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Hidden Markov Models

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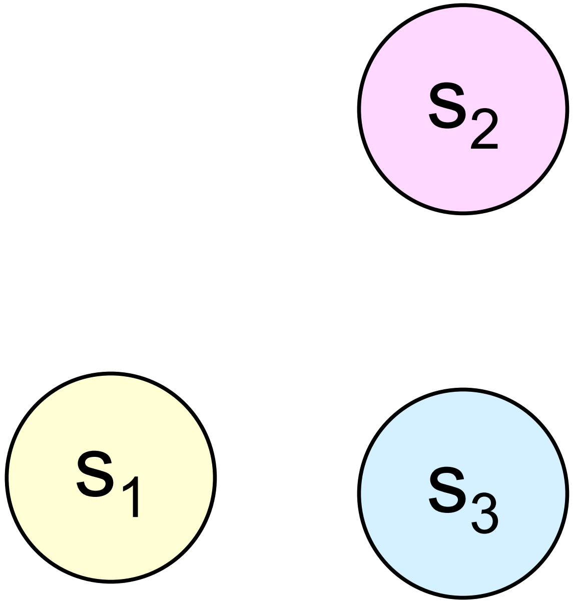
Nov 29th, 2001

A Markov System

Has *N* states, called s_1, s_2 .. s_N

There are discrete timesteps, *t=0, t=1, …*



N = 3

t=0

A Markov System

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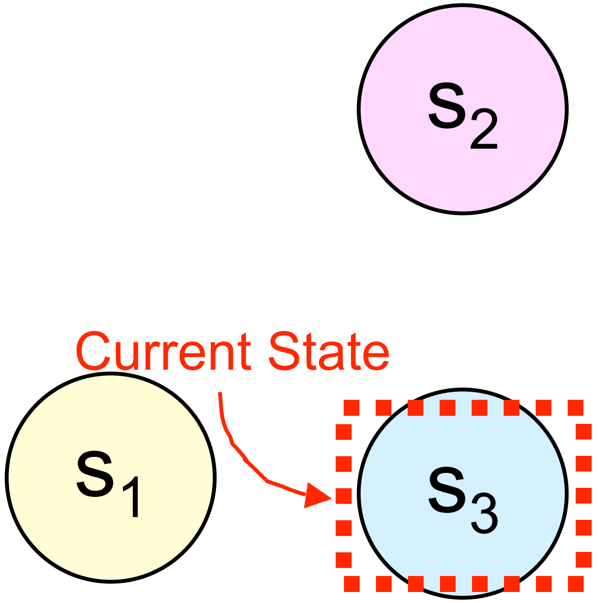
There are discrete timesteps, $t=0$, $t=1$, …

On the t’th timestep the system is in exactly one of the available states. Call it q_t

Note: $q_t \in \{s_1, s_2 .. s_N\}$

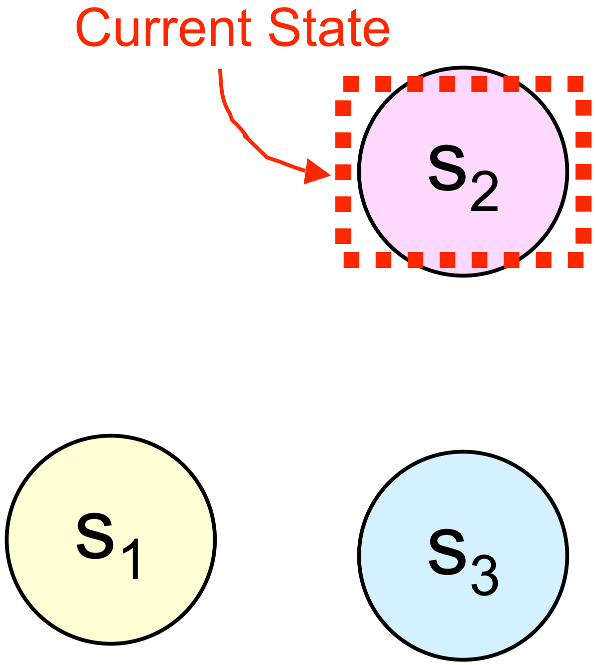


$N = 3$

$t=0$

$q_t=q_0=s_3$

A Markov System



$N = 3$

$t=1$

$q_t=q_1=s_2$

Has N states, called s_1, s_2 .. s_N

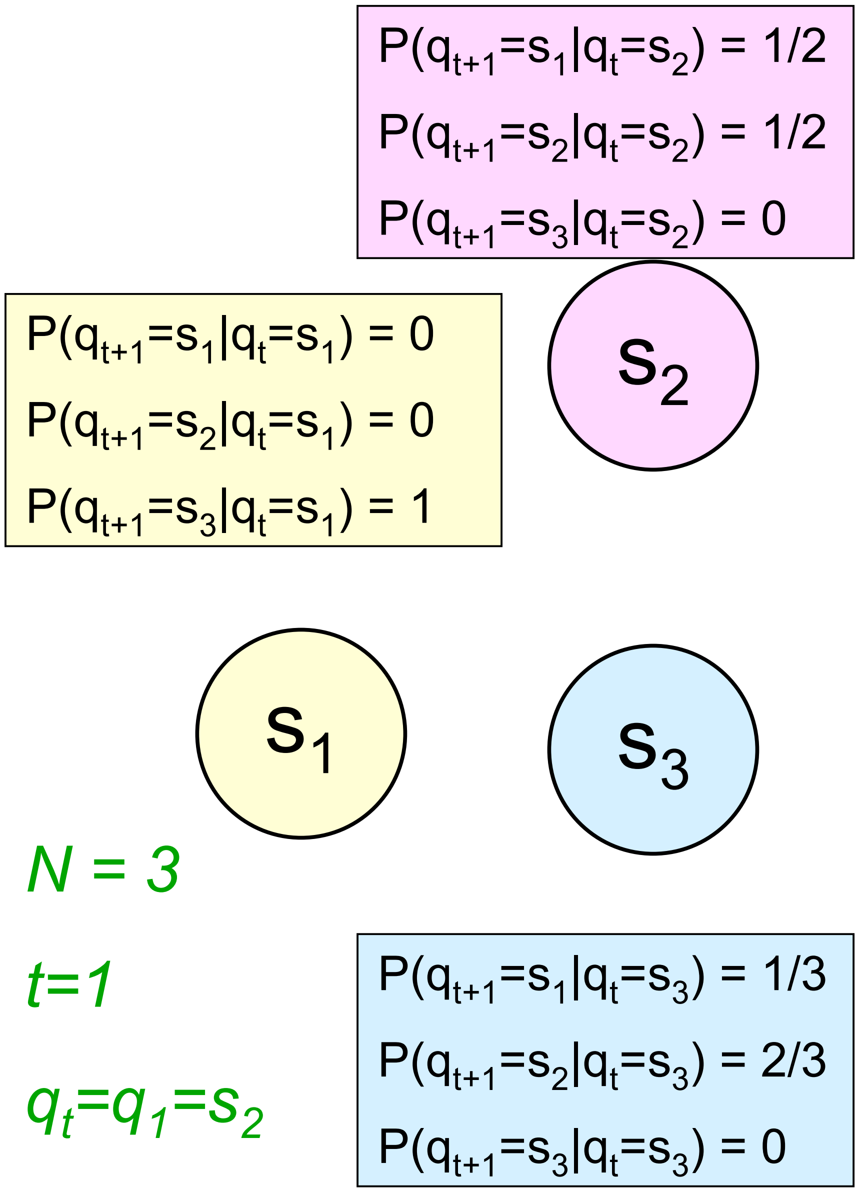
There are discrete timesteps, $t=0$, $t=1$, …

On the t'th timestep the system is in exactly one of the available states. Call it q_t

Note: $q_t \in \{s_1, s_2 .. s_N\}$

Between each timestep, the next state is chosen randomly.

A Markov System



$P(q_{t+1}=s_1|q_t=s_2) = 1/2$

$P(q_{t+1}=s_2|q_t=s_2) = 1/2$

$P(q_{t+1}=s_3|q_t=s_2) = 0$

$P(q_{t+1}=s_1|q_t=s_1) = 0$

$P(q_{t+1}=s_2|q_t=s_1) = 0$

$P(q_{t+1}=s_3|q_t=s_1) = 1$

$N = 3$

$t=1$

$q_t=q_1=s_2$

$P(q_{t+1}=s_1|q_t=s_3) = 1/3$

$P(q_{t+1}=s_2|q_t=s_3) = 2/3$

$P(q_{t+1}=s_3|q_t=s_3) = 0$

Has N states, called $s_1, s_2 .. s_N$

There are discrete timesteps, $t=0, t=1, \ldots$

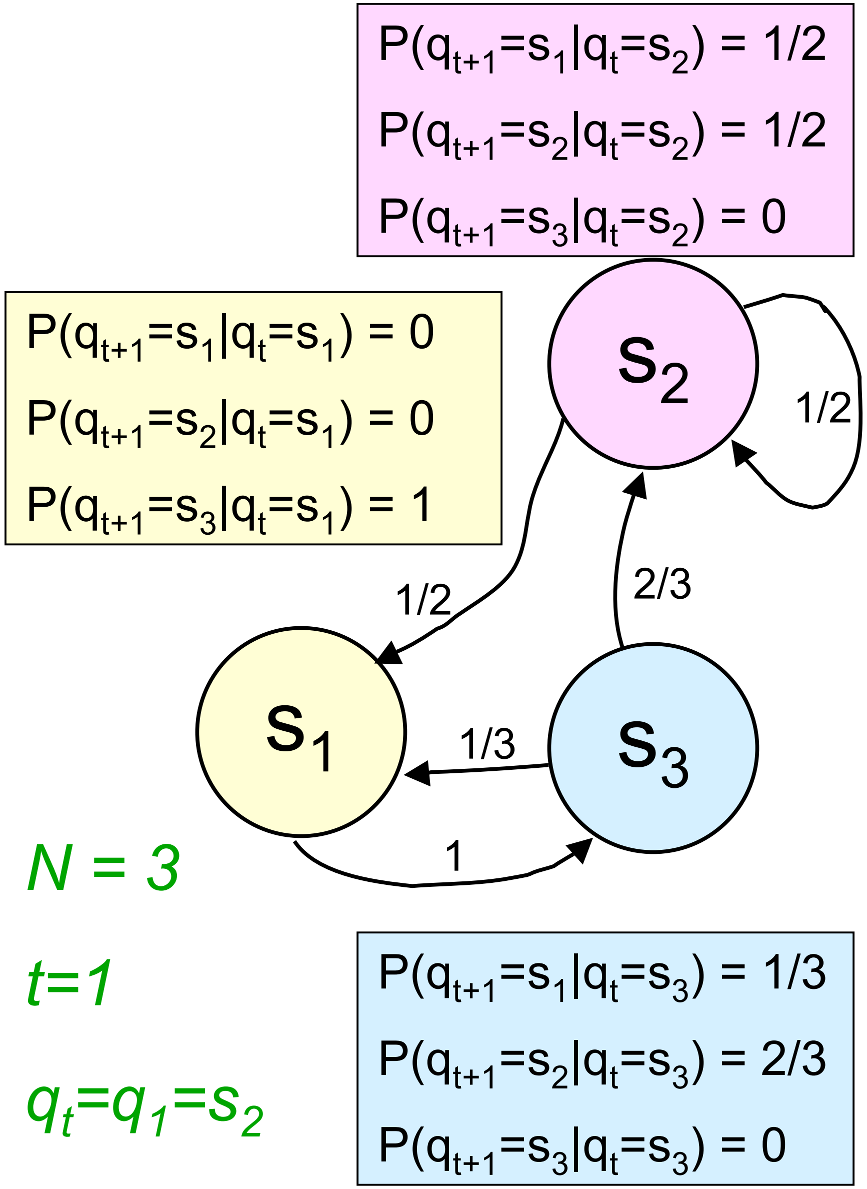
On the t'th timestep the system is in exactly one of the available states. Call it q_t

Note: $q_t \in \{s_1, s_2 .. s_N\}$

Between each timestep, the next state is chosen randomly.

The current state determines the probability distribution for the next state.

A Markov System



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$P(q_{t+1}=s_3|q_t=s_1) = 1$

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$P(q_{t+1}=s_3|q_t=s_3) = 0$

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There are discrete timesteps, $t=0, t=1, \ldots$

On the t'th timestep the system is in exactly one of the available states. Call it q_t

Note: $q_t \in \{s_1, s_2 .. s_N\}$

Between each timestep, the next state is chosen "randomly".

The current state determines the probability distribution for the next state.

Often notated with arcs between states

Markov Property

$P(q_{t+1}=s_1|q_t=s_2) = 1/2$

$P(q_{t+1}=s_2|q_t=s_2) = 1/2$

$P(q_{t+1}=s_3|q_t=s_2) = 0$

$P(q_{t+1}=s_1|q_t=s_1) = 0$

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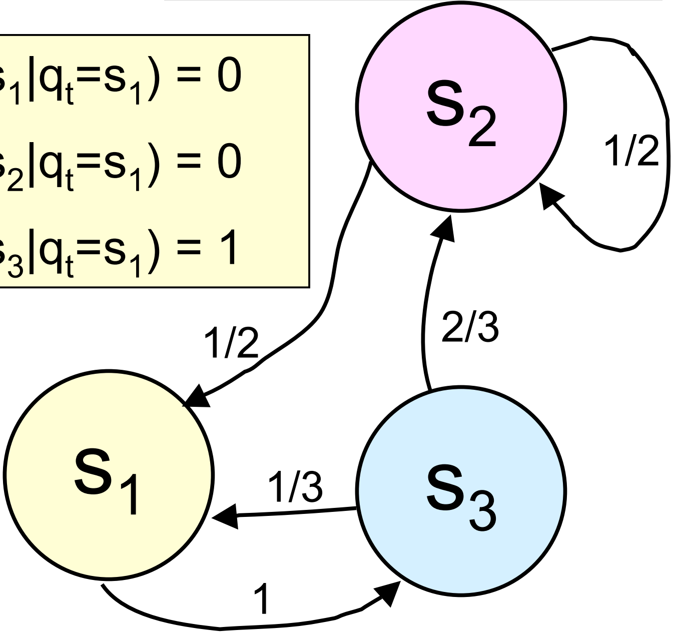

$N = 3$

$t=1$

$q_t=q_1=s_2$

$P(q_{t+1}=s_1|q_t=s_3) = 1/3$

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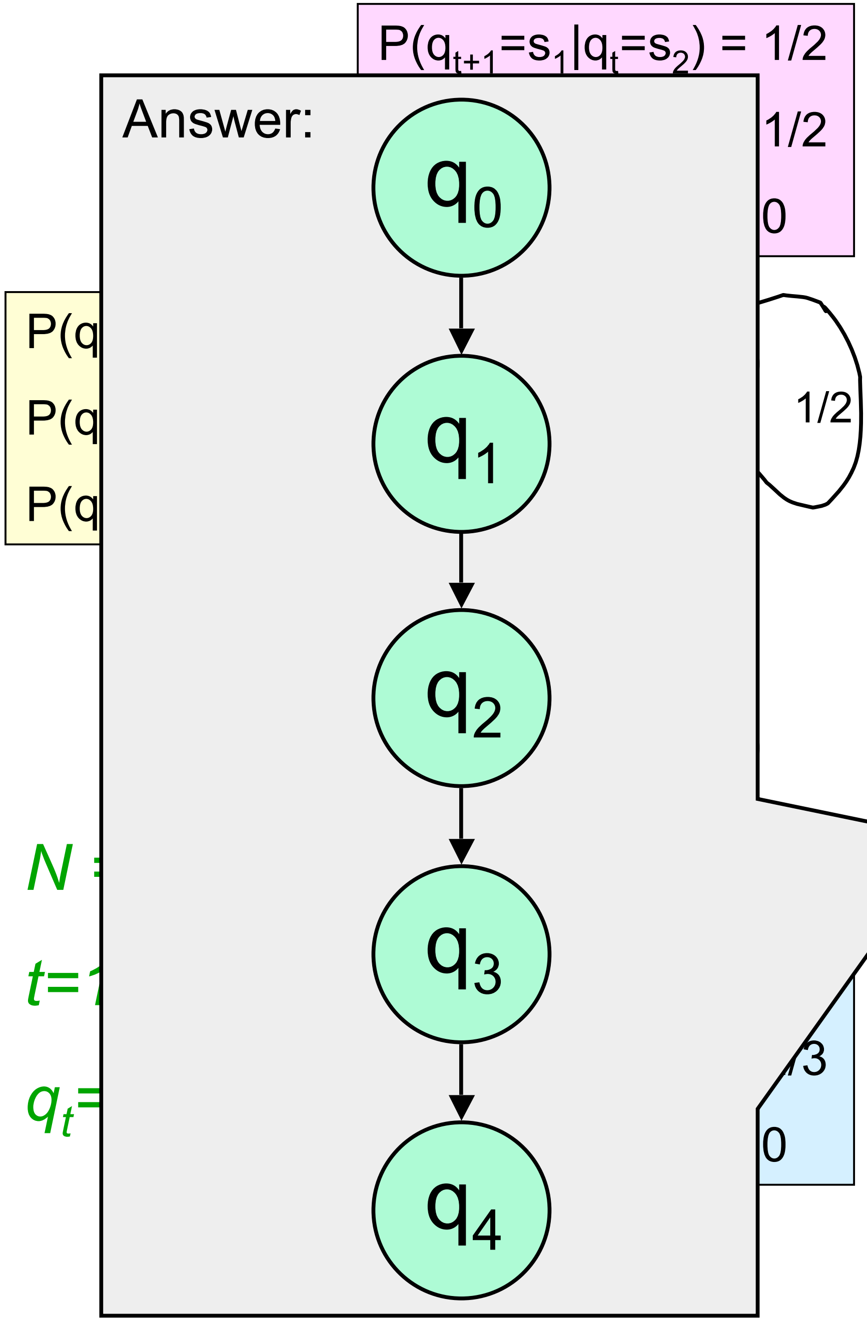
q_{t+1} is conditionally independent of $\{ q_{t-1}, q_{t-2}, \ldots q_1, q_0 \}$ given q_t.

In other words:

$P(q_{t+1} = s_j | q_t = s_i) =$

$P(q_{t+1} = s_j | q_t = s_i$,any earlier history)

Question: what would be the best Bayes Net structure to represent the Joint Distribution of (q_0, q_1, … q_3,q_4)?



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Markov Property

$P(q_{t+1}=s_1|q_t=s_2) = 1/2$

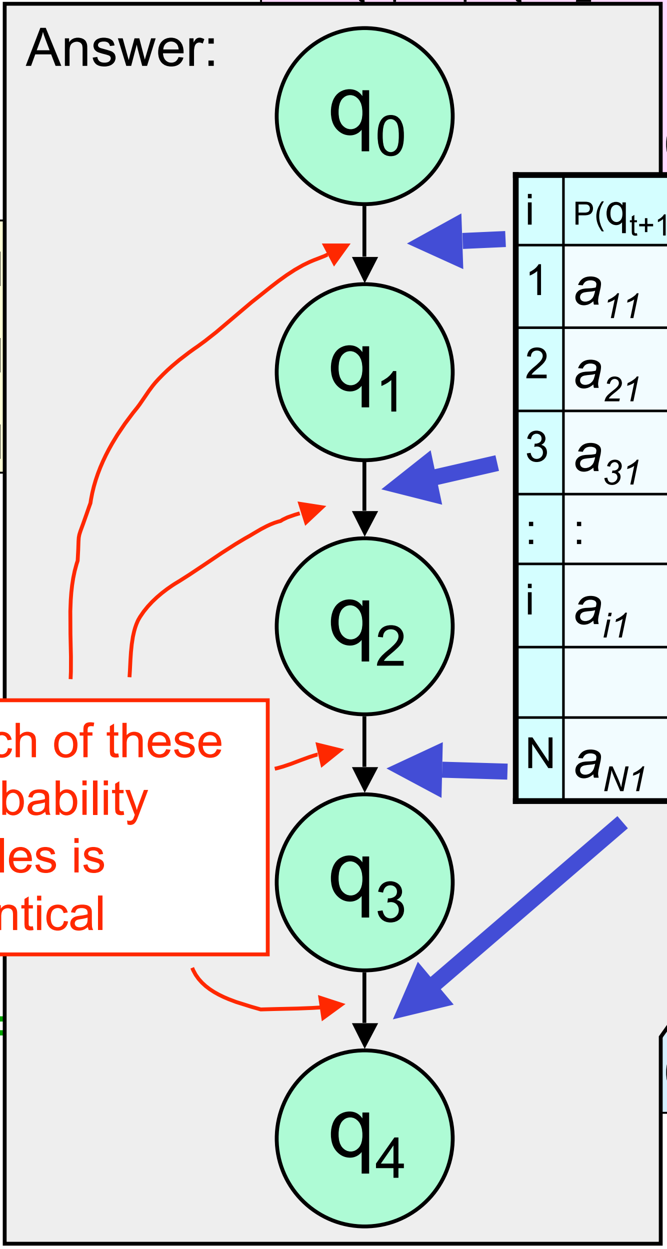
1/2

0

P(q

P(q

P(q



Each of these probability tables is identical

q_t=

/3

0

q_{t+1} is conditionally independent

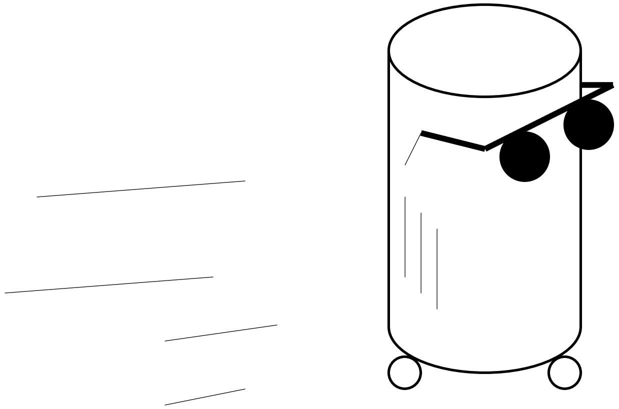
i	$P(q_{t+1}=s_1\|q_t=s_i)$	$P(q_{t+1}=s_2\|q_t=s_i)$	…	$P(q_{t+1}=s_j\|q_t=s_i)$	..	$P(q_{t+1}=s_N\|q_t=s_i)$
1	a_{11}	a_{12}	…	a_{1j}	…	a_{1N}
2	a_{21}	a_{22}	…	a_{2j}	…	a_{2N}
3	a_{31}	a_{32}	…	a_{3j}	…	a_{3N}
:	:	:	:	:	:	:
i	a_{i1}	a_{i2}	…	a_{ij}	…	a_{iN}
N	a_{N1}	a_{N2}	…	a_{Nj}	…	a_{NN}

the Joint Distribution of (q_0, q_1, q_2,q_3,q_4)?

Notation:

$$a_{ij} = P(q_{t+1} = s_j \mid q_t = s_i)$$

A Blind Robot



A human and a robot wander around randomly on a grid…

			R		
	H				

$$\textbf{STATE } q = \left\langle \begin{array}{l}\text{Location of Robot,}\\ \text{Location of Human}\end{array}\right\rangle$$

Note: N (num. states) = 18 * 18 = 324

Dynamics of System

q_0 =

					R
H					

Each timestep the human moves randomly to an adjacent cell. And robot also moves randomly to an adjacent cell.

Typical Questions:

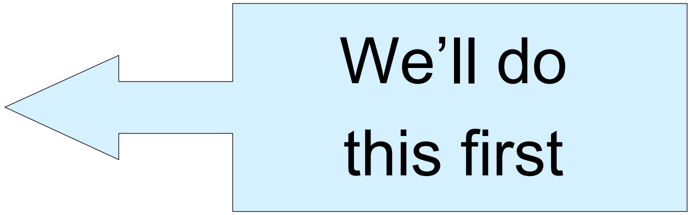
- “What’s the expected time until the robot hits the human?”
- “What’s the probability that the robot will hit the left wall before it hits the human?”
- “What’s the probability robot crushes human on next time step?”

Example Question

"It's currently time t, and human remains uncrushed. What's the probability of crushing occurring at time t + 1 ?"

If robot is blind:

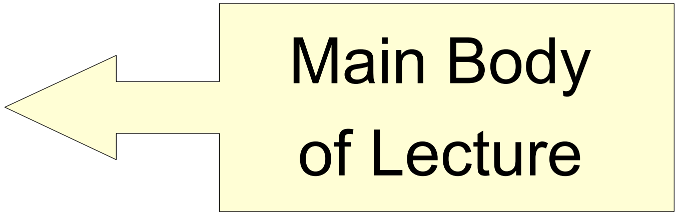
We can compute this in advance.



We'll do this first

If robot has some sensors, but incomplete state information …

Hidden Markov Models are applicable!



Main Body of Lecture

What is $P(q_t = s)$? slow, stupid answer

Step 1: Work out how to compute P(Q) for any path $Q = q_1 \, q_2 \, q_3 \, .. \, q_t$

Given we know the start state q_1 (i.e. $P(q_1)=1$)

$$
\begin{aligned}
P(q_1 \, q_2 \, .. \, q_t) &= P(q_1 \, q_2 \, .. \, q_{t-1}) \, P(q_t | q_1 \, q_2 \, .. \, q_{t-1}) \\
&= P(q_1 \, q_2 \, .. \, q_{t-1}) \, P(q_t | q_{t-1}) \qquad \textit{WHY?} \\
&= P(q_2|q_1)P(q_3|q_2)\ldots P(q_t|q_{t-1})
\end{aligned}
$$

Step 2: Use this knowledge to get $P(q_t = s)$

$$P(q_t = s) = \sum_{Q \in \text{Paths of length } t \text{ that end in } s} P(Q)$$

Computation is exponential in t

What is $P(q_t = s)$? Clever answer

- For each state s_i, define

  $p_t(i)$ = Prob. state is s_i at time t

  $= P(q_t = s_i)$

- Easy to do inductive definition

$$\forall i \quad p_0(i) =$$

$$\forall j \quad p_{t+1}(j) = P(q_{t+1} = s_j) =$$

What is P(q_t =s) ? Clever answer

- For each state s_i, define

  $p_t(i)$ = Prob. state is s_i at time t

  $= P(q_t = s_i)$
- Easy to do inductive definition

$$\forall i \quad p_0(i) = \begin{cases} 1 & \text{if } s_i \text{ is the start state} \\ 0 & \text{otherwise} \end{cases}$$

$$\forall j \quad p_{t+1}(j) = P(q_{t+1} = s_j) =$$

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$$\forall j \quad p_{t+1}(j) = P(q_{t+1} = s_j) =$$

$$\sum_{i=1}^{N} P(q_{t+1} = s_j \wedge q_t = s_i) =$$

What is $P(q_t = s)$? Clever answer

- For each state s_i, define

  $p_t(i)$ = Prob. state is s_i at time t

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$$\sum_{i=1}^{N} P(q_{t+1} = s_j \wedge q_t = s_i) =$$

$$\sum_{i=1}^{N} P(q_{t+1} = s_j \mid q_t = s_i) P(q_t = s_i) = \sum_{i=1}^{N} a_{ij} p_t(i)$$

Remember,

$$a_{ij} = P(q_{t+1} = s_j \mid q_t = s_i)$$

What is P(q_t =s) ? Clever answer

- For each state s_i, define

  $p_t(i)$ = Prob. state is s_i at time t

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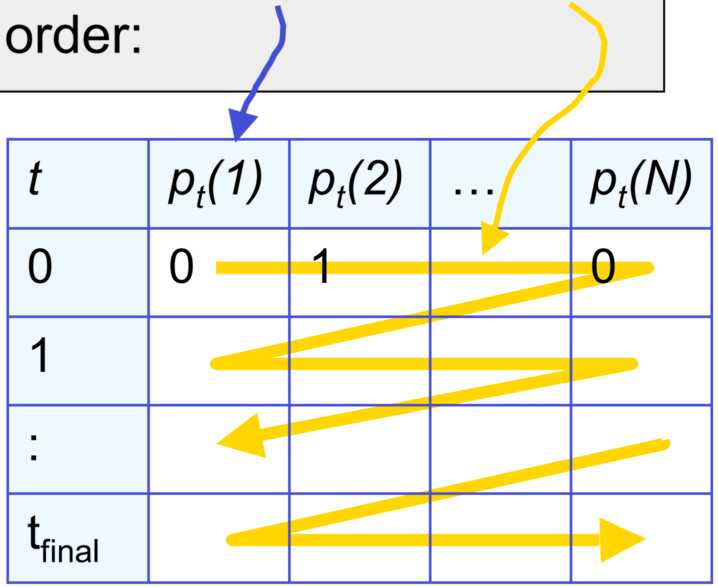
- Computation is simple.
- Just fill in this table in this order:



t	$p_t(1)$	$p_t(2)$	…	$p_t(N)$
0	0	1		0
1				
:				
t_{final}				

What is $P(q_t = s)$? Clever answer

- For each state s_i, define

  $p_t(i)$ = Prob. state is s_i at time t

  $= P(q_t = s_i)$

- Easy to do inductive definition

$$\forall i \quad p_0(i) = \begin{cases} 1 & \text{if } s_i \text{ is the start state} \\ 0 & \text{otherwise} \end{cases}$$

$$\forall j \quad p_{t+1}(j) = P(q_{t+1} = s_j) =$$

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$$\sum_{i=1}^{N} P(q_{t+1} = s_j \mid q_t = s_i) P(q_t = s_i) = \sum_{i=1}^{N} a_{ij} p_t(i)$$

- Cost of computing $P_t(i)$ for all states S_i is now $O(t\ N^2)$
- The stupid way was $O(N^t)$
- This was a simple example
- It was meant to warm you up to this trick, called *Dynamic Programming,* because HMMs do many tricks like this.

Hidden State

"It's currently time t, and human remains uncrushed. What's the probability of crushing occurring at time t + 1 ?"

If robot is blind:

We can compute this in advance.

We'll do this first

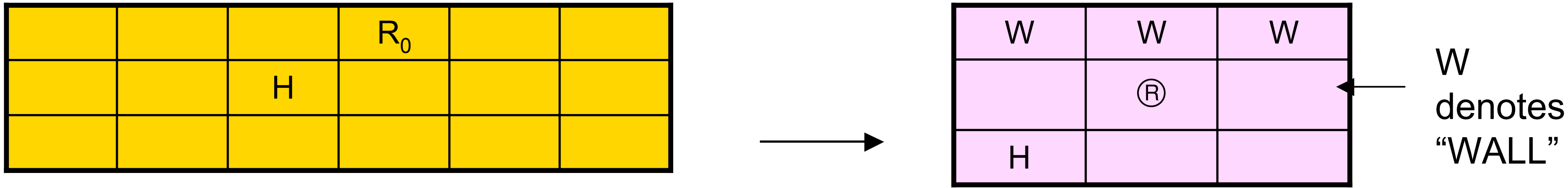
If robot has some sensors, but incomplete state information …

Hidden Markov Models are applicable!

Main Body of Lecture

Hidden State

- The previous example tried to estimate $P(q_t = s_i)$ unconditionally (using no observed evidence).
- Suppose we can observe something that's affected by the true state.
- Example: Proximity sensors. (tell us the contents of the 8 adjacent squares)

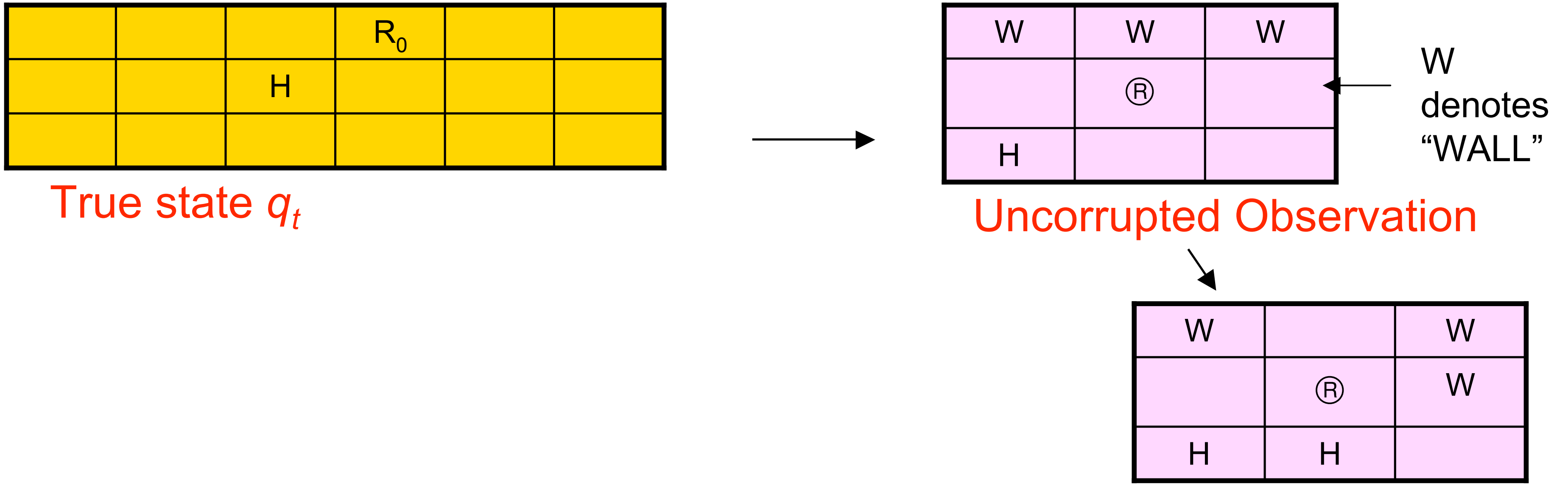


True state q_t

What the robot sees:
Observation O_t

Noisy Hidden State

- Example: Noisy Proximity sensors. (unreliably tell us the contents of the 8 adjacent squares)



			R_0		
		H			

True state q_t

W	W	W
	®	
H		

W denotes "WALL"

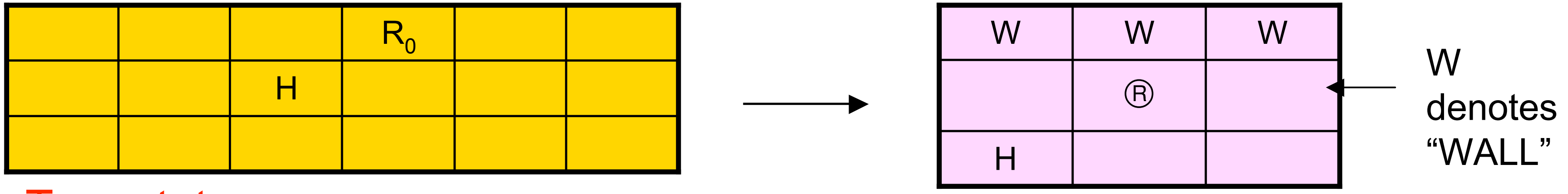
Uncorrupted Observation

W		W
	®	W
H	H	

What the robot sees: Observation O_t

Noisy Hidden State

- Example: Noisy Proximity sensors. (unreliably tell us the contents of the 8 adjacent squares)



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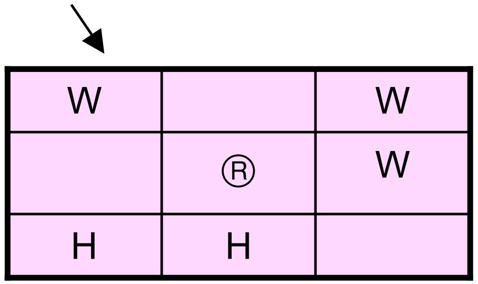


W		W
	®	W
H	H	

What the robot sees: Observation O_t

O_t is noisily determined depending on the current state.

Assume that O_t is conditionally independent of $\{q_{t-1}, q_{t-2}, \dots q_1, q_0, O_{t-1}, O_{t-2}, \dots O_1, O_0\}$ given q_t.

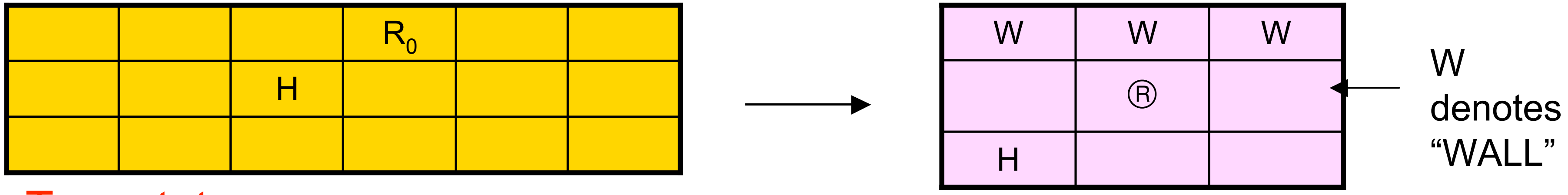
In other words:

$P(O_t = X | q_t = s_i) =$

$P(O_t = X | q_t = s_i, \text{any earlier history})$

Noisy Hidden State

- Example: Noisy Proximity sensors. (unreliably tell us the contents of the 8 adjacent squares)



			R_0		
		H			

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	Ⓡ	
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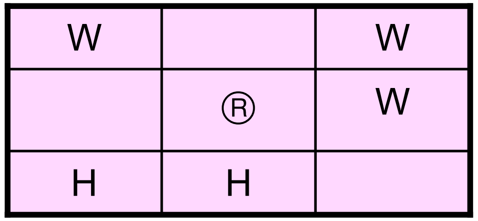
Uncorrupted Observation



W		W
	Ⓡ	W
H	H	

What the robot sees: Observation O_t

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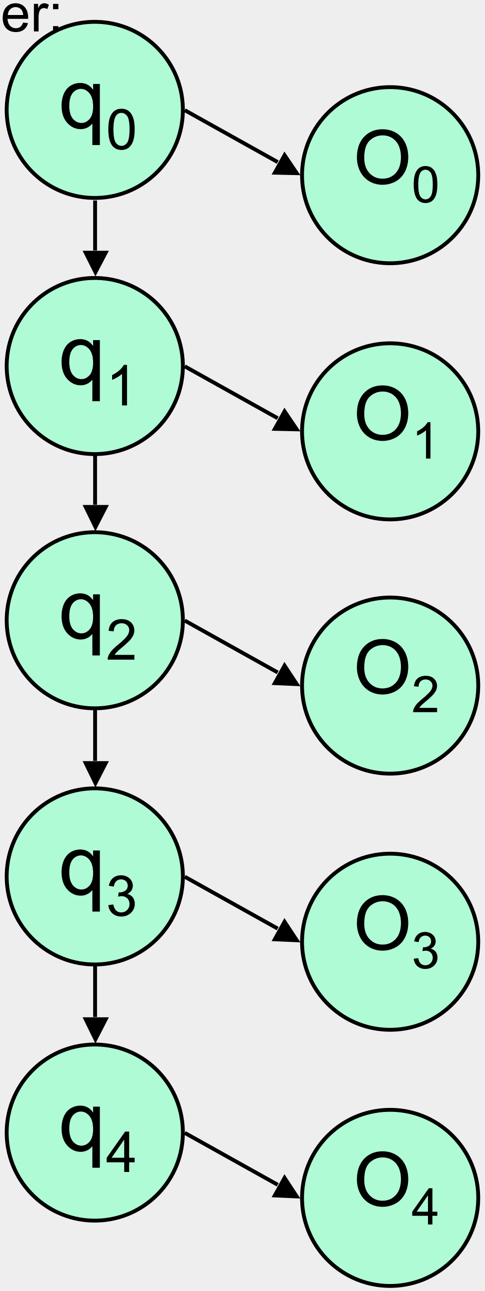
Question: what would be the best Bayes Net structure to represent the Joint Distribution of $(q_0, q_1, q_2, q_3, q_4, O_0, O_1, O_2, O_3, O_4)$?

Answer:



Hidden State

ximity sensors. (unreliably tell us
adjacent squares)

W	W	W
	®	
H		

W denotes "WALL"

Uncorrupted Observation

W		W
	®	W
H	H	

What the robot sees: Observation O_t

nding on

O_{t-1},

story)

Question: what'd be the best Bayes Net structure to represent the Joint Distribution of (q_0, q_1, q_2,q_3,q_4 ,O_0, O_1, O_2,O_3,O_4)?

Hidden State

ximity sens s

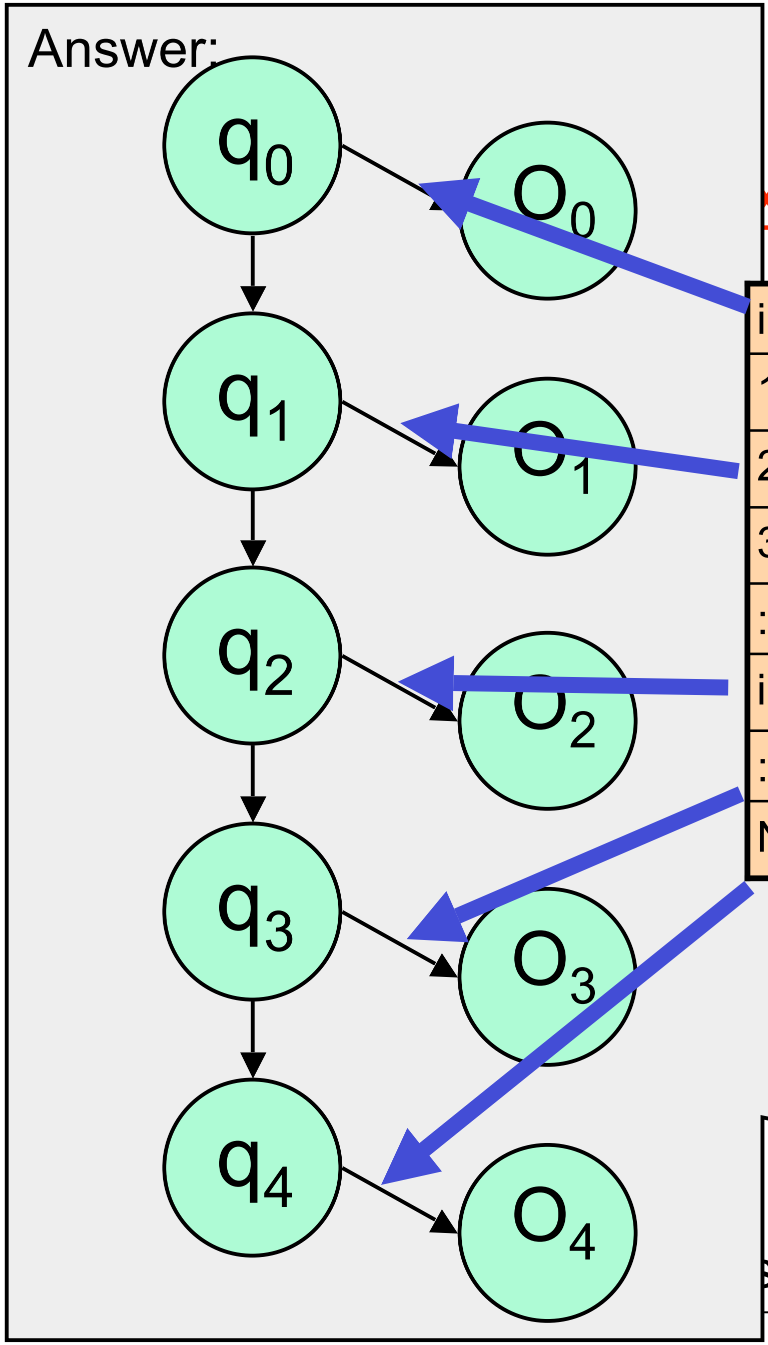
adjacent squa)

Notation:

$$b_i(k) = P(O_t = k \mid q_t = s_i)$$

i	$P(O_t=1 \mid q_t=s_i)$	$P(O_t=2 \mid q_t=s_i)$	…	$P(O_t=k \mid q_t=s_i)$	…	$P(O_t=M \mid q_t=s_i)$
1	$b_1(1)$	$b_1(2)$	…	$b_1(k)$	…	$b_1(M)$
2	$b_2(1)$	$b_2(2)$	…	$b_2(k)$	…	$b_2(M)$
3	$b_3(1)$	$b_3(2)$	…	$b_3(k)$	…	$b_3(M)$
:	:	:	:	:	:	:
i	$b_i(1)$	$b_i(2)$	…	$b_i(k)$	…	$b_i(M)$
:	:	:	:	:	:	:
N	$b_N(1)$	$b_N(2)$	…	$b_N(k)$	…	$b_N(M)$

Answer:



O_{t-1},

H	H	

story)

What the robot sees: Observation O_t

Question: what'd be the best Bayes Net structure to represent the Joint Distribution of (q_0, q_1, q_2,q_3,q_4 ,O_0, O_1, O_2,O_3,O_4)?

Hidden Markov Models

Our robot with noisy sensors is a good example of an HMM

- Question 1: State Estimation

  What is $P(q_T=S_i \mid O_1O_2\ldots O_T)$

  It will turn out that a Dynamic Programming “ trick” will get this for us.

- Question 2: Most Probable Path

  Given $O_1O_2\ldots O_T$, what is the most probable path that I took?

  And what is that probability?

  Yet another famous D.P. trick, the VITERBI algorithm, gets this.

- Question 3: Learning HMMs:

  Given $O_1O_2\ldots O_T$, what is the maximum likelihood HMM that could have produced this string of observations?

  Very very useful. Uses the E.M. Algorithm

Are H.M.M.s Useful?

You bet !!

- Robot planning + sensing when there's uncertainty (e.g. Reid Simmons / Sebastian Thrun / Sven Koenig)
- Speech Recognition/Understanding
  Phones → Words, Signal → phones
- Human Genome Project
  Complicated stuff your lecturer knows nothing about.
- Consumer decision modeling
- Economics & Finance.

Plus at least 5 other things I haven't thought of.

HMM Notation (from Rabiner's Survey)

*L. R. Rabiner, "A Tutorial on Hidden Markov Models and Selected Applications in Speech Recognition," Proc. of the IEEE, Vol.77, No.2, pp.257--286, 1989.

The states are labeled $S_1\ S_2\ ..\ S_N$

For a particular trial….

Let T be the number of observations

T is also the number of states passed through

$O = O_1\ O_2\ ..\ O_T$ is the sequence of observations

$Q = q_1\ q_2\ ..\ q_T$ is the notation for a path of states

$\lambda = \langle N, M, \{\pi_{i,}\}, \{a_{ij}\}, \{b_i(j)\}\rangle$ is the specification of an HMM

HMM Formal Definition

An HMM, λ, is a 5-tuple consisting of

- N the number of states
- M the number of possible observations
- $\{\pi_1, \pi_2, .. \pi_N\}$ The starting state probabilities

  $P(q_0 = S_i) = \pi_i$

This is new. In our previous example, start state was deterministic

- $$\left.\begin{array}{cccc} a_{11} & a_{12} & \dots & a_{1N} \\ a_{21} & a_{22} & \dots & a_{2N} \\ : & : & & : \\ a_{N1} & a_{N2} & \dots & a_{NN} \end{array}\right\}$$

The state transition probabilities

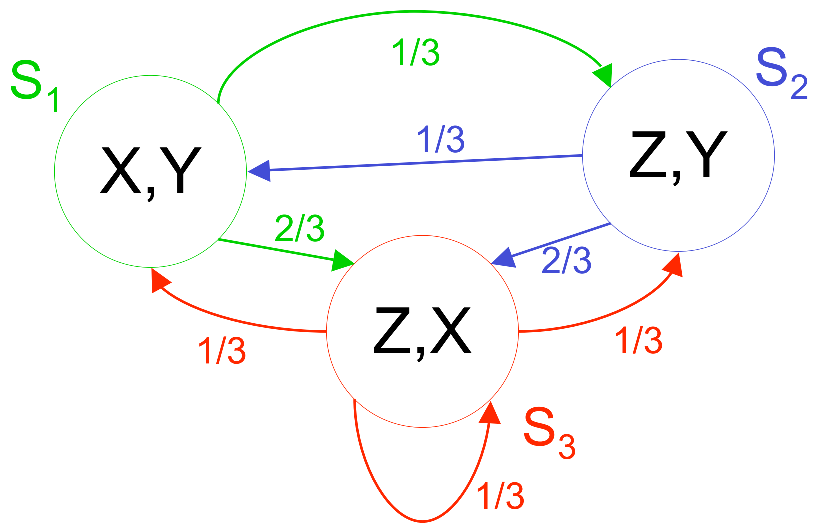
$P(q_{t+1}=S_j \mid q_t=S_i)=a_{ij}$

- $$\left.\begin{array}{cccc} b_1(1) & b_1(2) & \dots & b_1(M) \\ b_2(1) & b_2(2) & \dots & b_2(M) \\ : & : & & : \\ b_N(1) & b_N(2) & \dots & b_N(M) \end{array}\right\}$$

The observation probabilities

$P(O_t=k \mid q_t=S_i)=b_i(k)$

Here is an HMM



Start randomly in state 1 or 2

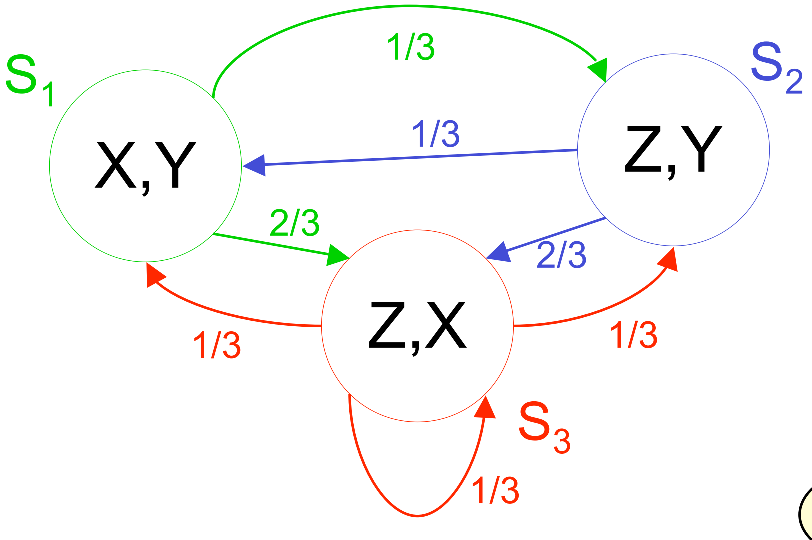
Choose one of the output symbols in each state at random.

$N = 3$

$M = 3$

$\pi_1 = 1/2$	$\pi_2 = 1/2$	$\pi_3 = 0$
$a_{11} = 0$	$a_{12} = 1/3$	$a_{13} = 2/3$
$a_{21} = 1/3$	$a_{22} = 0$	$a_{23} = 2/3$
$a_{31} = 1/3$	$a_{32} = 1/3$	$a_{33} = 1/3$
$b_1(X) = 1/2$	$b_1(Y) = 1/2$	$b_1(Z) = 0$
$b_2(X) = 0$	$b_2(Y) = 1/2$	$b_2(Z) = 1/2$
$b_3(X) = 1/2$	$b_3(Y) = 0$	$b_3(Z) = 1/2$

Here is an HMM



Start randomly in state 1 or 2

Choose one of the output symbols in each state at random.

Let's generate a sequence of observations:

$N = 3$

$M = 3$

$\pi_1 = 1/2$	$\pi_2 = 1/2$	$\pi_3 = 0$
$a_{11} = 0$	$a_{12} = 1/3$	$a_{13} = 2/3$
$a_{21} = 1/3$	$a_{22} = 0$	$a_{23} = 2/3$
$a_{31} = 1/3$	$a_{32} = 1/3$	$a_{33} = 1/3$
$b_1(X) = 1/2$	$b_1(Y) = 1/2$	$b_1(Z) = 0$
$b_2(X) = 0$	$b_2(Y) = 1/2$	$b_2(Z) = 1/2$
$b_3(X) = 1/2$	$b_3(Y) = 0$	$b_3(Z) = 1/2$

50-50 choice between S_1 and S_2

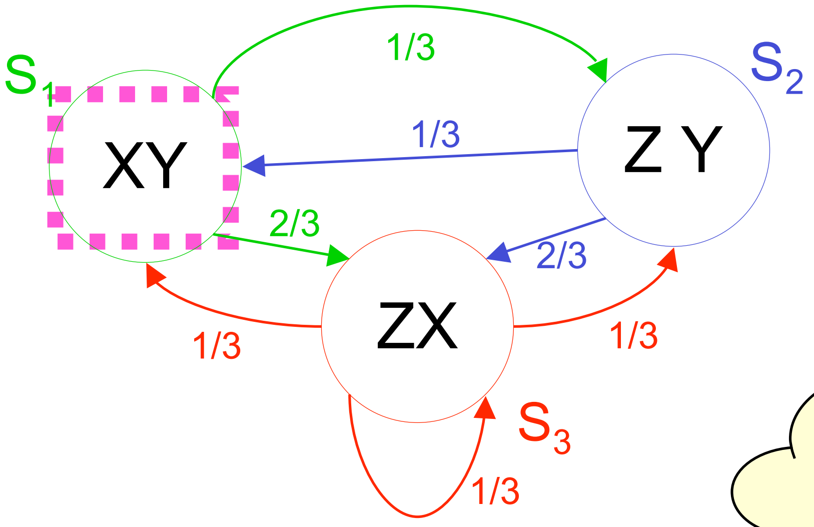
q_0=	__	O_0=	__
q_1=	__	O_1=	__
q_2=	__	O_2=	__

Here is an HMM

Start randomly in state 1 or 2

Choose one of the output symbols in each state at random.

Let's generate a sequence of observations:



N = 3

M = 3

$\pi_1 = 1/2$ $\pi_2 = 1/2$ $\pi_3 = 0$

$a_{11} = 0$ $a_{12} = 1/3$ $a_{13} = 2/3$

$a_{21} = 1/3$ $a_{22} = 0$ $a_{23} = 2/3$

$a_{31} = 1/3$ $a_{32} = 1/3$ $a_{33} = 1/3$

$b_1(X) = 1/2$ $b_1(Y) = 1/2$ $b_1(Z) = 0$

$b_2(X) = 0$ $b_2(Y) = 1/2$ $b_2(Z) = 1/2$

$b_3(X) = 1/2$ $b_3(Y) = 0$ $b_3(Z) = 1/2$

50-50 choice between X and Y

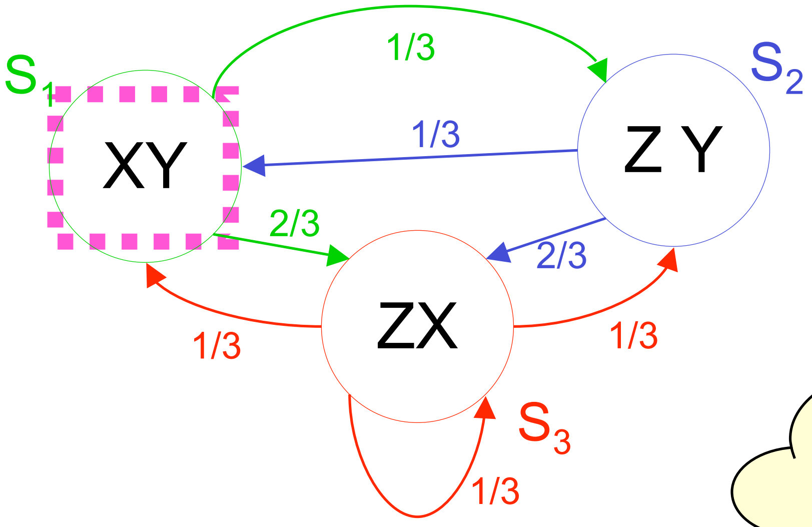
q_0=	S_1	O_0=	___
q_1=	___	O_1=	___
q_2=	___	O_2=	___

Here is an HMM

Start randomly in state 1 or 2

Choose one of the output symbols in each state at random.

Let's generate a sequence of observations:



$N = 3$

$M = 3$

$\pi_1 = 1/2$ $\quad$ $\pi_2 = 1/2$ $\quad$ $\pi_3 = 0$

$a_{11} = 0$ $\quad$ $a_{12} = 1/3$ $\quad$ $a_{13} = 2/3$

$a_{21} = 1/3$ $\quad$ $a_{22} = 0$ $\quad$ $a_{23} = 2/3$

$a_{31} = 1/3$ $\quad$ $a_{32} = 1/3$ $\quad$ $a_{33} = 1/3$

$b_1 (X) = 1/2$ $\quad$ $b_1 (Y) = 1/2$ $\quad$ $b_1 (Z) = 0$

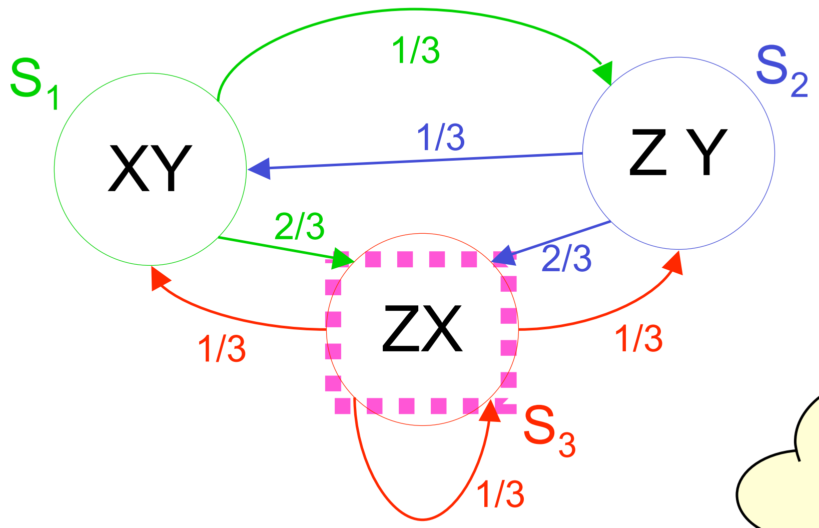
$b_2 (X) = 0$ $\quad$ $b_2 (Y) = 1/2$ $\quad$ $b_2 (Z) = 1/2$

$b_3 (X) = 1/2$ $\quad$ $b_3 (Y) = 0$ $\quad$ $b_3 (Z) = 1/2$

Goto S_3 with probability 2/3 or S_2 with prob. 1/3

q_0=	S_1	O_0=	X
q_1=	__	O_1=	__
q_2=	__	O_2=	__

Here is an HMM



N = 3

M = 3

$\pi_1 = 1/2$ $\pi_2 = 1/2$ $\pi_3 = 0$

$a_{11} = 0$ $a_{12} = 1/3$ $a_{13} = 2/3$

$a_{21} = 1/3$ $a_{22} = 0$ $a_{23} = 2/3$

$a_{31} = 1/3$ $a_{32} = 1/3$ $a_{33} = 1/3$

$b_1(X) = 1/2$ $b_1(Y) = 1/2$ $b_1(Z) = 0$

$b_2(X) = 0$ $b_2(Y) = 1/2$ $b_2(Z) = 1/2$

$b_3(X) = 1/2$ $b_3(Y) = 0$ $b_3(Z) = 1/2$

Start randomly in state 1 or 2

Choose one of the output symbols in each state at random.

Let's generate a sequence of observations:

50-50 choice between Z and X

q_0=	S_1	O_0=	X
q_1=	S_3	O_1=	__
q_2=	__	O_2=	__

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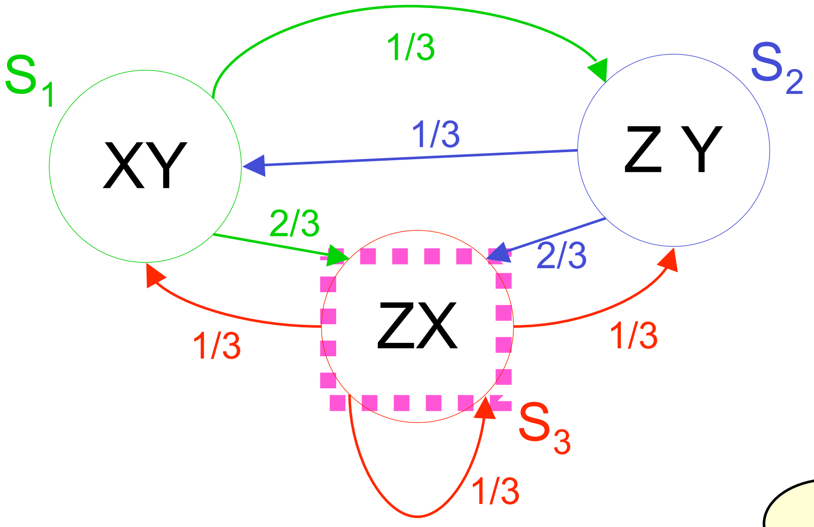
Here is an HMM



Start randomly in state 1 or 2

Choose one of the output symbols in each state at random.

Let's generate a sequence of observations:

Each of the three next states is equally likely

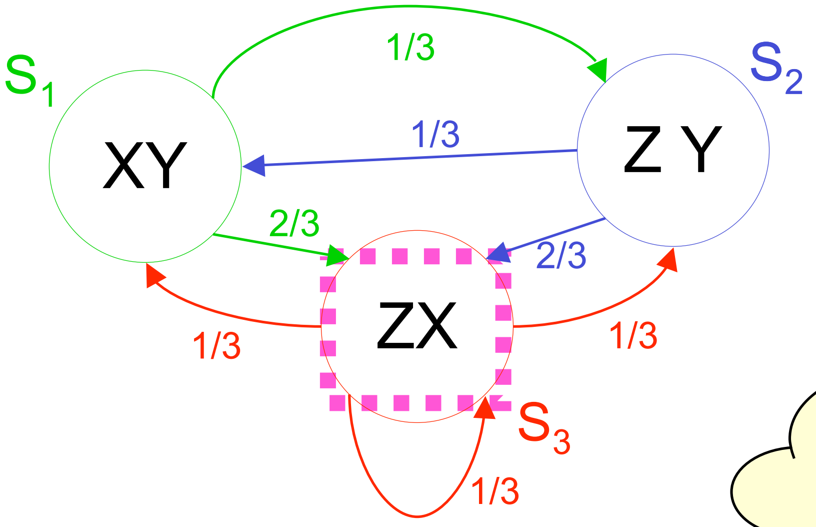
$N = 3$

$M = 3$

$\pi_1 = 1/2$	$\pi_2 = 1/2$	$\pi_3 = 0$
$a_{11} = 0$	$a_{12} = 1/3$	$a_{13} = 2/3$
$a_{12} = 1/3$	$a_{22} = 0$	$a_{13} = 2/3$
$a_{13} = 1/3$	$a_{32} = 1/3$	$a_{13} = 1/3$
$b_1 (X) = 1/2$	$b_1 (Y) = 1/2$	$b_1 (Z) = 0$
$b_2 (X) = 0$	$b_2 (Y) = 1/2$	$b_2 (Z) = 1/2$
$b_3 (X) = 1/2$	$b_3 (Y) = 0$	$b_3 (Z) = 1/2$

q_0=	S_1	O_0=	X
q_1=	S_3	O_1=	X
q_2=	__	O_2=	__

Here is an HMM



Start randomly in state 1 or 2

Choose one of the output symbols in each state at random.

Let's generate a sequence of observations:

50-50 choice between Z and X

$N = 3$

$M = 3$

$\pi_1 = 1/2$	$\pi_2 = 1/2$	$\pi_3 = 0$
$a_{11} = 0$	$a_{12} = 1/3$	$a_{13} = 2/3$
$a_{21} = 1/3$	$a_{22} = 0$	$a_{23} = 2/3$
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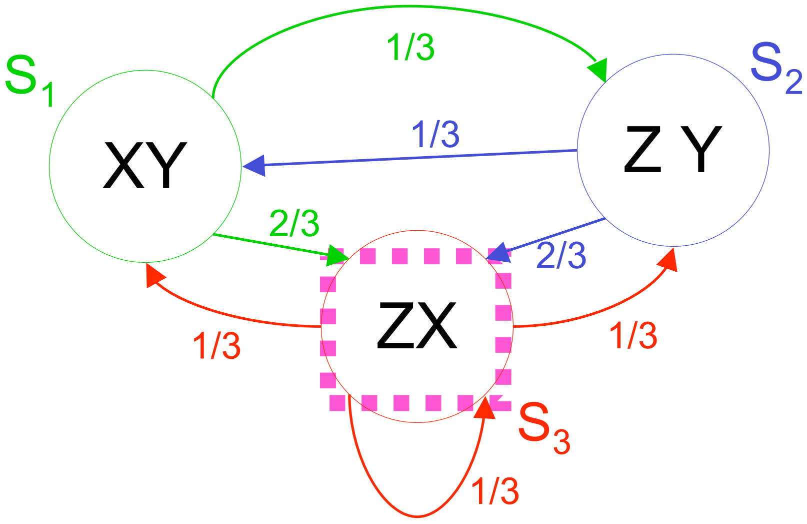
$b_1(X) = 1/2$	$b_1(Y) = 1/2$	$b_1(Z) = 0$
$b_2(X) = 0$	$b_2(Y) = 1/2$	$b_2(Z) = 1/2$
$b_3(X) = 1/2$	$b_3(Y) = 0$	$b_3(Z) = 1/2$

$q_0 =$	S_1	$O_0 =$	X
$q_1 =$	S_3	$O_1 =$	X
$q_2 =$	S_3	$O_2 =$	__

Here is an HMM



Start randomly in state 1 or 2

Choose one of the output symbols in each state at random.

Let’s generate a sequence of observations:

$N = 3$

$M = 3$

$\pi_1 = 1/2$ $\quad\pi_2 = 1/2$ $\quad\pi_3 = 0$

$a_{11} = 0$ $\quad a_{12} = 1/3$ $\quad a_{13} = 2/3$

$a_{12} = 1/3$ $\quad a_{22} = 0$ $\quad a_{13} = 2/3$

$a_{13} = 1/3$ $\quad a_{32} = 1/3$ $\quad a_{13} = 1/3$

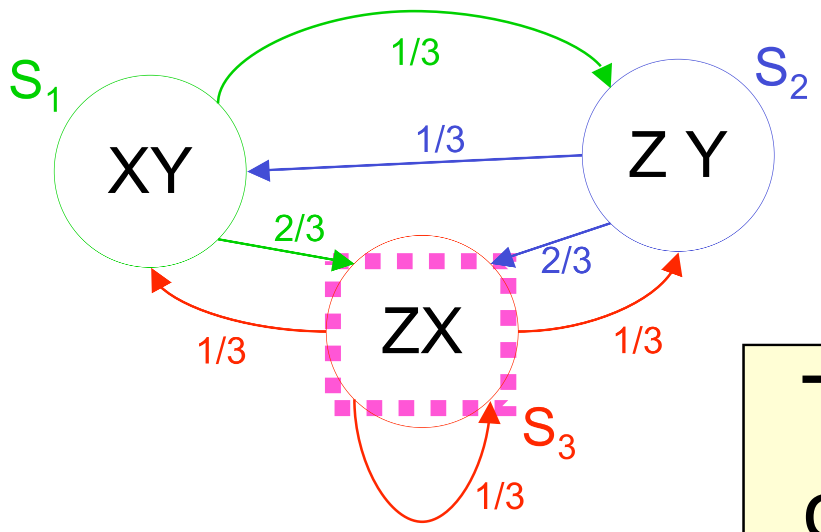
$b_1 (X) = 1/2$ $\quad b_1 (Y) = 1/2$ $\quad b_1 (Z) = 0$

$b_2 (X) = 0$ $\quad b_2 (Y) = 1/2$ $\quad b_2 (Z) = 1/2$

$b_3 (X) = 1/2$ $\quad b_3 (Y) = 0$ $\quad b_3 (Z) = 1/2$

q_0=	S_1	O_0=	X
q_1=	S_3	O_1=	X
q_2=	S_3	O_2=	Z

State Estimation



Start randomly in state 1 or 2

Choose one of the output symbols in each state at random.

Let's generate a sequence of observations:

This is what the observer has to work with…

$N = 3$

$M = 3$

$\pi_1 = 1/2$ $\quad$ $\pi_2 = 1/2$ $\quad$ $\pi_3 = 0$

$a_{11} = 0$ $\quad$ $a_{12} = 1/3$ $\quad$ $a_{13} = 2/3$

$a_{21} = 1/3$ $\quad$ $a_{22} = 0$ $\quad$ $a_{23} = 2/3$

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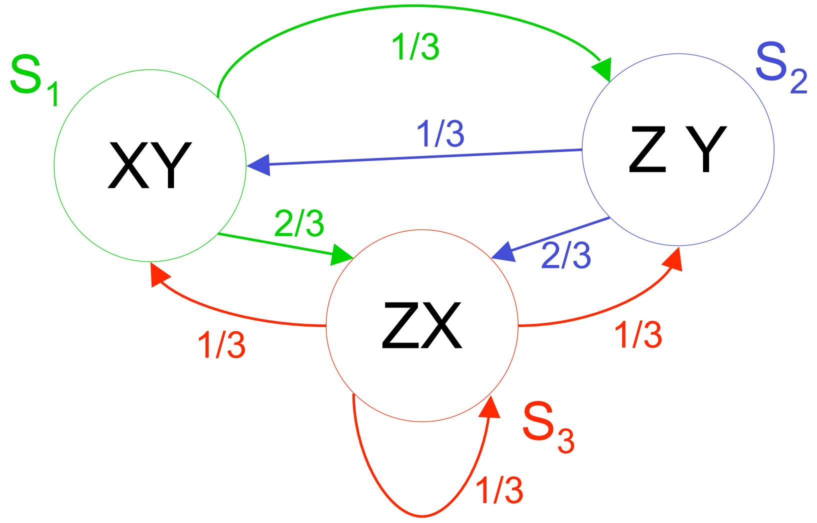
$b_2(X) = 0$ $\quad$ $b_2(Y) = 1/2$ $\quad$ $b_2(Z) = 1/2$

$b_3(X) = 1/2$ $\quad$ $b_3(Y) = 0$ $\quad$ $b_3(Z) = 1/2$

q_0=	?	O_0=	X
q_1=	?	O_1=	X
q_2=	?	O_2=	Z

Prob. of a series of observations

What is $P(\mathbf{O}) = P(O_1\ O_2\ O_3) = P(O_1 = X \wedge O_2 = X \wedge O_3 = Z)$?



Slow, stupid way:

$$P(\mathbf{O}) = \sum_{\mathbf{Q} \in \text{Paths of length 3}} P(\mathbf{O} \wedge \mathbf{Q})$$

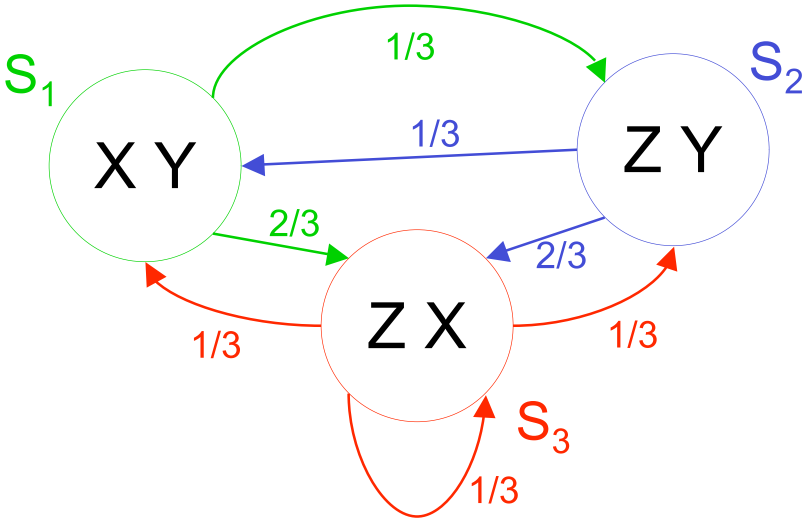
$$= \sum_{\mathbf{Q} \in \text{Paths of length 3}} P(\mathbf{O} \mid \mathbf{Q}) P(\mathbf{Q})$$

How do we compute P(Q) for an arbitrary path Q?

How do we compute P(O|Q) for an arbitrary path Q?

Prob. of a series of observations

What is $P(\mathbf{O}) = P(O_1\ O_2\ O_3) = P(O_1 = X \wedge O_2 = X \wedge O_3 = Z)$?



Slow, stupid way:

$$P(\mathbf{O}) = \sum_{\mathbf{Q} \in \text{Paths of length 3}} P(\mathbf{O} \wedge \mathbf{Q})$$

$$= \sum_{\mathbf{Q} \in \text{Paths of length 3}} P(\mathbf{O} | \mathbf{Q}) P(\mathbf{Q})$$

How do we compute P(Q) for an arbitrary path Q?

$P(Q)= P(q_1,q_2,q_3)$

$=P(q_1)\ P(q_2,q_3|q_1)$ (chain rule)

$=P(q_1)\ P(q_2|q_1)\ P(q_3|\ q_2,q_1)$ (Bayes)

$=P(q_1)\ P(q_2|q_1)\ P(q_3|\ q_2)$ (Markov)

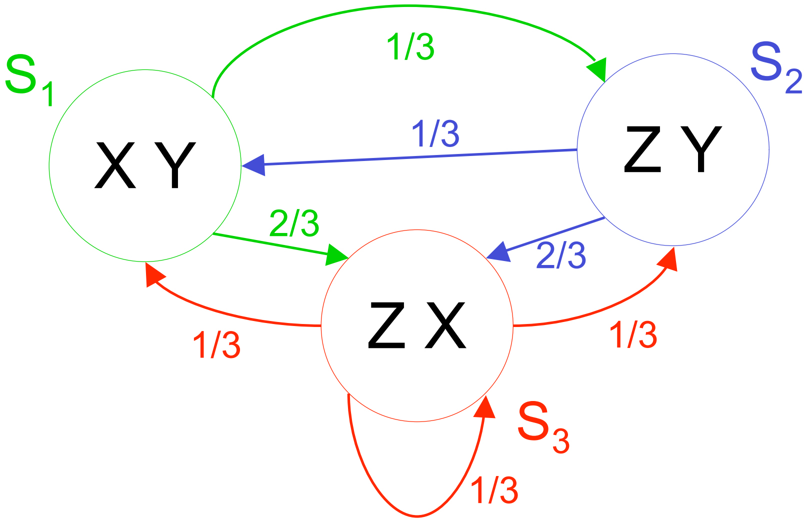
Example in the case $Q = S_1\ S_3\ S_3$:

$=1/2 * 2/3 * 1/3 = 1/9$

How do we compute P(O|Q) for an arbitrary path Q?

Prob. of a series of observations

What is $P(\mathbf{O}) = P(O_1\ O_2\ O_3) = P(O_1 = X \wedge O_2 = X \wedge O_3 = Z)$?



Slow, stupid way:

$$P(\mathbf{O}) = \sum_{\mathbf{Q} \in \text{Paths of length 3}} P(\mathbf{O} \wedge \mathbf{Q})$$

$$= \sum_{\mathbf{Q} \in \text{Paths of length 3}} P(\mathbf{O} \mid \mathbf{Q}) P(\mathbf{Q})$$

How do we compute P(Q) for an arbitrary path Q?

How do we compute P(O|Q) for an arbitrary path Q?

P(O|Q)

$= P(O_1\ O_2\ O_3 \mid q_1\ q_2\ q_3)$

$= P(O_1 \mid q_1)\ P(O_2 \mid q_2)\ P(O_3 \mid q_3)$

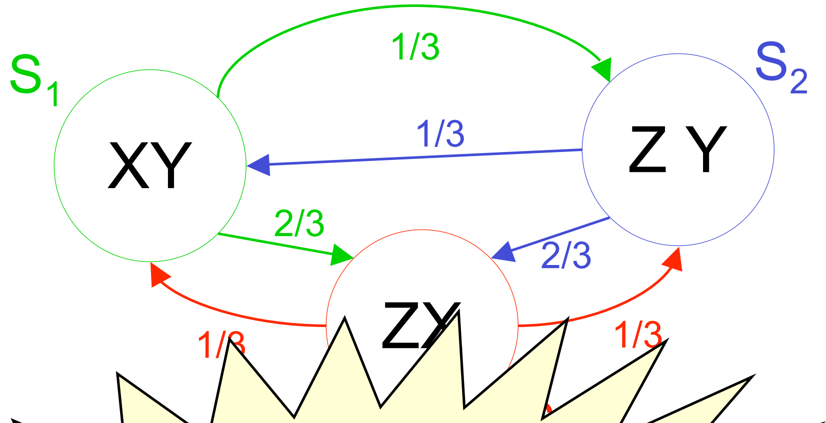
Example in the case $Q = S_1\ S_3\ S_3$:

$= P(X \mid S_1)\ P(X \mid S_3)\ P(Z \mid S_3) =$

$=1/2 * 1/2 * 1/2 = 1/8$

Prob. of a series of observations

What is $P(\mathbf{O}) = P(O_1\ O_2\ O_3) = P(O_1 = X \wedge O_2 = X \wedge O_3 = Z)$?



Slow, stupid way:

$$P(\mathbf{O}) = \sum_{\mathbf{Q} \in \text{Paths of length 3}} P(\mathbf{O} \wedge \mathbf{Q})$$

$$= \sum_{\mathbf{Q} \in \text{Paths of length 3}} P(\mathbf{O} \mid \mathbf{Q}) P(\mathbf{Q})$$

How do we compute P(Q) f
an arbitrary path Q?

How do we compute P(O|Q)
for an arbitrary path Q?

P(**O**) would need 27 P(Q) computations and 27 P(O|Q) computations

A sequence of 20 observations would need 3^{20} = 3.5 billion computations and 3.5 billion P(O|Q) computations

So let's be smarter…

The Prob. of a given series of observations, non-exponential-cost-style

Given observations $O_1\ O_2\ \ldots\ O_T$

Define

$$\alpha_t(i) = P(O_1\ O_2\ \ldots\ O_t\ \wedge q_t = S_i\ ) \qquad \text{where } 1 \leq t \leq T$$

$\alpha_t(i)$ = Probability that, in a random trial,

- we would have seen the first t observations
- we would have ended up in S_i as the t'th state visited.

In our example, what is $\alpha_2(3)$?

$\alpha_t(i)$: easy to define recursively

$\alpha_t(i) = P(O_1\ O_2\ \ldots\ O_T \wedge q_t = S_i)$

$$\alpha_1(i) = \mathrm{P}(O_1 \wedge q_1 = S_i)$$
$$= \mathrm{P}(q_1 = S_i)\mathrm{P}(O_1 | q_1 = S_i)$$

$$\alpha_{t+1}(j) = \mathrm{P}(O_1 O_2 \ldots O_t O_{t+1} \wedge q_{t+1} = S_j)$$
$$=$$

$\alpha_t(i)$: easy to define recursively

$\alpha_t(i) = P(O_1\ O_2\ \ldots\ O_T\ \wedge q_t = S_i)$

$$\alpha_1(i) = \mathrm{P}(O_1 \wedge q_1 = S_i)$$

$$= \mathrm{P}(q_1 = S_i)\mathrm{P}(O_1 | q_1 = S_i)$$

$$\alpha_{t+1}(j) = \mathrm{P}(O_1 O_2 \ldots O_t O_{t+1} \wedge q_{t+1} = S_j)$$

explain

$$= \sum_{i=1}^{N} \mathrm{P}(O_1 O_2 \ldots O_t \wedge q_t = S_i \wedge O_{t+1} \wedge q_{t+1} = S_j)$$

explain

$$= \sum_{i=1}^{N} \mathrm{P}(O_{t+1}, q_{t+1} = S_j | O_1 O_2 \ldots O_t \wedge q_t = S_i)\underbrace{\mathrm{P}(O_1 O_2 \ldots O_t \wedge q_t = S_i)}$$

$$= \sum_{i} \mathrm{P}(O_{t+1}, q_{t+1} = S_j | q_t = S_i)\ \alpha_t(i)$$

explain

$$= \sum_{i} \mathrm{P}(q_{t+1} = S_j | q_t = S_i)\mathrm{P}(O_{t+1} | q_{t+1} = S_j)\ \alpha_t(i)$$

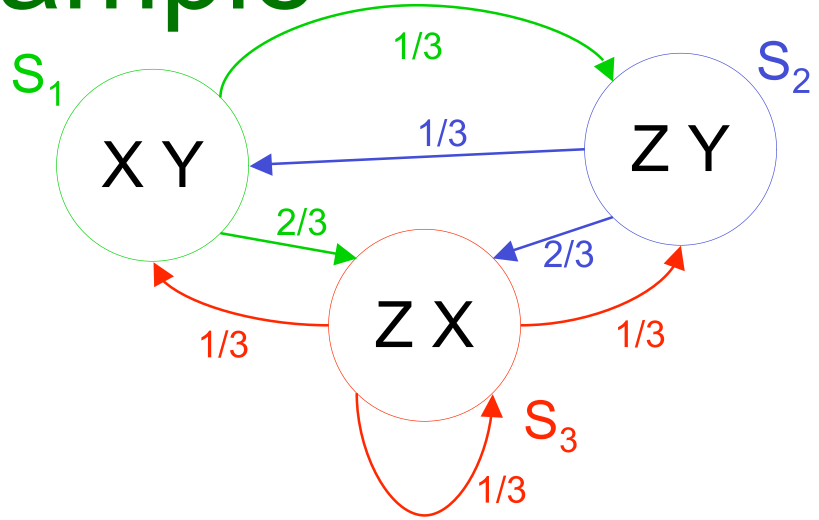
$$= \sum_{i} a_{ij} b_j(O_{t+1})\ \alpha_t(i)$$

in our example

$$\alpha_t(i) = \mathrm{P}(O_1 O_2 .. O_t \wedge q_t = S_i)$$

$$\alpha_1(i) = \ b_i(O_1)\pi_i$$

$$\alpha_{t+1}(j) = \sum_i a_{ij} b_j(O_{t+1}) \alpha_t(i)$$



WE SAW O_1 O_2 O_3 = X X Z

$\alpha_1(1) = \frac{1}{4}$ $\quad\alpha_1(2) = 0$ $\quad\alpha_1(3) = 0$

$\alpha_2(1) = 0$ $\quad\alpha_2(2) = 0$ $\quad\alpha_2(3) = \frac{1}{12}$

$\alpha_3(1) = 0$ $\quad\alpha_3(2) = \frac{1}{72}$ $\quad\alpha_3(3) = \frac{1}{72}$

Easy Question

We can cheaply compute

$$\alpha_t(i)=P(O_1O_2\ldots O_t\wedge q_t=S_i)$$

(How) can we cheaply compute

$$P(O_1O_2\ldots O_t) \quad ?$$

(How) can we cheaply compute

$$P(q_t=S_i|O_1O_2\ldots O_t)$$

Easy Question

We can cheaply compute

$$\alpha_t(i)=P(O_1 O_2 \ldots O_t \wedge q_t=S_i)$$

(How) can we cheaply compute

$$P(O_1 O_2 \ldots O_t) \quad ?$$

$$\boxed{\sum_{i=1}^{N} \alpha_t(i)}$$

(How) can we cheaply compute

$$P(q_t=S_i | O_1 O_2 \ldots O_t)$$

$$\boxed{\frac{\alpha_t(i)}{\sum_{j=1}^{N} \alpha_t(j)}}$$

Most probable path given observations

What is most probable path given $O_1 O_2 \ldots O_T$ i.e.

What is $\arg\max_Q P(Q \mid O_1 O_2 \ldots O_T)$?

Slow, stupid answer:

$$\arg\max_Q P(Q \mid O_1 O_2 \ldots O_T)$$

$$= \arg\max_Q \frac{P(O_1 O_2 \ldots O_T \mid Q) P(Q)}{P(O_1 O_2 \ldots O_T)}$$

$$= \arg\max_Q P(O_1 O_2 \ldots O_T \mid Q) P(Q)$$

Efficient MPP computation

We are going to compute the following variables:

$$\delta_t(i) = \max_{q_1 q_2 .. q_{t-1}} P(q_1\, q_2\, ..\, q_{t-1} \wedge q_t = S_i \wedge O_1\, ..\, O_t)$$

= probability of the path of length t-1 with the maximum chance of doing all these things:

…OCCURING

and

…ENDING UP IN STATE S_i

and

…PRODUCING OUTPUT $O_1 \ldots O_t$

DEFINE: $\text{mpp}_t(i)$ = that path

So: $\delta_t(i) = \text{Prob}(\text{mpp}_t(i))$

The Viterbi Algorithm

$$\delta_t(i) = \max_{q_1 q_2 \dots q_{t-1}} P(q_1 q_2 \dots q_{t-1} \wedge q_t = S_i \wedge O_1 O_2 \dots O_t)$$

$$mmp_t(i) = \arg\max_{q_1 q_2 \dots q_{t-1}} P(q_1 q_2 \dots q_{t-1} \wedge q_t = S_i \wedge O_1 O_2 \dots O_t)$$

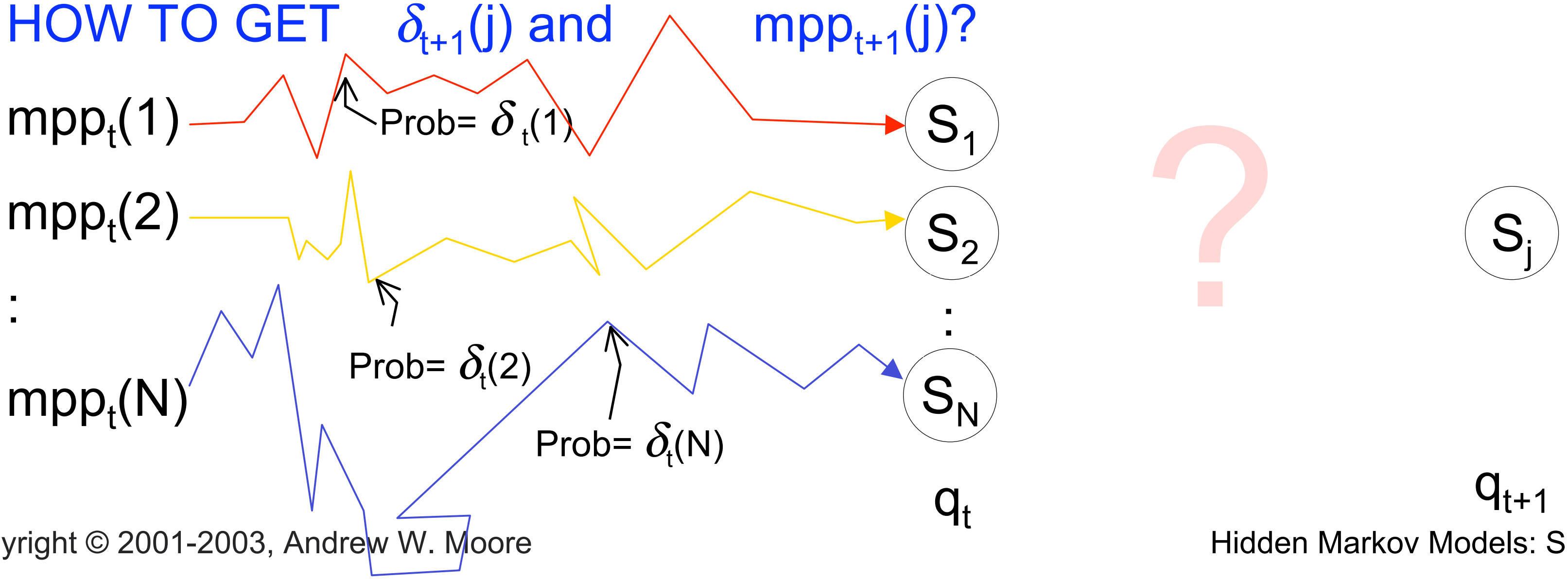
$$\delta_1(i) = \max_{q_0} P(q_1 = S_i \wedge O_1)$$

explain

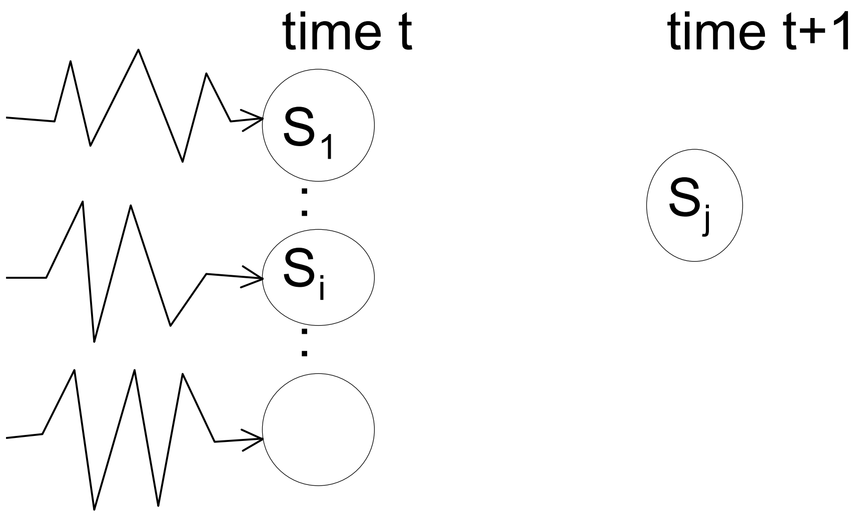
$$= P(q_1 = S_i) P(O_1 \mid q_1 = S_i) = \pi_i b_i(O_1)$$

Now, suppose we have all the δ_t(i)'s and mpp$_t$(i)'s for all i.

HOW TO GET δ_{t+1}(j) and mpp$_{t+1}$(j)?



The Viterbi Algorithm

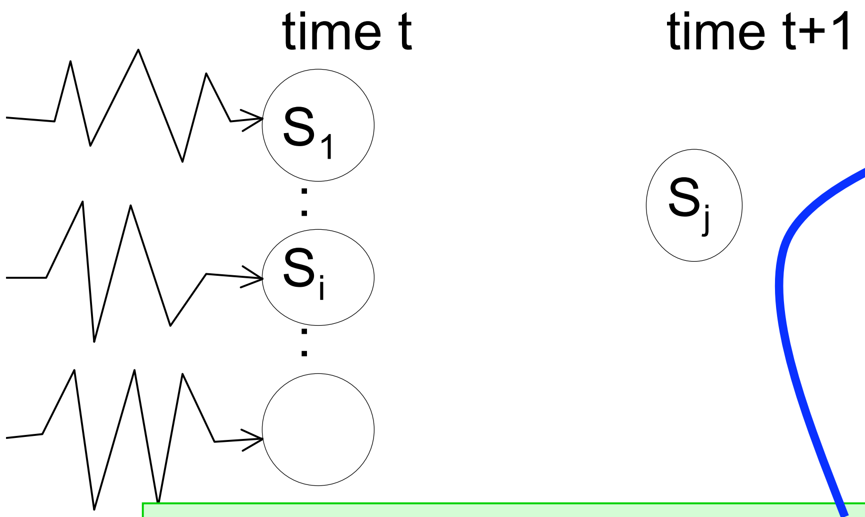


The most prob path with last two states S_i S_j

is

the most prob path to S_i , followed by transition $S_i \rightarrow S_j$

The Viterbi Algorithm



The most prob path with last two states S_i S_j

is

the most prob path to S_i , followed by transition $S_i \rightarrow S_j$

What is the prob of that path?

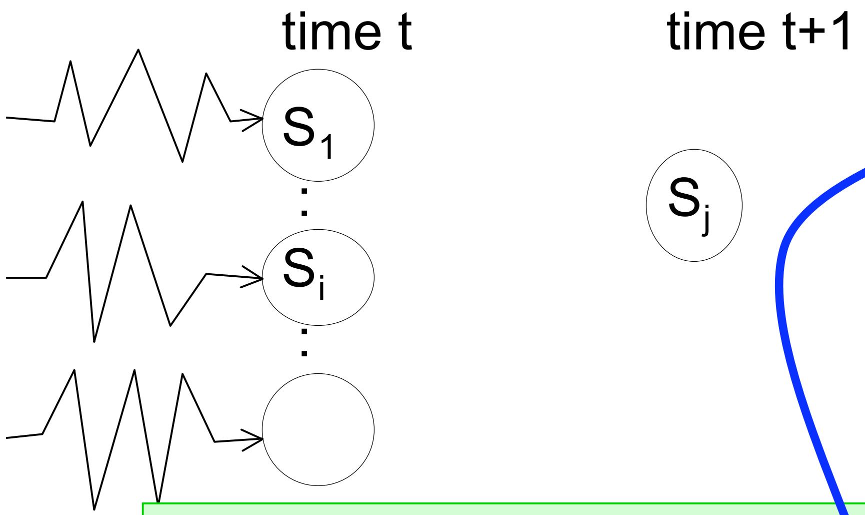
$$\delta_t(i) \text{ x } P(q_{t+1} = S_j \wedge O_{t+1} | q_t = S_i)$$

$$= \quad \delta_t(i) a_{ij} b_j (O_{t+1})$$

SO The most probable path to S_j has S_{i^*} as its penultimate state

where $i^* = \underset{i}{\operatorname{argmax}} \, \delta_t(i) a_{ij} b_j (O_{t+1})$

The Viterbi Algorithm



The most prob path with last two states S_i S_j

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What is the prob of that path?

$\delta_t(i)$ x $P(q_{t+1}= S_j \wedge$

= $\delta_t(i)\ a_{ij}\ b_j\ (O_{t+1})$

SO The most probable

S_{i^*} as its penultimate state

where $i^*=\underset{i}{\operatorname{argmax}}\ \delta_t(i)\ a_{ij}\ b_j\ (O_{t+1})$

Summary:

$$\left.\begin{array}{l} \delta_{t+1}(j) = \delta_t(i^*)\ a_{ij}\ b_j\ (O_{t+1}) \\ mpp_{t+1}(j) = mpp_{t+1}(i^*)S_j \end{array}\right\}$$

with i* defined to the left

What’s Viterbi used for?

Classic Example

Speech recognition:

Signal → words

HMM → observable is signal

→ Hidden state is part of word formation

What is the most probable word given this signal?

UTTERLY GROSS SIMPLIFICATION

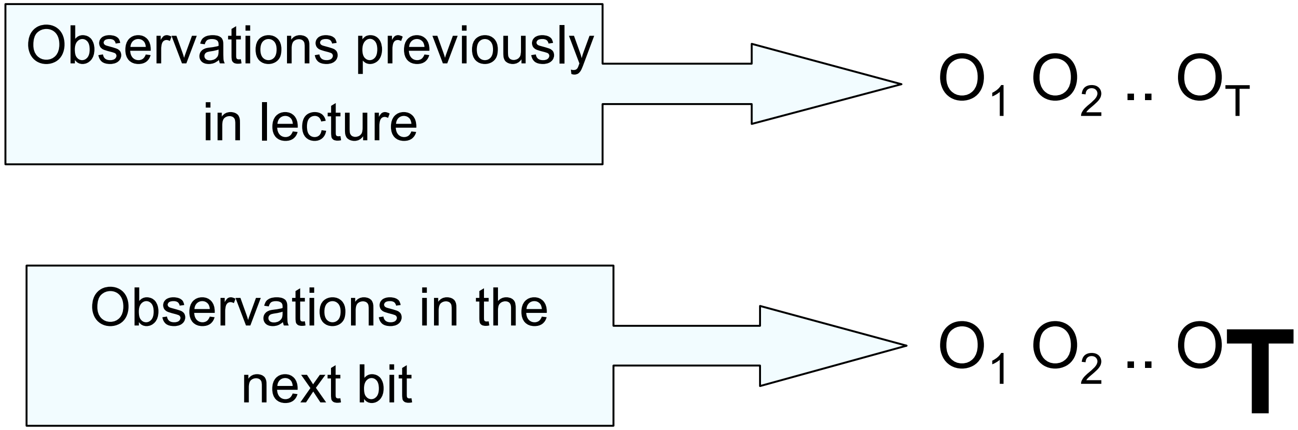
In practice: many levels of inference; not one big jump.

HMMs are used and useful

But how do you design an HMM?

Occasionally, (e.g. in our robot example) it is reasonable to deduce the HMM from first principles.

But usually, especially in Speech or Genetics, it is better to infer it from large amounts of data. O_1 O_2 .. O_T with a big “T”.



Observations previously in lecture → O_1 O_2 .. O_T

Observations in the next bit → O_1 O_2 .. O**T**

Inferring an HMM

Remember, we've been doing things like

$$P(O_1\ O_2\ ..\ O_T \mid \lambda\ )$$

That "λ" is the notation for our HMM parameters.

<u>Now</u> We have some observations and we want to estimate λ from them.

AS USUAL: We could use

(i) MAX LIKELIHOOD $\lambda = \underset{\lambda}{\operatorname{argmax}}\ P(O_1\ ..\ O_T \mid \lambda)$

(ii) BAYES

Work out $P(\lambda \mid O_1\ ..\ O_T\ )$

and then take $E[\lambda]$ or $\underset{\lambda}{\max}\ P(\lambda \mid O_1\ ..\ O_T\ )$

Max likelihood HMM estimation

Define

$\gamma_t(i) = P(q_t = S_i \mid O_1 O_2 \ldots O_T , \lambda)$

$\varepsilon_t(i,j) = P(q_t = S_i \wedge q_{t+1} = S_j \mid O_1 O_2 \ldots O_T , \lambda)$

$\gamma_t(i)$ and $\varepsilon_t(i,j)$ can be computed efficiently $\forall i,j,t$

(Details in Rabiner paper)

$\sum_{t=1}^{T-1} \gamma_t(i) =$ Expected number of transitions out of state i during the path

$\sum_{t=1}^{T-1} \varepsilon_t(i,j) =$ Expected number of transitions from state i to state j during the path

$$\gamma_t(i) = \mathrm{P}(q_t = S_i | O_1 O_2 .. O_T, \lambda)$$

$$\varepsilon_t(i,j) = \mathrm{P}(q_t = S_i \wedge q_{t+1} = S_j | O_1 O_2 .. O_T, \lambda)$$

$$\sum_{t=1}^{T-1} \gamma_t(i) = \text{expected number of transitions out of state i during path}$$

$$\sum_{t=1}^{T-1} \varepsilon_t(i,j) = \text{expected number of transitions out of i and into j during path}$$

HMM estimation

$$\text{Notice } \frac{\sum_{t=1}^{T-1} \varepsilon_t(i,j)}{\sum_{t=1}^{T-1} \gamma_t(i)} = \frac{\begin{pmatrix}\text{expected frequency} \\ \text{i} \rightarrow \text{j}\end{pmatrix}}{\begin{pmatrix}\text{expected frequency} \\ \text{i}\end{pmatrix}}$$

$$= \text{Estimate of Prob}(\text{Next state S}_\text{j} | \text{This state S}_\text{i})$$

We can re-estimate

$$\text{a}_\text{ij} \leftarrow \frac{\sum \varepsilon_t(i,j)}{\sum \gamma_t(i)}$$

We can also re-estimate

$$\text{b}_\text{j}(O_k) \leftarrow \cdots \qquad \text{(See Rabiner)}$$

EM for HMMs

Expectation Maximization

If we knew λ we could estimate EXPECTATIONS of quantities such as

Expected number of times in state i

Expected number of transitions i → j

If we knew the quantities such as

Expected number of times in state i

Expected number of transitions i → j

We could compute the MAX LIKELIHOOD estimate of

$$\lambda = \langle \{a_{ij}\},\{b_i(j)\}, \pi_i \rangle$$

→ EM Algorithm…

EM for HMMs

1. Get your observations $O_1 \dots O_T$
2. Guess your first λ estimate $\lambda(0)$, k=0
3. k = k+1
4. Given $O_1 \dots O_T$, λ (k) compute
   $\gamma_t(i)$, $\varepsilon_t(i,j)$ $\forall 1 \le t \le T$, $\forall 1 \le i \le N$, $\forall 1 \le j \le N$
5. Compute expected freq. of state i, and expected freq. $i \rightarrow j$
6. Compute new estimates of a_{ij}, $b_j(k)$, π_i accordingly. Call them λ (k+1)
7. Goto 3, unless converged.

Known (for the HMM case) as the BAUM-WELCH algorithm.

Bad News

- There are lots of local minima

Good News

- The local minima are usually adequate models of the data.

Notice

- EM does not estimate the number of states. That must be given.
- Often, HMMs are forced to have some links with zero probability. This is done by setting $a_{ij}=0$ in initial estimate $\lambda(0)$
- Easy extension of everything seen today: HMMs with real valued outputs

Bad News

- There are lots o
- The local minim data.

Trade-off between too few states (inadequately modeling the structure in the data) and too many (fitting the noise).

Thus #states is a regularization parameter.

Blah blah blah… bias variance tradeoff…blah blah…cross-validation…blah blah….AIC, BIC….blah blah (same ol' same ol')

Notice

- EM does not estimate the number of states. That must be given.
- Often, HMMs are forced to have some links with zero probability. This is done by setting $a_{ij}=0$ in initial estimate $\lambda(0)$
- Easy extension of everything seen today: HMMs with real valued outputs

What You Should Know

- What is an HMM ?
- Computing (and defining) $\alpha_t(i)$
- The Viterbi algorithm
- Outline of the EM algorithm
- To be very happy with the kind of maths and analysis needed for HMMs
- Fairly thorough reading of Rabiner* up to page 266* [Up to but not including "IV. Types of HMMs"].

DON'T PANIC: starts on p. 257.

*L. R. Rabiner, "A Tutorial on Hidden Markov Models and Selected Applications in Speech Recognition," Proc. of the IEEE, Vol.77, No.2, pp.257--286, 1989.