Chapter 1.2 - Space and Time

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Space

Every point in 3-dimensional space can be assigned a position vector $\vec{r}$ which specifies its distance and direction from a chosen origin. The components of $\vec{r}$ depend on the chosen coordinate system.

Let us begin with Cartesian coordinates such that

$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z} \tag{1}$$

where $\hat{x}, \hat{y}$ and $\hat{z}$ are the unit vectors of this system. There are other notations that can be used to represent the same vector in Equation (1). For example

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

is a notation that uses differently labelled unit vectors such that $\hat{x} \to \hat{i}$ and so on. We can also write

$$\vec{r} = (x, y, z)$$

where it is commonly understood that x is the $\hat{x}$ component of the vector and so on. As equations and problems grow more complex it becomes advantageous to adopt summation notation. Letting

$$x = r_1 \qquad y = r_2 \qquad z = r_3$$

and for the unit vectors

$$\hat{x} = \vec{e}_1 \qquad \hat{y} = \vec{e}_2 \qquad \hat{z} = \vec{e}_3$$

we can state the following.

$$\vec{r} = r_1\vec{e}_1 + r_2\vec{e}_2 + r_3\vec{e}_3 = \sum_{i=1}^{3} r_i\vec{e}_i \tag{2}$$

All of the above notations of $\vec{r}$ are perfectly valid and the chosen form typically will depend on the context of the problem at hand.

Vector Operations

Let us start our analysis with two 3-dimensional vectors $\vec{r}$ and $\vec{s}$.

$$\vec{r} = (r_1, r_2, r_3) \qquad \vec{s} = (s_1, s_2, s_3) \tag{3}$$

Vectors can be added simply by adding like components.

$$\vec{r} + \vec{s} = (r_1 + s_1, r_2 + s_2, r_3 + s_3) \tag{4}$$

We can also multiply a vector by a scalar c and the scalar simply distributes itself to each component.

$$c\vec{r} = (cr_1, cr_2, cr_3) \tag{5}$$

When this is done, the new vector $c\vec{r}$ points either parallel (or anti-parallel) to the original vector $\vec{r}$. We mention anti-parallel, for if $c < 0$ then each component gets reversed along its respective direction.

Therefore subtracting vectors is only adding the negative of one vector to another.

$$\vec{r} - \vec{s} = \vec{r} + (-1\vec{s}) = (r_1, r_2, r_3) + (-s_1, -s_2, -s_3) = (r_1 - s_1, r_2 - s_2, r_3 - s_3) \tag{6}$$

Vectors can be multiplied to each other in two different ways. The first being the scalar product

$$\vec{r} \cdot \vec{s} = rs\cos(\theta) \tag{7}$$

$$\vec{r} \cdot \vec{s} = \sum_n r_n s_n = r_1 s_1 + r_2 s_2 + r_3 s_3 \tag{8}$$

which we can represent in two ways. In Equation (9) we take the sum of like components. In Equation (8), θ is the angle between the two vectors while r and s are the magnitudes of $\vec{r}$ and $\vec{s}$ respectively.

The scalar product allows us to define the magnitude of the vector. Consider the following:

$$\vec{r} \cdot \vec{r}$$

Using Equation (8) and recognizing that $\theta = 0$ (since the vectors are the same) we have

$$\vec{r} \cdot \vec{r} = r^2 \cos(0) = r^2.$$

Using Equation (9) we have

$$\vec{r} \cdot \vec{r} = r_1^2 + r_2^2 + r_3^2.$$

Setting these two results equal gives us our result.

$$r = \sqrt{r_1^2 + r_2^2 + r_3^2} = |\vec{r}| \tag{9}$$

Its also useful to know the following common notation.

$$\vec{r} \cdot \vec{r} = \vec{r}^2 \tag{10}$$

With the magnitude of a vector now defined let us note that by definition of a unit vector

$$|\hat{x}| = |\hat{y}| = |\hat{z}| = 1. \tag{11}$$

Notice from Equations (8) and (9) that the results are both scalars, hence the name being scalar product. There is also the vector product of two vectors where as the name suggests, the result is a vector.

$$\vec{r} \times \vec{s} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ r_1 & r_2 & r_3 \\ s_1 & s_2 & s_3 \end{vmatrix} = (r_2 s_3 - r_3 s_2)\hat{x} - (r_1 s_3 - r_3 s_1)\hat{y} + (r_1 s_2 - r_2 s_1)\hat{z} = \vec{t} \tag{12}$$

The result of a vector product is perpendicular to both vectors multiplied to yield it. Consider the resultant vector $\vec{t}$ in Equation (12) and let us use Equation (9) to take its scalar product with $\vec{r}$.

$$\begin{aligned} \vec{t} \cdot \vec{r} &= (r_2 s_3 - r_3 s_2) r_1 - (r_1 s_3 - r_3 s_1) r_2 + (r_1 s_2 - r_2 s_1) r_3 \\ &= r_1 r_2 s_3 - r_1 r_2 s_3 + r_2 r_3 s_1 - r_2 r_3 s_1 + r_1 r_3 s_2 - r_1 r_3 s_2 = 0 \end{aligned}$$

Since neither t or r has a magnitude of zero Equation (8) yields the implication that

$$\vec{t} \cdot \vec{r} = tr\cos(\theta) = 0 \quad \Longrightarrow \quad \cos(\theta) = 0 \tag{13}$$

and this is only true when two vectors are perpendicular. Finally, the magnitude of the vector product can be written as

$$|\vec{r} \times \vec{s}| = rs\sin(\theta) \tag{14}$$

where again θ is the angle between the two vectors. Equations (8) and (14) bring some important and interesting observations:

1. The scalar product of two perpendicular vectors is zero.
2. The vector product of two parallel (or anti-parallel) vectors is a zero vector.

Differentiation of Vectors

Recall the usual definition of a derivative (with respect to time) of a scalar function $x = x(t)$.

$$\frac{dx}{dt} = \lim_{\Delta t \to \infty} \frac{x(t + \Delta t) - x(t)}{\Delta t} = \lim_{\Delta t \to \infty} \frac{\Delta x}{\Delta t} \tag{15}$$

Differentiating a vector $\vec{r} = \vec{r}(t)$ takes on a nearly identical form.

$$\frac{d\vec{r}}{dt} = \lim_{\Delta t \to \infty} \frac{\vec{r}(t + \Delta t) - \vec{r}(t)}{\Delta t} = \lim_{\Delta t \to \infty} \frac{\Delta \vec{r}}{\Delta t} \tag{16}$$

In the scope of vector differentiation the derivative is still a linear operator. This means

$$\frac{d(\vec{r}(t) + \vec{s}(t))}{dt} = \frac{d\vec{r}}{dt} + \frac{d\vec{s}}{dt} \tag{17}$$

the derivative distributes in a sum of vectors and for a scalar function $f(t)$

$$\frac{d(f(t)\vec{r}(t))}{dt} = \frac{df}{dt}\vec{r} + f\frac{d\vec{r}}{dt} \tag{18}$$

the derivative adheres to the product rule. An important thing to understand is that the unit vectors for a Cartesian coordinate system $\hat{x}, \hat{y}$ and $\hat{z}$ are spatially-fixed. This means that as we analyze a body moving through this coordinate frame, the unit vectors do not change over time. Therefore

$$\frac{d\hat{x}}{dt} + \frac{d\hat{y}}{dt} + \frac{d\hat{z}}{dt} = \vec{0}. \tag{19}$$

With that in mind let us now attach the math to the physics! If a particle has a time-dependent position vector of

$$\vec{r}(t) = x(t)\hat{x} + y(t)\hat{y} + z(t)\hat{z} \tag{20}$$

we can take its time derivative to express the particle's velocity vector

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{x} + \frac{dy}{dt}\hat{y} + \frac{dz}{dt}\hat{z} = v_x\hat{x} + v_y\hat{y} + v_z\hat{z} \tag{21}$$

where the components of $\vec{v}$ are just the individual derivatives of the respective components of $\vec{r}$. Similarly we can define the particle's acceleration vector

$$\vec{a}(t) = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2} = \frac{d^2x}{dt^2}\hat{x} + \frac{d^2y}{dt^2}\hat{y} + \frac{d^2z}{dt^2}\hat{z} = a_x\hat{x} + a_y\hat{y} + a_z\hat{z} \tag{22}$$

where the components of $\vec{a}$ are just the individual second derivatives of the respective components of $\vec{r}$.

Notice these definitions (in Cartesian coordinates) are true because of Equation (19). For example when we differentiate in Equation (21) technically we are adhering to the product rule

$$\frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{x} + x\frac{d\hat{x}}{dt} + \ldots$$

but from Equation (19) we know that the time derivative of each unit vector is zero.

We will find that in other coordinate systems the unit vectors are not spatially-fixed, but rather bodily-fixed, which means they will in fact be time-dependent.