Chapter 1.4 - Newton's First and Second Laws; Inertial Frames

Textbook: Classical Mechanics by John R. Taylor

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Newton's First Law

We will begin our analysis by considering the mechanics of a point particle. Later we can apply this analysis to the mechanics of an extended body by treating it as a collection of many point particles.

Newton's First Law, known as the law of inertia, states that if the net force on a particle is zero, it will move with constant velocity $\vec{v}$. Of course trivially if there are no forces acting on the particle it will have a constant velocity of zero (i.e. it won't move!).

However let us assume there is a force $\vec{F}_1$ acting on a moving particle in the x-direction with a magnitude of f

$$\vec{F}_1 = f\hat{x}$$

and also a force $\vec{F}_2$ with equal magnitude but in the opposite direction.

$$\vec{F}_2 = -f\hat{x} = -\vec{F}_1$$

Thus even though forces are acting on the particle, the net force

$$\vec{F} = \vec{F}_1 + \vec{F}_2 = f\hat{x} - f\hat{x} = \vec{0}$$

vanishes and thus the particle's velocity stays constant.

Newton's Second Law

Newton's Second Law says that the acceleration $\vec{a}$ of a particle is directly proportional to the net force $\vec{F}$ on the particle, with its mass m being the proportionality constant.

$$\vec{F} = m\vec{a} \tag{1}$$

Knowing that in terms of velocity

$$\vec{a} = \frac{d\vec{v}}{dt} = \dot{\vec{v}} \tag{2}$$

and thus in terms of position

$$\vec{a} = \frac{d}{dt}\vec{v} = \frac{d}{dt}\frac{d\vec{r}}{dt} = \frac{d^2\vec{r}}{dt^2} = \ddot{\vec{r}} \tag{3}$$

(note the use of dot notation; a dot above represents taking a derivative with respect to time) we can use Equations (2) and (3) to express the law in terms of the particle's position or velocity vectors.

$$\vec{F} = m\ddot{\vec{r}} = m\dot{\vec{v}} \tag{4}$$

In words, if the net force on the particle is zero then the time-derivative of $\vec{v}$ is zero, meaning $\vec{v}$ is constant, which is exactly what Newton's First Law expressed!

In reality, Equation (1) is actually a simplification of Newton's Second Law. Truthfully it is an expression that uses the particle's momentum linear $\vec{p}$

$$\vec{F} = \dot{p} \tag{5}$$

where

$$\vec{p} = m\vec{v}. \tag{6}$$

If we take the time-derivative of Equation (6)

$$\dot{p} = \frac{d}{dt}(m\vec{v}) = \dot{m}\vec{v} + m\dot{\vec{v}} \tag{7}$$

we get two terms. In most cases we assume the particle has a constant mass, meaning $\dot{m} = 0$ and thus

$$\vec{F} = m\dot{\vec{v}} = m\vec{a}$$

we arrive back at Equation (1). However in cases where the mass can change Equation (5) is the general form of the law.

$$\vec{F} = \dot{\vec{p}} = \dot{m}\vec{v} + m\vec{a} \tag{8}$$

Differential Equations

When we look at the position vector expression of Equation (4) what we see is a second-order differential equation. Our goal is typically to find $\vec{r}(t)$ from knowing what the net force on the particle is. With $\vec{r}(t)$ known we can then calculate any other dynamical quantity of interest.

Let us consider and solve for a simple 1-Dimensional case where $\vec{r}(t) = \vec{x}(t) = x(t)\hat{x}$ and $\vec{F} = F_0\hat{x}$ where F_0 is constant. Newton's Second Law says

$$F_0 = m\ddot{x}(t) \qquad \Longrightarrow \qquad \ddot{x}(t) = \frac{F_0}{m}. \tag{9}$$

First let us integrate Equation (9) over time (keep in mind we use dummy variables when integrating in such a way).

$$\int_0^t \ddot{x}dt' = \int_0^t \frac{F_0}{m}dt' \quad \rightarrow \quad \dot{x}(t) - \dot{x}(0) = \frac{F_0}{m}t$$
$$\dot{x}(t) = v_0 + \frac{F_0}{m}t$$

Above, $\dot{x}(0) = v_0$ is the particle's initial velocity. Now integrate again.

$$\int_0^t \dot{x}dt' = \int_0^t \left(v_0 + \frac{F_0}{m}t'\right)dt' \quad \rightarrow \quad x(t) - x(0) = v_0 t + \frac{F_0}{2m}t^2$$
$$x(t) = x_0 + v_0 t + \frac{F_0}{2m}t^2$$

Above, $x(0) = x_0$ is the particle's initial position. Therefore we have solved for the particle's position $x(t)$. You may even recognize this as one the kinematic equation learned in Freshman Physics.

Inertial Frames

Upon introducing Newton's First and Second Laws it is important to emphasize that they only hold in inertial (non-accelerating and non-rotating) frames. In fact, inertial frames are defined as frames where Newton's First Law (the law of inertia) hold true.

The frames in which Newton's First Law doesn't hold true, the one which are accelerating or rotating with respect to an inertial frame, are called non-inertial frames.

Many frames can be considered inertial up to very good approximation, but eventually we will also begin to work with non-inertial frames and witness the potential benefits of working with them.