Complex Numbers in Simple and Quantum Harmonic Motion

Introduction:

In Simple Harmonic Motion (SHM) complex numbers are used to represent oscillatory motion mathematically; instead of using trigonometric functions like sine and cosine separately, complex exponential functions provide a combined representation i.e., the displacement of an object undergoing SHM can be expressed as: 1

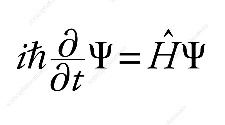
x(t) = a cos ωt + b sin ωt =A·e^(iωt)

where A is the amplitude, ω is the angular frequency, and i is the imaginary unit.

In quantum mechanics, and more specifically quantum harmonic motion, the wavefunction [ψ(x)] can be represented using complex numbers; the amplitude and phase of the complex wavefunction account for the particle's position and momentum.2

Simple versus Quantum Harmonic Motion:

Energy Levels:

The energy of a simple harmonic oscillator (such as a pendulum or a mass-spring system) is continuous data so it can be any value. The total energy is equal to KE + GPE and proportional to different amplitudes and velocities of oscillation.3 However, in quantum mechanics, the energy of a harmonic oscillator is quantized due to the wave-particle duality of quantum particles (there can only be a whole number of particles, and therefore some multiple of e as the energy) and proved by Schrödinger equation's**\*** solutions for the harmonic oscillator potential.4 The lowest possible energy level, known as the ground state, is nonzero. If energy was equal to zero, it would imply position and momentum are zero, which contradicts the Heisenberg uncertainty principle.5

**\***Figure 1: Schrödinger’s Equation

Wavefunction:

In SHM, there is no concept of a wavefunction; the position and momentum of the oscillator are described by classical variables that can be precisely determined at any given time. For a harmonic oscillator, these quantities can be described by straightforward equations of motion, e.g. derivations of Newton's laws.6 In quantum mechanics, the uncertainty principle causes limits on the precision with which certain pairs of physical properties (i.e. position and momentum) can be known at any given time; instead, quantum mechanics uses a wavefunction, which is typically denoted as ψ (psi), to describe the state of the system.7 This wavefunction provides the probabilistic nature of quantum systems - the probability amplitude of finding the oscillator in a particular position/momentum state. The square of the wavefunction's magnitude gives the probability density of finding the particle at a specific location.8

Operator Formalism:

Classical mechanics, including SHM, uses equations of motion, such as Newton's laws of motion, to describe the behaviour of harmonic oscillators. These equations relate the forces acting on a particle, to its motion.9 In contrast, quantum mechanics uses the formalism of operators such as the position and momentum operators, and their evolution over time. Operators act on the wavefunction to produce measurable results. The Schrödinger equation, a key equation in quantum mechanics, depends on time to show the evolution of the wavefunction under the influence of these operators.10

Why complex numbers are used:

Complex numbers are used in SHM to model the motion of an object that oscillates back and forth around an equilibrium position since it has a sinusoidal pattern.11 The motion can be described using real numbers (e.g. position, velocity, and acceleration), and trigonometric functions like sine and cosine, but complex numbers are used to make calculations easier, as they model oscillatory motion using one function- an exponential function including complex numbers.12

Euler's formula also uses complex numbers to represent SHM:

e^(iθ) = cos(θ) + i sin(θ)

'e' is Euler's number, 'i' is √ (-1), and θ represents the angle.13

Damped Harmonic Motion is when oscillations gradually decrease in amplitude due to energy dissipation. Complex numbers are used in modelling and analysing damped oscillations; by introducing damping factors in complex equations, the solutions of real and complex roots can be compared, providing insights into the behaviour of damped systems and lets physicists predict how fast oscillations will decay over time.11,12

One use of complex numbers in SHM and quantum harmonic motion is through phasors - rotating arrows on a graph; the length of the arrow represents the amplitude of the oscillation, and the angle represents the phase of the oscillation.14 Phasors use complex numbers; the real part corresponds to the cosine component of the motion and the imaginary part corresponds to the sine component.15

Quantum harmonic motion uses complex numbers to account for the wave-particle duality of quantum particles, i.e. in Schrödinger's equation - a partial differential equation, whose solutions are wavefunctions. These wavefunctions describe the behaviour of particles and change over time representing the changing state of the particle. 16

When the magnitude of the wavefunction is squared, it gives the probability density - the likelihood of finding a particle in a specific region of space or with a certain momentum: 17

∣ψ(x)∣2

the probability density of finding the particle at a specific position x. 18

Appendix:

<https://colab.research.google.com/drive/1QZZFOXxHaJi_gHz6hVD7Ww9PD4dXxdgj?usp=drive_link>

<https://colab.research.google.com/drive/1cOrq28CjmeQZ0wMf8bTvYNTGCvfCr3U8?usp=drive_link>

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