Chapter 1.5 - The Third Law and Conservation of Momentum

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Newton's Third Law

In considering two objects, if object 1 exerts a force $\vec{F}_{21}$ on object 2, then object 2 exerts an equal but opposite reaction force $\vec{F}_{12}$ back on object 1.

In terms of the variables

$$\vec{F}_{12} = -\vec{F}_{21} \tag{1}$$

where we use the following notation.

$$\vec{F}_{12} = \text{force exerted by object 2 on object 1}$$
$$\vec{F}_{21} = \text{force exerted by object 1 on object 2}$$

For example when the earth exerts a gravitational force on the moon, the moon exerts a force back on the earth which is equal in magnitude but opposite in direction.

Consider drawing a straight line from the center of object 1 to the center of object 2. Force vectors that lie upon that line are called central forces, and many forces we encounter in physics have this property.

Conservation of Linear Momentum (2-Particle System)

Let us consider 2 particles in a system that are exerting forces on each other. Let us also assume there are external forces acting on each particle.

We first can write out the net force $\vec{F}_1$ on particle 1

$$\vec{F}_1 = \vec{F}_{12} + \vec{F}_1^{\text{ext}} \tag{2}$$

where Equation (2) contains both the force from particle 2 and any external forces. Similarly we can do the same for the net force acting on particle 2.

$$\vec{F}_2 = \vec{F}_{21} + \vec{F}_2^{\text{ext}} \tag{3}$$

Now, recall the true form of Newton's Second Law which relates the net force on a particle to the time-derivative of its linear momentum.

$$\vec{F} = \dot{\vec{p}} \tag{4}$$

Therefore let us use Equation (4) to re-express Equations (2) and (3) in terms of each particle's linear momentum.

$$\dot{\vec{p}}_1 = \vec{F}_1 = \vec{F}_{12} + \vec{F}_1^{\text{ext}} \tag{5}$$
$$\dot{\vec{p}}_2 = \vec{F}_2 = \vec{F}_{21} + \vec{F}_2^{\text{ext}} \tag{6}$$

The system's total linear momentum $\vec{P}$ is just the vector sum of each particle's linear momentum

$$\vec{P} = \vec{p}_1 + \vec{p}_2 \tag{7}$$

and we can easily take the time-derivative of Equation (7)

$$\dot{\vec{P}} = \dot{\vec{p}}_1 + \dot{\vec{p}}_2 \tag{8}$$

to get the time-derivative of the system's total momentum. Let us now plug Equations (5) and (6) into (8)

$$\dot{\vec{P}} = \vec{F}_1 + \vec{F}_2 = \vec{F}_{12} + \vec{F}_1^{\text{ext}} + \vec{F}_{21} + \vec{F}_2^{\text{ext}}. \tag{9}$$

The sum of external forces on each particle

$$\vec{F}_1^{\text{ext}} + \vec{F}_2^{\text{ext}} = \vec{F}^{\text{ext}} \tag{10}$$

equals the total net external force on the system and thus by utilizing Newton's Third Law ($\vec{F}_{12} = -\vec{F}_{21}$) and substituting Equations (1) and (10) into (9) we arrive at the following result.

$$\dot{\vec{P}} = \vec{F}^{\text{ext}} \tag{11}$$

We can draw two important conclusions from Equation (11). First, the internal particle forces have no effect in the analysis of the system's total linear momentum.

Second, consider the case when the net external force is zero.

$$\vec{F}^{\text{ext}} = \vec{0} \quad \Longrightarrow \quad \dot{\vec{P}} = \vec{0} \quad \Longrightarrow \quad \vec{P} = \text{constant}$$

When this is the case, the system's total momentum is constant; this is the Conservation of Linear Momentum.

Conservation of Linear Momentum (N-Particle System)

Let us now consider a system of N particles. We shall label the particles with α and β indices where each can take the values of $1, 2, \ldots, N$.

Thus for example, an arbitrary α particle has a net force of $\vec{F}_\alpha$ acting on it and has a mass of m_α and a momentum of $\vec{p}_\alpha$.

In much the same way as the 2-Particle system let us begin by writing out the net force exerted on an α particle.

$$\vec{F}_\alpha = \sum_{\beta, \beta \neq \alpha} \vec{F}_{\alpha\beta} + \vec{F}_\alpha^{\text{ext}} \tag{12}$$

In Equation (12), $\vec{F}_\alpha^{\text{ext}}$ is the net external force acting on the α particles. Note that out of the N particles, only $N - 1$ of them exert a force on the α particle. This is because a particle cannot exert a force on itself, hence why the sum ranges over all β except for when $\beta = \alpha$ (for there cannot be a $\vec{F}_{\alpha\alpha}$ force).

Newton's Second Law tells us that Equation (12) equals the time-derivative of the α particle's linear momentum.

$$\dot{\vec{p}}_\alpha = \vec{F}_\alpha = \sum_{\beta, \beta \neq \alpha} \vec{F}_{\alpha\beta} + \vec{F}_\alpha^{\text{ext}} \tag{13}$$

We can generalize Equations (7) and (8) quite easily first by writing out the system's total linear momentum

$$\vec{P} = \sum_\alpha \vec{p}_\alpha \tag{14}$$

which is just a sum of every particle's linear momentum and then taking its time-derivative.

$$\dot{\vec{P}} = \sum_\alpha \dot{\vec{p}}_\alpha \tag{15}$$

Let us now plug Equation (13) into (15).

$$\dot{\vec{P}} = \sum_{\alpha} \sum_{\beta,\beta\neq\alpha} \vec{F}_{\alpha\beta} + \sum_{\alpha} \vec{F}_{\alpha}^{\text{ext}} \tag{16}$$

Using some summation arithmetic on the double sum in Equation (16) we recognize that instead of summing over β where $\beta \neq \alpha$, we can instead sum over all β where $\beta > \alpha$, but we must include then the opposite force in each term.

$$\dot{\vec{P}} = \sum_{\alpha} \sum_{\beta,\beta>\alpha} \left(\vec{F}_{\alpha\beta} + \vec{F}_{\beta\alpha}\right) + \sum_{\alpha} \vec{F}_{\alpha}^{\text{ext}} \tag{17}$$

To make sense of this change, consider the case when $\alpha = 1$. If our double sum only included $\vec{F}_{\alpha\beta}$ we could have the force, for example, of $\vec{F}_{12}$ when $\beta = 2$, but how would we ever account for $\vec{F}_{21}$ for when $\alpha = 2$, since β cannot equal 1 since $\beta > \alpha$. That is why under this change we must add $\vec{F}_{\beta\alpha}$ into the sum directly.

If the change from the double sums in Equation (17) to (16) still has you in disbelief, write out all the terms for when $N = 3$ for a bit of concrete proof.

The beauty here is that from Newton's Third Law, $\vec{F}_{\beta\alpha} = -\vec{F}_{\alpha\beta}$, the entire double sum in Equation (17) vanishes. Therefore

$$\dot{\vec{P}} = \sum_{\alpha} \vec{F}_{\alpha}^{\text{ext}} = \vec{F}^{\text{ext}} \tag{18}$$

and thus we see that, just like in the 2-Particle system, the internal forces have no consequence on the evolution of the system's total linear momentum.

Also as before, in an N-particle system, the case when the net external force is zero yields the same result.

$$\vec{F}^{\text{ext}} = \vec{0} \quad \Longrightarrow \quad \dot{\vec{P}} = \vec{0} \quad \Longrightarrow \quad \vec{P} = \text{constant}$$

When this is the case, the system's total momentum is constant; this is the Conservation of Linear Momentum but now proven in the general case.