ISMT S-136 Time Series Analysis with Python

Harvard Summer School

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Lecture 8

Contents

Contents

Definition of *Pure* Seasonal ARMA

Def.

x_t is said to follow *pure* Seasonal ARMA, denoted by ARMA$(P, Q)_s$, if

$$\underbrace{\Phi_P(B^s)}_{\substack{\text{seasonal}\\ \text{AR}\\ \text{operator}}} x_t = \underbrace{\Theta_Q(B^s)}_{\substack{\text{seasonal}\\ \text{MA}\\ \text{operator}}} w_t,$$

where

$$\begin{aligned}\Phi_P(B^s) =& 1 - \Phi_1 B^s - \Phi_2 B^{2s} - \ldots - \Phi_P B^{Ps},\\ \Theta_Q(B^s) =& 1 + \Theta_1 B^s + \Theta_2 B^{2s} + \ldots + \Theta_Q B^{Qs}.\end{aligned}$$

Contents

Definition of *Mixed* Seasonal ARMA

Def.

x_t is said to follow *mixed* Seasonal ARMA, denoted by ARMA$(p, q) \times (P, Q)_s$, if

$$\underbrace{\Phi_P(B^s)}_{\substack{\text{seasonal}\\ \text{AR}\\ \text{operator}}} \phi(B) x_t = \underbrace{\Theta_Q(B^s)}_{\substack{\text{seasonal}\\ \text{MA}\\ \text{operator}}} \theta(B) w_t,$$

where

$$\begin{aligned}\Phi_P(B^s) =& 1 - \Phi_1 B^s - \Phi_2 B^{2s} - \ldots - \Phi_P B^{Ps}, \\ \Theta_Q(B^s) =& 1 + \Theta_1 B^s + \Theta_2 B^{2s} + \ldots + \Theta_Q B^{Qs}\end{aligned}$$

and

$$\begin{aligned}\phi(B) =& 1 - \phi_1 B - \phi_2 B^2 - \ldots - \phi_p B^p, \\ \theta(B) =& 1 + \theta_1 B + \theta_2 B^2 + \ldots + \theta_q B^q.\end{aligned}$$

Contents

Definition of *Multiplicative* Seasonal ARIMA (SARIMA)

Def.

x_t is said to follow *multiplicative* Seasonal ARIMA (SARIMA), denoted by ARIMA$(p, d, q) \times (P, D, Q)_s$, if

$$\underbrace{\Phi_P(B^s)}_{\text{seasonal AR operator}} \phi(B) \underbrace{(1 - B^s)^D}_{\nabla_s^D} \underbrace{(1 - B)^d}_{\nabla^d} x_t = \delta + \underbrace{\Theta_Q(B^s)}_{\text{seasonal MA operator}} \theta(B) w_t,$$

where

$$\begin{aligned}
\Phi_P(B^s) =& 1 - \Phi_1 B^s - \Phi_2 B^{2s} - \ldots - \Phi_P B^{Ps}, \\
\Theta_Q(B^s) =& 1 + \Theta_1 B^s + \Theta_2 B^{2s} + \ldots + \Theta_Q B^{Qs}
\end{aligned}$$

and

$$\begin{aligned}
\phi(B) =& 1 - \phi_1 B - \phi_2 B^2 - \ldots - \phi_p B^p, \\
\theta(B) =& 1 + \theta_1 B + \theta_2 B^2 + \ldots + \theta_q B^q.
\end{aligned}$$

Contents

Motivation

We notice that if x_t is a causal AR(1) process:

$$x_t = \phi x_{t-1} + w_t, \quad \text{where} \quad w_t \overset{\text{iid}}{\sim} \mathcal{N}(0, \sigma_w^2),$$

then

$$\begin{aligned}
\mathrm{E}[x_t \,|\, x_{t-1}, x_{t-2}, \ldots] &= \mathrm{E}[\phi x_{t-1} + w_t \,|\, x_{t-1}, x_{t-2}, \ldots] = \phi x_{t-1}, \\
\mathrm{Var}(x_t \,|\, x_{t-1}, x_{t-2}, \ldots) &= \mathrm{Var}(\phi x_{t-1} + w_t \,|\, x_{t-1}, x_{t-2}, \ldots) = \sigma_w^2,
\end{aligned}$$

i.e. we assume constant conditional variance ("conditional homoskedasticity").

Can we modify the model to introduce non-constant conditional variance ("conditional heteroskedasticity")?

Answer:

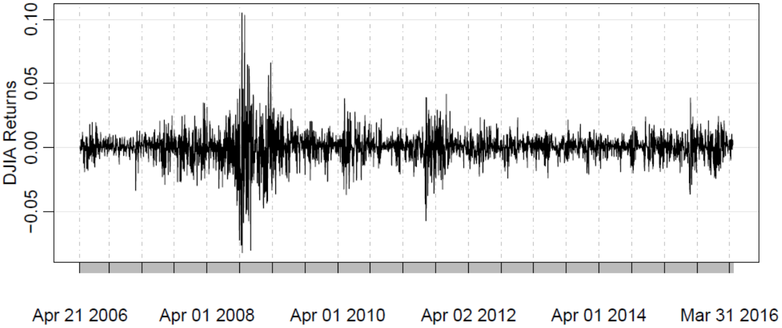
1. Autoregressive Conditionally Heteroscedastic (ARCH)
2. Generalized Autoregressive Conditionally Heteroscedastic (GARCH)
3. Stochastic Volatility (SV) Models

Example: Returns of Dow Jones Industrial Average (DJIA)

Return (relative gain) of an asset at time t is defined as

$$r_t = \frac{x_t - x_{t-1}}{x_{t-1}} \approx \nabla \ln x_t,$$

where x_t denotes the price of the asset at time t.



Source: *Time Series Analysis and Its Applications: With R Examples* by R. Shumway and D. Stoffer

Contents

Definition of ARCH(1)

Def.

r_t is said to follow ARCH(1) if

$$\underbrace{r_t}_{\text{“}w_t\text{”}} = \sigma_t \varepsilon_t, \quad \text{where} \quad \varepsilon_t \overset{\text{iid}}{\sim} \mathcal{N}(0,1),$$

$$\sigma_t^2 = \alpha_0 + \alpha_1 r_{t-1}^2,$$

where $\alpha_0, \alpha_1 \geq 0$.

Note:

1. If r_t is ARCH(1), then $r_t \,|\, r_{t-1} \sim \mathcal{N}(0, \alpha_0 + \alpha_1 r_{t-1}^2)$
2. If $\alpha_1 = 0$, then ARCH(1) is ARMA($0,0$) with $w_t \overset{\text{iid}}{\sim} \mathcal{N}(0, \alpha_0)$.

Difference Equation for ARCH(1) Squared

Suppose r_t is ARCH(1):

$$\begin{aligned} r_t &= \sigma_t \varepsilon_t, \quad \text{where} \quad \varepsilon_t \overset{\text{iid}}{\sim} \mathcal{N}(0,1), \\ \sigma_t^2 &= \alpha_0 + \alpha_1 r_{t-1}^2, \end{aligned}$$

then one can equivalently write

$$\begin{aligned} r_t^2 &= \sigma_t^2 \varepsilon_t^2, \\ 0 &= \alpha_0 + \alpha_1 r_{t-1}^2 - \sigma_t^2, \end{aligned}$$

and therefore:

$$r_t^2 = \alpha_0 + \alpha_1 r_{t-1}^2 + \underbrace{\sigma_t^2(\varepsilon_t^2 - 1)}_{v_t},$$

where $\varepsilon_t^2 \sim \chi_1^2$.

Properties of ARCH(1)

Properties:

Let r_t be ARCH(1):

$$r_t = \sigma_t \varepsilon_t, \quad \text{where} \quad \varepsilon_t \overset{\text{iid}}{\sim} \mathcal{N}(0,1),$$
$$\sigma_t^2 = \alpha_0 + \alpha_1 r_{t-1}^2,$$

then

1. $r_t \,|\, r_{t-1} \sim \mathcal{N}(0, \alpha_0 + \alpha_1 r_{t-1}^2)$.
2. $\mathrm{E}[r_t] = E[\mathrm{E}[r_t \,|\, r_{t-1}, r_{t-2}, \ldots]] = \mathrm{E}[\mathrm{E}[r_t \,|\, r_{t-1}]] = 0.$
3. For any $h > 0$:

$$\begin{aligned}\mathrm{Cov}(r_{t+h}, r_t) = \mathrm{E}[r_t r_{t+h}] &= \mathrm{E}[\mathrm{E}[r_t r_{t+h} \,|\, r_{t+h-1}, r_{t+h-2}, \ldots]] \\ &= \mathrm{E}[r_t \mathrm{E}[r_{t+h} \,|\, r_{t+h-1}, r_{t+h-2}, \ldots]] = \mathrm{E}[r_t \cdot 0] = 0\end{aligned}$$

4. Unconditional $\mathrm{E}[r_t]$ and $\mathrm{Var}[r_t]$ are constants with respect to time t.

Parameter Estimation: Likelihood for ARCH(1)

The likelihood function is

$$\begin{aligned} &L(\alpha_0, \alpha_1) \\ &\quad = f(r_1, r_2, \dots, r_n \,|\, \alpha_0, \alpha_1) \\ &\quad = f(r_1 \,|\, \alpha_0, \alpha_1) f(r_2 \,|\, r_1 \alpha_0, \alpha_1) \dots f(r_n \,|\, \alpha_0, \alpha_1) \\ &\quad = f(r_1 \,|\, \alpha_0, \alpha_1) \prod_{t=2}^{n} f(r_t \,|\, r_{t-1}, \alpha_0, \alpha_1) \end{aligned}$$

and since $r_t \,|\, r_{t-1} \sim \mathcal{N}(0, \underbrace{\alpha_0 + \alpha_1 r_{t-1}^2}_{\sigma_t^2})$, we get

$$L(\alpha_0, \alpha_1) = f(r_1 \,|\, \alpha_0, \alpha_1) \prod_{t=2}^{n} \frac{1}{\sqrt{2\pi}\sigma_t} e^{-\frac{r_t^2}{2\sigma_t^2}}.$$

Given r_1, the conditional likelihood is

$$L(\alpha_0, \alpha_1 \,|\, r_1) = \prod_{t=2}^{n} \frac{1}{\sqrt{2\pi}\sigma_t} e^{-\frac{r_t^2}{2\sigma_t^2}}$$

Parameter Estimation: Likelihood for ARCH(1)

Given r_1, the conditional log-likelihood is

$$
\begin{aligned}
& l(\alpha_0, \alpha_1 \,|\, r_1) \\
& \quad = \ln\left[L(\alpha_0, \alpha_1 \,|\, r_1)\right] \\
& \quad = \ln\left[\prod_{t=2}^{n} \frac{1}{\sqrt{2\pi}\sigma_t} e^{-\frac{r_t^2}{2\sigma_t^2}}\right] \\
& \quad = \ln\left[\prod_{t=2}^{n} \frac{1}{\sqrt{2\pi(\alpha_0 + \alpha_1 r_{t-1}^2)}} e^{-\frac{r_t^2}{2(\alpha_0 + \alpha_1 r_{t-1}^2)}}\right] \\
& \quad = -\frac{n-1}{2}\ln(2\pi) - \frac{1}{2}\sum_{t=2}^{n} \ln(\alpha_0 + \alpha_1 r_{t-1}^2) - \sum_{t=2}^{n} \frac{r_t^2}{2(\alpha_0 + \alpha_1 r_{t-1}^2)}.
\end{aligned}
$$

Estimates of parameters α_0 and α_1 are obtained by maximizing $l(\alpha_0, \alpha_1 \,|\, r_1)$.

Contents

AR(1)-ARCH(1)

One can combine
AR(1) model for x_t with
ARCH(1) model for errors r_t as follows:

Def.:
x_t is said to follow AR(1)-ARCH(1) if

$$x_t = \phi x_{t-1} + \underbrace{r_t}_{\text{``}w_t\text{''}}$$

where

$$\begin{aligned} r_t &= \sigma_t \varepsilon_t, \quad \text{where} \;\; \varepsilon_t \overset{\text{iid}}{\sim} \mathcal{N}(0,1), \\ \sigma_t^2 &= \alpha_0 + \alpha_1 r_{t-1}^2. \end{aligned}$$

Note: r_t is unobserved in this case.

Contents

Definition of ARCH(p)

Def.

r_t is said to follow ARCH(p) if

$$\underbrace{r_t}_{\text{“}w_t\text{”}} = \sigma_t \varepsilon_t, \quad \text{where} \quad \varepsilon_t \overset{\text{iid}}{\sim} \mathcal{N}(0,1),$$

$$\sigma_t^2 = \alpha_0 + \alpha_1 r_{t-1}^2 + \ldots + \alpha_p r_{t-p}^2,$$

where $\alpha_0, \alpha_1, \ldots, \alpha_p \geq 0$.

Note:

If r_t is ARCH(p), then

$r_t \,|\, r_{t-1}, \ldots, r_{t-p} \sim \mathcal{N}(0, \alpha_0 + \alpha_1 r_{t-1}^2 + \ldots + \alpha_p r_{t-p}^2)$

Parameter Estimation: Likelihood for ARCH(p)

The likelihood function is

$$
\begin{aligned}
&L(\alpha_0, \alpha_1, \ldots, \alpha_p) \\
&\quad = f(r_1, r_2, \ldots, r_n \,|\, \alpha_0, \alpha_1, \ldots, \alpha_p) \\
&\quad = f(r_1 \,|\, \alpha_0, \alpha_1, \ldots, \alpha_p) \ldots f(r_{p-1} \,|\, r_{p-2}, \ldots, r_1, \alpha_0, \alpha_1, \ldots, \alpha_p) \\
&\qquad f(r_{p+1} \,|\, r_p, \ldots, r_1, \alpha_0, \alpha_1, \ldots, \alpha_p) \ldots f(r_n \,|\, r_{n-1}, \ldots, r_{n-p}, \alpha_0, \alpha_1, \ldots, \alpha_p) \\
&\quad = f(r_1 \,|\, \alpha_0, \alpha_1, \ldots, \alpha_p) \ldots f(r_{p-1} \,|\, r_{p-2}, \ldots, r_1, \alpha_0, \alpha_1, \ldots, \alpha_p) \\
&\qquad \prod_{t=p+1}^{n} f(r_t \,|\, r_{t-1}, \ldots, r_{t-p}, \alpha_0, \alpha_1, \ldots, \alpha_p)
\end{aligned}
$$

and since $r_t \,|\, r_{t-1}, \ldots, r_{t-p} \sim \mathcal{N}(0, \underbrace{\alpha_0 + \alpha_1 r_{t-1}^2 + \ldots + \alpha_p r_{t-p}^2}_{\sigma_t^2})$, we get

$$
\begin{aligned}
L(\alpha_0, \alpha_1, \ldots, \alpha_p) = &f(r_1 \,|\, \alpha_0, \alpha_1, \ldots, \alpha_p) \ldots f(r_{p-1} \,|\, r_{p-2}, \ldots, r_1, \alpha_0, \alpha_1, \ldots, \alpha_p) \\
&\prod_{t=p+1}^{n} \frac{1}{\sqrt{2\pi}\sigma_t} e^{-\frac{r_t^2}{2\sigma_t^2}}.
\end{aligned}
$$

Parameter Estimation: Likelihood for ARCH(p)

Given $r_1, \dots, r_p$, the conditional likelihood is

$$L(\alpha_0, \alpha_1, \dots, \alpha_p \,|\, r_1, \dots, r_p) = \prod_{t=p+1}^{n} \frac{1}{\sqrt{2\pi}\sigma_t} e^{-\frac{r_t^2}{2\sigma_t^2}}$$

and the conditional log-likelihood is

$$\begin{aligned}
l(\alpha_0, \alpha_1, \dots, \alpha_p \,|\, r_1, \dots, r_p) &= \ln\left[L(\alpha_0, \alpha_1, \dots, \alpha_p \,|\, r_1, \dots, r_p)\right] \\
&= \ln\left[\prod_{t=p+1}^{n} \frac{1}{\sqrt{2\pi}\sigma_t} e^{-\frac{r_t^2}{2\sigma_t^2}}\right] \\
&= -\frac{n-p}{2}\ln(2\pi) - \frac{1}{2}\sum_{t=p+1}^{n} \ln(\alpha_0 + \alpha_1 r_{t-1}^2 + \ldots + \alpha_p r_{t-p}^2) \\
&\quad - \sum_{t=p+1}^{n} \frac{r_t^2}{2(\alpha_0 + \alpha_1 r_{t-1}^2 + \ldots + \alpha_p r_{t-p}^2)}.
\end{aligned}$$

Estimates of parameters $\alpha_0, \alpha_1 \dots, \alpha_p$ are obtained by maximizing $l(\alpha_0, \alpha_1, \dots, \alpha_p \,|\, r_1, \dots, r_p)$.

Contents

Generalized Autoregressive Conditionally Heteroscedastic (GARCH)

Can ARCH(p) model for errors

$$r_t = \sigma_t \varepsilon_t, \quad \text{where} \quad \varepsilon_t \overset{\text{iid}}{\sim} \mathcal{N}(0,1),$$

$$\sigma_t^2 = \alpha_0 + \alpha_1 r_{t-1}^2 + \ldots + \alpha_p r_{t-p}^2 = \alpha_0 + \sum_{j=1}^{p} \alpha_j r_{t-j}^2$$

be extended further?

Answer:

<u>Def.</u>

r_t is said to follow GARCH(p, q) if

$$\underbrace{r_t}_{\text{``}w_t\text{''}} = \sigma_t \varepsilon_t, \quad \text{where} \quad \varepsilon_t \overset{\text{iid}}{\sim} \mathcal{N}(0,1),$$

$$\sigma_t^2 = \alpha_0 + \sum_{j=1}^{p} \alpha_j r_{t-j}^2 + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2,$$

where $\alpha_0, \alpha_1, \ldots, \alpha_p \geq 0$.

Contents

Example: AR(1)-GARCH(2,3)

One can combine, for example,
AR(1) model for x_t with, for example,
ARCH(2,3) model for errors r_t as follows:

Def.:
x_t is said to follow AR(1)-GARCH(2,3) if

$$x_t = \phi x_{t-1} + \underbrace{r_t}_{\text{``}w_t\text{''}}$$

where

$$\begin{aligned} r_t &= \sigma_t \varepsilon_t, \quad \text{where} \quad \varepsilon_t \overset{\text{iid}}{\sim} \mathcal{N}(0,1), \\ \sigma_t^2 &= \underbrace{\alpha_0 + \alpha_1 r_{t-1}^2 + \alpha_2 r_{t-1}^2}_{\text{here, } p=2} + \underbrace{\beta_1 \sigma_{t-1}^2 + \beta_1 \sigma_{t-2}^2 + \beta_1 \sigma_{t-3}^2}_{\text{here, } q=3}. \end{aligned}$$

Note: r_t is unobserved in this case.

Contents

Parameter Estimation: Likelihood for GARCH(p,q)

If we have GARCH(p,q) model one can write the likelihood function similarly to ARCH(p).