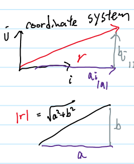
Untitled page

Module 2

Tuesday, April 21, 2020 11:08 PM

Linear Algebra Module #2

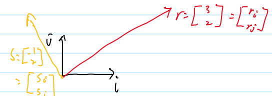
- Length of a vector
- dot product of a vector



coordinate system

$a\hat{i} + b\hat{j} = \begin{bmatrix} a \\ b \end{bmatrix}$

$|r| = \sqrt{a^2+b^2}$



$r = \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} r_i \\ r_j \end{bmatrix}$

$s = \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} s_i \\ s_j \end{bmatrix}$

dot product of

r and s $= r \cdot s = r_i s_i + r_j s_j$

$= 3 \times (-1) + 2 \times 2 = -3 + 4 = 1$

properties of dot product

1. Commutative $r \cdot s = s \cdot r$
2. Distributive over addition $r \cdot (s+t) = r \cdot s + r \cdot t$

$r \cdot (as) = a(r \cdot s)$

$r_i(as_i) + r_j(as_j) = a(r_i s_i + r_j s_j)$

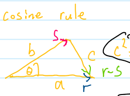
associative over scalar multiplication

$r \cdot r = r_i r_i + r_j r_j$

$= r_i^2 + r_j^2$

$r \cdot r = |r|^2$

cosine rule



$c^2 = a^2 + b^2 - 2ab\cos\theta$

$\Downarrow$

$|r-s|^2 = |r|^2 + |s|^2 - 2|r||s|\cos\theta$

using the fact that $r \cdot r = |r|^2$

we can replace $|r-s|^2$ with $(r-s) \cdot (r-s)$

$\Downarrow$

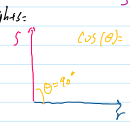
$r \cdot r - s \cdot r - s \cdot r + s \cdot s$

$\cancel{|r|^2} + \cancel{|s|^2} - 2|r||s|\cos\theta = \cancel{|r|^2} - 2s \cdot r + \cancel{|s|^2}$

$\cancel{-2}|r||s|\cos\theta = \cancel{-2}s \cdot r$

$s \cdot r = |r||s|\cos\theta$

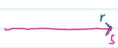
insights:



$\cos(\theta) = \cos(90°) = 0$

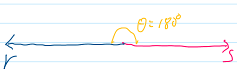
$s \cdot r = |r||s| \cdot 0$

that is, if r and s are orthogonal to each other, then the dot product $s \cdot r$ will also equal to 0.



if $\theta = 0$, $\cos(\theta) = 1$

$s \cdot r = |r||s|$



$\cos(\theta) = -1$

$r \cdot s = -|r||s|$

Projection

$\cos(\theta) = \frac{adj}{hyp} = \frac{adj}{|s|}$

$\cos\theta |s| = adj$

$|r||s| = |r| \underbrace{|s|\cos\theta}_{\substack{adj \\ projection}}$

adj is a shadow casting from s

projection

$\frac{|r||s|}{|r|} = |s|\cos\theta$

Scalar projection

Vector Projection

$r \cdot \frac{r \cdot s}{|r||r|} = \frac{r \cdot s}{r \cdot r} \cdot r$

$|r|^2 = r \cdot r$

number

vector

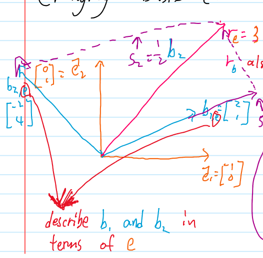
Note: $\frac{r}{|r|} = r$ divided by its length $|r|$

intuition: $\frac{r \cdot s}{|r|}$ = Scalar Projection

$\frac{r}{|r|}$ is a vector going in the direction of r by has been normalized to have a length 1. (i.e. a unit vector)

Vector Projection is a number multiply by a unit vector

Changing basis (co-ordinate systems) example using the projection product



$r_e = 3\vec{e}_1 + 4\vec{e}_2 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

r_b also exist

using b_1 and b_2, we can describe r in terms of b_1 and b_2 in which the numbers to represent will be different.

we call e and b base vectors.

r only have any meaning when we define e and b

describe b_1 and b_2 in terms of e

$\hat{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

describe b_1 and b_2 in terms of e

$b_{1,e} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2e_1 \\ 1e_2 \end{bmatrix}$

we call e and b base vectors. r only have any meaning when we define e and b

r_e = describe r in base of e

$$\frac{r_e \cdot b_1}{|b_1|^2} = \frac{3 \times 2 + 4 \times 1}{2^2 + 1^2} = \frac{6+4}{5} = \frac{10}{5} = 2$$

$$\frac{r_e \cdot b_1}{|b_1|^2} \cdot b_1 = 2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$$\frac{r_e \cdot b_2}{|b_2|^2} = \frac{3 \times (-2) + 4 \times (4)}{(-2)^2 + (4)^2} = \frac{-6+16}{4+16} = \frac{10}{20} = \frac{1}{2}$$

$$\frac{r_e \cdot b_2}{|b_2|^2} \cdot b_2 = \frac{1}{2} \begin{bmatrix} -2 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

adding 2 vector projections together: $\begin{bmatrix} 4 \\ 2 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} = r_e$ in terms of base b

So r_b is $\begin{bmatrix} 2 \\ 1/2 \end{bmatrix}$

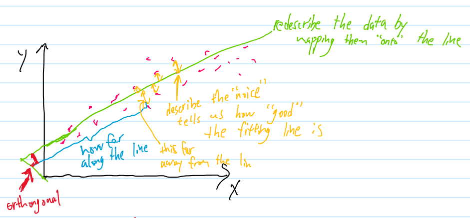
By using dot products and vector projections, we are able to "describe" a vector in different bases

(Note: The new bases have to be orthogonal to each other)

Definitions:

Basis: a set of n vectors that
(i) are not linear combinations of each other (linearly independent)
(ii) span the space
the space is then n-dimensional

example:



we can use dot products to do projection to map from x and y onto the line

Summary:

Vectors can be think of as objects that describe where we are in the space (physical space, space of data, ...)

projection of vectors onto different basis in the case that the new basis are orthongonal (90°) to each other.

Matrices, Vectors and Solving Simultaneous Equation Problems

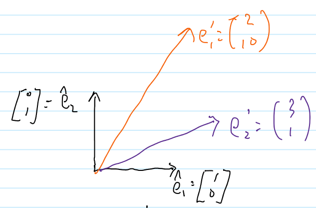
apple and banana example revist:

$$\left.\begin{aligned} 2a + 3b &= 8 \\ 10a + 1b &= 13 \end{aligned}\right\} \text{Simultaneous Equation}$$

can be rewritten as:

$$\begin{pmatrix} 2 & 3 \\ 10 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 8 \\ 13 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 3 \\ 10 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 10 \end{pmatrix}$$

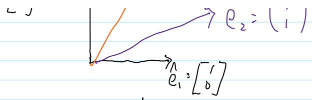


$$\begin{pmatrix} 2 & 3 \\ 10 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

basis vector

what this matrix does is it move/transform basis vector is some way. changes space

a function that



because we can make vectors using combinations of $\hat{e}_1$ and $\hat{e}_2$ the result of the transformation (i.e. e_1', e_2') is just gonna be some sum of $\hat{e}_1$ and $\hat{e}_2$.

what this matrix does is it move/transform basis vector is some way.
changes space

a function that operates on input vector and gives other output vector

How matrices transform space

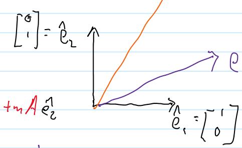
$$\underset{A}{\begin{pmatrix} 2 & 3 \\ 10 & 1 \end{pmatrix}} \underset{r}{\begin{pmatrix} a \\ b \end{pmatrix}} = \underset{r'}{\begin{bmatrix} 8 \\ 13 \end{bmatrix}}$$

$$A(nr) = nr'$$

$$A(r+s) = Ar + As$$

$$A(n\hat{e}_1 + m\hat{e}_2) = nA\hat{e}_1 + mA\hat{e}_2$$

$$= n e_1' + m e_2'$$



$$e_1' = \begin{pmatrix} 2 \\ 10 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 10 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$e_2' = \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 10 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\underbrace{\begin{bmatrix} 2 & 3 \\ 10 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 12 \\ 32 \end{bmatrix}}$$

$$\begin{bmatrix} 2 & 3 \\ 10 & 1 \end{bmatrix} \left[3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right] = 3 \underbrace{\left(\begin{bmatrix} 2 & 3 \\ 10 & 1 \end{bmatrix} \underset{\hat{e}_1}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} \right)}_{e_1' = \begin{pmatrix} 2 \\ 10 \end{pmatrix}} + 2 \underbrace{\left(\begin{bmatrix} 2 & 3 \\ 10 & 1 \end{bmatrix} \underset{\hat{e}_2}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} \right)}_{e_2' = \begin{pmatrix} 3 \\ 1 \end{pmatrix}}$$

$$= 3 \begin{pmatrix} 2 \\ 10 \end{pmatrix} + 2 \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 30 \end{pmatrix} + \begin{pmatrix} 6 \\ 2 \end{pmatrix} = \begin{pmatrix} 12 \\ 32 \end{pmatrix}$$

Matrix multiplication: multiplication of the vector sum of transformed basis vectors.
e_1' e_2'