• Statically balanced robots equivalent to the bicycle.

There are 2 kind of robots:

→ Tricycle robot is demonstrated in the figure 1.

In Figure 1, arrow A points the differential's position if it is RWD model.

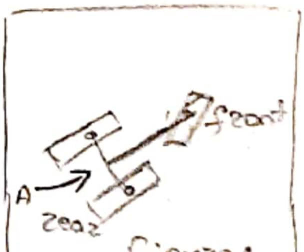

Figure 1

In Figure 2, car-like robot is given. This kind of robots have 1 differential:

→ front → FWD

→ rear → RWD

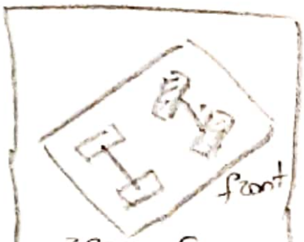

Figure 2

Now we can investigate ICR of car-like robot. Assume that the robot has such pose as given in Figure 3. It's obvious that, if front wheels are parallel, then there will be no ICR, because zero motion lines of right and left front wheels will not intersect. If there is no ICR, it leads us to know this robot is slipping (i.e. No ICR → slippage)

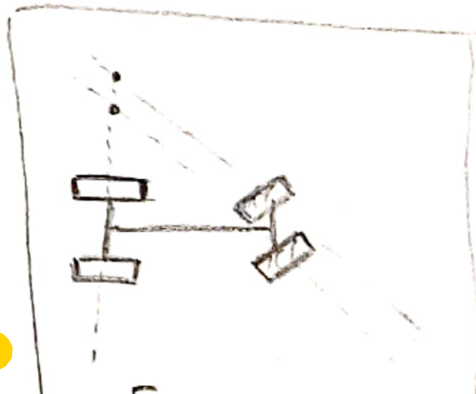
Figure 3

On the other hand, if front wheels are not parallel, then there will be ICR. In this case front wheels have different rotations. (Figure 4). This case is called as Ackermann steering.

Figure 4

Ackerman steering allows two wheels to assume different rotations were required. This mechanism allows wheels to have different orientations and exact ICR point. In this case, the robot never slips (i.e. Ackermann steering -> no slippage)

General coordinates of those robots will be analyzed as following:

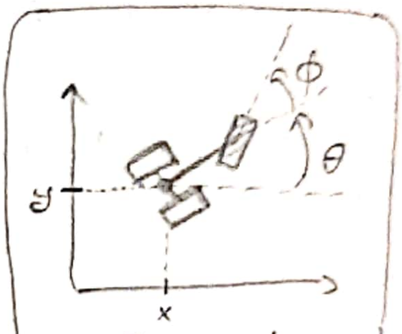

Figure4

Tricycle :

According to figure4 :

$$q = \begin{pmatrix} x \\ y \\ \theta \\ \phi \end{pmatrix} \quad (1)$$

Kinematic models (RWD, FWD) will be same as bicycle

<u>Car like robots.</u>

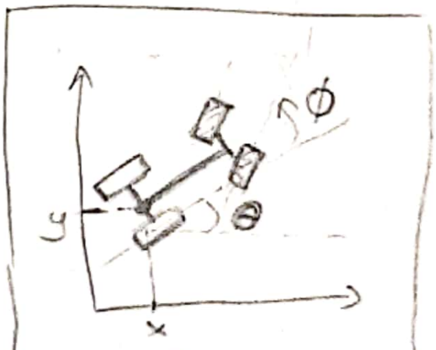

Figure5

According to Figure 5, q will be like equation 1;

Kinematic models (RWD, FWD) will be same as bicycle.

exercises

- Tricycle with trailer

The model is given in Figure 6a. In order to define Kinematic Model we will generalize the model Figure 6b In order to generate Kinematic Model of robot we will follow 2 steps are below:

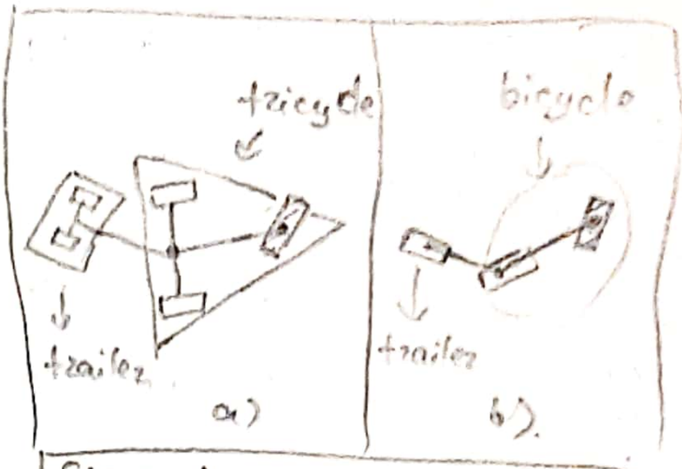

Figure 6

- write the kinematic constraints
- derive kinematic model.

In order to write kinematic constraints we'll analyze Figure 6b in Figure 7.

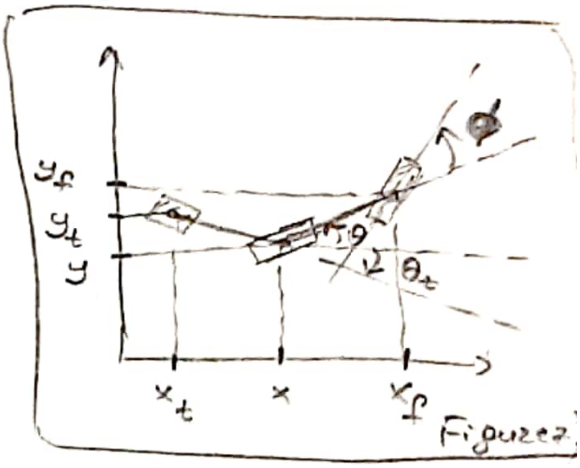

Figure 7

According to Figure 7 generalized coordinates will be:

$$q = \begin{pmatrix} x \\ y \\ \theta \\ \phi \\ \theta_t \end{pmatrix} \quad (2)$$

According to the (2), we see that $n = 5$. On the other hand, there are 3 wheels which means 3 RW/S constraints. We can write them wrt each wheel's coordinates, as below:

$$\dot{x} \sin\theta - \dot{y} \cos\theta = 0 \quad (3)$$

$$\dot{x}_f \sin(\theta + \phi) - \dot{y}_f \cos(\theta + \phi) = 0 \quad (4)$$

$$\dot{x}_t \sin\theta_t - \dot{y}_t \cos\theta_t = 0 \quad (5)$$

(3) → rear wheel, (4) → front wheel and (5) → trailer wheel.

In next step we will define (4) and (5) wrt generalized coordinates by using the following equations, which are derived from Figure 2.

$$x_f = x + l\cos\theta \quad (6)$$
$$y_f = y + l\sin\theta \quad (7)$$
$$x_t = x - l_t\cos\theta_t \quad (8)$$
$$y_t = y - l_t\sin\theta_t \quad (9)$$

Note: l and l_t are rigid bodies' length → constant.

From (6), (7), (8) and (9), we get (10), (11), (12) and (13), respectively.

$$\dot{x}_f = \dot{x} - l\dot{\theta}\sin\theta \quad (10)$$
$$\dot{y}_f = \dot{y} + l\dot{\theta}\cos\theta \quad (11)$$
$$\dot{x}_t = \dot{x} + l_t\dot{\theta}_t \sin\theta_t \quad (12)$$
$$\dot{y}_t = \dot{y} - l_t\dot{\theta}_t \cos\theta_t \quad (13)$$

Transforming the definitions of constraints wrt generalized coordinates.

- Using (10) and (11) in (4):

$$(\dot{x} - l\dot{\theta}\sin\theta)\cos(\theta+\phi) - (\dot{y} + l\dot{\theta}\cos\theta)\cos(\theta+\phi) = 0 \quad (a)$$

$$(a) \Rightarrow \quad \dot{x}\cos(\theta+\phi) - \dot{y}\cos(\theta+\phi) - l\dot{\theta}\underbrace{\cos(\theta+\phi-\theta)}_{\cos\phi} = 0 \quad (14)$$

(14) is front wheel RWS constraints in terms of q.

- Using (12) and (13) in (5):

$$(\dot{x} + l\sin\theta_t\dot{\theta}_t)\sin\theta_t - (\dot{y} - l_t\cos\theta_t)\cos\theta_t = 0 \quad (b)$$

$$(b) \Rightarrow \quad \dot{x}\sin\theta_t - \dot{y}\cos\theta_t + l_t\dot{\theta}_t\underbrace{(\sin^2\theta_t + \cos^2\theta_t)}_{1} = 0. \quad (15)$$

$$(15) \Rightarrow \quad \dot{x}\sin\theta_t - \dot{y}\cos\theta_t + l_t\dot{\theta}_t = 0 \quad (15)$$

(15) ⇒ is trailer wheel RWS constraint in terms of q.

AMR Lectures (part 2.)

In this step we can generate matrix form of constraints.

$$A^T(q)\,\dot{q} = 0 \quad (16)$$

$$\begin{pmatrix} \sin(\theta+\phi) & -\cos(\theta+\phi) & \ell\cos\phi & 0 & 0 \\ \sin\theta & -\cos\theta & 0 & 0 & 0 \\ \sin\theta_t & -\cos\theta_t & 0 & 0 & \ell_t \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi} \\ \dot{\theta}_t \end{pmatrix} = 0 \quad (17).$$

Notice that 1st row is front wheel, 2nd row is rear wheel and 3rd row is trailer wheel constraints. $A^T(q)$ has (3×5) dimension. If we analyze $A^T(q)$ matrix its first 2 rows and 4 columns give us such submatrix that is constraint matrix of bicycle.

After having this knowledge we can write Kinematic model.

→ We have 3 constraints and $n=5$: $m = 5-3 = 2$

It means we'll have 2 vector fields in order to derive the basis of $N(A^T(q))$. We know that these vectors will define kinematic model of robot:

$$\dot{q} = g_1(q)u_1 + g_2(q)u_2 \quad (18)$$

In order to generate these vector fields we will need to remember the bicycle's kinematic models (RWD & FWD).

Vector fields that generate kinematic of bicycle are given as below: Note: It is RWD model.

RWD => $$g_1^{bicycle} = \begin{pmatrix} \cos\theta \\ \sin\theta \\ \tan\phi/e \\ 0 \end{pmatrix} \quad (19) \qquad g_2^{bicycle} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad (20)$$

It's obvious that if we expand each vector field we can define vector fields of our robot.

$$g_1 = \begin{pmatrix} \cos\theta \\ \sin\theta \\ \tan\phi/e \\ 0 \\ \circledast \end{pmatrix} \quad (21) \qquad g_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad (22)$$

We know that g_1 and g_2 must give us zero as a result of multiplication with each row of $A^T(q)$. (22) satisfies all of them. (21) has already satisfy the first 2 rows. Thus we need to choose such value for $\circledast$ in order to satisfy the 3rd row, too.

$$\sin\theta_t \cos\theta - \cos\theta_t \sin\theta + l_t \circledast = \sin(\theta_t - \theta) + l_t \circledast = 0 \quad (c)$$

$$(c) \Rightarrow \quad \circledast = -\frac{\sin(\theta_t - \theta)}{l_t} = \frac{\sin(\theta - \theta_t)}{l_t} \quad (23).$$

Thus, we can rewrite (21) as below:

$$g_1 = \begin{pmatrix} \cos\theta \\ \sin\theta \\ (\tan\phi)/e \\ 0 \\ \frac{\sin(\theta - \theta_t)}{e_t} \end{pmatrix} \quad (24)$$

Note: If we want add another trailer it would not be particular thing. We can solve that model expanding the dimension.

Notice that if we choose another generalized coordinates at the beginning, ~~if~~ we would have different Kinematic model.

Exercise (Home)

Assume robot has such pose as in Figure 8. Instead of using θ_t as 5^{th} generalized coordinates we'll get the angle of bicycle wrt trailer (which is δ). Try to define Kinematic Model wrt new generalized coordinates

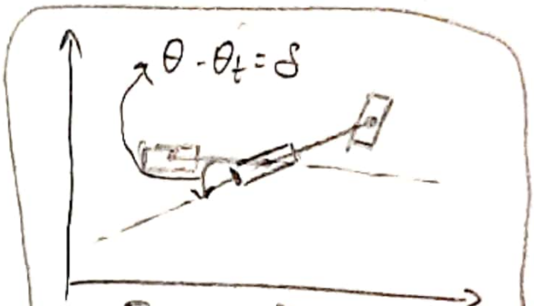

Figure 8.

- Unicycle with wheel orientation

The problem is described with Figures.

We have unicycle robot (9a) and wheel has also tick on it. In some scenarios, we also need to know what is the orientation of the wheel (i.e. where is the tick). We define the orientation of wheel as ϕ. Generalized coordinates will be:

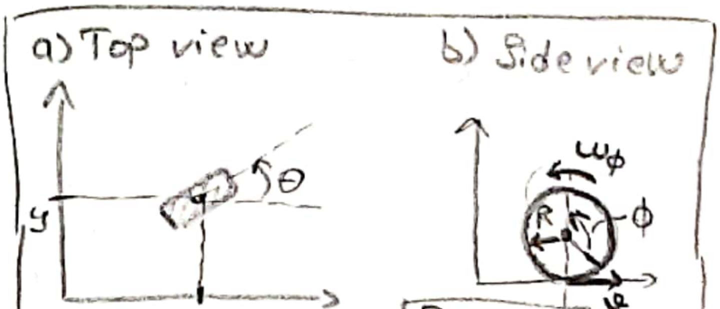

Figure 9

$$q = \begin{pmatrix} x \\ y \\ \theta \\ \phi \end{pmatrix} \quad (25)$$

We will do following steps on that model:

→ a) Kinematic model

→ b) Prove controllability

→ c) Build a maneuver for going from q_s (start configuration) to q_g (goal configuration), for $\forall q_s, q_g$.

a) Kinematic model:

1st approach: Using unicycle model and augmenting it

- Augmentation approach

$$\dot{q} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix} v + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \omega \quad (26)$$

(26) is Kinematic model of Unicycle.

v in the equation (26) is the same v in Figure 9b.

If we call angular velocity of wheel according to ϕ as ω_ϕ, we'll get the following equation:

$$v = R\omega_\phi \quad (27)$$

If we define u_1 as ω_ϕ, which is $\dot{\phi}$, then we'll have the following Kinematic Model.

$$\dot{q} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi} \end{pmatrix} = \begin{pmatrix} R\cos\theta \\ R\sin\theta \\ 0 \\ 1 \end{pmatrix} \omega_\phi + \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \omega_\theta \quad (28)$$

(28) is the Kinematic model we were looking for.

AMR Lectures part 3

2^{nd} approach: Canonical approach

→ Define constraints:

1^{st}: we have 1 wheel → 1 RWS constraint:

$$\dot{x}\sin\theta - \dot{y}\cos\theta = 0 \quad (29)$$

2^{nd}: We already found that there are 2 vf's. It means we should find another constraint

We'll imply kinematic constraint:

$$v - R\dot{\phi} = 0 \quad (30)$$

$(30) \Rightarrow \pm\sqrt{\dot{x}^2+\dot{y}^2} - R\dot{\phi} = 0 \quad (31) \Rightarrow$ But it is not PFAFFIAN form.

Because (31) is not in Pfaffian form, we need to define (30) with different way:

$$v = v(\cos^2\theta + \sin^2\theta) = (\underbrace{v\cos\theta}_{\dot{x}})\cos\theta + (\underbrace{v\sin\theta}_{\dot{y}})\sin\theta =$$

$$= \dot{x}\cos\theta + \dot{y}\sin\theta \quad (32)$$

Using (32) in (30):

$$\dot{x}\cos\theta + \dot{y}\sin\theta - R\dot{\phi} = 0 \quad (33)$$

(33) is our second constraint (which is kinematic constraint). Now we can use equations (29) and (33) in the form of (16).

$$\begin{pmatrix} \sin\theta & -\cos\theta & 0 & 0 \\ \cos\theta & \sin\theta & 0 & R \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \\ \dot{\phi} \end{pmatrix} = 0 \quad (34)$$

If we find the Basis of $N(A^T(q))$ we'll have exactly same results as (28).

b) Prove controllability.

We get g_1 and g_2 from (28)

$$(28) \Rightarrow \quad g_1 = \begin{pmatrix} R\cos\theta \\ R\sin\theta \\ 0 \\ 1 \end{pmatrix} \quad (35) \qquad g_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad (36)$$

Using g_1 and g_2 we'll compute Lie-Bracket :

$$g_3 \triangleq [g_1, g_2] = \frac{\partial g_2}{\partial q} g_1 - \frac{\partial g_1}{\partial q} g_2 = -\begin{pmatrix} 0 & 0 & -R\sin\theta & 0 \\ 0 & 0 & R\cos\theta & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} R\sin\theta \\ -R\cos\theta \\ 0 \\ 0 \end{pmatrix} \quad (37)$$

Because $n=4$, we'll compute second order Lie Bracket:
If we compute $[g_1, g_3]$, it will give us zero vector as result.
It is not giving us extra dimension thus we'll check $[g_2, g_3]$:

$$g_4 \triangleq [g_2, g_3] = \begin{pmatrix} 0 & 0 & R\cos\theta & 0 \\ 0 & 0 & R\sin\theta & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} R\cos\theta \\ R\sin\theta \\ 0 \\ 0 \end{pmatrix} \quad (38)$$

Accessibility Distribution computation will be:

$$\Delta_A = \{ g_1, g_2, g_3, g_4, \underbrace{\ldots}_{\text{we don't need them.}} \} \quad (39)$$

We also know that, changing columns' positions doesn't affect the rank of matrix:

$$\text{rank}\begin{pmatrix} | & | & | & | \\ g_1 & g_2 & g_3 & g_4 \\ | & | & | & | \end{pmatrix} = \text{rank}\begin{pmatrix} | & | & | & | \\ g_3 & g_4 & g_1 & g_2 \\ | & | & | & | \end{pmatrix} =$$

$$\text{rank}\begin{pmatrix} R\sin\theta & R\cos\theta & R\cos\theta & 0 \\ -R\cos\theta & R\sin\theta & R\sin\theta & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = 4 \quad (40)$$

→ determinant ≠ 0.

→ $\det = -R^2$

Note: We don't care about determinant, We compute it in order to check rank. I.E:
If $\det \neq 0 \rightarrow$ matrix has full rank.

c) maneuver

The maneuver is described in Figure 10. There are 4 phases are given as below:

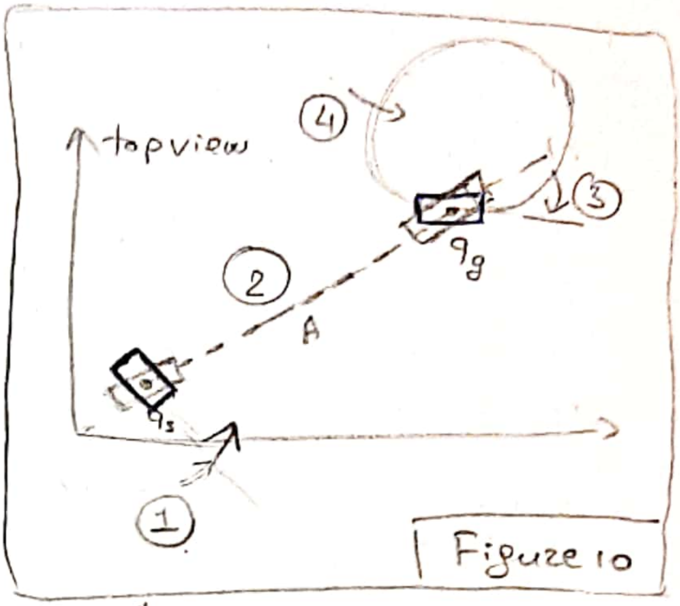

Figure 10

① We need to find rotation of q_s wrt q_g. We need to rotate wheel until it is alligned with the line A (Figure 10).

② Rolling until connection poits of wheel and q_s collides.

③ Rotate the wheel wrt q_s.

④ We are not sure that orientation of wheel is exactly same as q_s. Thus we need to move it along a circle in order to get exact pose as q_s, which the circle has r radius.

On the other hand, the Length of the way that wheel passes has relation with the angle change as below:

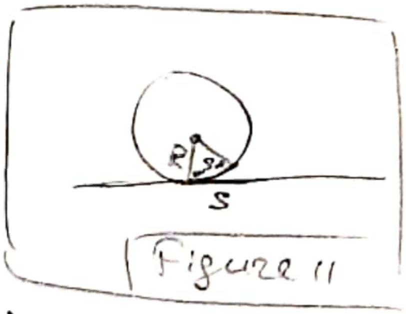

Figure 11

$$\Delta\phi = \frac{R}{S} \quad (41)$$

In (41), R is the radius of the wheel. Additionally, we know that wheel must finish route along the wheel in order to get the exact pose. Thus the length of route is $2\pi r$.

If we define the last orientation of wheel before the route of circle as ϕ_s and goal orientation as ϕ_g, then:

$$\Delta\phi = \phi_g - \phi_s \quad (42)$$

By using this knowledge, we can define the radius of the circle that we need to pass as below:

$$r = \frac{(\phi_g - \phi_s) R}{2\varphi} \quad (43)$$