Optimization Problem Types

Definition of "Linear" and "Convex"

1. Linear

A function $f: \mathbb{R}^n \to \mathbb{R}$ is called linear if it can be expressed as:

$$f(x) = a^T x + b$$

Where $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$.
In simpler terms, a linear function is a straight line in $\mathbb{R}$ or a plane in higher dimensions.

2. Convex Function

A function $f: \mathbb{R}^n \to \mathbb{R}$ is convex if for any two points $x_1, x_2 \in \mathbb{R}^n$ and any $\lambda \in [0,1]$, the following inequality holds:

$$f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda) f(x_2)$$

This means that the line segment connecting any two points on the graph of f lies above or on the graph.

Intuition

The "linear" nature of the function, which makes it a straight line or plane, ensures that it inherently satisfies the definition of "Convexity". Therefore, all linear functions are convex because they naturally meet the criteria for a convex fuction.

Linear Programming Problems

1. Definition
LP problem is one which the objective and all of the constraints are linear functions of the decision variables.
ex. $ax_1 + bx_2 + cx_3$

where $x_1, x_2, \text{and } x_3$ are decision variables, and the variables are multiplied by coefficients ($a, b, \text{and } c$) that are constant in optimization problem.

2. Characteristics

- Since all linear functions are convex, linear programming problems are intrinsically easier to solve than general (NLP) problems.

- LP has at most one feasible region and the optimal solution will always be found at a 'corner point' where the constraints intersect.

Quadratic Programming (QP) Problems

1. Definition

QP problem has an objective which is a quadratic of the decision variables, and constraints which are all linear functions of the variables.

ex. $2x_1^2 + 3x_2^2 + 4x_1x_2$

Where, x_1 and x_2 are decision variables.

A widely used QP problem is the Markowitz mean-variable portfolio optimization problem, where the quadratic objective is the portfolio variance (sum of the variances and covariances of individual securities), and the linear constraints specify a lower bound for portfolio return.

2. Characteristics

- Have only one feasible region with "flat surfaces" on its surface (due to the linear constraints), but the optimal solution may be found anywhere within the region or on its surface.

- The "best" quadratics have Hessians that are positive definite (in a minimization problem) or negative definite (in a maximization problem). - You can picture the graph of these functions as having a "round bowl" shape with a single bottom (or top).

Solving LP and QP problems

1. LP problems

- Simplex method (Dantzig, 1948)
- Newton-Barrier method (Interior Point Method) (Karmarkar, 1984)

2. QP problems

- QP problems = special case of smooth non-linear problem → can be solved by a smooth nonlinear optimization method such as GRG or SQP method.

- However, a faster and more reliable way to solve a QP problem is to use an extension of the simplex method or an extension of the interior point or Barrier method.