THEORY OF FINANCE

Solution Sheet on Problem Set 3

Asset Pricing Models & Portfolio Choice

Deadline: 30.11.2021

Solved by: Jonas Husmann, Niklas Kampe, Cyril Janak

Task		Points Earned

1. Analyzing Beta Sorted Portfolios

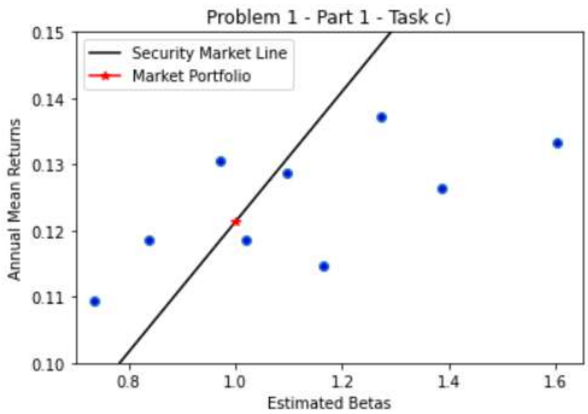
a) Annualized portfolio statistics (4 points)

Index	beta1	beta2	beta3	beta4	beta5	beta6	beta7	beta8	beta9	beta10
Annual_Mean_Return	0.112186	0.109293	0.118481	0.130431	0.118619	0.128726	0.114752	0.137202	0.126393	0.133314
Annual_Std	0.119012	0.131352	0.140925	0.158933	0.165454	0.178226	0.189436	0.208976	0.23108	0.273389
Annual_Excess_Return	0.0702929	0.0672964	0.0768125	0.0891893	0.0769554	0.0874232	0.07295	0.0962018	0.0850071	0.092175
Annual_Excess_Std	0.119012	0.131352	0.140925	0.158933	0.165454	0.178226	0.189436	0.208976	0.23108	0.273389
Annual_Sharpe_Ratio	0.590637	0.512338	0.545061	0.561174	0.465116	0.490519	0.385091	0.460349	0.367868	0.337157

b) CAPM regression & report of statistics (8 points)

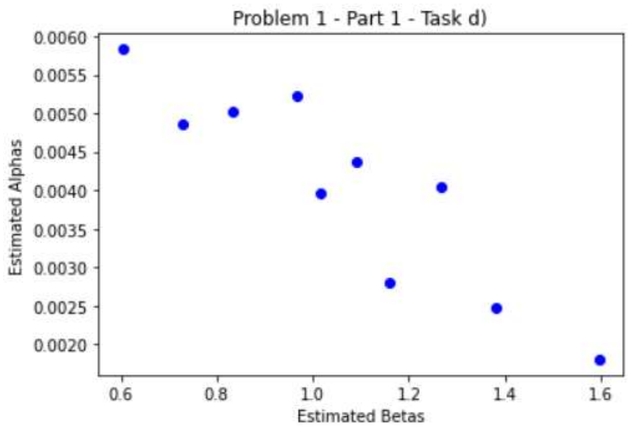
Index	beta1	beta2	beta3	beta4	beta5	beta6	beta7	beta8	beta9	beta10
Annual_Mean_Return	0.112186	0.109293	0.118481	0.130431	0.118619	0.128726	0.114752	0.137202	0.126393	0.133314
Annual_Std	0.119012	0.131352	0.140925	0.158933	0.165454	0.178226	0.189436	0.208976	0.23108	0.273389
Annual_Excess_Return	0.0702929	0.0672964	0.0768125	0.0891893	0.0769554	0.0874232	0.07295	0.0962018	0.0850071	0.092175
Annual_Excess_Std	0.119012	0.131352	0.140925	0.158933	0.165454	0.178226	0.189436	0.208976	0.23108	0.273389
Annual_Sharpe_Ratio	0.590637	0.512338	0.545061	0.561174	0.465116	0.490519	0.385091	0.460349	0.367868	0.337157
Alpha	0.00210575	0.00113274	0.00129356	0.00150664	0.00024222	0.000650653	-0.000920661	0.000322387	-0.00124333	-0.0019215
P-Value Alpha	0.00940924	0.125773	0.0417338	0.0123245	0.678947	0.318684	0.182276	0.693145	0.207695	0.143884
Beta	0.609035	0.734604	0.838369	0.972658	1.02071	1.09513	1.16492	1.27263	1.38671	1.602
P-Value Beta	7.46831e-149	4.22352e-206	4.03425e-274	0	0	0	0	1.47436e-316	7.25632e-290	4.00752e-254
Coefficient Of Determination	0.622251	0.741575	0.835417	0.883997	0.898389	0.891207	0.892327	0.875801	0.85174	0.81207

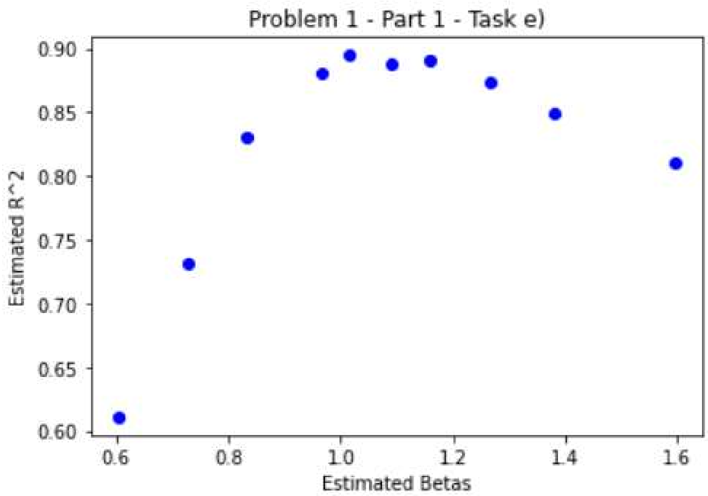
c) Plot and interpret returns and betas (8 points)



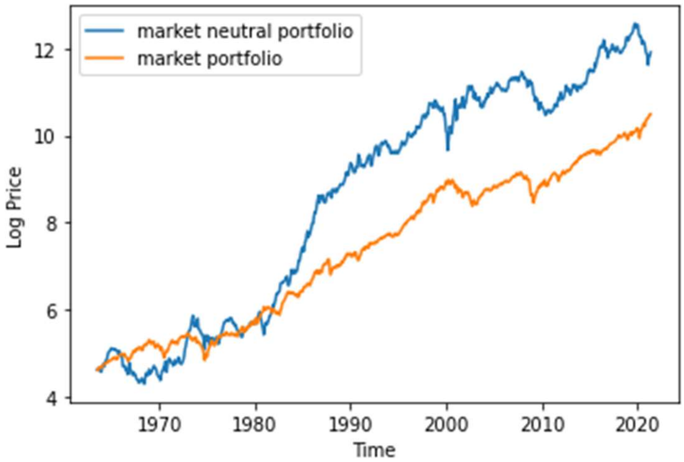
Intercept (Risk Free Rate): 0.02295

Slope: 0.09839

	The security market line distinguishes between Portfolios above and under the line. Portfolios above the line are outperforming the market, whereas Portfolios under the line are underperforming the market. The market portfolio is equal to β = 1. For our dataset, we observe that the assets with a β > 1 are underperforming, whereas the β < 1 are outperforming the market. Recalling that a β > 1 means that the stock's price swings more wildly (i.e. is more volatile) than the overall market, we can conclude, that the higher β do not perform strong enough, compared to the market (i.e. the returns are not high enough, given the higher volatility). This is in line with our results in part a). Recall the formula for the Sharpe Ratio: $$Sharpe\ Ratio\ SR = \frac{R_p - R_f}{\sigma_p}$$ The analysis in part a) gave us a decreasing Sharpe Ratio from $\beta 1$ to $\beta 10$. When we look into the parameters, which are defining the Sharpe Ratio, we see that the Returns are – more or less – increasing from $\beta 1$ to $\beta 10$. However, the volatility (i.e. the Standard Deviation) is increasing much more, in relation to the increasing return. This leads to the decrease in the Sharpe Ratio, which is, eventually, why they are underperforming the market.	
d) Plot and interpret alphas and betas (8 points)	 We compare how volatile a stock's price is in comparison to the overall stock market (β, x-axis) versus the investment strategy's ability to beat the market (α, y-axis). Also called "excess return" or "abnormal rate of return". α is created by active investing, where as β can be earned through passive investing.	

	Our plot shows a decreasing relationship for α and β. Meaning, $\beta 1$ has a – relatively – high abnormal return, while having a low volatility (in comparison with the market). Based on our dataset, the lower CAPM-β have earned higher abnormal returns, while having lower volatility than the market. On the other side, the higher CAPM- β have earned clearly lower abnormal returns, while having much higher volatility (again, in comparison with the market). Recall that in CAPM with β close to zero, the return should also be close to zero. When β are low, the return can be explained by active investing (i.e. high α). On the other side, when β are high, returns can be explained by the higher volatility (i.e. higher β), which is a risk premium for pro cyclical assets.	
e) Plot and interpret R squared and betas (8 points)	 Lastly, we compare again our β on the x-axis with the Estimated R^2, the measure of how well observed outcomes are replicated by the model, on the y-axis. Obviously, the higher our R^2, the more accurate the linear relation in our CAPM-Regression. We observe an increase in R^2 up to $\beta = 1$ and then a (weaker compared to the increase) decreases in higher β. Our dataset basically shows that for high/low Beta, the linear relation between returns and market returns is weaker, compared to medium- β. Which means, that our model is less accurate, when we analyze high/low β Portfolios.	

<table>
<tr>
<td>f)
Build beta-neutral portfolio and plot results vs. market
(6 points)</td>
<td>The required weight for the long position in beta 1 is 2.646.


</td>
<td></td>
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<tr>
<td>g)
Performance comparison of beta-neutral portfolio to market
(6 points)</td>
<td>Monthly results:

market neutral portfolio mean return: 0.0136
market portfolio mean return: 0.0095

market neutral portfolio sharpe ratio: 0.1251
market portfolio sharpe ratio: 0.1309

<table>
<tr><th><u>Correlation Matrix</u></th><th>returns_mkt_neutral</th><th>market_return</th></tr>
<tr><td>returns_mkt_neutral</td><td>1.000000</td><td>0.006087</td></tr>
<tr><td>market_return</td><td>0.006087</td><td>1.000000</td></tr>
</table>

Per the above correlation matrix the market neutral portfolio is indeed still correlated to the market returns, however only very slightly. Moreover, given we only rebalance the portfolio once and not e.g. every month the low correlation in this case can be considered to be market neutral.</td>
<td></td>
</tr>
</table>

h) regressions on beta and Fama-French models (14 points)

For the following regressions:
x1 = Mkt_RF
x2 = SMB
x3 = HML
x4 = RMW
x5 = CMA

CAPM regression:

```
                            OLS Regression Results
==============================================================================
Dep. Variable:                      y   R-squared:                       0.000
Model:                            OLS   Adj. R-squared:                 -0.001
Method:                 Least Squares   F-statistic:                  0.007204
Date:                Mon, 29 Nov 2021   Prob (F-statistic):              0.932
Time:                        22:52:57   Log-Likelihood:                 776.27
No. Observations:                 696   AIC:                            -1549.
Df Residuals:                     694   BIC:                            -1539.
Df Model:                           1
Covariance Type:            nonrobust
==============================================================================
                 coef    std err          t      P>|t|      [0.025      0.975]
------------------------------------------------------------------------------
const          0.0099      0.003      3.260      0.001       0.004       0.016
x1             0.0057      0.068      0.085      0.932      -0.127       0.139
==============================================================================
Omnibus:                       35.042   Durbin-Watson:                   1.910
Prob(Omnibus):                  0.000   Jarque-Bera (JB):              101.569
Skew:                          -0.142   Prob(JB):                     8.80e-23
Kurtosis:                       4.850   Cond. No.                         22.5
==============================================================================
```

Fama-French 3-factor regression:

```
                            OLS Regression Results
==============================================================================
Dep. Variable:                      y   R-squared:                       0.332
Model:                            OLS   Adj. R-squared:                  0.329
Method:                 Least Squares   F-statistic:                     114.5
Date:                Mon, 29 Nov 2021   Prob (F-statistic):           3.15e-60
Time:                        22:52:57   Log-Likelihood:                 916.58
No. Observations:                 696   AIC:                            -1825.
Df Residuals:                     692   BIC:                            -1807.
Df Model:                           3
Covariance Type:            nonrobust
==============================================================================
                 coef    std err          t      P>|t|      [0.025      0.975]
------------------------------------------------------------------------------
const          0.0089      0.003      3.566      0.000       0.004       0.014
x1             0.3886      0.059      6.566      0.000       0.272       0.505
x2            -1.4095      0.085    -16.581      0.000      -1.576      -1.243
x3             0.7774      0.087      8.949      0.000       0.607       0.948
==============================================================================
Omnibus:                       24.117   Durbin-Watson:                   2.014
Prob(Omnibus):                  0.000   Jarque-Bera (JB):               58.622
Skew:                           0.047   Prob(JB):                     1.86e-13
Kurtosis:                       4.419   Cond. No.                         37.1
==============================================================================
```

Fama-French 5-factor regression

```
                            OLS Regression Results
==============================================================================
Dep. Variable:                      y   R-squared:                       0.384
Model:                            OLS   Adj. R-squared:                  0.380
Method:                 Least Squares   F-statistic:                     86.16
Date:                Mon, 29 Nov 2021   Prob (F-statistic):           2.42e-70
Time:                        22:52:57   Log-Likelihood:                 945.08
No. Observations:                 696   AIC:                            -1878.
Df Residuals:                     690   BIC:                            -1851.
Df Model:                           5
Covariance Type:            nonrobust
==============================================================================
                 coef    std err          t      P>|t|      [0.025      0.975]
------------------------------------------------------------------------------
const          0.0045      0.002      1.815      0.070      -0.000       0.009
x1             0.5314      0.060      8.800      0.000       0.413       0.650
x2            -1.2419      0.086    -14.419      0.000      -1.411      -1.073
x3             0.3060      0.112      2.724      0.007       0.085       0.527
x4             0.7058      0.119      5.923      0.000       0.472       0.940
x5             1.0264      0.173      5.946      0.000       0.687       1.365
==============================================================================
Omnibus:                        6.622   Durbin-Watson:                   2.008
Prob(Omnibus):                  0.036   Jarque-Bera (JB):                8.870
Skew:                          -0.057   Prob(JB):                       0.0119
Kurtosis:                       3.541   Cond. No.                         82.4
==============================================================================
```

In the CAPM regression we get a very high p value for the beta coefficient (and also a non-existent R^2). Keeping in mind that the portfolio that was used for this regression was specifically created to be market neutral (i.e. beta = 0) this does not surprise and is reasonable.

With the Fama-French 3-factor and 5-factor regression models the excess return of the market portfolio can be better described (adj R^2 of 0.329 and 0.380, respectively). It can also be seen that the goodness of fit of the model increases by adding the two additional independent variables of RMW and CMA.

Moreover, irrespective whether the 3-factor or 5-factor model was used, the p values for all independent variables are extremely low, indicating that they are all significant and have an influence on the excess return of the market neutral portfolio.

For both the 3-factor and 5-factor model we get very low constants (alpha) implying that almost all the return can be explained by the 3-factors or 5-factors respectively.

We tested for multicollinearity using the VIF. Given that the VIF is below for 2.4 for all independent variables multicollinearity is not a problem in this case. Considering how the factors are chosen and created this is sensible.

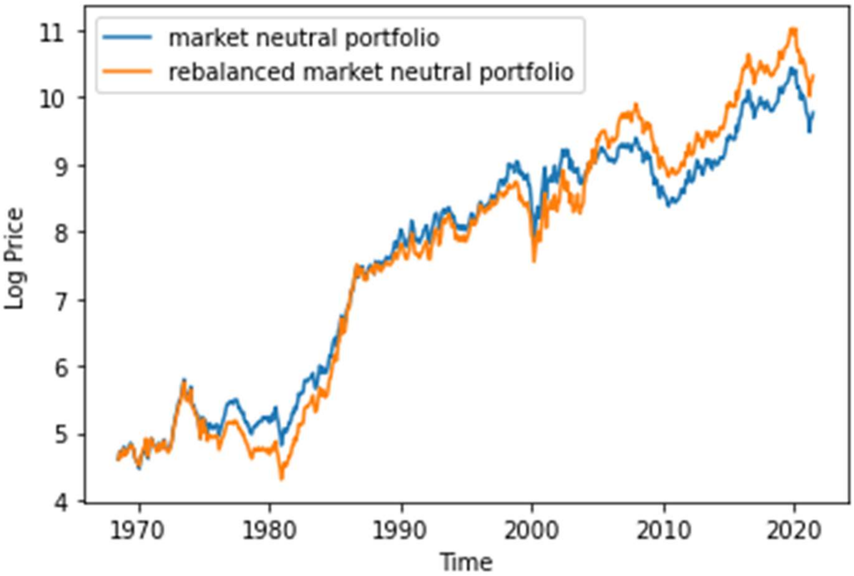
i) Performance analysis of rebalanced portfolio (10 points)

French-Fama 5-factor regression on the monthly rebalanced portfolio:

```
                            OLS Regression Results
==============================================================================
Dep. Variable:                      y   R-squared:                       0.341
Model:                            OLS   Adj. R-squared:                  0.336
Method:                 Least Squares   F-statistic:                     65.22
Date:                Mon, 29 Nov 2021   Prob (F-statistic):           7.29e-55
Time:                        23:51:40   Log-Likelihood:                 798.12
No. Observations:                 636   AIC:                            -1584.
Df Residuals:                     630   BIC:                            -1558.
Df Model:                           5
Covariance Type:            nonrobust
==============================================================================
                 coef    std err          t      P>|t|      [0.025      0.975]
------------------------------------------------------------------------------
const          0.0061      0.003      2.106      0.036       0.000       0.012
x1             0.4761      0.068      6.952      0.000       0.342       0.611
x2            -1.2620      0.100    -12.588      0.000      -1.459      -1.065
x3             0.2630      0.128      2.061      0.040       0.012       0.513
x4             0.6319      0.136      4.661      0.000       0.366       0.898
x5             1.1176      0.201      5.550      0.000       0.722       1.513
==============================================================================
Omnibus:                        8.412   Durbin-Watson:                   1.935
Prob(Omnibus):                  0.015   Jarque-Bera (JB):               11.159
Skew:                           0.125   Prob(JB):                      0.00377
Kurtosis:                       3.599   Cond. No.                         82.0
==============================================================================
```

With monthly rebalancing we get a fairly similar outcome vs if we had just done an initial weighting (cf. question 1 h). Some slight shifts in the coefficients of the independent variables can be observed, and the adjusted R^2 is slightly lower here.

For further comparison we also plotted the log price development of market neutral portfolio (from 1 f) versus the log price development of the monthly rebalanced market neutral portfolio.



2. Factor Rotation
a) Formulate and solve the optimization function
(8 points)

$max\ E[r_p] + (1-\gamma) * \frac{1}{2} * Var(r_p)$ s.t. $1' * v + v_{rf} = 1$ (i)

→ v = 4x1 weight vector on risky factor portfolios, v_{rf} = weight on risk-free asset, r_p = log-return of portfolio of risky factor portfolios and risk-free asset

$E[r_p] = v * \mu_{risky} + v_{rf} * \mu_{rf}$ (ii)

$Var(r_p) = v' * \Sigma * \mathrm{v}$ (iii)

→ μ_{risky} , μ_{rf} = mean log returns over 2-year time horizon, Σ = variance-covariance matrix

(ii) and (iii) into (i):

$max\ v * \mu_{risky} + v_{rf} * \mu_{rf} + (1-\gamma) * \frac{1}{2} * v' * \Sigma * \mathrm{v}$ s.t. $1' * v + v_{rf} = 1$

Lagrange formulization:

$L = v * \mu_{risky} + v_{rf} * \mu_{rf} + (1-\gamma) * \frac{1}{2} * v' * \Sigma * \mathrm{v} + \lambda * [1' * v + v_{rf} - 1]$

FOCs:

$\frac{\delta L}{\delta v}$: $\mu^e + \frac{1}{2} * diag(\Sigma) - \gamma * \Sigma * \mathrm{v} + \lambda = 0_{\mathrm{nx1}}$ (from 14.9) (iv)

$\frac{\delta L}{\delta v_{rf}}$: $\mu_{rf} + \lambda = 0$ (v)

$\frac{\delta L}{\delta \lambda}$: $1' * v + v_{rf} - 1 = 0$ (vi)

Solving for Lambda:

$\lambda = -\mu_{rf}$ (vii)

(vii) into (iv):

$\mu^e + \frac{1}{2} * diag(\Sigma) - \gamma * \Sigma * \mathrm{v} - \mu_{rf} = 0_{\mathrm{nx1}}$

$\gamma * \Sigma * \mathrm{v} = \mu^e + \frac{1}{2} * diag(\Sigma) - \mu_{rf}$

$\mathbf{v} = \frac{\mathbf{1}}{\boldsymbol{\gamma}} * \boldsymbol{\Sigma}^{-1} * \left(\boldsymbol{\mu}^e + \frac{1}{2} * \boldsymbol{diag}(\boldsymbol{\Sigma}) - \boldsymbol{\mu}_{rf}\right)$ (viii) → **Optimal risky weights**

(viii) into (vi):

$1' * \frac{1}{\gamma} * \Sigma^{-1} * \left(\mu^e + \frac{1}{2} * diag(\Sigma) - \mu_{rf}\right) + v_{rf} - 1 = 0$

$\boldsymbol{v}_{rf} = \mathbf{1} - \mathbf{1}' * \frac{\mathbf{1}}{\boldsymbol{\gamma}} * \boldsymbol{\Sigma}^{-1} * \left(\boldsymbol{\mu}^e + \frac{1}{2} * \boldsymbol{diag}(\boldsymbol{\Sigma}) - \boldsymbol{\mu}_{rf}\right)$ (iv) → **Optimal risk-free weight**

b) Calculate time-varying weights of optimal portfolio (10 points)

Time-varying weights of factor model:

based on the optimization solution:

risky weight vector: $\mathbf{v} = \frac{1}{\gamma} * \mathbf{\Sigma}^{-1} * \left(\boldsymbol{\mu}^{e} + \frac{1}{2} * \boldsymbol{diag}(\mathbf{\Sigma}) - \boldsymbol{\mu}_{rf}\right)$

risk-free weight: $\boldsymbol{v}_{rf} = \mathbf{1} - \mathbf{1}' * \frac{1}{\gamma} * \mathbf{\Sigma}^{-1} * \left(\boldsymbol{\mu}^{e} + \frac{1}{2} * \boldsymbol{diag}(\mathbf{\Sigma}) - \boldsymbol{\mu}_{rf}\right)$

c) Report performance measures and run Fama-French 5-factor regression (10 points)

Performance measures:

Mean Return: 0.45%
Volatility: 0.86%
Sharpe Ratio: 46.3%

Fama-French 5-factor regression:

```
                            OLS Regression Results
==============================================================================
Dep. Variable:                      y   R-squared:                       0.068
Model:                            OLS   Adj. R-squared:                  0.028
Method:                 Least Squares   F-statistic:                     1.694
Date:                Tue, 30 Nov 2021   Prob (F-statistic):              0.142
Time:                        20:32:20   Log-Likelihood:                 411.08
No. Observations:                 122   AIC:                            -810.2
Df Residuals:                     116   BIC:                            -793.3
Df Model:                           5
Covariance Type:            nonrobust
==============================================================================
                 coef    std err          t      P>|t|      [0.025      0.975]
------------------------------------------------------------------------------
const          0.0043      0.001      5.168      0.000       0.003       0.006
x1            -0.0340      0.022     -1.580      0.117      -0.077       0.009
x2             0.0596      0.039      1.543      0.125      -0.017       0.136
x3             0.0111      0.034      0.321      0.748      -0.057       0.079
x4             0.0388      0.052      0.749      0.455      -0.064       0.141
x5            -0.1380      0.060     -2.307      0.023      -0.256      -0.020
==============================================================================
Omnibus:                       41.547   Durbin-Watson:                   1.533
Prob(Omnibus):                  0.000   Jarque-Bera (JB):               90.741
Skew:                           1.394   Prob(JB):                     1.98e-20
Kurtosis:                       6.174   Cond. No.                         82.7
==============================================================================
```

	Based on the regression results, we can conclude that the degree of explanation of the portfolio returns, based on the Fama-French 5-factor model, is very low which can be seen at the R-squared of 6.8% and the p-value of 0.142, while the latter is normally significant when assuming confidence levels between 1-10%. Thus, the model can't explain most of the variance in the portfolio returns and leads to insignificant and unreliable results.	