EDIN01 – Project 2

Tony Jin (to1643ji-s@student.lu.se)

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1 Home exercise 1

$p(x) = x^4 + x^2 + 1 \implies \alpha^4 = \alpha^2 + 1$. Then, $\alpha^6 = 1$ and the polynomial is not primitive. $(x^2 + x + 1)^2 = x^4 + x^2 + 1$, thus the polynomial $p(x)$ is reducible.

$p(x) = x^3 + x + 1$ As $p(1) = 0$, we know there's a polynomial factor $(x + 2)$ in $p(x)$, thus not irreducible and not primitive.

$p(x) = x^2 + \alpha^5 x + 1$, where $\alpha^4 + \alpha + 1 = 0$. As

$$\begin{aligned} p(\alpha^6) &= (\alpha^6)^2 + \alpha^5(\alpha^6) + 1 \\ &= \alpha^{12} + \alpha^{11} + 1 \\ &= (\alpha^3 + \alpha^2 + \alpha + 1) + (\alpha^3 + \alpha^2 + \alpha) + 1 \\ &= 0, \end{aligned}$$

we see that the polynomial $p(x)$ is not irreducible, and thus not primitive either.

2 Laboratory exercise 1

```
> Primitive(x^23 + x^5 + 1) mod 2
> true

> Primitive(x^23 + x^6 + 1) mod 2
> false
> Irreduc(x^23 + x^6 + 1) mod 2
> false

> Primitive(x^18 + x^3 + 1) mod 2
> false
> Irreduc(x^18 + x^3 + 1) mod 2
> true

> Primitive(x^8 + x^6 + 1) mod 7
> false
> irreduc(x^8 + x^6 + 1) mod 7
> false
```

3 Home exercise 2

$|F_{2^4}| = 16 \implies \alpha^{15} \equiv 1 \pmod{\alpha^4 + \alpha + 1}$, so the possible orders are $15, 5, 3, 1$.

$$\begin{aligned}
\alpha &\equiv \alpha \pmod{\alpha^4+\alpha+1}\\
\alpha^3 &\equiv \alpha^3 \pmod{\alpha^4+\alpha+1}\\
\alpha^5 &\equiv \alpha(\alpha+1) \pmod{\alpha^4+\alpha+1}\\
\alpha^{15} &\equiv 1 \pmod{\alpha^4+\alpha+1}
\end{aligned}$$

$$\begin{aligned}
\alpha^2 &\equiv \alpha^2 \pmod{\alpha^4+\alpha+1}\\
(\alpha^2)^3 &\equiv \alpha^3+\alpha^2 \pmod{\alpha^4+\alpha+1}\\
(\alpha^2)^5 &\equiv \alpha^2+\alpha+1 \pmod{\alpha^4+\alpha+1}\\
(\alpha^2)^{15} &\equiv 1 \pmod{\alpha^4+\alpha+1}
\end{aligned}$$

$$\begin{aligned}
\alpha^3 &\equiv \alpha^3 \pmod{\alpha^4+\alpha+1}\\
(\alpha^3)^3 &\equiv \alpha^3+\alpha \pmod{\alpha^4+\alpha+1}\\
(\alpha^3)^5 &\equiv 1 \pmod{\alpha^4+\alpha+1}
\end{aligned}$$

$$(\alpha^5)^3 \equiv \alpha^{15} \equiv 1 \pmod{\alpha^4+\alpha+1}$$

Thus, the orders are

$$\begin{aligned}
ord(\alpha) &= 15\\
ord(\alpha^2) &= 15\\
ord(\alpha^3) &= 5\\
ord(\alpha^5) &= 3.
\end{aligned}$$

4 Laboratory exercise 2

```
> G18 := GF(2, 18, alpha^18 + alpha^3 + 1)
> ...

> a := G18:-ConvertIn(alpha)
> 189
> a := G18:-ConvertIn(alpha^2)
> 189
> a := G18:-ConvertIn(alpha^3)
> 63
> a := G18:-ConvertIn(alpha^3 + alpha)
> 262143
```

5 Home exercise 3

Given $x^4 + x^2 + 1 = (x^2 + x + 1)^2$, then

$$C(D) = (1 + D + D^2)^2 \quad \text{and} \quad C_1(D) = 1 + D + D^2.$$

The order of $C_1(D)$ is 3, thus we get that $T_1 = 3$, $T_2 = 2^2 T_1$, as $p = 2$ and $m = 2$ s.t. $2^{m-1} < 2 \leq 2^m$ holds. Then, by plugging in, we get,

$$1(1) \oplus \frac{(2^2-1)}{3}(3) \oplus \frac{2^2(2^2-1)}{6}(6) = 1(1) \oplus 1(3) \oplus 2(6).$$

Given $x^3 + x + 1 = (x + 2)(x^2 + x + 2)$, then

$$C(D) = (x + 2)(x^2 + x + 2) \implies C_1(D) = (x + 2) \text{ and } C_2(D) = (x^2 + x + 2).$$

The order of $C_1(D)$ is 3 $\implies T_1 = 3$. Then, the cycle set of $C_1(D)$ is

$$1(1) \oplus \frac{3^1 - 1}{1}(1) = 1(1) \oplus 2(1).$$

The order of $C_2(D)$ is 8 $\implies T_1 = 8$. Then, the cycle set of $C_2(D)$ is

$$1(1) \oplus \frac{3^2 - 1}{8}(8) = 1(1) \oplus 1(8).$$

Thus, the cycle set of $C(D)$ is

$$\begin{aligned}[1(1) \oplus 2(1)] \times [1(1) \oplus 1(8)] &= 1(1) \oplus 2(1) \oplus 1(8) \oplus 2(8) \\ &= 3(1) \oplus 3(8).\end{aligned}$$

6 Laboratory exercise 3

```
> G := GF(2, 23, alpha^23 + alpha^5 + 1)
> a := G:-ConvertIn(alpha)
> G:-order(a)
> 8388607
```

Given that the order of the primitive polynomial $x^23 + x^5 + 1$ is 8388607, the cycle set is then

$$1(1) \oplus \frac{2^{23} - 1}{8388607}(8388607) = 1(1) \oplus 1(8388607).$$

```
> Factor(x^23 + x^6 + 1) mod 2
> (x^16 + x^15 + x^13 + x^12 + x^8 + x^6 + x^4 + x^3 + x^2 + x + 1)...
>
> ‘mod‘(Primitive(x^4 + x^3 + 1), 2)
> true
> ‘mod‘(Primitive(x^3 + x + 1), 2)
> true
> ‘‘(Primitive(x^16 + x^15 + x^13 + x^12 + x^8 + x^6 + x^4 + x^3 + x^2 + x + 1), 2)
> false
>
> G := GF(2, 16, x^16 + x^15 + x^13 + x^12 + x^8 + x^6 + x^4 + x^3 + x^2 + x + 1)
> a := G:-ConvertIn(x)
> G:-order(a)
> 21845
```

Given the polynomial

$$(x^{16} + x^{15} + x^{13} + x^{12} + x^8 + x^6 + x^4 + x^3 + x^2 + x + 1)(x^4 + x^3 + 1)(x^3 + x + 1),$$

we get that the cycle set of $(x^4 + x^3 + 1)$ is

$$1(1) \oplus \frac{2^4 - 1}{15}(15) = 1(1) \oplus 1(15),$$

and the cycle set of $(x^3 + x + 1)$ is

$$1(1) \oplus \frac{2^3 - 1}{7}(7) = 1(1) \oplus 1(7).$$

The cycle set of $(x^{16} + x^{15} + x^{13} + x^{12} + x^8 + x^6 + x^4 + x^3 + x^2 + x + 1)$ is

$$1(1) \oplus \frac{2^16 - 1}{21845}(21845) = 1(1) \oplus 3(21845).$$

Thus,

$$\begin{aligned}
&[1(1) \oplus 1(7)] \times [1(1) \oplus 1(15)] \times [1(1) \oplus 3(21845)] \\
&= [1(1) \oplus 3(21845)] \times [1(1) \oplus 1(7) \oplus 1(15)] \\
&= 1(1) \oplus 1(7) \oplus 1(15) \oplus 1(105) \oplus 3(21845) \oplus 3(152915) \oplus 15(21845) \oplus 15(458745) \\
&= 1(1) \oplus 1(7) \oplus 1(15) \oplus 1(105) \oplus 18(21845) \oplus 3(152915) \oplus 15(458745).
\end{aligned}$$

7 Home exercise 4

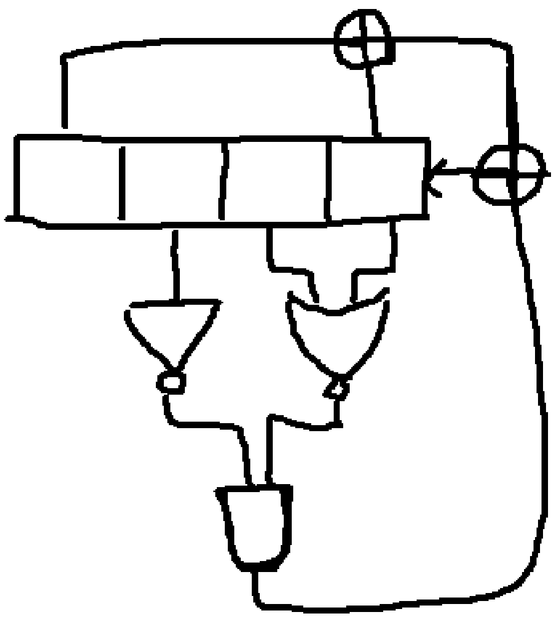
$x^4 + x + 1$.

8 Laboratory exercise 4

```
for i from 0 to 4 do
  for j from 0 to 4 do
    for k from 0 to 4 do
      for l from 0 to 4 do
        if `mod`(Primitive(l*x^4 + k*x^3 + j*x^2 + i*x + 1), 5) then print(l, k, j, i, 1)
        end if
      end do
    end do
  end do
end do
```

$2x^4 + 2x^3 + x^2 + 1$ is a primitive polynomial in $\mathbb{F}_5$.

9 Home exercise 5



10 Laboratory exercise 5

Let $f : \mathbb{Z}_2^4 \times \mathbb{Z}_5^4 \to \mathbb{Z}_{10}^4$ be defined as

$$f(x, y) = 625x + y,$$

where $x \in [0, 15]$ and $y \in [0, 624]$. Then, f is a bijective function.

Proof. Let y be fixed, then

$$\begin{aligned} f(x_1, y) &= f(x_2, y) \\ 625x_1 + y &= 625x_2 + y \\ 625x_1 &= 625x_2 \\ x_1 &= x_2 \end{aligned}$$

Let x be fixed, then

$$\begin{aligned} f(x, y_1) &= f(x, y_2) \\ 625x + y_1 &= 625x + y_2 \\ y_1 &= y_2 \end{aligned}$$

Thus, the function f is injective.

Let $z = [0, 9999] \subset \mathbb{Z}$. Then,

$$z = z_1 \cup \cdots \cup z_{15},$$

where $z_i = [625i, 625i + 624]$ and $0 \leq i \leq 15$. Clearly, the intervals z_i are all disjoint.

Let $z^* \in z_1 \cup \cdots \cup z_{15}$. Then, z^* can only be in one sub-interval z_i of z, as all sub-intervals are disjoint, and z^* can then be written as $z^* = 625i + c$ where $c \in [0, 624]$ and $i \in [0, 15]$.

Thus, picking $y = c$ and $x = i$ gives us a solution, and the function is then surjective. $\square$

Let $f : \mathbb{Z}_2 \times \mathbb{Z}_5 \to \mathbb{Z}_{10}$. Then, defining $f(x, y) = 5x + y$ creates a bijective function.

```
package main

import (
  "fmt"
  "io"
  "os"
  "strings"
)

func WriteStringToFile(filepath, s string) error {
  fo, err := os.Create(filepath)
  if err != nil {
    return err
  }
  defer fo.Close()

  _, err = io.Copy(fo, strings.NewReader(s))
  if err != nil {
    return err
  }

  return nil
}

func mod(a int, n int) int {
  if a % n < 0 {
```

```
    return (a % n) + n
  } else {
    return a % n
  }
}


// Non-linear FSRs with 0-state
func LFSR2(poly []int, state[]int, n int) (out int, in int) {
  for i := 0; i < len(poly); i++ {
    in = in - poly[i] * state[i]
  }

  if (state[0] != 0 &&
    state[1] == 0 &&
    state[2] == 0 &&
    state[3] == 0) {
    return 1, 0
  } else if (state[1] == 0 &&
    state[2] == 0 &&
    state[3] == 0) {
    return 0, 1
  } else {
    return state[0], mod(in, n)
  }
}

func LFSR5(poly []int, state[]int, n int) (out int, in int) {
  for i := 0; i < len(poly); i++ {
    in = in - poly[i] * state[i]
  }

  if (
    state[0] == 2 &&
    state[1] == 0 &&
    state[2] == 0 &&
    state[3] == 0) {
    return 2, 0
  } else if (
    state[0] == 0 &&
    state[1] == 0 &&
    state[2] == 0 &&
    state[3] == 0) {
    return 0, 1
  } else {
    return state[0], mod(in, n)
  }
}

// Bijective function phi: Z_2 x Z_5 -> Z_10
func phi(x int, y int) int {
  return 5 * x + y
}

func main() {
  p := []int{1, 0, 0, 1}
  q := []int{2, 2, 1, 0}
  state2 := []int{0, 0, 0, 0}
  state5 := []int{0, 0, 0, 0}
```

```
  seq := make([]int, 0)

  for i := 0; i < 10003; i++ {
    out2, in2 := LFSR2(p, state2, 2)
    out5, in5 := LFSR5(q, state5, 5)

    state2 = append(state2[1:], in2)
    state5 = append(state5[1:], in5)

    seq = append(seq, phi(out2, out5))
  }

  if err := WriteStringToFile("input", strings.Trim(strings.Join(strings.Fields(fmt.Sprint(se
    panic(err)
  }
}
```