where v is the *driving* velocity of the rear wheel.[5] In this case, one has

$$\boldsymbol{g}_3(\boldsymbol{q}) = [\boldsymbol{g}_1, \boldsymbol{g}_2](\boldsymbol{q}) = \begin{bmatrix} 0 \\ 0 \\ -\dfrac{1}{\ell \cos^2\phi} \\ 0 \end{bmatrix} \quad \boldsymbol{g}_4(\boldsymbol{q}) = [\boldsymbol{g}_1, \boldsymbol{g}_3](\boldsymbol{q}) = \begin{bmatrix} -\dfrac{\sin\theta}{\ell \cos^2\phi} \\ \dfrac{\cos\theta}{\ell \cos^2\phi} \\ 0 \\ 0 \end{bmatrix},$$

again linearly independent from $\boldsymbol{g}_1(\boldsymbol{q})$ and $\boldsymbol{g}_2(\boldsymbol{q})$. Hence, the rear-wheel drive bicycle is also controllable with degree of nonholonomy $\kappa = 3$.

Like the unicycle, the bicycle is also unstable in static conditions. Kinematically equivalent vehicles that are mechanically balanced are the *tricycle* and the *car-like* robot, introduced in Sect. 1.2.2 and shown respectively in Fig. 1.15 and 1.16. In both cases, the kinematic model is given by (11.18) or by (11.19) depending on the wheel drive being on the front or the rear wheels. In particular, (x, y) are the Cartesian coordinates of the midpoint of the rear wheel axle, θ is the orientation of the vehicle, and ϕ is the steering angle.

11.3 Chained Form

The possibility of transforming the kinematic model (11.10) of a mobile robot in a canonical form is of great interest for solving planning and control problems with efficient, systematic procedures. Here, the analysis is limited to systems with two inputs, like the unicycle and bicycle models.

A $(2, n)$ *chained form* is a two-input driftless system

$$\dot{\boldsymbol{z}} = \boldsymbol{\gamma}_1(\boldsymbol{z})v_1 + \boldsymbol{\gamma}_2(\boldsymbol{z})v_2,$$

whose equations are expressed as

$$\begin{aligned} \dot{z}_1 &= v_1 \\ \dot{z}_2 &= v_2 \\ \dot{z}_3 &= z_2 v_1 \\ &\vdots \\ \dot{z}_n &= z_{n-1} v_1. \end{aligned} \tag{11.20}$$

Using the following notation for a 'repeated' Lie bracket:

$$\mathrm{ad}_{\boldsymbol{\gamma}_1}\boldsymbol{\gamma}_2 = [\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2] \qquad \mathrm{ad}^k_{\boldsymbol{\gamma}_1}\boldsymbol{\gamma}_2 = [\boldsymbol{\gamma}_1, \mathrm{ad}^{k-1}_{\boldsymbol{\gamma}_1}\boldsymbol{\gamma}_2],$$

[5] Note that the kinematic model (11.19) is no longer valid for $\phi = \pm\pi/2$, where the first vector field is not defined. This corresponds to the mechanical jam in which the front wheel is orthogonal to the sagittal axis of the vehicle. This singularity does not arise in the front-wheel drive bicycle (11.18), that in principle can still pivot around the rear wheel contact point in such a situation.

one has for system (11.20)

$$\boldsymbol{\gamma}_1 = \begin{bmatrix} 1 \\ 0 \\ z_2 \\ z_3 \\ \vdots \\ z_{n-1} \end{bmatrix} \quad \boldsymbol{\gamma}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \Rightarrow \quad \mathrm{ad}^k_{\boldsymbol{\gamma}_1}\boldsymbol{\gamma}_2 = \begin{bmatrix} 0 \\ \vdots \\ (-1)^k \\ \vdots \\ 0 \end{bmatrix},$$

where $(-1)^k$ is the $(k+2)$-th component. This implies that the system is controllable, because the accessibility distribution

$$\Delta_{\mathcal{A}} = \mathrm{span}\,\{\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2, \mathrm{ad}_{\boldsymbol{\gamma}_1}\boldsymbol{\gamma}_2, \ldots, \mathrm{ad}^{n-2}_{\boldsymbol{\gamma}_1}\boldsymbol{\gamma}_2\}$$

has dimension n. In particular, the degree of nonholonomy is $\kappa = n-1$.

There exist necessary and sufficient conditions for transforming a generic two-input driftless system

$$\dot{\boldsymbol{q}} = \boldsymbol{g}_1(\boldsymbol{q})u_1 + \boldsymbol{g}_2(\boldsymbol{q})u_2 \tag{11.21}$$

in the chained form (11.20) via coordinate and input transformations

$$\boldsymbol{z} = \boldsymbol{T}(\boldsymbol{q}) \qquad \boldsymbol{v} = \boldsymbol{\beta}(\boldsymbol{q})\boldsymbol{u}. \tag{11.22}$$

In particular, it can be shown that systems like (11.21) with dimension n not larger than 4 can *always* be put in chained form. This applies, for example, to the kinematic models of the unicycle and the bicycle.

There also exist sufficient conditions for transformability in chained form that are relevant because they are constructive. Define the distributions

$$\begin{aligned} \Delta_0 &= \mathrm{span}\,\{\boldsymbol{g}_1, \boldsymbol{g}_2, \mathrm{ad}_{\boldsymbol{g}_1}\boldsymbol{g}_2, \ldots, \mathrm{ad}^{n-2}_{\boldsymbol{g}_1}\boldsymbol{g}_2\} \\ \Delta_1 &= \mathrm{span}\,\{\boldsymbol{g}_2, \mathrm{ad}_{\boldsymbol{g}_1}\boldsymbol{g}_2, \ldots, \mathrm{ad}^{n-2}_{\boldsymbol{g}_1}\boldsymbol{g}_2\} \\ \Delta_2 &= \mathrm{span}\,\{\boldsymbol{g}_2, \mathrm{ad}_{\boldsymbol{g}_1}\boldsymbol{g}_2, \ldots, \mathrm{ad}^{n-3}_{\boldsymbol{g}_1}\boldsymbol{g}_2\}. \end{aligned}$$

Assume that, in a certain set, it is $\dim \Delta_0 = n$, Δ_1 and Δ_2 are involutive, and there exists a scalar function $h_1(\boldsymbol{q})$ whose differential $d\boldsymbol{h}_1$ satisfies

$$d\boldsymbol{h}_1 \cdot \Delta_1 = 0 \qquad d\boldsymbol{h}_1 \cdot \boldsymbol{g}_1 = 1,$$

where the symbol $\cdot$ denotes the inner product between a row vector and a column vector — in particular, $\cdot\Delta_1$ is the inner product with any vector generated by distribution Δ_1. In this case, system (11.21) can be put in the form (11.20) through the coordinate transformation[6]

$$z_1 = h_1$$

[6] This transformation makes use of the Lie derivative (see Appendix D).

$$
\begin{aligned}
z_2 &= L_{\boldsymbol{g}_1}^{n-2} h_2 \\
&\vdots \\
z_{n-1} &= L_{\boldsymbol{g}_1} h_2 \\
z_n &= h_2,
\end{aligned}
$$

where h_2 must be chosen independent of h_1 and such that $dh_2 \cdot \Delta_2 = 0$. The input transformation is given by

$$
\begin{aligned}
v_1 &= u_1 \\
v_2 &= \left(L_{\boldsymbol{g}_1}^{n-1} h_2\right) u_1 + \left(L_{\boldsymbol{g}_2} L_{\boldsymbol{g}_1}^{n-2} h_2\right) u_2.
\end{aligned}
$$

In general, the coordinate and input transformations are not unique.

Consider the kinematic model (11.13) of the unicycle. With the change of coordinates

$$
\begin{aligned}
z_1 &= \theta \\
z_2 &= x\cos\theta + y\sin\theta \\
z_3 &= x\sin\theta - y\cos\theta
\end{aligned}
\tag{11.23}
$$

and the input transformation

$$
\begin{aligned}
v &= v_2 + z_3 v_1 \\
\omega &= v_1,
\end{aligned}
\tag{11.24}
$$

one obtains the (2,3) chained form

$$
\begin{aligned}
\dot{z}_1 &= v_1 \\
\dot{z}_2 &= v_2 \\
\dot{z}_3 &= z_2 v_1.
\end{aligned}
\tag{11.25}
$$

Note that, while z_1 is simply the orientation θ, coordinates z_2 and z_3 represent the position of the unicycle in a moving reference frame whose z_2 axis is aligned with the sagittal axis of the vehicle (see Fig. 11.3).

As for mobile robots with bicycle-like kinematics, consider for example the model (11.19) corresponding to the rear-wheel drive case. Using the change of coordinates

$$
\begin{aligned}
z_1 &= x \\
z_2 &= \frac{1}{\ell}\sec^3\theta\tan\phi \\
z_3 &= \tan\theta \\
z_4 &= y
\end{aligned}
$$

and the input transformation

$$v = \frac{v_1}{\cos\theta}$$
$$\omega = -\frac{3}{\ell}\, v_1 \sec\theta \sin^2\!\phi + \frac{1}{\ell}\, v_2 \cos^3\!\theta \cos^2\!\phi,$$

the (2,4) chained form is obtained:

$$\begin{aligned}
\dot{z}_1 &= v_1\\
\dot{z}_2 &= v_2\\
\dot{z}_3 &= z_2 v_1\\
\dot{z}_4 &= z_3 v_1.
\end{aligned}$$

This transformation is defined everywhere in the configuration space, with the exception of points where $\cos\theta = 0$. The equivalence between the two models is then subject to the condition $\theta \neq \pm k\pi/2$, with $k = 1, 2, \ldots$.

11.4 Dynamic Model

The derivation of the dynamic model of a mobile robot is similar to the manipulator case, the main difference being the presence of nonholonomic constraints on the generalized coordinates. An important consequence of nonholonomy is that exact linearization of the dynamic model via feedback is no longer possible. In the following, the Lagrange formulation is used to obtain the dynamic model of an n-dimensional mechanical system subject to $k < n$ kinematic constraints in the form (11.3), and it is shown how this model can be partially linearized via feedback.

As usual, define the Lagrangian $\mathcal{L}$ of the mechanical system as the difference between its kinetic and potential energy:

$$\mathcal{L}(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \mathcal{T}(\boldsymbol{q}, \dot{\boldsymbol{q}}) - \mathcal{U}(\boldsymbol{q}) = \frac{1}{2}\dot{\boldsymbol{q}}^T \boldsymbol{B}(\boldsymbol{q})\dot{\boldsymbol{q}} - \mathcal{U}(\boldsymbol{q}), \tag{11.26}$$

where $\boldsymbol{B}(\boldsymbol{q})$ is the (symmetric and positive definite) inertia matrix of the mechanical system. The Lagrange equations are in this case

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\boldsymbol{q}}}\right)^T - \left(\frac{\partial \mathcal{L}}{\partial \boldsymbol{q}}\right)^T = \boldsymbol{S}(\boldsymbol{q})\boldsymbol{\tau} + \boldsymbol{A}(\boldsymbol{q})\boldsymbol{\lambda}, \tag{11.27}$$

where $\boldsymbol{S}(\boldsymbol{q})$ is an $(n \times m)$ matrix mapping the $m = n - k$ external inputs $\boldsymbol{\tau}$ to generalized forces performing work on $\boldsymbol{q}$, $\boldsymbol{A}(\boldsymbol{q})$ is the transpose of the $(k \times n)$ matrix characterizing the kinematic constraints (11.3), and $\boldsymbol{\lambda} \in \mathbb{R}^m$ is the vector of *Lagrange multipliers*. The term $\boldsymbol{A}(\boldsymbol{q})\boldsymbol{\lambda}$ represents the vector of reaction forces at the generalized coordinate level. It has been assumed that the number of available inputs matches the number of DOFs (*full actuation*),