Chapter 1.2 - Space and Time - Problem Solutions

Textbook: Classical Mechanics by John R. Taylor

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Problem 1.1

We have the two following vectors.

$$\vec{b} = \hat{x} + \hat{y}$$
$$\vec{c} = \hat{x} + \hat{z}$$

a) Add like components.

$$\vec{b} + \vec{c} = (1+1)\hat{x} + (1+0)\hat{y} + (0+1)\hat{z} = 2\hat{x} + \hat{y} + \hat{z}$$

b) Scalar multiply each vector and then add like components.

$$5\vec{b} = 5\hat{x} + 5\hat{y}$$
$$2\vec{c} = 2\hat{x} + 2\hat{z}$$
$$5\vec{b} + 2\vec{c} = (5+2)\hat{x} + (5+0)\hat{y} + (0+2)\hat{z} = 7\hat{x} + 5\hat{y} + 2\hat{z}$$

c) Sum the products of like components.

$$\vec{b} \cdot \vec{c} = \sum_i b_i c_i = (1 \cdot 1) + (1 \cdot 0) + (0 \cdot 1) = 1$$

d) Take the determinant.

$$\vec{b} \times \vec{c} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = (1-0)\hat{x} - (1-0)\hat{y} + (0-1)\hat{z} = \hat{x} - \hat{y} - \hat{z}$$

Problem 1.2

We have the two following vectors.

$$\vec{b} = (1, 2, 3)$$
$$\vec{c} = (3, 2, 1)$$

a) Add like components.

$$\vec{b} + \vec{c} = (1+3, 2+2, 3+1) = (4, 4, 4)$$

b) Scalar multiply each vector and then add like components.

$$5\vec{b} = (5, 10, 15)$$
$$2\vec{c} = (6, 4, 2)$$
$$5\vec{b} - 2\vec{c} = (5-6, 10-4, 15-2) = (-1, 6, 13)$$

c) Sum the products of like components.

$$\vec{b} \cdot \vec{c} = \sum_i b_i c_i = (1 \cdot 3) + (2 \cdot 2) + (3 \cdot 1) = 3 + 4 + 3 = 10$$

d) Take the determinant.

$$\vec{b} \times \vec{c} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 1 & 2 & 3 \\ 3 & 2 & 1 \end{vmatrix} = (2-6)\hat{x} - (1-9)\hat{y} + (2-6)\hat{z} = -4\hat{x} + 8\hat{y} - 4\hat{z}$$

Problem 1.3

First let us define a vector $\vec{r}_{xy}$ that lies flat in the xy-plane.

$$\vec{r}_{xy} = (x, y, 0)$$
$$|\vec{r}_{xy}| = r_{xy} = \sqrt{x^2 + y^2} \quad \rightarrow \quad r_{xy}^2 = x^2 + y^2$$

Now let us define a one-dimensional vector $\vec{r}_z$.

$$\vec{r}_z = (0, 0, z)$$
$$|\vec{r}_z| = r_z = z \quad \rightarrow \quad r_z^2 = z^2$$

Notice

$$\vec{r}_{xy} \cdot \vec{r}_z = 0$$

and now let us define $\vec{r}$ as the sum of these two vectors

$$\vec{r} = \vec{r}_{xy} + \vec{r}_z = (x, y, z)$$

and thus by expanding out the scalar product and plugging in our previous results

$$r^2 = \vec{r} \cdot \vec{r} = (\vec{r}_{xy} + \vec{r}_z) \cdot (\vec{r}_{xy} + \vec{r}_z) = r_{xy}^2 + r_z^2 + 2\vec{r}_{xy} \cdot \vec{r}_z = x^2 + y^2 + z^2$$

we arrive at the correct value for r^2.

Problem 1.4

We have the following two vectors.

$$\vec{b} = (1, 2, 4)$$
$$\vec{c} = (4, 2, 1)$$

First calculate their inner product.

$$\vec{b} \cdot \vec{c} = \sum_i b_i c_i = (1 \cdot 4) + (2 \cdot 2) + (4 \cdot 1) = 12$$

Now calculate each vector's magnitude.

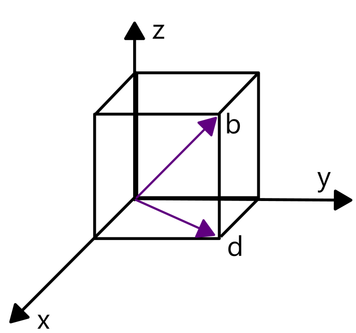
$$|\vec{b}| = b = \sqrt{\vec{b} \cdot \vec{b}} = \sqrt{1^2 + 2^2 + 4^2} = \sqrt{21}$$
$$|\vec{c}| = c = \sqrt{\vec{c} \cdot \vec{c}} = \sqrt{4^2 + 2^2 + 1^2} = \sqrt{21}$$

Finally plug everything in to find θ.

$$\vec{b} \cdot \vec{c} = bc\cos(\theta) \quad \rightarrow \quad 12 = 21\cos(\theta)$$
$$\theta = \arccos\left(\frac{12}{21}\right) \approx 0.963 \text{ radians}$$

Problem 1.5

By using the cube



we have the following two vectors

$$\vec{b} = (1, 1, 1)$$
$$\vec{d} = (1, 1, 0)$$

where $\vec{b}$ is the body vector which goes from the origin to the opposite body corner and $\vec{d}$ is the diagonal vector which lies only in the xy-plane, travelling diagonally across one of the cube's faces.

First calculate their inner product.

$$\vec{b} \cdot \vec{d} = \sum_i b_i d_i = (1 \cdot 1) + (1 \cdot 1) + (1 \cdot 0) = 2$$

Now calculate each vector's magnitude.

$$|\vec{b}| = b = \sqrt{\vec{b} \cdot \vec{b}} = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$$
$$|\vec{d}| = d = \sqrt{\vec{d} \cdot \vec{d}} = \sqrt{1^2 + 1^2 + 0^2} = \sqrt{2}$$

Finally plug everything in to find θ.

$$\vec{b} \cdot \vec{d} = bd\cos(\theta) \quad \rightarrow \quad 2 = \sqrt{3}\sqrt{2}\cos(\theta)$$
$$\theta = \arccos\left(\sqrt{\frac{2}{3}}\right) \approx 0.615 \text{ radians}$$

Problem 1.6

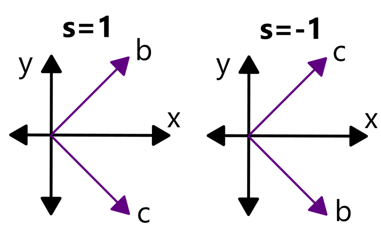
We have the following two vectors.

$$\vec{b} = \hat{x} + s\hat{y}$$
$$\vec{c} = \hat{x} - s\hat{y}$$

Set their scalar product to zero to solve for s.

$$0 = \vec{b} \cdot \vec{c} = (1 \cdot 1) + (s \cdot -s) = 1 - s^2 \quad \rightarrow \quad s = \pm 1$$

The scalar s can be ± 1 because for either value, the pair of vectors remains the same (even though each individual vector changes) and keeps the same orthogonal orientation.

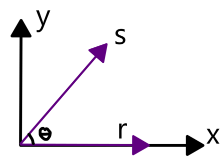


Problem 1.7

Though it will be proven in Problem 1.16, note now that the scalar product is rotationally invariant. This means the scalar product between two vectors remains the same no matter how we rotate the frame.

This means we can rotate our frame in such a way that $\vec{r}$ aligns with the x-axis and $\vec{s}$ lies in only the xy-plane.

$$\vec{r} = (r_x, 0, 0)$$
$$\vec{s} = (s_x, s_y, 0)$$



Based on this orientation θ is both the angle between the two vectors and between $\vec{s}$ and the x-axis. Using simple trigonometry we can rewrite $\vec{s}$.

$$\vec{s} = (s_x, s_y, 0) = (s\cos(\theta), s\sin(\theta), 0)$$

Let us also write out the magnitude of $\vec{r}$.

$$|\vec{r}| = r = r_x$$

Now we can write out both forms of the scalar product and plug in our above results

$$\vec{r}\cdot\vec{s} = \sum_i r_i s_i = r_x s_x$$
$$\vec{r}\cdot\vec{s} = rs\cos(\theta) = r_x s\cos(\theta) = r_x s_x$$

and we see that they are equivalent. Since we can always rotate our frame to achieve this orientation of the two vectors, this proof holds in the general case.

Problem 1.8

a) Let us define a vector $\vec{w}$ which is the sum of $\vec{u}$ and $\vec{v}$ and show what its components equal.

$$\vec{w} = (w_1, w_2, w_3) = \vec{u} + \vec{v} = (u_1, u_2, u_3) + (v_1, v_2, v_3) = (u_1 + v_1, u_2 + v_2, u_3 + v_3)$$
$$w_i = u_i + v_i \text{ where } i = 1, 2, 3$$

Now carry out the calculations using summation notation.

$$\vec{r}\cdot(\vec{u}+\vec{v}) = \vec{r}\cdot\vec{w} = \sum_i r_i w_i = \sum_i r_i(u_i + v_i) = \sum_i r_i u_i + \sum_i r_i v_i = \vec{r}\cdot\vec{u} + \vec{r}\cdot\vec{v}$$

b) Use summation notation and apply the derivative to the time-dependent vector components.

$$\frac{d}{dt}(\vec{r}\cdot\vec{s}) = \frac{d}{dt}\sum_i r_i s_i = \sum_i \frac{d}{dt}(r_i s_i) = \sum_i \left(r_i\frac{ds_i}{dt} + \frac{dr_i}{dt}s_i\right)$$
$$= \sum_i r_i\frac{ds_i}{dt} + \sum_i \frac{dr_i}{dt}s_i = \vec{r}\cdot\frac{d\vec{s}}{dt} + \frac{d\vec{r}}{dt}\cdot\vec{s}$$

Problem 1.9

Recall for vectors that

$$\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$$
$$\vec{a}\cdot\vec{a} = a^2$$

and using these identities we only need to analyze $(\vec{a}-\vec{b})^2$ to achieve the law of cosines.

$$(\vec{a}-\vec{b})^2 = (\vec{a}-\vec{b})\cdot(\vec{a}-\vec{b}) = \vec{a}\cdot\vec{a} - \vec{b}\cdot\vec{a} - \vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{b} = a^2 + b^2 - 2\vec{a}\cdot\vec{b}$$

Problem 1.10

We are given the particle's position vector.

$$\vec{r}(t) = R\cos(\omega t)\hat{x} + R\sin(\omega t)\hat{y}$$

First, we need to confirm that $r(t) = R$ for all t.

$$r(t) = |\vec{r}| = \sqrt{\vec{r}\cdot\vec{r}} = \sqrt{R^2\cos^2(\omega t) + R^2\sin^2(\omega t)} = \sqrt{R^2} = R$$

Now we need to confirm that the particle starts only on the x-axis at $t = 0$.

$$\vec{r}(0) = R\cos(0)\hat{x} + R\sin(0)\hat{y} = R\hat{x}$$

Lastly we need to confirm it is moving counter-clockwise. Let us analyze the equation at increments of growing times, $n\pi/2\omega$ where $n = 1, 2, 3, 4$.

$$\begin{aligned}
\vec{r}\left(\frac{\pi}{2\omega}\right) &= R\cos\left(\frac{\pi}{2}\right)\hat{x} + R\sin\left(\frac{\pi}{2}\right)\hat{y} = R\hat{y} \\
\vec{r}\left(\frac{\pi}{\omega}\right) &= R\cos(\pi)\hat{x} + R\sin(\pi)\hat{y} = -R\hat{x} \\
\vec{r}\left(\frac{3\pi}{2\omega}\right) &= R\cos\left(\frac{3\pi}{2}\right)\hat{x} + R\sin\left(\frac{3\pi}{2}\right)\hat{y} = -R\hat{y} \\
\vec{r}\left(\frac{2\pi}{\omega}\right) &= R\cos(2\pi)\hat{x} + R\sin(2\pi)\hat{y} = R\hat{x}
\end{aligned}$$

Thus we see that as time progresses, the particle goes from the positive x-axis, to the positive y-axis, to negative the x-axis, to the negative y-axis and then back to its starting position all while maintaining a constant radius from the origin.

Velocity and acceleration are calculated by taking time derivatives.

$$\vec{v}(t) = \dot{\vec{r}}(t) = -R\omega\sin(\omega t)\hat{x} + R\omega\cos(\omega t)$$

$$\vec{a}(t) = \dot{\vec{v}}(t) = -R\omega^2\cos(\omega t)\hat{x} - R\omega^2\sin(\omega t)\hat{y} = -\omega^2\left[R\cos(\omega t)\hat{x} + R\sin(\omega t)\hat{y}\right] = -\omega^2\vec{r}(t)$$

We see that acceleration is just the position vector multiplied by a constant. Whereas the position vector points radially outward, the negative sign indicates that acceleration points radially inward.

Its magnitude is $|\vec{a}| = \omega^2|\vec{r}| = \omega^2 R$. These are the results of centripetal acceleration regarding uniform circular motion.

Problem 1.11

We are given the particle's position vector.

$$\vec{r}(t) = b\cos(\omega t)\hat{x} + c\sin(\omega t)\hat{y}$$

Let us analyze the equation at increments of growing times, $n\pi/2\omega$ where $n = 0, 1, 2, 3, 4$.

$$\begin{aligned}
\vec{r}(0) &= b\cos(0)\hat{x} + c\sin(0)\hat{y} = b\hat{x} \\
\vec{r}\left(\frac{\pi}{2\omega}\right) &= b\cos\left(\frac{\pi}{2}\right)\hat{x} + c\sin\left(\frac{\pi}{2}\right)\hat{y} = c\hat{y} \\
\vec{r}\left(\frac{\pi}{\omega}\right) &= b\cos(\pi)\hat{x} + c\sin(\pi)\hat{y} = -b\hat{x} \\
\vec{r}\left(\frac{3\pi}{2\omega}\right) &= b\cos\left(\frac{3\pi}{2}\right)\hat{x} + c\sin\left(\frac{3\pi}{2}\right)\hat{y} = -c\hat{y} \\
\vec{r}\left(\frac{2\pi}{\omega}\right) &= b\cos(2\pi)\hat{x} + c\sin(2\pi)\hat{y} = b\hat{x}
\end{aligned}$$

The particle's orbit is very similar to circular orbit, however it does not maintain a constant distance from the origin. We do see though while on the axes, its either $\pm b$ or $\pm c$ away depending on which axis its currently on. Thus $\vec{r}(t)$ is the parametric equation for an elliptical orbit centered on the origin.

Problem 1.12

We are given the particle's position vector.

$$\vec{r}(t) = b\cos(\omega t)\hat{x} + c\sin(\omega t)\hat{y} + v_0 t\hat{z}$$

Let us analyze the equation at increments of growing times, $n\pi/2\omega$ where $n = 0, 1, 2, 3, 4$.

$$\begin{aligned}
\vec{r}(0) &= b\cos(0)\hat{x} + c\sin(0)\hat{y} + 0\hat{z} = b\hat{x} \\
\vec{r}\left(\frac{\pi}{2\omega}\right) &= b\cos\left(\frac{\pi}{2}\right)\hat{x} + c\sin\left(\frac{\pi}{2}\right)\hat{y} + \frac{\pi v_0}{2\omega}\hat{z} = c\hat{y} + \frac{\pi v_0}{2\omega}\hat{z} \\
\vec{r}\left(\frac{\pi}{\omega}\right) &= b\cos(\pi)\hat{x} + c\sin(\pi)\hat{y} + \frac{\pi v_0}{\omega}\hat{z} = -b\hat{x} + \frac{\pi v_0}{\omega}\hat{z} \\
\vec{r}\left(\frac{3\pi}{2\omega}\right) &= b\cos\left(\frac{3\pi}{2}\right)\hat{x} + c\sin\left(\frac{3\pi}{2}\right)\hat{y} + \frac{3\pi v_0}{2\omega}\hat{z} = -c\hat{y} + \frac{3\pi v_0}{2\omega}\hat{z} \\
\vec{r}\left(\frac{2\pi}{\omega}\right) &= b\cos(2\pi)\hat{x} + c\sin(2\pi)\hat{y} + \frac{2\pi v_0}{\omega}\hat{z} = b\hat{x} + \frac{2\pi v_0}{\omega}\hat{z}
\end{aligned}$$

The particle's orbit is very similar to circular orbit, however it does not maintain a constant distance from the origin. Its also very similar to a flat elliptical orbit, except that it is moving linearly upward in the z-direction. Thus $\vec{r}(t)$ is the parametric equation for an elliptical spiral.

Problem 1.13

By definition

$$u = |\vec{u}| = 1$$

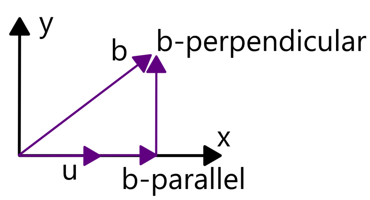
since $\vec{u}$ is a unit vector. Thus let us carry out the calculations.

$$\begin{aligned}
(\vec{u}\cdot\vec{b})^2 + (\vec{u}\times\vec{b})^2 &= (ub\cos(\theta))^2 + \left[(\vec{u}\times\vec{b})\cdot(\vec{u}\times\vec{b})\right] = b^2\cos^2(\theta) + |\vec{u}\times\vec{b}|^2 \\
&= b^2\cos^2(\theta) + u^2b^2\sin^2(\theta) = b^2\left[\cos^2(\theta) + \sin^2(\theta)\right] = b^2
\end{aligned}$$

The scalar product gives the component of $\vec{b}$ that is parallel to $\vec{u}$. The vector product gives the perpendicular component. Thus

$$\vec{u}\cdot\vec{b} = b_{\text{parallel}}$$
$$|\vec{u}\times\vec{b}| = b_{\text{perpendicular}}$$
$$b^2 = b^2_{\text{parallel}} + b^2_{\text{perpendicular}}$$

the proven relationship is a form of the Pythagorean Theorem.



Problem 1.14

Given the two vectors

$$\vec{a} = (a_1, a_2, a_3) \quad\rightarrow\quad a = \sqrt{a_1^2 + a_2^2 + a_3^2}$$
$$\vec{b} = (b_1, b_2, b_3) \quad\rightarrow\quad b = \sqrt{b_1^2 + b_2^2 + b_3^2}$$

we can expand the following quantities. First

$$\begin{aligned}|\vec{a}+\vec{b}|^2 = |(a_1+b_1, a_2+b_2, a_3+b_3)|^2 &= (a_1+b_1)^2+(a_2+b_2)^2+(a_3+b_3)^2 \\ &= a_1^2+a_2^2+a_3^2+b_1^2+b_2^2+b_3^3+2a_1b_1+2a_2b_2+2a_3b_3 \\ &= a^2+b^2+2(\vec{a}\cdot\vec{b}) = a^2+b^2+2ab\cos(\theta)\end{aligned}$$

and then

$$(a+b)^2 = a^2+b^2+2ab$$

gives us everything we need. Now we work through the inequality arithmetic.

$$\begin{gathered}|\vec{a}+\vec{b}| \le (a+b) \\ |\vec{a}+\vec{b}|^2 \le (a+b)^2 \\ a^2+b^2+2ab\cos(\theta) \le a^2+b^2+2ab \\ \cos(\theta) \le 1\end{gathered}$$

Of course the cosine of an angle is less than or equal to 1 and thus the claim is proven. This is called the triangle inequality because $(a+b)$ represents the sum of lengths of two sides of a triangle.

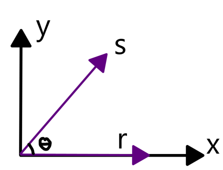
However $(\vec{a}+\vec{b})$ represents the third side of that same triangle and so $|\vec{a}+\vec{b}|$ represents the length of that third side. Therefore with those definitions, the inequality presented is exactly synonymous with the triangle inequality stating that the sum of lengths for any two sides is always greater than or equal to the length of the third side.

Problem 1.15

Though it will be proven in Problem 1.16, note now that the scalar product is rotationally invariant. This means the scalar product between two vectors remains the same no matter how we rotate the frame.

This means we can rotate our frame in such a way that $\vec{r}$ aligns with the x-axis and $\vec{s}$ lies in only the xy-plane.

$$\begin{gathered}\vec{r} = (r_x, 0, 0) \\ \vec{s} = (s_x, s_y, 0)\end{gathered}$$



Based on this orientation θ is both the angle between the two vectors and between $\vec{s}$ and the x-axis. Using simple trigonometry we can rewrite $\vec{s}$.

$$\vec{s} = (s_x, s_y, 0) = (s\cos(\theta), s\sin(\theta), 0)$$

Let us also write out the magnitude of $\vec{r}$.

$$|\vec{r}| = r = r_x$$

Now we can write out both forms of the cross product magnitudes and plug in our above results

$$\begin{gathered}|\vec{r}\times\vec{s}| = |\vec{p}| = |p_x\hat{x}+p_y\hat{y}+p_z\hat{z}| = |r_xs_y\hat{z}| = r_xs_y \\ |\vec{r}\times\vec{s}| = rs\sin(\theta) = r_xs\sin(\theta) = r_xs_y\end{gathered}$$

and we see that they are equivalent. Since we can always rotate our frame to achieve this orientation of the two vectors, this proof holds in the general case.

Lastly notice that in the above analysis

$$\vec{p} = r_x s_y \hat{z} = \vec{r} \times \vec{s}$$

showing that the resultant vector $\vec{p}$ (only in the z-dimension) is perpendicular to both $\vec{r}$ (only in x-dimension) and $\vec{y}$ (only in xy-dimensions).

Problem 1.16

a) The Pythagorean Theorem tells us that

$$r^2 = r_1^2 + r_2^2 + r_3^2 \quad \rightarrow \quad r = \sqrt{r_1^2 + r_2^2 + r_3^2}$$

and using summation notation we can manipulate this to achieve the sought after relation.

$$r = \sqrt{r_1^2 + r_2^2 + r_3^2} = \sqrt{\sum_i r_i r_i} = \sqrt{\vec{r} \cdot \vec{r}}$$

b) The previously proved relation $r = \sqrt{\vec{r} \cdot \vec{r}}$ guarantees that $\vec{r} \cdot \vec{r}$ is the same in any coordinate frame. Obviously this is true too for $\vec{s} \cdot \vec{s}$ or any other vector. Thus consider a vector

$$\vec{t} = \vec{r} + \vec{s}$$

and we know $\vec{t} \cdot \vec{t}$ is the same in any frame. Therefore by carrying out the scalar product

$$\vec{t} \cdot \vec{t} = (\vec{r} + \vec{s}) \cdot (\vec{r} + \vec{s}) = \vec{r} \cdot \vec{r} + \vec{s} \cdot \vec{s} + 2\vec{r} \cdot \vec{s}$$

we know $\vec{r} \cdot \vec{s}$ must also be the same in any frame since every other quantity in the equation is.

Problem 1.17

a) Let us define a vector $\vec{w}$ which is the sum of $\vec{u}$ and $\vec{v}$ and show what its components equal.

$$\vec{w} = (w_1, w_2, w_3) = \vec{u} + \vec{v} = (u_1, u_2, u_3) + (v_1, v_2, v_3) = (u_1 + v_1, u_2 + v_2, u_3 + v_3)$$
$$w_i = u_i + v_i \text{ where } i = 1, 2, 3$$

Now carry out the calculations.

$$\begin{gathered} \vec{r} \times (\vec{u} + \vec{v}) = \vec{r} \times \vec{w} = (r_2 w_3 - r_3 w_2)\hat{x} + (r_3 w_1 - r_1 w_3)\hat{y} + (r_1 w_2 - r_2 w_1)\hat{z} \\ = (r_2 u_3 - r_3 u_2)\hat{x} + (r_3 u_1 - r_1 u_3)\hat{y} + (r_1 u_2 - r_2 u_1)\hat{z} + (r_2 v_3 - r_3 v_2)\hat{x} + (r_3 v_1 - r_1 v_3)\hat{y} + (r_1 v_2 - r_2 v_1)\hat{z} \\ = \vec{r} \times \vec{u} + \vec{r} \times \vec{v} \end{gathered}$$

b) Apply the derivative to the time-dependent vector components.

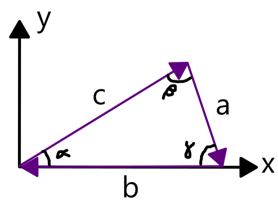
$$\begin{aligned} \frac{d}{dt}(\vec{r} \times \vec{s}) = \frac{d}{dt}&[(r_2 s_3 - r_3 s_2)\hat{x} + (r_3 s_1 - r_1 s_3)\hat{y} + (r_1 s_2 - r_2 s_1)\hat{z}] \\ &= (r_2 \dot{s}_3 - r_3 \dot{s}_2)\hat{x} + (r_3 \dot{s}_1 - r_1 \dot{s}_3)\hat{y} + (r_1 \dot{s}_2 - r_2 \dot{s}_1)\hat{z} \\ &+ (\dot{r}_2 s_3 - \dot{r}_3 s_2)\hat{x} + (\dot{r}_3 s_1 - \dot{r}_1 s_3)\hat{y} + (\dot{r}_1 s_2 - \dot{r}_2 s_1)\hat{z} \\ &= \vec{r} \times \frac{d\vec{s}}{dt} + \frac{d\vec{r}}{dt} \times \vec{s} \end{aligned}$$

Problem 1.18

a) Recall the simple formula for the area of a triangle.

$$\text{area } = \frac{1}{2} \times \text{ base } \times \text{ height}$$

From the diagram given



the height can be found simply by calculating the vertical component of the vector on the left. Thus for the given diagram

$$\text{area } = \frac{1}{2} \times |\vec{b}| \times |\vec{c}| |\sin(\alpha)| = \frac{1}{2}|bc\sin(\alpha)| = \frac{1}{2}|\vec{b} \times \vec{c}|$$

where $c\sin(\alpha)$ is the vertical component of $\vec{c}$ found using simple trigonometry. We see that the expression for the area of a triangle leads exactly to the definition of the magnitude of the cross product of two vectors, where the angle is the angle between the two vectors.

By simply rotating the triangle such that each side lies on the bottom, we can generate alternate expressions in the same way.

$$\begin{aligned}
\text{area } &= \frac{1}{2} \times |\vec{a}| \times |\vec{b}| |\sin(\gamma)| = \frac{1}{2}|ab\sin(\gamma)| = \frac{1}{2}|\vec{a} \times \vec{b}| \\
\text{area } &= \frac{1}{2} \times |\vec{c}| \times |\vec{a}| |\sin(\beta)| = \frac{1}{2}|ca\sin(\beta)| = \frac{1}{2}|\vec{c} \times \vec{a}|
\end{aligned}$$

All expressions represent the same quantity.

$$\text{area } = \frac{1}{2}|\vec{a} \times \vec{b}| = \frac{1}{2}|\vec{b} \times \vec{c}| = \frac{1}{2}|\vec{c} \times \vec{a}|$$

b) Let us rewrite the equality found in the previous part.

$$\begin{gathered}
\frac{1}{2}|\vec{a} \times \vec{b}| = \frac{1}{2}|\vec{b} \times \vec{c}| = \frac{1}{2}|\vec{c} \times \vec{a}| \\
ab\sin(\gamma) = bc\sin(\alpha) = ca\sin(\beta)
\end{gathered}$$

Taking the first two quantities

$$a\sin(\gamma) = c\sin(\alpha) \quad \to \quad \frac{a}{\sin(\alpha)} = \frac{c}{\sin(\gamma)}$$

and taking the last two quantities

$$b\sin(\alpha) = a\sin(\beta) \quad \to \quad \frac{b}{\sin(\beta)} = \frac{a}{\sin(\alpha)}$$

we can combine the equivalencies.

$$\frac{a}{\sin(\alpha)} = \frac{b}{\sin(\beta)} = \frac{c}{\sin(\gamma)}$$

Problem 1.19

First distribute the time-derivative.

$$\begin{aligned}\frac{d}{dt}[\vec{a}\cdot(\vec{v}\times\vec{r})] &= \dot{\vec{a}}\cdot(\vec{v}\times\vec{r}) + \vec{a}\cdot\frac{d}{dt}(\vec{v}\times\vec{r}) \\ &= \dot{\vec{a}}\cdot(\vec{v}\times\vec{r}) + \vec{a}\cdot(\dot{\vec{v}}\times\vec{r}+\vec{v}\times\dot{\vec{r}}) \\ &= \dot{\vec{a}}\cdot(\vec{v}\times\vec{r}) + \vec{a}\cdot(\vec{a}\times\vec{r}) + \vec{a}\cdot(\vec{v}\times\vec{v})\end{aligned}$$

The last term vanishes because a vector crossed with itself is zero.

$$(\vec{v}\times\vec{v}) = \vec{0}$$

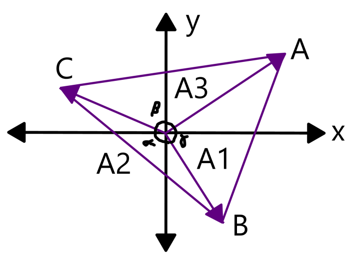
$$\frac{d}{dt}[\vec{a}\cdot(\vec{v}\times\vec{r})] = \dot{\vec{a}}\cdot(\vec{v}\times\vec{r}) + \vec{a}\cdot(\vec{a}\times\vec{r})$$

The new last term vanishes because $(\vec{a}\times\vec{r})$ is perpendicular to both $\vec{a}$ and $\vec{r}$. And so the scalar product of $\vec{a}$ with a resultant vector perpendicular to it is zero. Thus only one term survives.

$$\frac{d}{dt}[\vec{a}\cdot(\vec{v}\times\vec{r})] = \dot{\vec{a}}\cdot(\vec{v}\times\vec{r})$$

Problem 1.20

From the image



it is clear that the larger triangle is made up of three smaller triangles. Therefore the total area is just the sum of all three areas.

$$\text{Area} = A_1 + A_2 + A_3$$

We can use the results from Problem 1.18 to find each area using the vectors and angles involved in each triangle.

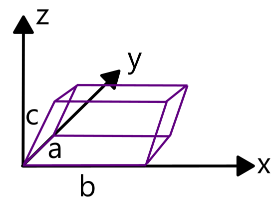
$$\begin{aligned}A_1 &= \frac{1}{2}AB\sin(\gamma) = \frac{1}{2}|\vec{A}\times\vec{B}| \\ A_2 &= \frac{1}{2}BC\sin(\alpha) = \frac{1}{2}|\vec{B}\times\vec{C}| \\ A_3 &= \frac{1}{2}CA\sin(\beta) = \frac{1}{2}|\vec{C}\times\vec{A}|\end{aligned}$$

Thus sum all areas.

$$\text{Area} = \frac{1}{2}|\vec{A}\times\vec{B}| + \frac{1}{2}|\vec{B}\times\vec{C}| + \frac{1}{2}|\vec{C}\times\vec{A}| = \frac{1}{2}|(\vec{A}\times\vec{B}) + (\vec{B}\times\vec{C}) + (\vec{C}\times\vec{A})|$$

Problem 1.21

First we will align the object as shown below.



Let us begin by finding the area of the side that's in the xz-plane. The magnitude of the vector product gives that area.

$$\text{area} = \text{base} \times \text{height} = bc\sin(\theta) = |\vec{b} \times \vec{c}|$$

The right hand rule tells us that the resultant vector points in the y direction.

$$(\vec{b} \times \vec{c}) = |\vec{b} \times \vec{c}|\hat{y}$$

To find the volume we simply need to multiple the area by the length.

$$\text{volume} = \text{length} \times A = a|\vec{b} \times \vec{c}|$$

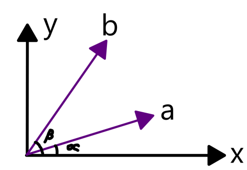
Knowing that $\vec{a} = a\hat{y}$ we can now substitute in vector operations into our expression

$$\text{volume} = a|\vec{b} \times \vec{c}| = a|\vec{b} \times \vec{c}|\cos(0) = a \cdot (\vec{b} \times \vec{c}) = |a \cdot (\vec{b} \times \vec{c})|$$

where we include the magnitude into the final step since volume must be a non-negative quantity. For example we could have calculated the area using $\vec{c} \times \vec{b}$ and that would have given a vector pointing in the $-y$ direction, thus the magnitude in the final step is necessary.

Problem 1.22

Consider the following diagram.



First write out each vector's components.

$$\begin{aligned} \vec{a} &= a\cos(\alpha)\hat{x} + a\sin(\alpha)\hat{y} \quad &\rightarrow \quad a = |\vec{a}| \\ \vec{b} &= b\cos(\beta)\hat{x} + b\sin(\beta)\hat{y} \quad &\rightarrow \quad b = |\vec{b}| \end{aligned}$$

a) Perform the scalar product by sum products of like components.

$$\vec{a} \cdot \vec{b} = ab\cos(\alpha)\cos(\beta) + ab\sin(\alpha)\sin(\beta)$$

The angle between the vectors is $\theta = \beta - \alpha$, thus we can write the scalar product using its other form as well.

$$\vec{a} \cdot \vec{b} = ab\cos(\theta) = ab\cos(\beta - \alpha)$$

Now simply equate the two expressions and simplify.

$$\begin{gathered} \vec{a} \cdot \vec{b} = ab\cos(\alpha)\cos(\beta) + ab\sin(\alpha)\sin(\beta) = ab\cos(\beta - \alpha) \\ \cos(\beta - \alpha) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta) \end{gathered}$$

Since the cosine function is symmetric we can also write the solution in another form.

$$\cos(\alpha-\beta)=\cos(\alpha)\cos(\beta)+\sin(\alpha)\sin(\beta)$$

b) Perform the cross product via determinant and take its magnitude.

$$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ a\cos(\alpha) & a\sin(\alpha) & 0\\ b\cos(\beta) & b\sin(\beta) & 0\end{vmatrix}=ab[\cos(\alpha)\sin(\beta)-\cos(\beta)\sin(\alpha)]\hat{z}$$
$$|\vec{a}\times\vec{b}|=ab[\cos(\alpha)\sin(\beta)-\cos(\beta)\sin(\alpha)]$$

Write down the alternate expression for the cross product magnitude as well.

$$|\vec{a}\times\vec{b}|=ab\sin(\theta)=ab\sin(\beta-\alpha)$$

Now simply equate the two expressions and simplify.

$$|\vec{a}\times\vec{b}|=ab[\cos(\alpha)\sin(\beta)-\cos(\beta)\sin(\alpha)]=ab\sin(\beta-\alpha)$$
$$\sin(\beta-\alpha)=\cos(\alpha)\sin(\beta)-\cos(\beta)\sin(\alpha)$$

Since the sine function is anti-symmetric we can write out the solution in another form.

$$\sin(\alpha-\beta)=\cos(\beta)\sin(\alpha)-\cos(\alpha)\sin(\beta)$$

Problem 1.23

Consider the well known BAC-CAB vector identity .

$$\vec{a}\times(\vec{b}\times\vec{c})=\vec{b}(\vec{a}\cdot\vec{c})-\vec{c}(\vec{a}\cdot\vec{b})$$

Let us now write this in a way such that we only use the problem vectors $\vec{b}$ and $\vec{v}$.

$$\vec{b}\times(\vec{v}\times\vec{b})=\vec{v}(\vec{b}\cdot\vec{b})-\vec{b}(\vec{b}\cdot\vec{v})$$

Considering the given identities

$$\vec{b}\cdot\vec{v}=\lambda$$
$$\vec{b}\times\vec{v}=-(\vec{v}\times\vec{b})=\vec{c}$$

we have all we need to solve for $\vec{v}$.

$$\vec{b}\times(-\vec{c})=b^2\vec{v}-\lambda\vec{b}$$
$$b^2\vec{v}=\lambda\vec{b}-\vec{b}\times\vec{c}$$
$$\vec{v}=\frac{\lambda\vec{b}-\vec{b}\times\vec{c}}{b^2}$$