Wrap up

a robot subject to K kinematic constraints

$$K \left\{ A(q)\,\dot{q} = 0 \right. \quad (1)$$

where $A(q)$ has n-dimensions and $q \in C$, C has n dimensions, too.

There are 2 possibilities → (*) Holonomic
(*) Non-holonomic

Mobilities: local mobility; global mobility.

If we make a table:

	local mobility	global mobility
Holonomic	restricted	restricted
Non Holonomic	restricted	unrestricted ($q \in C$)

- Why holonomic × local mobility is restricted?

$\dot{q}$ must be in the null space of $A^T(q)$:

$$\dot{q} \in N(A^T(q)) \quad (2)$$

$N(A^T(q))$ is linear subspace of C of dimension $n-K$ (it's also same for NonHolonomic × local mobility)

- Why holonomic × global mobility is restricted?

$q \in C$ must satisfy the integral of (1) and it is a subset of C of dimension $n-K$.

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- Kinematic Models of WMR

$q \in C$, n-dimensional subject to $A(1)$

$$(1) \Rightarrow A^T(q)\dot{q} = 0 \quad \} \text{ K-constraints}$$

i.e. $\dot{q} \in N(A^T(q)) \rightarrow m = n - K$ dimensions

$N(A^T(q))$ is linear space. Because of that we can always find basis for that space which can be given as below:

$$\{ g_1(q), \ldots, g_{n-k}(q) \} \quad (2)$$

(2) is a span for $N(A^T(q))$ (a basis of $N(A^T(q))$).

Additionally, because constraints depend on q, vectors $g_i(q)$, where $i = 1 \div n-k$, are also dependent from q. This leads us to think that, these vectors are not constant. They can change wrt configuration (figure 1). These kind of vectors generate vector fields.

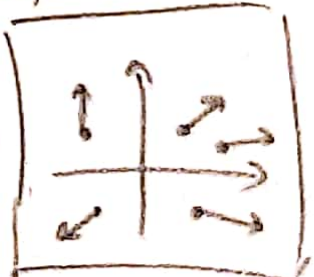

Figure 1. Vector field

Thus we can say each $g_i(q)$ is a vector field.

In order to show linear combinations of vector fields we can write the following equation:

$$\dot{q} \in N(A^T(q)) \Leftrightarrow \dot{q} = \sum_{j=1}^{m} g_j(q)\, u_j \quad (3)$$

where $m \triangleq n - K$ (4)

Equation (3) will lead us to express linear combination of vector fields that generate basis of $N(A^T(q))$, where u_j is scalar coefficients.

On the other hand, the equation (3) is the Kinematic Model of robot expresses all the admissible velocities that robot can have at q configuration.

We can demonstrate it in figure 2.

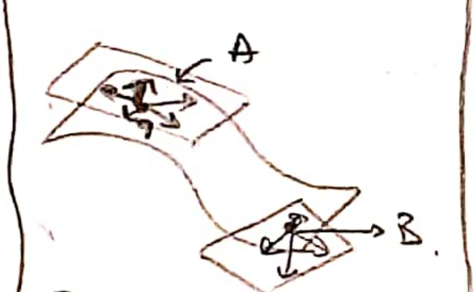

Figure 2.

Assume that robot is in config. q_A. Then all possible velocities that robot can have are only linear combinations of the vectors on tangent plane to the surface at point A. Then, if we move robot to point B, it will be in the config. q_B. All possible velocities will be the linear combinations of all vectors on that tangent plane which coincides with the surface at point B.

Note: All velocities that we're talking about are generalized velocities.

Additionally we can rewrite the equation (5) as following:

$$\dot{q} = \underbrace{\begin{pmatrix} g_1(q) & \cdots & g_m(q) \end{pmatrix}}_{G(q)} \underbrace{\begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix}}_{u} \quad (5)$$

$$\dot{q} = G(q)_{n\times m} \, u_{m\times 1} \quad (6)$$

Each $g_i(q)$ has n-dimensions. Notice that because of (4), $n > m$ which leads us to know that $G(q)$ is tall matrix.

This system is dynamical system. General description of dynamical systems is expressed as below:

$$\left.\begin{aligned} &\dot{x} = Ax + Bu \rightarrow \text{linear} \\ &\dot{x} = f(x) + \sum_{i=1} g_i(x) u_i \end{aligned}\right\} (7)$$

$\rightarrow$ nonlinear

In (7) x is state and u is input.

If we analyze equations (3) and (5) what will we have?

→ In our case q is the state of system.
$\boxed{q\,(n\text{-dim})}$

→ In dynamical systems, generally, inputs are considered such variables that can be chosen arbitrarily. In our case those are u_j coefficients. : $\boxed{u\,(m\text{-dim})}$

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To sum up, it is obvious that our system is dynamical system. Why?
Because,

→ It tells us how the configurations of the system can enroll what are the
- possible velocities ($\dot{q}$)
- given the current value of config. (q)
- and given the coefficients that we choose (u)

According to the equations (3), (7), we see that the system is non-linear system (because of $g(q)$ function).

Additionally, it is driftless system. Why?
Let's check the equation (7):

$$(7) \Rightarrow \dot{x} = f(x) + \sum_{i=1} g_i(x) u_i$$

If input is zero ($u_i = 0$) $\dot{x} = f(x)$ which leads the system to move. Thus we call $f(x)$ as a drift.

If we analyze the equation (3), we'll see that if $u_i = 0$, then system will stop as well. Thus the dynamical model that we have is driftless.

In other words, driftless systems have not motion if they have not inputs.

If a robot is unconstrained (i.e. it has not got any kinematic constraints), $\dot{q}$ can be arbitrary (because there will not be any null space that $\dot{q}$ has to belong to). We can write it as below:

$$\dot{q} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} u_1 + \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} u_2 + \dots \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} u_n \quad (8)$$

Explanation of (8):

$\dot{q}$ can be expressed in any combination of arbitrary u_i $(i = 1 \div n)$ in n-dimensional configuration space, because there aren't any constraints.

We can rewrite (8) as below:

$$\dot{q} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} = I u = u \quad (9)$$

According to (9), it's seen that $\dot{q}$ can be any u. Additionally we can show (9) as below:

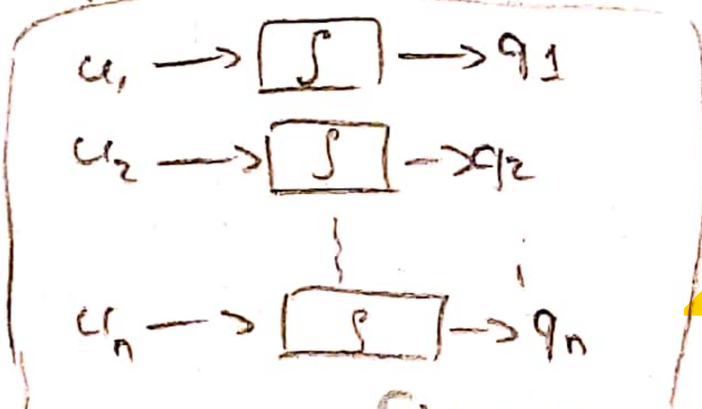

Figure 3.

⇒ the Kinematic model of an unconstrained robot is a set of n simple integrators

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Note: There is not a single Kinematic Model that can express our system. Thus we can say $G(q)$ and u aren't chosen uniquely.

Unicycle robots (the robot with 1 cycle)

The model can be designed as "rolling coin".

Note: It's ideal case. We know that with single wheel it can fall. But we assume it will not fall.

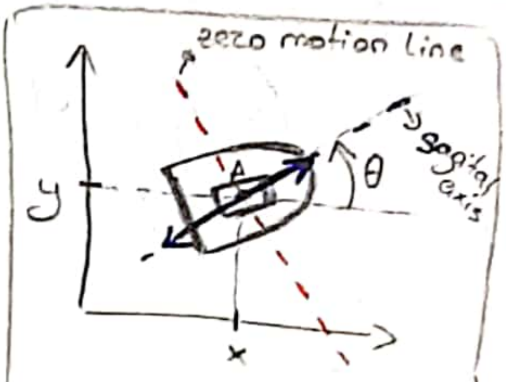

Figure 4: Unicycle

Note 2: A is contact point, and we model our system wrt to it.

$$q = \begin{pmatrix} x \\ y \\ \theta \end{pmatrix} \quad (9) \qquad ; \quad n = 3 \quad (10)$$

$$\mathcal{C} = \mathbb{R}^2 \times SO(1)$$ → we have planar motion in 3-1 3dimensional

We will have same definition.

→ robot cannot move along the zero motion line (with red in fig 4)

→ robot can move along the "sagital axis) with direction of blue arrows in fig 4.

→ cycle's move is rolling without slipping (or pure rolling)

↳ It's a constraint that we can write:

$$\dot{x} \sin\theta - \dot{y} \cos\theta = 0 \quad (11)$$

$$(11) \Rightarrow \begin{pmatrix} \sin\theta & -\cos\theta & 0 \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix} = 0 \quad (12)$$

From (12):

$$A^T(q) = (\sin\theta \quad -\cos\theta \quad 0) \quad (13)$$

We have 1 constraint (A^T has 1 row).
Thus $\dot{q} \in N(A^T(q))$, where $N(A^T(q))$ is a 2 dimensional linear space. We can implement the equation (3) as below:

$$\dot{q} = g_1(q)u_1 + g_2(q)u_2 \quad (14)$$

As we said before choice of $G(q)$ matrix (i.e. $g(q_i)$ where $i = 1, \dots, m$) is not unique. We can choose $g_1(q)$, $g_2(q)$ as following:

$$g_1(q) = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix} (15) \quad g_2(q) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} (16)$$

We can check (15) and (16) in such way that:
If we multiply → (13) and (15) it'll be zero
↳ (13) and (16) it'll be zero.

Thus our kinematic model will be:

$$\dot{q} = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix} u_1 + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u_2 = \underbrace{\begin{pmatrix} \cos\theta & 0 \\ \sin\theta & 0 \\ 0 & 1 \end{pmatrix}}_{G} \underbrace{\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}}_{u} \quad (17)$$

(17) is Kinematic model of unicycle.

(17) can be written also as following:

$$\left.\begin{aligned} \dot{x} &= \cos\theta\, u_1 \\ \dot{y} &= \sin\theta\, u_1 \\ \dot{\theta} &= u_2 \end{aligned}\right\} (18)$$

We can analyze (17) as below,

$$(17) :> \quad \dot{q} = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix} u_1 + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u_2 .$$

Why the blue arrow (vector) is $g_1(q)$?

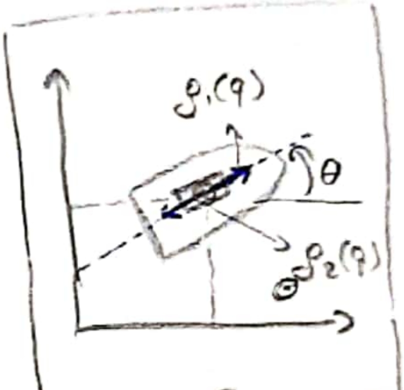

Figure 5

→ It drives the robot $\sin\theta$ in y, $\cos\theta$ in x direction. The 3rd element is zero, because the motion is planar.

→ $g_1(q)$ is unit vector of sagital axis

→ $g_1(q)$ is called as Drive Vector Field

What is $g_2(q)$?

→ $g_2(q)$ is the unit vector coming out of the page.

→ If We apple only u_2, it will be,

→ 0 motion on x, 0 motion on y, we change only θ.

→ we only change the direction of unicycle without driving !

→ $g_2(q)$ is called Steer Vector Field

Now we'll discuss limitations of u_1 and u_2.

Interpretation of u_1, u_2:

$$\dot{x}^2 + \dot{y}^2 = u_1^2 \quad (19) \Rightarrow u_1 = \pm\sqrt{\dot{x}^2 + \dot{y}^2} \quad (20)$$

(20) is the cartesian velocity vector's expression (Figure 6).

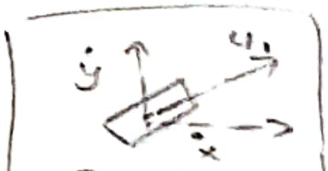

Figure 6

Thus, u_1 is the magnitude (with sign) of the Cartesian velocity of the **contact point**.

We can get angular velocity of wheel too, by using this information. (Figure 7)

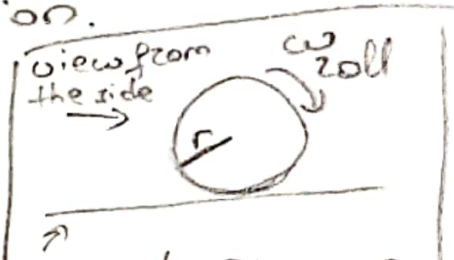

Figure 7.

$$\sqrt{\dot{x}^2 + \dot{y}^2} = u_1 = \omega_{roll}\, r \quad (21)$$

In (21) : $\omega_{roll} \rightarrow$ ang. vel of rolling
$r \rightarrow$ radius of wheel.

Thus we call u_1 as **driving Velocity.**

What is u_2?

$u_2 = \dot{\theta} \rightarrow$ the angular velocity of the wheel around the vertical axis $\rightarrow$ We call it **Steering Velocity**!

We can rewrite **Kinematic Model as below**:

$$\dot{q} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix} v + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \omega \quad (22)$$

In (22) : $\begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix} \rightarrow$ drive vf ; $v \rightarrow$ driving velocity ; $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \rightarrow$ steer vf ; $\omega \rightarrow$ steering velocity

In order to distinguish v, ω and generalized velocities, v, ω are called as PSEUDO VELOCITIES. Because, actual velocity of robot (generalized) is $\dot{q}$.

v, ω are Velocity Inputs!

$\dot{q}$ is generalized velocity (acc. to config.)

Now we can discuss that is this model Holonomic or Non-Holonomic? It depends on the integrability of the "rolling without slipping" constraint. Then we can ask this:

→ Is the rolling without slipping constraint holonomic or nonholonomic?

→ We already know it is Non Holonomic! (constructive controllability)

• <u>Bicycle model</u>

The model is demonstrated in Figure 8.

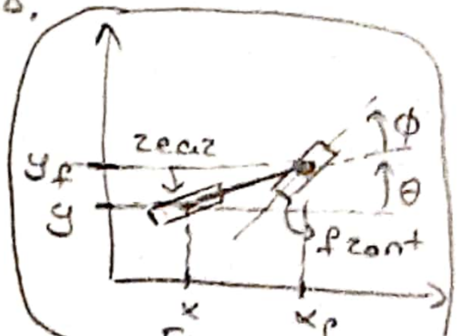

Figure 8

We define general coordinates of model as following:

$$q = \begin{pmatrix} x \\ y \\ \theta \\ \phi \end{pmatrix} \quad (23)$$

For this model $n = 4$. The configuration space of this model is cartesian product of $\mathbb{R}^2$ (because of x, y) and $SO(2)$ (because of θ) and $SO(2)$ (because of ϕ).

• Constraints:

1st → Rear wheel cannot slip
2nd → Front wheel cannot slip } 2 constraints.

We can define the 1st constraint as below:

RWS for rear ⇒ $\dot{x}\sin\theta - \dot{y}\cos\theta = 0.$ (24)

We can define the 2nd constraint as below

RWS for front → $\dot{x}_f \sin(\theta+\phi) - \dot{y}_f \cos(\theta+\phi) = 0$ (25)

According to the equation (23) we need 4 variables. Thus we need to define $\dot{x}_f$ and $\dot{y}_f$ in terms of our general coordinates. According to the Figure 8:

$$x_f = x + l\cos\theta \quad (26)$$
$$y_f = y + l\sin\theta \quad (27)$$

Then (25) can be rewritten as below:

(25) ⇒ $(\dot{x} - l\dot{\theta}\sin\theta)\sin(\theta+\phi) - (\dot{y} - l\dot{\theta}\cos\theta)\cos(\theta+\phi) = 0$ (28)

(28) ⇒ $\dot{x}\sin(\theta+\phi) - \dot{y}\cos(\theta+\phi) - l\dot{\theta}\cos(\theta+\phi-\theta) =$

$= \dot{x}\sin(\theta+\phi) - \dot{y}\cos(\theta+\phi) - l\dot{\theta}\cos\phi = 0$ (29).

(29) is the compact form of RWS for front wheel

We can rewrite these constraints in matrix form in order to get the form of equation (1):

$$\underbrace{\begin{pmatrix} \sin\theta & -\cos\theta & 0 & 0 \\ \sin(\theta+\phi) & \cos(\theta+\phi) & -l\cos\phi & 0 \end{pmatrix}}_{A^T(q)} \underbrace{\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi} \end{pmatrix}}_{\dot{q}} = 0 \quad (30)$$

We can generate the kinematic model according to the null space of $A^T(q)$.
$A^T(q)$ has 2 rows and 4 columns. Thus $N(A^T(q))$ will have a basis with 2 vectors.

A basis for $N(A^T(q))$:

$$g_1(q) = \begin{pmatrix} \cos\theta \\ \sin\theta \\ \tan\phi/\ell \\ 0 \end{pmatrix} \quad (31) \qquad g_2(q) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad (32)$$

Why we define (31) like that?

We need 4 elements in $g_1(q)$ such that gives us zero for 2 rows of $A^T(q)$. 3^{rd} element can be anything in order to get 0 while we multiply it with the 1^{st} row of $A^T(q)$. However we need such value that can also give us zero while we multiply it with the 2^{nd} row of $A^T(q)$ matrix. Thus we wrote $\frac{\tan\phi}{\ell}$ for the 3^{rd} element of $g_1(q)$

Additionally, $g_1(q)$ is DRIVE vf and $g_2(q)$ is STEER vf.

To sum up we can write kinematic model as following 2 equations:

$$\dot{q} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \\ (\tan\phi)/\ell \\ 0 \end{pmatrix} u_1 + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} u_2 \quad (33)$$

or

$$\dot{q} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} \cos\theta & 0 \\ \sin\theta & 0 \\ (\tan\phi)/\ell & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \quad (34)$$

What are u_1 and u_2?

$$u_1^2 = \dot{x}^2 + \dot{y}^2 \quad (35) \quad \text{or} \quad u_1 = \pm\sqrt{\dot{x}^2 + \dot{y}^2} \quad (36)$$

According to (35) and (36) u_1 is a good choice for REAR WHEEL DRIVE. Thus u_1 is driving velocity:

$$u_1 = v \quad (37)$$

Additionally, from (33) it's seen that

$$u_2 = \dot{\phi} \quad (38)$$

From (38), u_2 is steering velocity.

<u>exercise</u>: Derive Kinematic model of FWD (Forward Wheel Drive).