol{\theta}_0 \end{bmatrix}$$

$$\hat{\boldsymbol{B}}(\hat{\boldsymbol{q}})\ddot{\hat{\boldsymbol{q}}} + \hat{\boldsymbol{C}}(\hat{\boldsymbol{q}}, \dot{\hat{\boldsymbol{q}}})\dot{\hat{\boldsymbol{q}}} + \hat{\boldsymbol{e}}(\hat{\boldsymbol{q}}) = \hat{\boldsymbol{\tau}} + \hat{\boldsymbol{J}}^T(\hat{\boldsymbol{q}})\boldsymbol{f}_{ext}$$

formally similar

$$\boldsymbol{B}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q}, \dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{e}(\boldsymbol{q}) = \boldsymbol{\tau} + \boldsymbol{J}^T(\boldsymbol{q})\boldsymbol{f}_{ext}$$

Lagrangian dynamics

dynamic equations (general form)

$$\boldsymbol{B}(\boldsymbol{q})\ddot{\boldsymbol{q}} + \boldsymbol{C}(\boldsymbol{q},\dot{\boldsymbol{q}})\dot{\boldsymbol{q}} + \boldsymbol{e}(\boldsymbol{q}) = \boldsymbol{\tau} + \boldsymbol{J}^T(\boldsymbol{q})\boldsymbol{f}_{ext}$$

but here we have a special structure

$$\boldsymbol{B}(\boldsymbol{q})\left(\begin{bmatrix} \ddot{\hat{\boldsymbol{q}}} \\ \ddot{\boldsymbol{x}}_0 \\ \ddot{\boldsymbol{\theta}}_0 \end{bmatrix} + \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{g} \\ \boldsymbol{0} \end{bmatrix}\right) + \boldsymbol{n}(\boldsymbol{q},\dot{\boldsymbol{q}}) = \begin{bmatrix} \boldsymbol{\tau} \\ \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix} + \sum_i \boldsymbol{J}_i^T(\boldsymbol{q})\boldsymbol{f}_i$$

where $-\boldsymbol{g}$ is the (Cartesian) gravity acceleration vector and $\boldsymbol{J}_i$ is the Jacobian matrix associated to the i-th contact force $\boldsymbol{f}_i$

Lagrangian dynamics

$$\boldsymbol{B}(\boldsymbol{q})\left(\begin{bmatrix}\ddot{\hat{\boldsymbol{q}}}\\ \ddot{\boldsymbol{x}}_0\\ \ddot{\boldsymbol{\theta}}_0\end{bmatrix}+\begin{bmatrix}\boldsymbol{0}\\ \boldsymbol{g}\\ \boldsymbol{0}\end{bmatrix}\right)+\boldsymbol{n}(\boldsymbol{q},\dot{\boldsymbol{q}})=\begin{bmatrix}\boldsymbol{\tau}\\ \boldsymbol{0}\\ \boldsymbol{0}\end{bmatrix}+\sum_i \boldsymbol{J}_i^T(\boldsymbol{q})\boldsymbol{f}_i$$

mass/ inertia

accelerations

forces/torques

- centrifugal/Coriolis terms
- joint torques
- contact forces

joint torques only affect joint coordinates!
to move $\boldsymbol{x}_0$ (i.e., the position of the reference body) the contact forces are necessary

Newton-Euler equations

$$\boldsymbol{B}(\boldsymbol{q})\left(\begin{bmatrix} \ddot{\hat{\boldsymbol{q}}} \\ \ddot{\boldsymbol{x}}_0 \\ \ddot{\boldsymbol{\theta}}_0 \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \boldsymbol{g} \\ \mathbf{0} \end{bmatrix}\right) + \boldsymbol{n}(\boldsymbol{q},\dot{\boldsymbol{q}}) = \begin{bmatrix} \boldsymbol{\tau} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} + \sum_i \boldsymbol{J}_i^T(\boldsymbol{q})\boldsymbol{f}_i$$

the second and third rows of the Lagrangian dynamics express the linear and rotational dynamics of the whole robot

these correspond to the **Newton-Euler equations**, obtained by balancing **forces** and **moments** acting on the robot as a whole

Newton-Euler equations

Newton equation:
variation of linear momentum = force balance

$$M\ddot{\boldsymbol{c}} = \sum_i \boldsymbol{f}_i - M\boldsymbol{g}$$

$\boldsymbol{c}$: CoM position
M : total mass of the system

hence: we need contact forces to move the CoM in a direction different from that of gravity!

Newton-Euler equations

Euler equation:
variation of angular momentum = moment balance

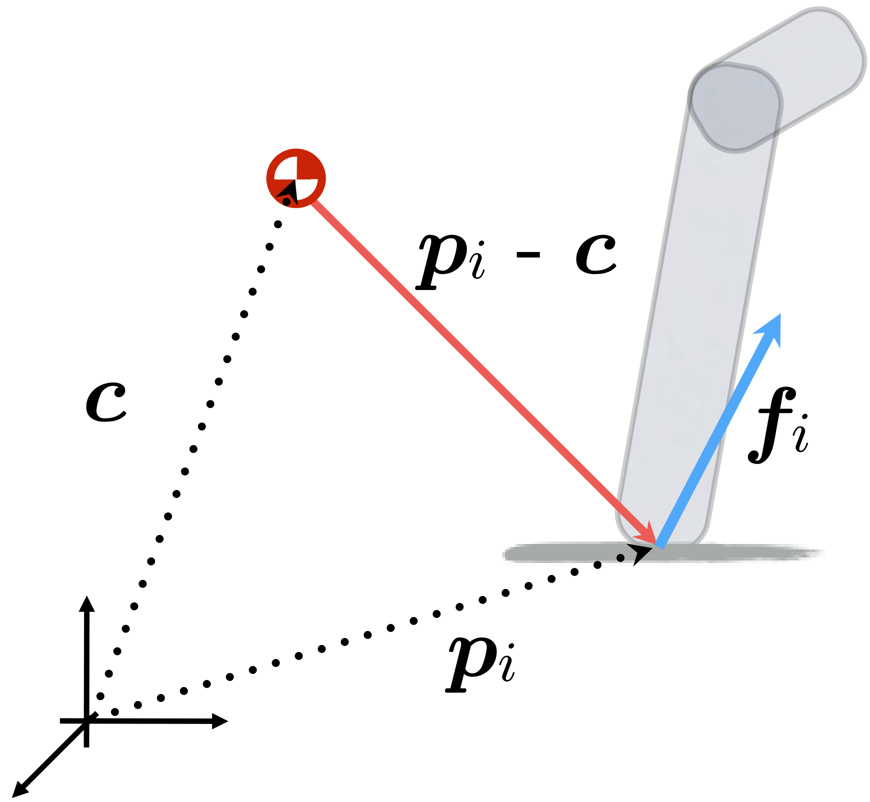
$$(\boldsymbol{c}-\boldsymbol{o}) \times M\ddot{\boldsymbol{c}} + \dot{\boldsymbol{L}} = \sum_i (\boldsymbol{p}_i - \boldsymbol{o}) \times \boldsymbol{f}_i - (\boldsymbol{c}-\boldsymbol{o}) \times M\boldsymbol{g}$$

moments are computed wrt to a generic point $\boldsymbol{o}$

$\boldsymbol{p}_i$: position of the contact point of force $\boldsymbol{f}_i$
$\boldsymbol{L}$: angular momentum of the robot wrt its CoM

Newton-Euler equations

recall: the moment of a force (or torque) is a measure of its tendency to cause a body to rotate about a specific point or axis



$$(\boldsymbol{p}_i - \boldsymbol{c}) \times \boldsymbol{f}_i$$

moment generated by the contact force $\boldsymbol{f}_i$ around the CoM

angular momentum around the CoM:
sum of the angular momentum of each robot link

$$\boldsymbol{L} = \sum_k (\boldsymbol{x}_k - \boldsymbol{c}) \times m_k \dot{\boldsymbol{x}}_k + I_k \boldsymbol{\omega}_k$$

$\boldsymbol{\omega}_k$: angular velocity of the k-th link

Zero Moment Point

in the equation of moment balance

$$(\boldsymbol{c}-\boldsymbol{o}) \times M\ddot{\boldsymbol{c}} + \dot{\boldsymbol{L}} = \sum_i (\boldsymbol{p}_i - \boldsymbol{o}) \times \boldsymbol{f}_i - (\boldsymbol{c}-\boldsymbol{o}) \times M\boldsymbol{g}$$

choose the point $\boldsymbol{o}$ so that $\sum_i (\boldsymbol{p}_i - \boldsymbol{o}) \times \boldsymbol{f}_i$ is zero

this is the Zero Moment Point (ZMP), i.e., the point wrt to which the moment of the contact forces is zero

we denote this point by $\boldsymbol{z}$

Newton-Euler on flat ground

combine the Newton and Euler equations: divide the Euler equation by the z-component of Newton equation

$$M(\ddot{c}^z + g^z) = \sum_i f_i^z$$

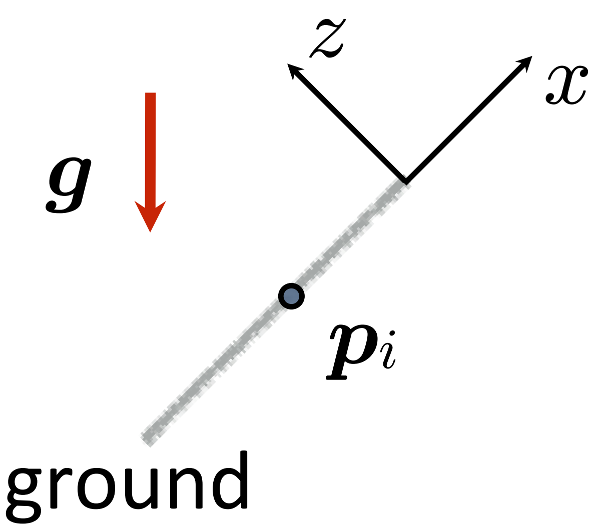
leads to

$$\frac{M(\boldsymbol{c} - \boldsymbol{z}) \times (\ddot{\boldsymbol{c}} + \boldsymbol{g}) + \dot{\boldsymbol{L}}}{m(\ddot{c}^z + g)} = \frac{\sum_i (\boldsymbol{p}_i - \boldsymbol{z}) \times \boldsymbol{f}^i}{\sum_i f_i}$$

flat ground hypothesis (not necessarily horizontal)

$$p_i^z = 0$$

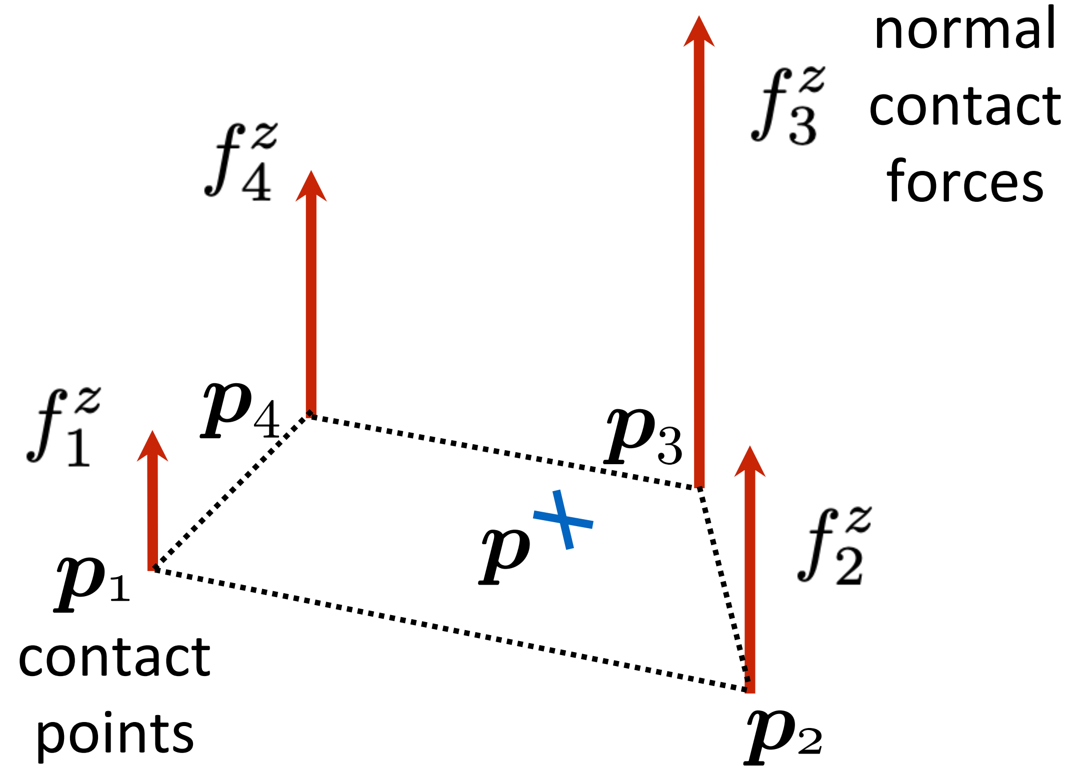
and we may have $\boldsymbol{g}^{x,y} \neq 0$



Center of Pressure

the **Center of Pressure (CoP)** is a point defined for a set of forces acting on a flat surface

$$p^{x,y} = \frac{\sum_i p_i^{x,y} f_i^z}{\sum_i f_i^z}$$



flat ground: the CoP corresponds to the point of application of the **Ground Reaction Force vector** (GRF)

note: GRF can also have a horizontal component (friction)

Center of Pressure

on flat ground, the moment balance equation tells us that the CoP and the ZMP coincide

$$\frac{M(\boldsymbol{c}-\boldsymbol{z})\times(\ddot{\boldsymbol{c}}+\boldsymbol{g})+\dot{\boldsymbol{L}}}{m(\ddot{c}^z+g)}=\frac{\sum_i(\boldsymbol{p}_i-\boldsymbol{z})\times\boldsymbol{f}^i}{\sum_i f_i}=\boldsymbol{p}-\boldsymbol{z}=0$$

$f_i^z \geq 0$ the vertical component of the contact forces can only be positive (**unilateral** force)

therefore the CoP/ZMP must belong to the convex hull of the contact points, i.e. the **Support Polygon**

sufficient condition for balance

Newton-Euler on flat ground

flat ground $p_i^z = 0$ first two components (x and y)

$$c^y - \frac{c^z}{\ddot{c}^z + g^z}(\ddot{c}^y + g^y) + \frac{\dot{\boldsymbol{L}}^x}{M(\ddot{c}^z + g^z)} = \frac{\sum_i f_i^z p_i^y}{\sum_i f_i^z}$$

$$c^x - \frac{c^z}{\ddot{c}^z + g^z}(\ddot{c}^x + g^x) - \frac{\dot{\boldsymbol{L}}^y}{M(\ddot{c}^z + g^z)} = \frac{\sum_i f_i^z p_i^x}{\sum_i f_i^z}$$

or in compact form

$$\boxed{\boldsymbol{c}^{x,y} - \frac{c^z}{\ddot{c}^z + g^z}(\ddot{\boldsymbol{c}}^{x,y} + \boldsymbol{g}^{x,y}) + \frac{\boldsymbol{S}\dot{\boldsymbol{L}}^{x,y}}{M(\ddot{c}^z + g^z)} = \frac{\sum_i f_i^z \boldsymbol{p}_{\boldsymbol{i}}^{x,y}}{\sum_i f_i^z}}$$

with $\boldsymbol{S} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

more on the CoP

the Center of Pressure (CoP) $\boldsymbol{z}$ is usually defined as the point on the ground where the resultant of the ground reaction force acts

we have 2 types of interaction forces at the foot/ground interface: **normal forces** f_i^z and tangential forces $\boldsymbol{f}_i^{x,y}$

the CoP may be defined as the point $\boldsymbol{z}$ where the resultant of the normal forces f_i^z acts

$$f^z = \sum_i f_i^z$$

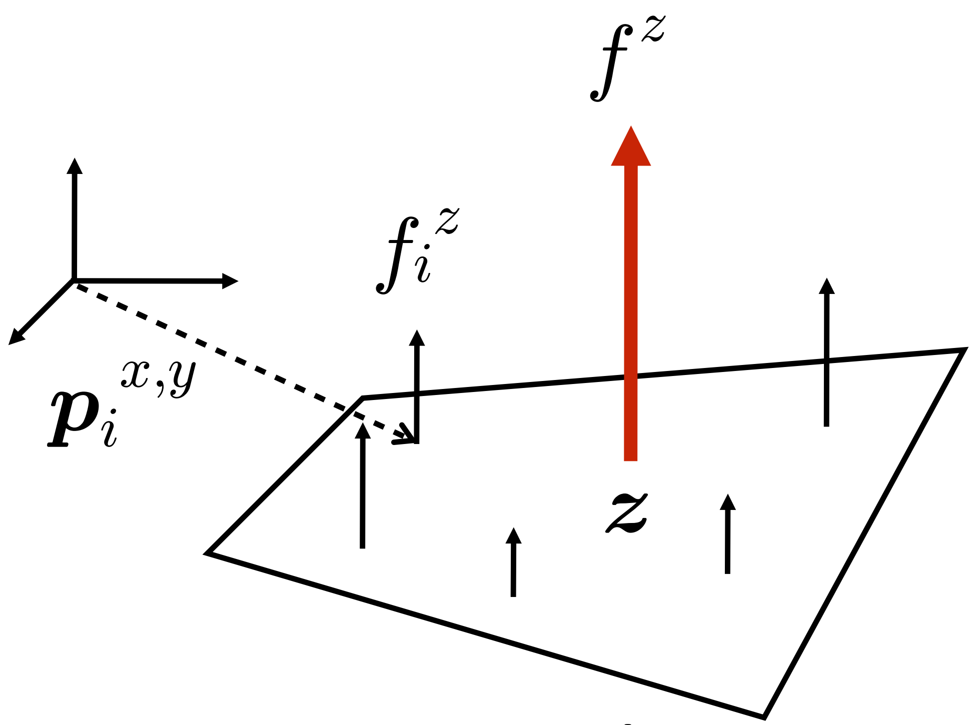
the resultant of the tangential forces may be represented at $\boldsymbol{z}$ by a force $\boldsymbol{f}^{x,y}$ and a moment M_t

$$\boldsymbol{f}^{x,y} = \sum_i \boldsymbol{f}_i^{x,y} \qquad\qquad M_t = \sum_i \boldsymbol{r}_i \times \boldsymbol{f}_i^{x,y}$$

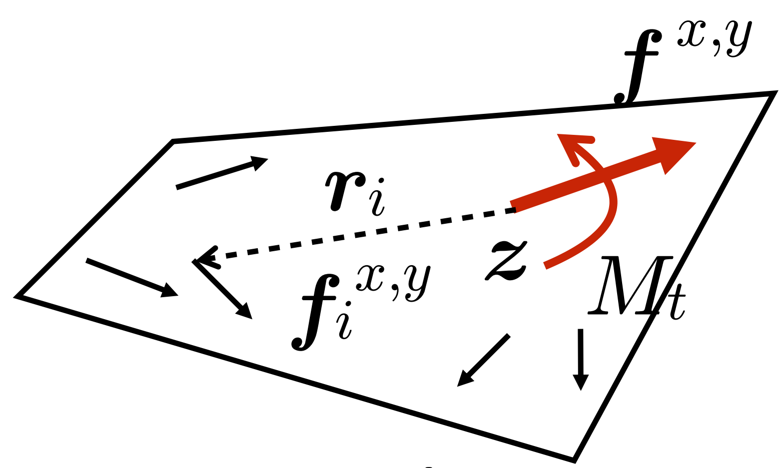
where $\boldsymbol{r}_i$ is the vector from z to the point of application of $\boldsymbol{f}_i^{x,y}$

more on the CoP

the sum of the normal and tangential components gives the resulting GRF

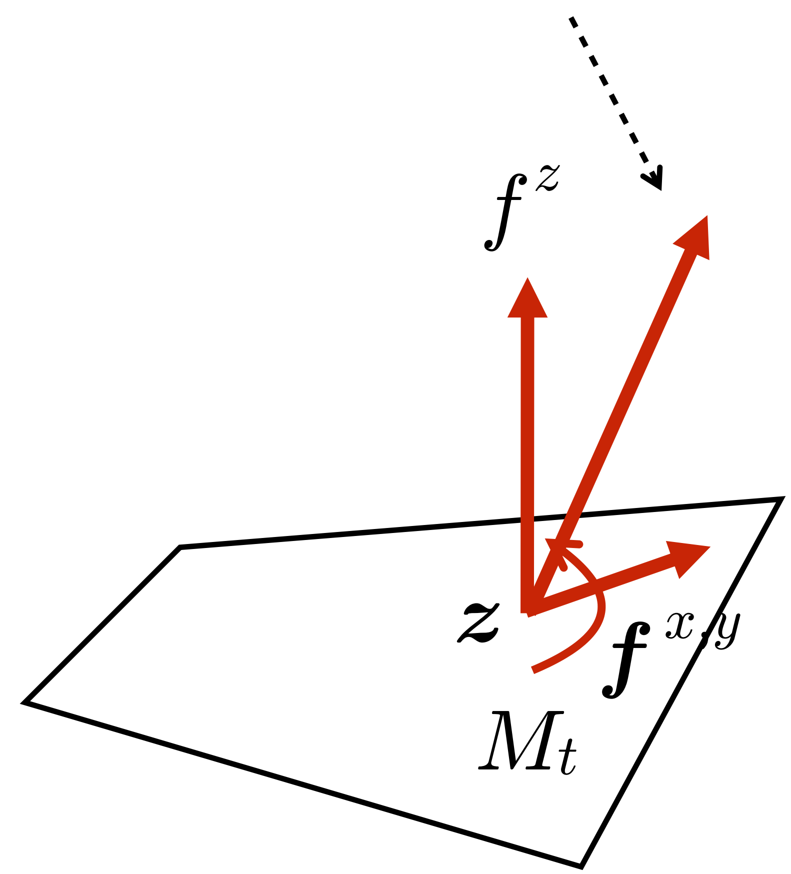


normal
forces



tangential
forces

resulting
GRF



$$f^z = \sum_i f_i^z$$

$$\frac{\sum_i f_i^z \boldsymbol{p}_i^{x,y}}{\sum_i f_i^z} = \boldsymbol{z}^{x,y}$$

$$\boldsymbol{f}^{x,y} = \sum_i \boldsymbol{f}_i^{x,y}$$

$$M_t = \sum_i \boldsymbol{r}_i \times \boldsymbol{f}_i^{x,y}$$

Lagrangian dynamics: multi-body system

$$\boldsymbol{c}^{x,y} - \frac{c^z}{\ddot{c}^z + g^z}(\ddot{\boldsymbol{c}}^{x,y} + \boldsymbol{g}^{x,y}) + \frac{\boldsymbol{S}\dot{\boldsymbol{L}}^{x,y}}{M(\ddot{c}^z + g^z)} = \frac{\sum_i f_i^z \boldsymbol{p}_i^{x,y}}{\sum_i f_i^z}$$

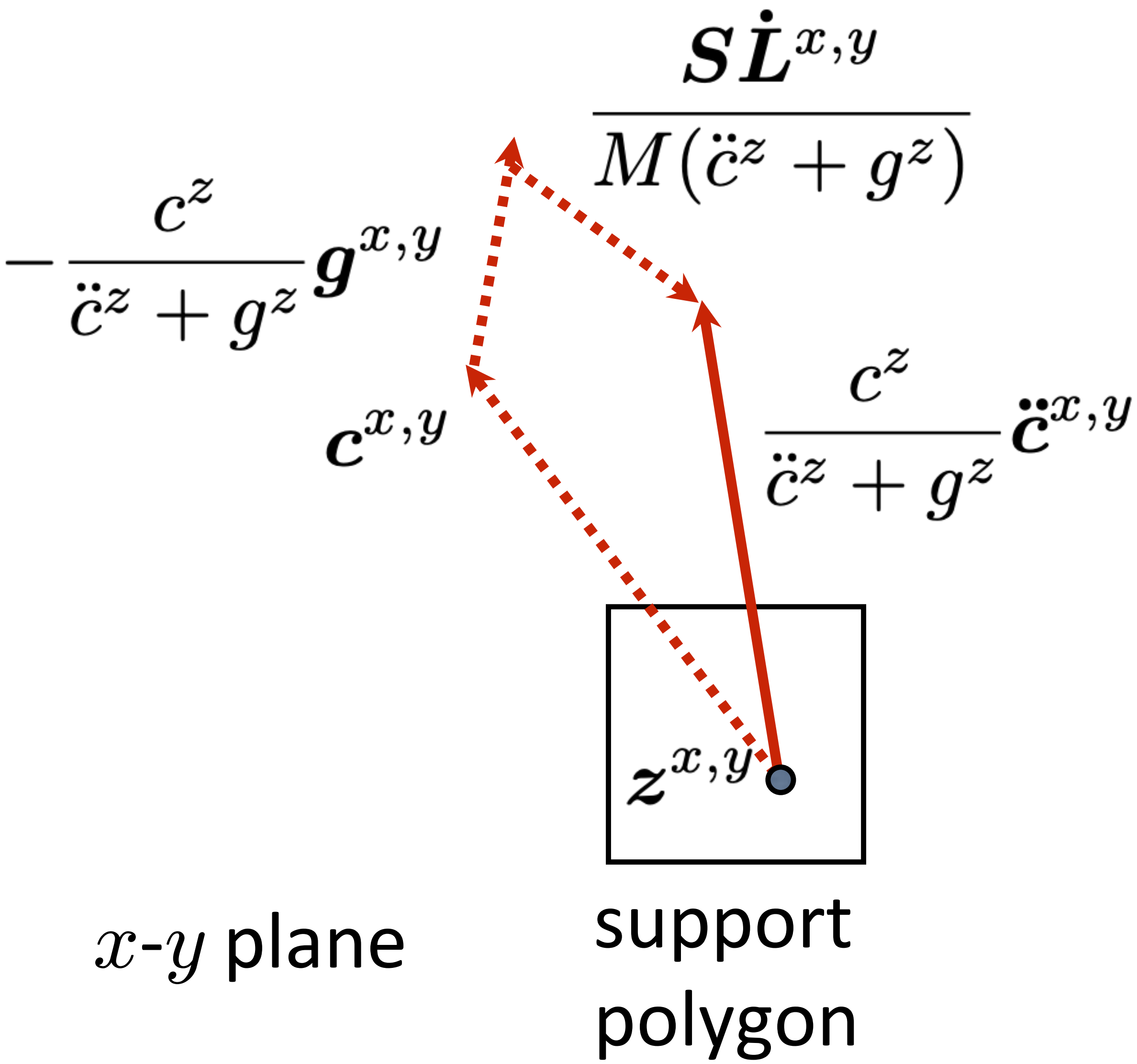
$\Big\downarrow$ rewritten as

$$\frac{c^z}{\ddot{c}^z + g^z}(\ddot{\boldsymbol{c}}^{x,y} + \boldsymbol{g}^{x,y}) = (\boldsymbol{c}^{x,y} - \boldsymbol{z}^{x,y}) + \frac{\boldsymbol{S}\dot{\boldsymbol{L}}^{x,y}}{M(\ddot{c}^z + g^z)}$$

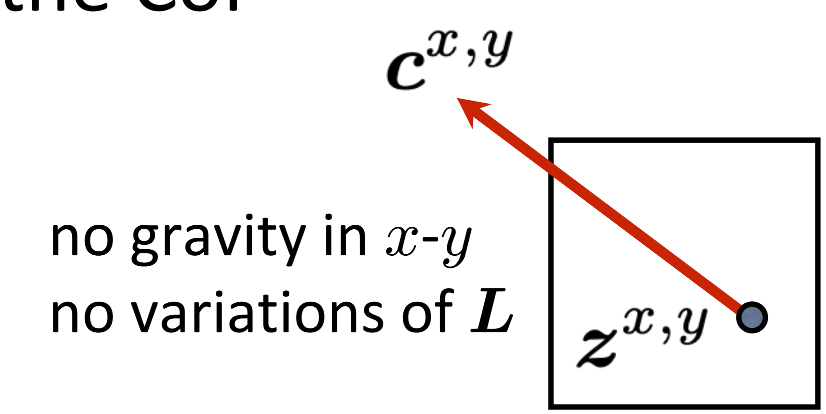
we can analyze the effect of the various terms on the CoM horizontal acceleration (horizontal = in the x-y plane)

Lagrangian dynamics: multi-body system

$$\frac{c^z}{\ddot{c}^z + g^z}\ddot{\boldsymbol{c}}^{x,y} = -\frac{c^z}{\ddot{c}^z + g^z}\boldsymbol{g}^{x,y} + (\boldsymbol{c}^{x,y} - \boldsymbol{z}^{x,y}) + \frac{\boldsymbol{S}\dot{\boldsymbol{L}}^{x,y}}{M(\ddot{c}^z + g^z)}$$



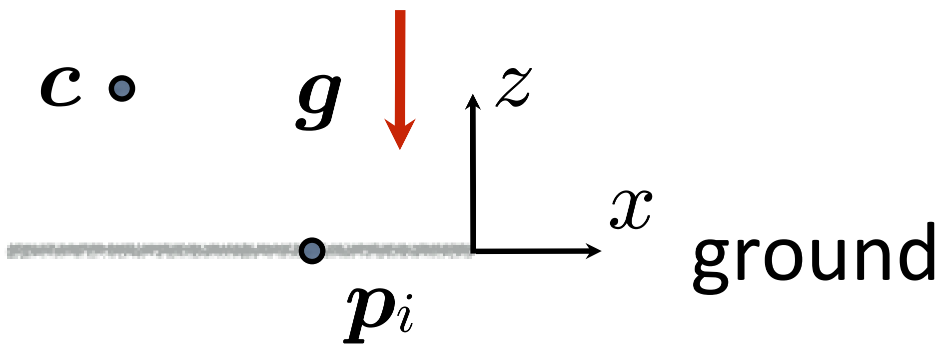
aside from the effect of gravity (horizontal components) and variations of the angular momentum, the CoM horizontal acceleration is the result of a force pushing the CoM away from the CoP



Lagrangian dynamics: approximations

on **horizontal** flat ground + CoM at **constant height** + neglect $\dot{\boldsymbol{L}}^{x,y}$

$$\boldsymbol{g}^{x,y} = 0 \qquad\qquad c^z = \text{constant}$$



$$\frac{c^z}{\cancel{\ddot{c}^z} + g^z}(\ddot{\boldsymbol{c}}^{x,y} + \cancel{\boldsymbol{g}^{x,y}}) = (\boldsymbol{c}^{x,y} - \boldsymbol{z}^{x,y}) + \cancel{\frac{\boldsymbol{S}\dot{\boldsymbol{L}}^{x,y}}{M(\ddot{c}^z + g^z)}}$$

$$\boxed{\boldsymbol{c}^{x,y} - \frac{c^z}{g^z}\ddot{\boldsymbol{c}}^{x,y} = \boldsymbol{z}^{x,y}}$$

or

$$\boxed{\ddot{\boldsymbol{c}}^{x,y} = \frac{g^z}{c^z}(\boldsymbol{c}^{x,y} - \boldsymbol{z}^{x,y})}$$

Linear Inverted Pendulum (LIP)

Linear Inverted Pendulum interpretation

2 independent equations

$$\boxed{\ddot{\boldsymbol{c}}^{x,y} = \frac{g^z}{c^z}\left(\boldsymbol{c}^{x,y} - \boldsymbol{z}^{x,y}\right)}$$

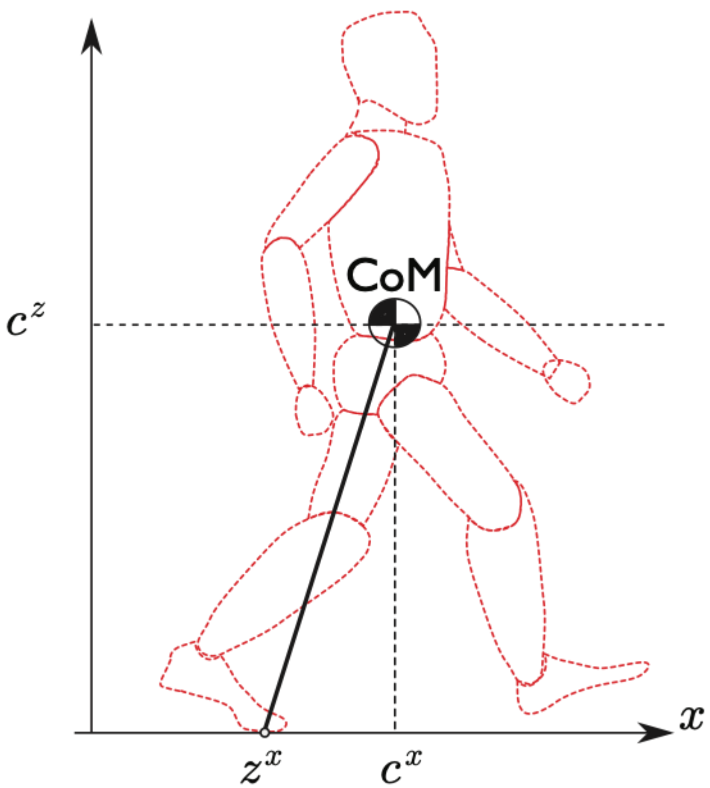
how the CoM moves in

$$\ddot{c}^x = \frac{g^z}{c^z}\left(c^x - z^x\right)$$

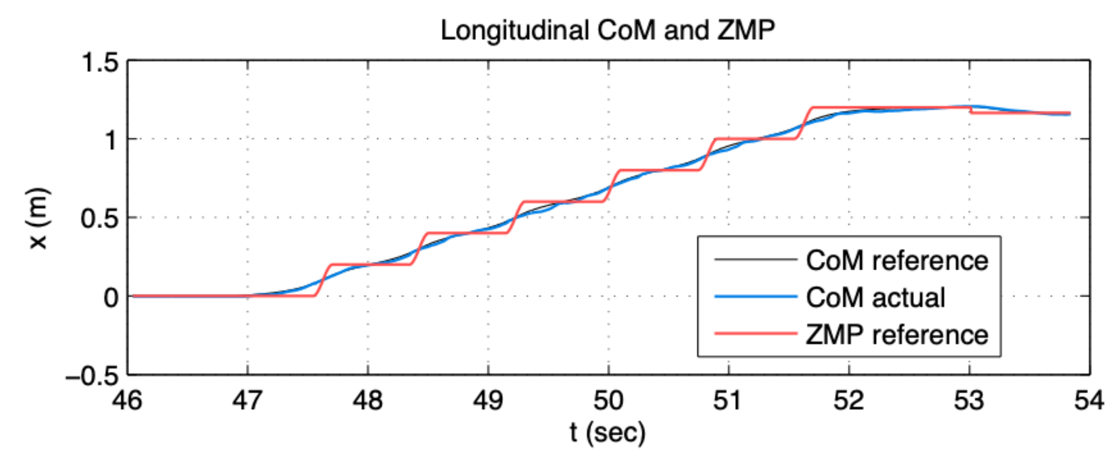
longitudinal direction (sagittal plane)

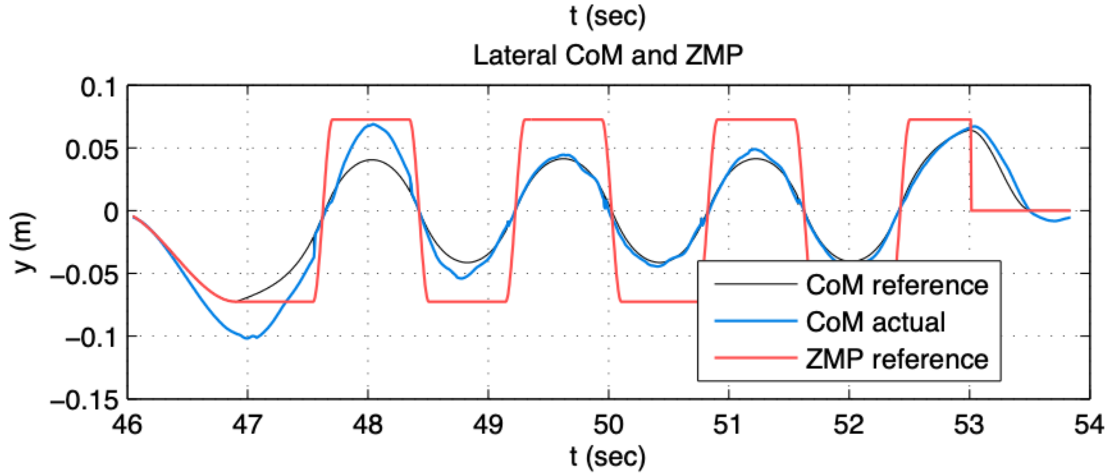
$$\ddot{c}^y = \frac{g^z}{c^z}\left(c^y - z^y\right)$$

lateral direction



typical behaviors

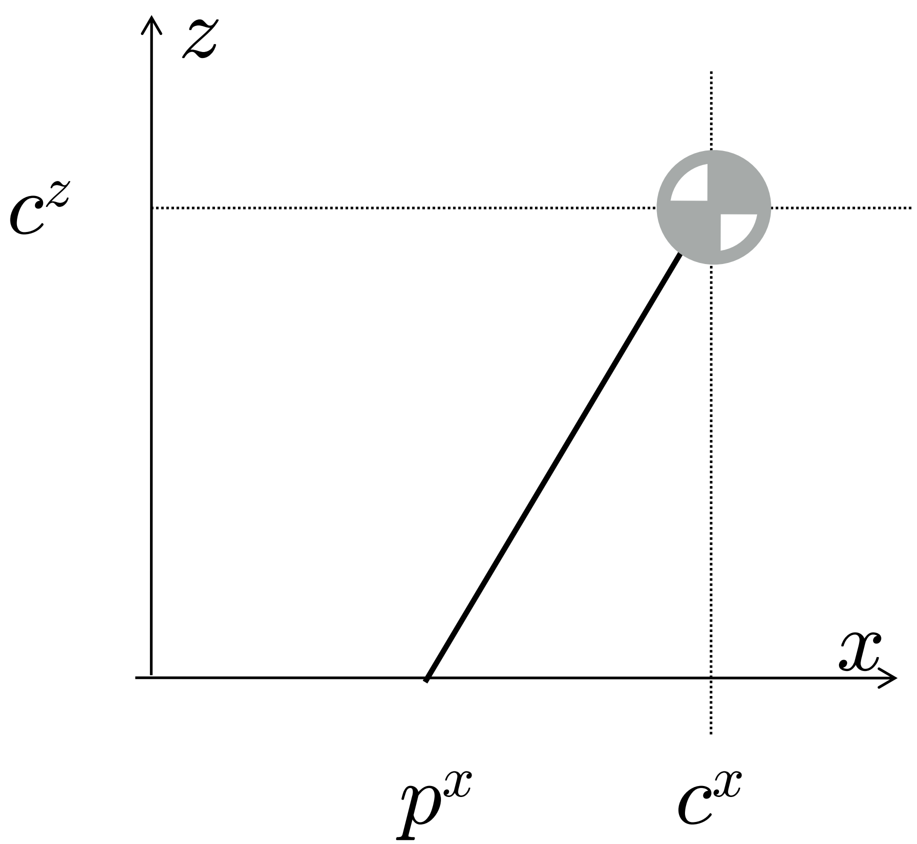




Linear Inverted Pendulum interpretation

- Point foot

  the simplest interpretation of the LIP is that of a telescoping (so to remain at a constant height) massless leg in contact with the ground at p^x (point of contact)



we can interpret the (**longitudinal** direction) LIP equation as a moment balance around p^x

$$M\ddot{c}^x c^z - Mg(c^x - p^x) = 0$$

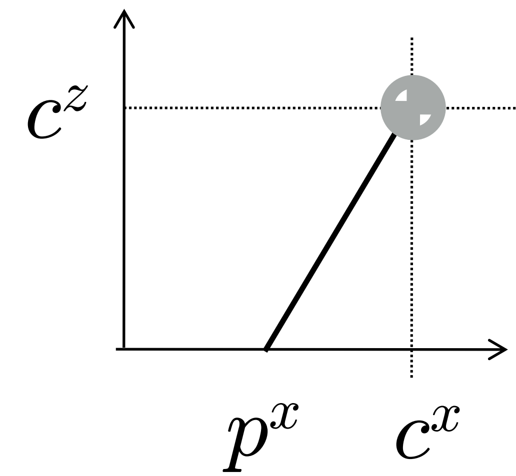
i.e.
$$\ddot{c}^x = \frac{g^z}{c^z}(c^x - p^x)$$

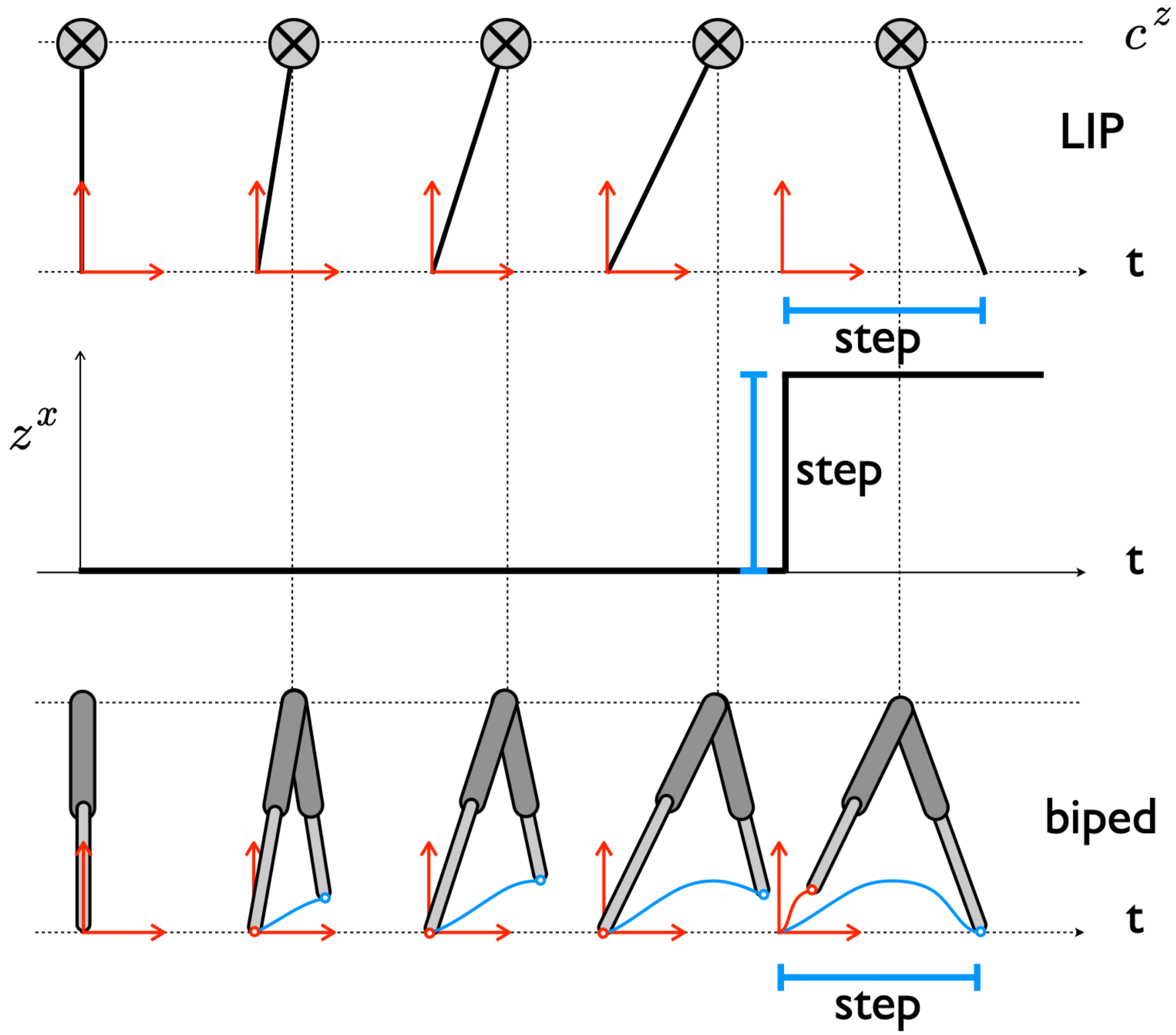
in this case the ZMP z^x coincides with the point of contact p^x of the fictitious leg

$$p^x = z^x$$

Linear Inverted Pendulum interpretation

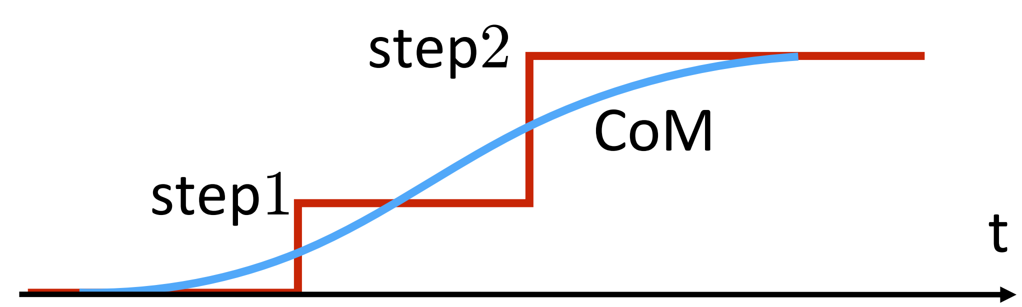
- Point foot (longitudinal direction)





typical footsteps and CoM

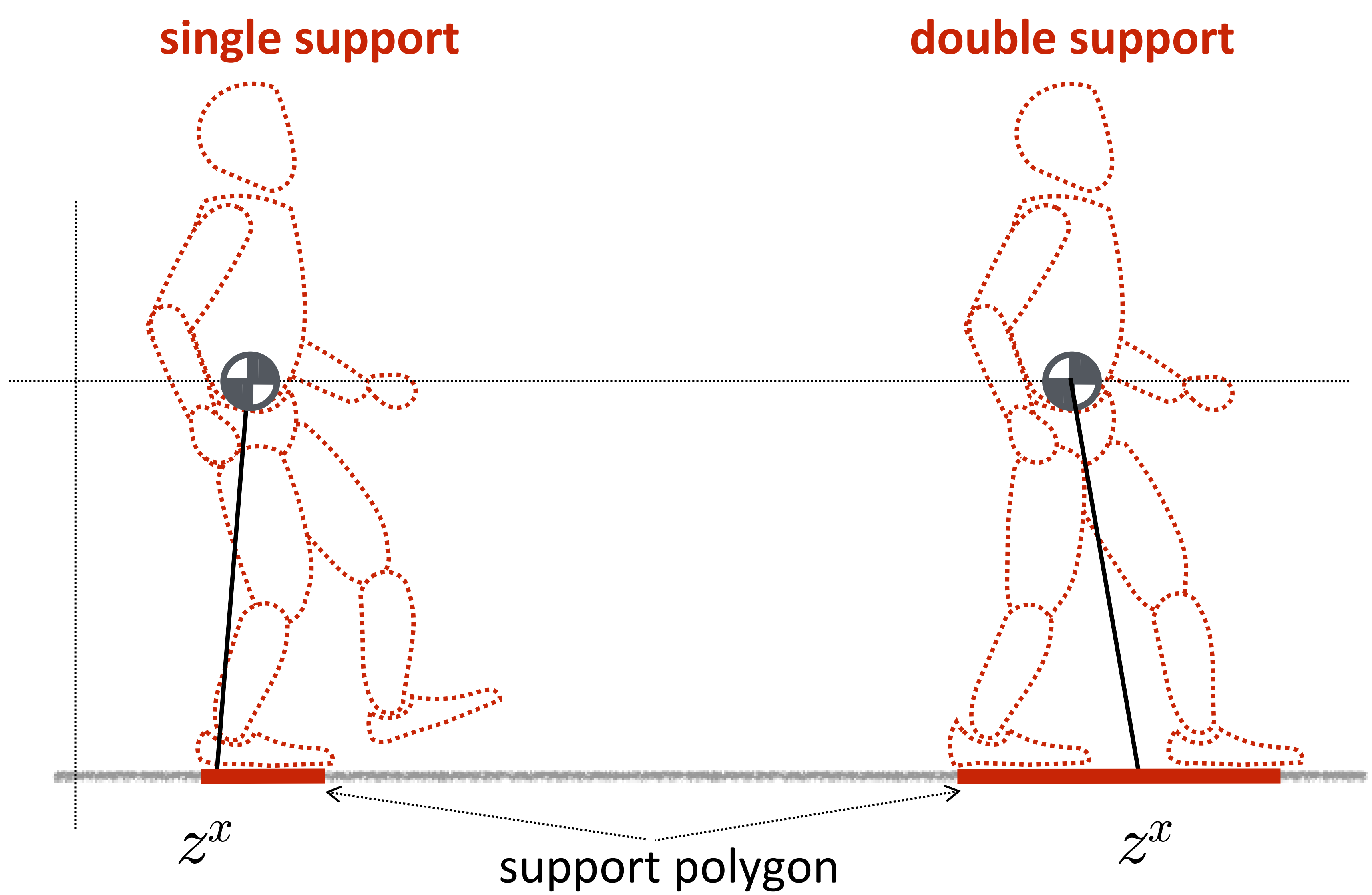
$p^x = z^x$ c^x



may also be seen as a compass biped with only one leg touching the ground at the same time

Linear Inverted Pendulum interpretation

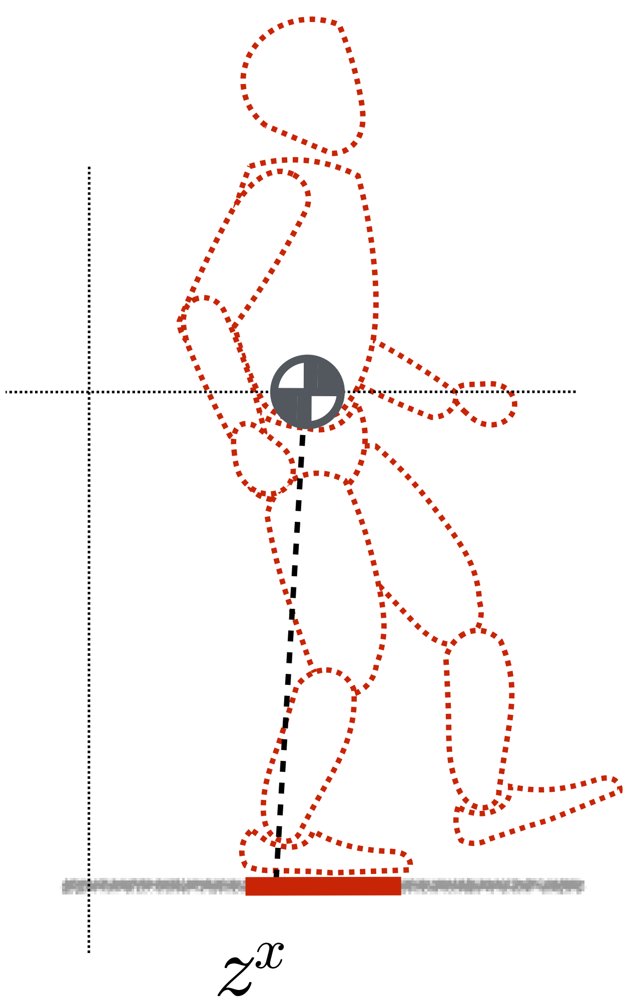
- Finite sized foot (with ankle torque τ_y)
  since z^x represents the ZMP location, there is no difficulty in extending the interpretation of the LIP considering both single and double support phases with a finite foot dimension



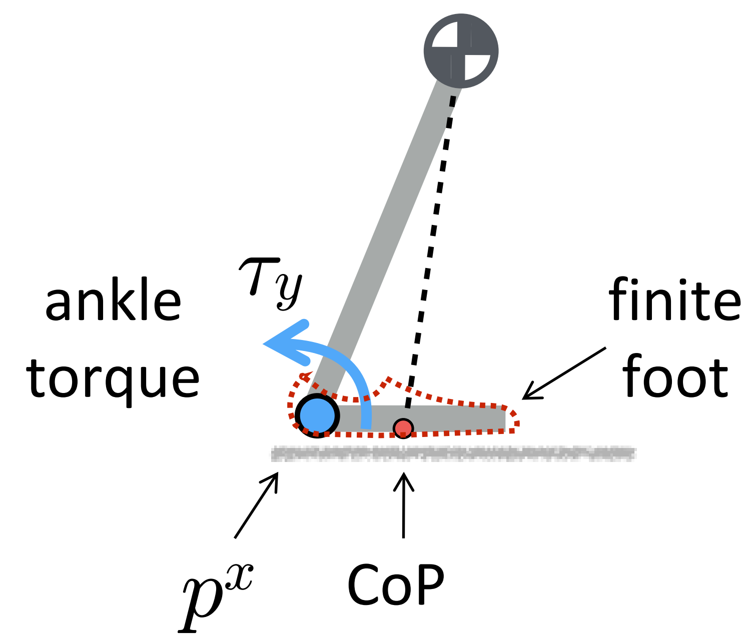
Linear Inverted Pendulum interpretation

- Finite sized foot (with ankle torque τ_y)

  we can see the single support phase from the stance foot point of view i.e. with the dynamics of the rest of the humanoid represented by an equivalent fictitious leg. A way to keep the CoM balanced is using an equivalent ankle torque (the real joint torques are such that an equivalent ankle torque is applied)



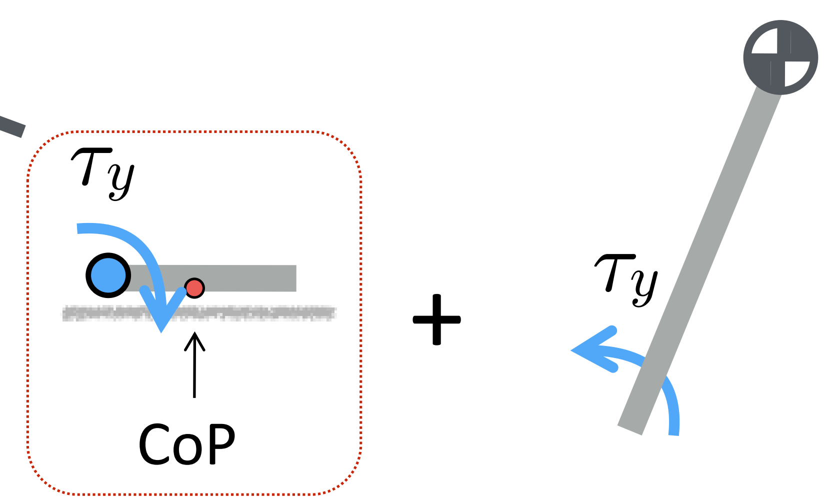
finite foot with
equivalent massless
leg plus ankle torque



moment w.r.t. z^x (CoP) $= 0$

$$Mg(z^x - p^x) - \tau_y = 0$$

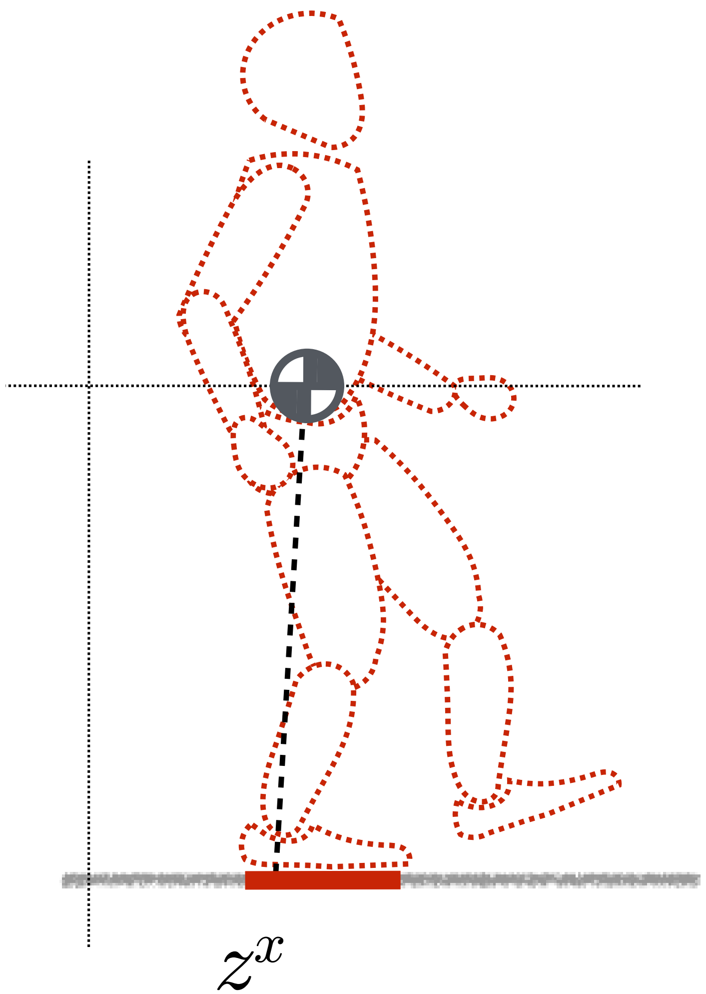
$$z^x = p^x + \frac{\tau_y}{Mg}$$



Linear Inverted Pendulum interpretation

- Finite sized foot (with ankle torque τ_y)
  note: it is possible to move the CoP through the ankle torque τ_y without stepping

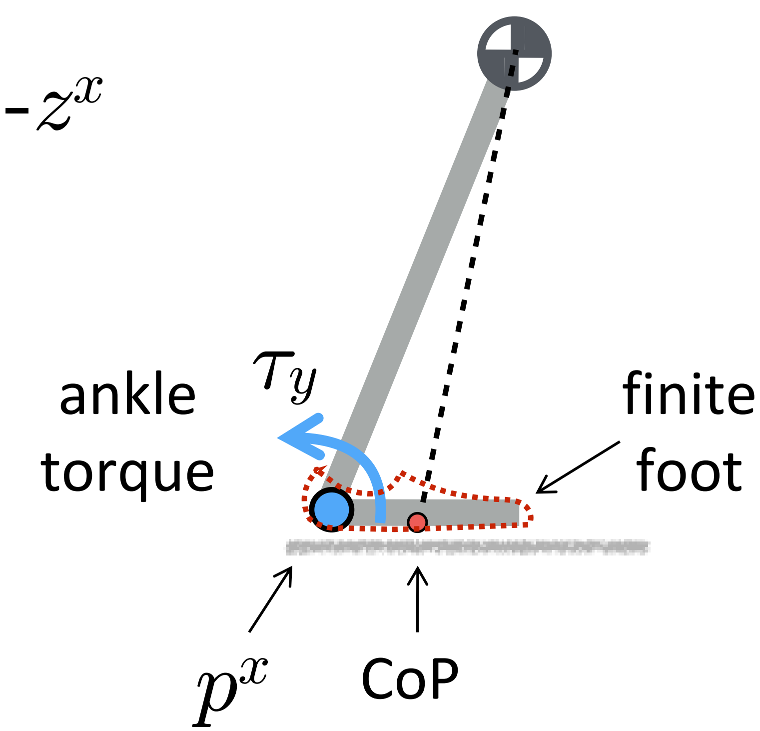
single support



$$M\ddot{c}^x c^z - Mg(c^x - p^x) + \tau_y = 0$$

$$\ddot{c}^x = \frac{g^z}{c^z}\left(c^x \underbrace{- p^x - \frac{\tau_y}{Mg}}_{-\,\text{CoP} = -z^x}\right)$$

finite foot with
equivalent massless
leg plus ankle torque



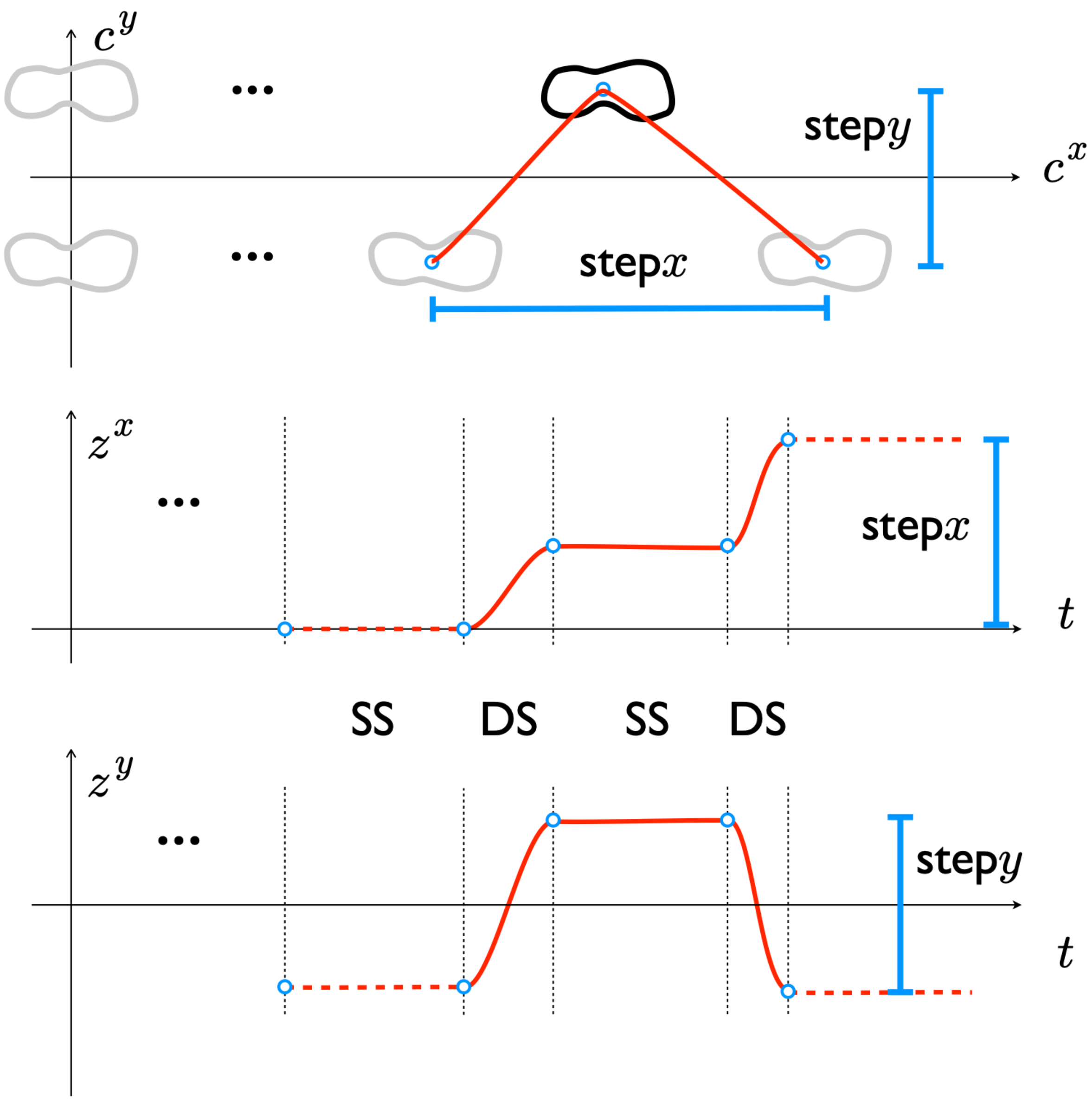
Linear Inverted Pendulum interpretation

- Finite sized foot (with ankle torque τ_y)

with $\quad z^x = p^x + \dfrac{\tau_y}{Mg}$

$$\boxed{\ddot{c}^x = \frac{g^z}{c^z}\left(c^x - z^x\right)}$$

longitudinal direction



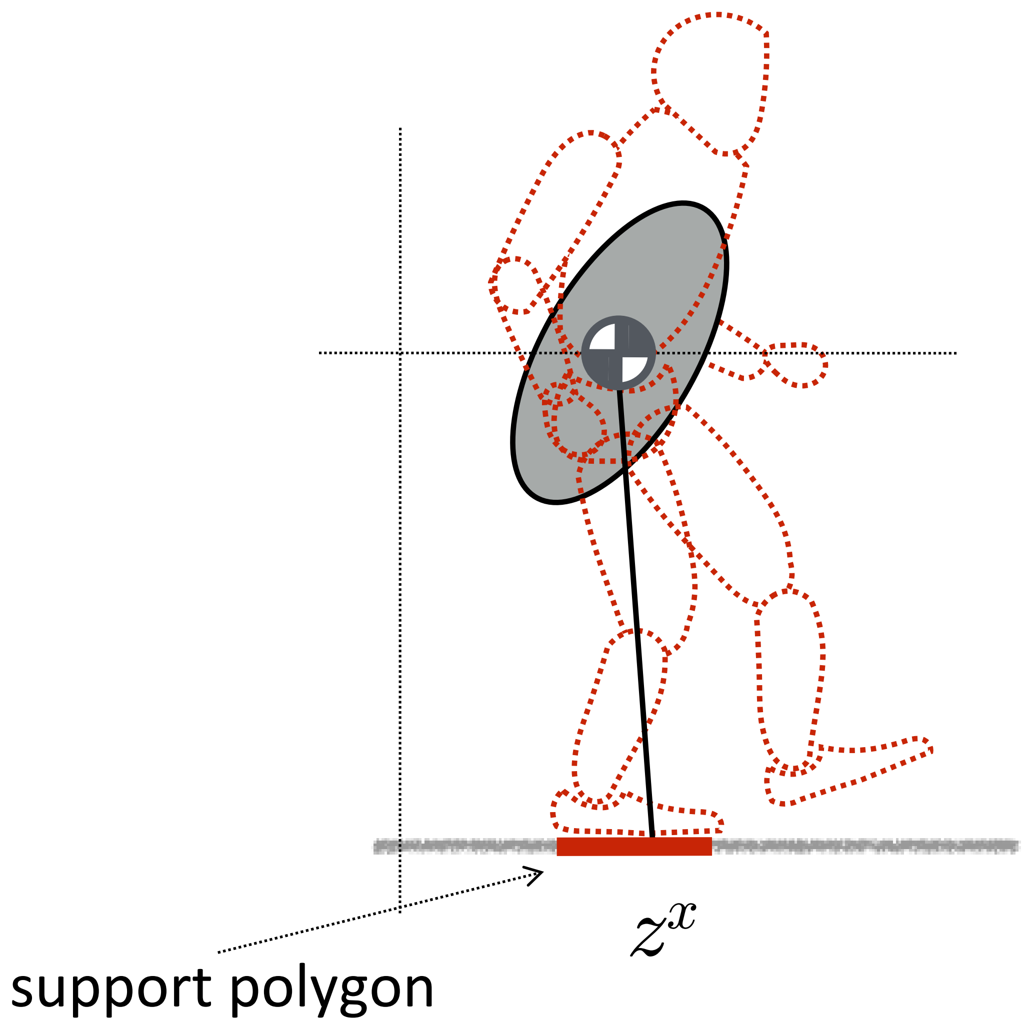
typical footsteps with single and double support:
for example, in the first single support (- -) the left foot is swinging;
as soon as the right foot touches the ground the double support starts (—) and the ZMP moves from the left to the right foot
(longitudinal and lateral motions)

SS: single support
DS: double support

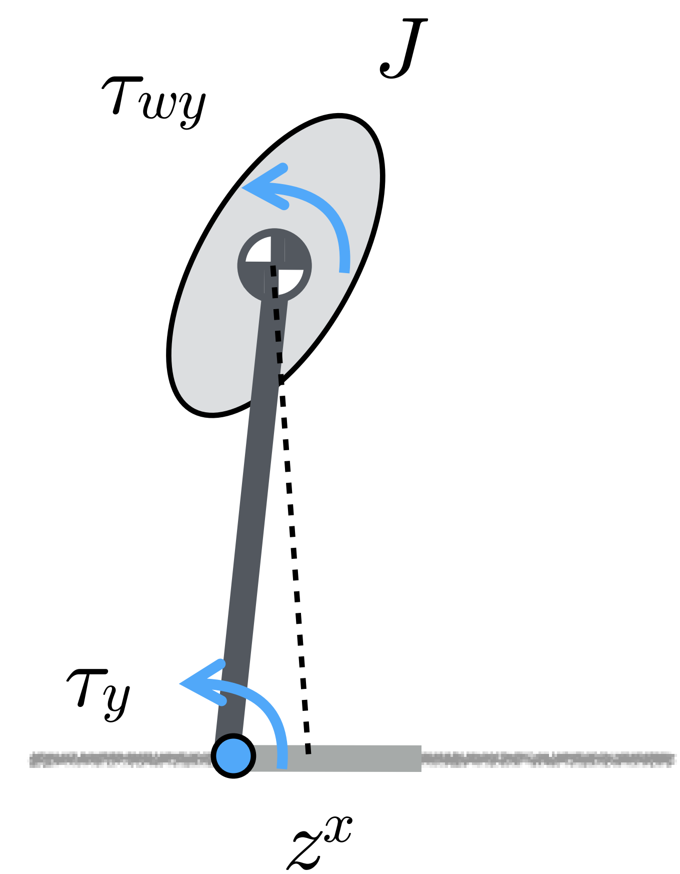
Linear Inverted Pendulum interpretation

- Finite sized foot and reaction mass

  it is also possible to extend the point-mass to be a rigid body with its rotational inertia so that also a hip movement can be modelled



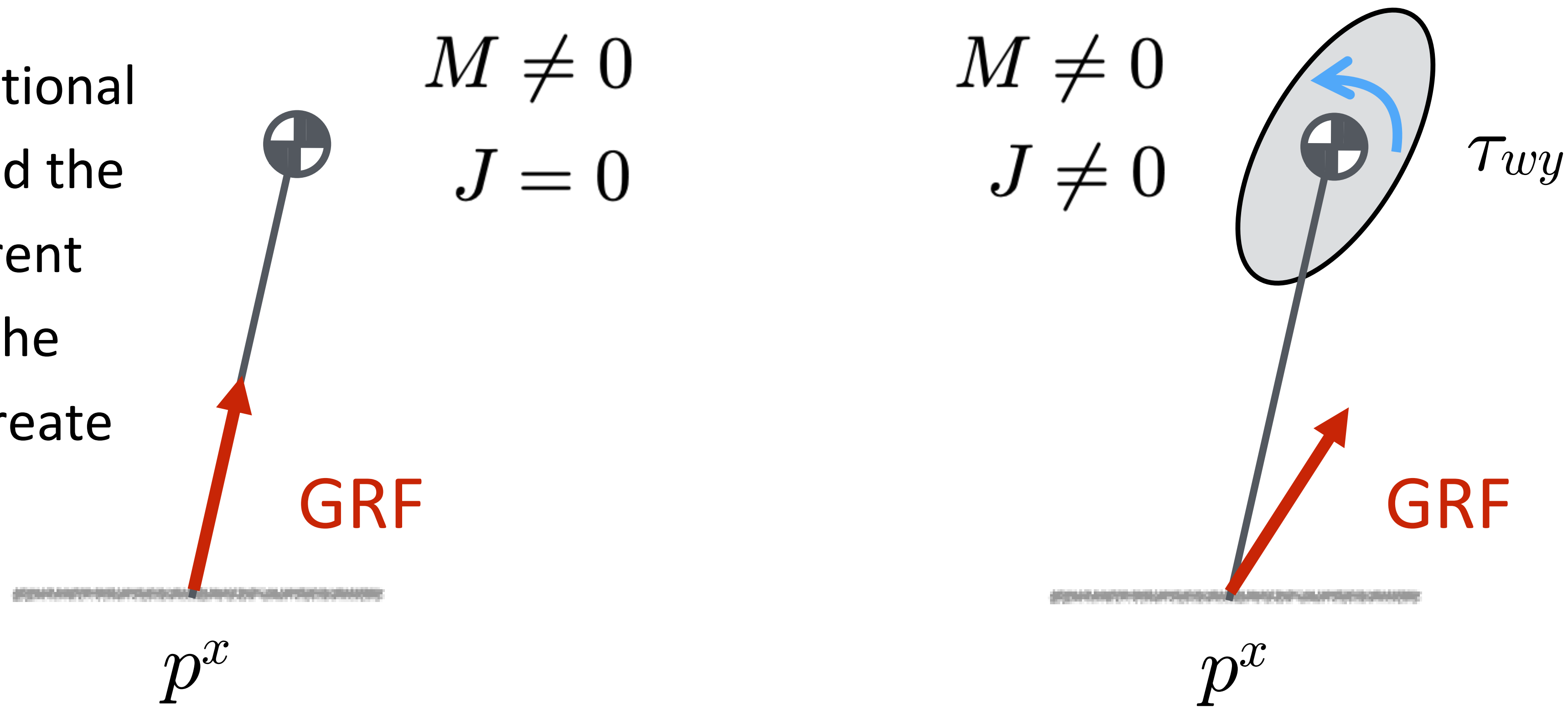
finite foot with equivalent massless leg plus ankle torque and reaction mass



Linear Inverted Pendulum interpretation

- Finite sized foot and reaction mass
  what is the effect of the rotating inertia around the CoM?

since no rotational inertia around the CoM, a different direction of the GRF would create a non-zero moment

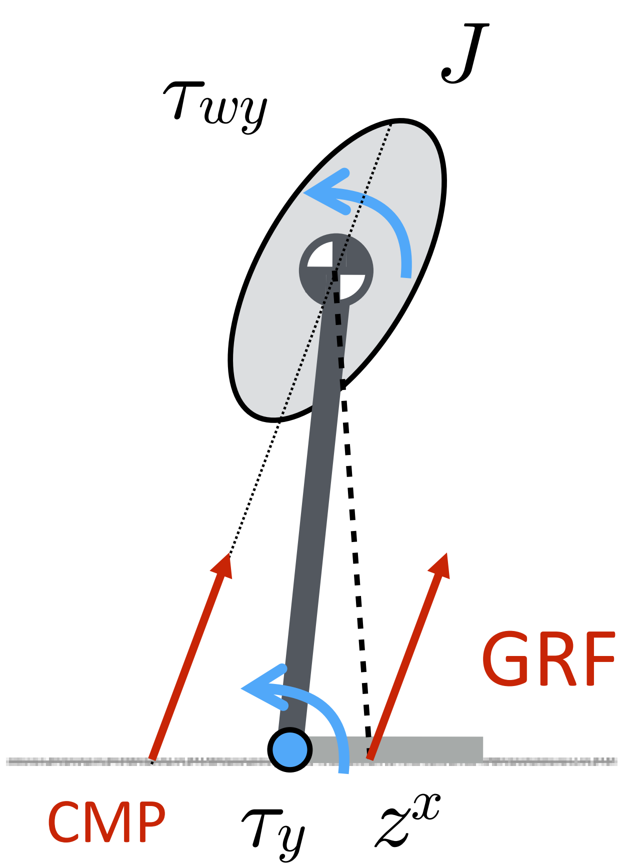


a reaction mass type pendulum, by virtue of its non-zero rotational inertia, allows the ground reaction force to deviate from the lean line. This has important implication in gait and balance.

Linear Inverted Pendulum interpretation

- Finite sized foot and reaction mass

moment around ankle $$M\ddot{c}^x c^z - Mg(c^x - p^x) + \tau_y - \tau_{wy} = 0$$



i.e. $$\ddot{c}^x = \frac{g^z}{c^z}\left(c^x - \underbrace{z^x - \frac{\tau_{wy}}{Mg}}_{\text{- CMP}}\right)$$

with $$z^x = p^x + \frac{\tau_y}{Mg}$$

ankle torque moves the CoP while the reaction mass torque changes the GRF direction

to highlight the presence of a non-zero moment around the CoM a new point named Centroidal Moment Pivot (CMP) is introduced and defined as the point where a line parallel to the ground reaction force, passing through the CoM, intersects with the external contact surface

Linear Inverted Pendulum: basic scope

$$\boxed{\ddot{\boldsymbol{c}}^{x,y} = \frac{g^z}{c^z}\left(\boldsymbol{c}^{x,y} - \boldsymbol{z}^{x,y}\right)}$$

Although extremely simplified, the LIP equation describes in first approximation the time evolution of the CoM trajectory. Moreover

- it defines a differential relationship between the CoM trajectory and the ZMP (or CMP) time evolution
- it is easier to design a controller which makes the actual CoM follow a desired behaviour
- dynamic balancing will be characterized in terms of the ZMP
- the problem will then be to understand which CoM trajectory, solution of the LIP equation, guarantees that dynamic balancing is achieved