ISMT S-136 Time Series Analysis with Python

Harvard Summer School

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Lecture 9

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Motivation

Recall: Let $\phi(B) = 1 - \phi_1 B - \ldots - \phi_m B^m$, then

$$\text{AR}(m)\text{-ARCH}(p): \quad \phi(B)x_t = r_t$$
$$r_t = \sigma_t \varepsilon_t, \quad \text{where} \quad \varepsilon_t \overset{\text{iid}}{\sim} \mathcal{N}(0,1),$$
$$\sigma_t^2 = \alpha_0 + \sum_{j=1}^{p} \alpha_j r_{t-j}^2.$$

$$\text{AR}(m)\text{-GARCH}(p, q): \quad \phi(B)x_t = r_t$$
$$r_t = \sigma_t \varepsilon_t, \quad \text{where} \quad \varepsilon_t \overset{\text{iid}}{\sim} \mathcal{N}(0,1),$$
$$\sigma_t^2 = \alpha_0 + \sum_{j=1}^{p} \alpha_j r_{t-j}^2 + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2.$$

Note that, given $x_{t-1}, x_{t-2}, \ldots$, volatility σ_t^2 is conditionally nonstochastic. Can GARCH be modified to also include (conditional) stochastisity of σ_t^2?

Answer: Stochastic Volatility (SV) Model.

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Definition of Stochastic Volatility (SV) Model

Let's consider GARCH$(1, 1)$:

$$r_t = \sigma_t \varepsilon_t, \quad \text{where} \quad \varepsilon_t \overset{\text{iid}}{\sim} \mathcal{N}(0, 1),$$
$$\sigma_t^2 = \alpha_0 + \alpha_1 r_{t-1}^2 + \beta_1 \sigma_{t-1}^2,$$

or equivalently:

$$\ln r_t^2 = \ln \sigma_t^2 + \ln \varepsilon_t^2, \quad \text{where} \quad \varepsilon_t \overset{\text{iid}}{\sim} \mathcal{N}(0, 1),$$
$$\sigma_t^2 = \alpha_0 + \alpha_1 r_{t-1}^2 + \beta_1 \sigma_{t-1}^2.$$

Let's modfiy the model and introduce stochastic volatility:

Def.
Stochastic volatility (SV) model for r_t is defined as follows:

$$\ln r_t^2 = \ln \sigma_t^2 + \ln \varepsilon_t^2, \quad \text{where} \quad \varepsilon_t \overset{\text{iid}}{\sim} \mathcal{N}(0, 1),$$
$$\ln \sigma_t^2 = \phi_0 + \phi_1 \ln \sigma_{t-1}^2 + w_t, \quad \text{where} \quad w_t \overset{\text{iid}}{\sim} \mathcal{N}(0, \sigma_w^2).$$

Definition of Stochastic Volatility (SV) Models

Remark:
The stochastic volatility model,

$$\ln r_t^2 = \ln \sigma_t^2 + \ln \varepsilon_t^2, \quad \text{where} \quad \varepsilon_t \overset{\text{iid}}{\sim} \mathcal{N}(0,1),$$
$$\underbrace{\ln \sigma_t^2}_{v_t} = \phi_0 + \phi_1 \underbrace{\ln \sigma_{t-1}^2}_{v_{t-1}} + w_t, \quad \text{where} \quad w_t \overset{\text{iid}}{\sim} \mathcal{N}(0,\sigma_w^2),$$

can be rewritten as follows:

$$r_t = e^{\frac{v_t}{2}} \varepsilon_t, \quad \text{where} \quad \varepsilon_t \overset{\text{iid}}{\sim} \mathcal{N}(0,1),$$
$$v_t = \phi_0 + \phi_1 v_{t-1} + w_t, \quad \text{where} \quad w_t \overset{\text{iid}}{\sim} \mathcal{N}(0,\sigma_w^2),$$

where we introduced $v_t = \ln \sigma_t^2$.

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Parameter Estimation: Likelihood for SV Model

We notice that in

$$r_t = e^{\frac{v_t}{2}} \varepsilon_t, \quad \text{where} \quad \varepsilon_t \overset{\text{iid}}{\sim} \mathcal{N}(0, 1),$$

$$v_t = \phi_0 + \phi_1 v_{t-1} + w_t, \quad \text{where} \quad w_t \overset{\text{iid}}{\sim} \mathcal{N}(0, \sigma_w^2),$$

both v_t and ε_t are unobserved.

In order to obtain estimates of the parameters, one needs to maximize the likelihood function, given observations $r_1, r_2, \ldots, r_n$.

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Cross-Covariance and Joint Stationary

Def.
Given two time series x_t and y_t,

$$\gamma_{xy}(s,t) = \mathrm{Cov}(x_s, y_t) = \mathrm{E}[(x_s - \mu_{xs})(y_t - \mu_{yt})]$$

is called *cross-covariance function* between the two series x_t and y_t.

Def.
Two time series x_t and y_t are called *jointly stationary* if
they are stationary and the cross-covariance function depends on $s-t$ only.

Notation:
$\gamma_{xy}(s,t)$ will be denoted by $\gamma_{xy}(s-t)$ in case of jointly stationary x_t and y_t.

Remark:
$\gamma_{xy}(h)$ is generally not an even function.

Cross-Correlation Function (CCF) between x_t and y_t

Def.

Given two time series x_t and y_t,

$$\begin{aligned}\rho_{xy}(s,t) =& \mathrm{Corr}(x_s, y_t)\\ =& \frac{\mathrm{Cov}(x_s, y_t)}{\sqrt{\mathrm{Var}(x_s)\mathrm{Var}(y_t)}}\\ =& \frac{\gamma_{xy}(s,t)}{\sqrt{\gamma_x(s,s)\gamma_y(t,t)}}\end{aligned}$$

is called *cross-correlation function* (CCF) between the two series x_t and y_t.

Notation:

If x_t and y_t are jointly stationary, $\rho_{xy}(s,t)$ depends on $s-t$ only and will be denoted by $\rho_{xy}(s-t)$ then.

Remark:

$\rho_{xy}(h)$ is generally not an even function.

Example 1.23 Joint Stationarity

Let

$$
\begin{aligned}
x_t =& w_t + w_{t-1}, \\
y_t =& w_t - w_{t-1},
\end{aligned}
$$

where $w_t \overset{\text{iid}}{\sim} \mathcal{N}(0, \sigma_w^2)$.

Then

$$
\rho_{xy}(h) = \begin{cases} 0, & h = 0, \\ \frac{1}{2}, & h = 1, \\ -\frac{1}{2}, & h = -1, \\ 0, & |h| \geq 2. \end{cases}
$$

Example 1.24 Prediction Using Cross-Correlation

Let x_t be a stationary time series.
Assume that x_t "leads" y_t (equivalently, y_t "lags" x_t):

$$y_t = ax_{t-5} + w_t,$$

where $w_t \overset{\text{iid}}{\sim} \mathcal{N}(0, \sigma_w^2)$.

Then

$$\begin{aligned}\gamma_{yx}(h) =& \mathrm{Cov}(y_{t+h}, x_t)\\ =& \mathrm{Cov}(ax_{t+h-5} + w_{t+h}, x_t)\\ =& \mathrm{Cov}(ax_{t+h-5}, x_t)\\ =& a\gamma_x(h-5),\end{aligned}$$

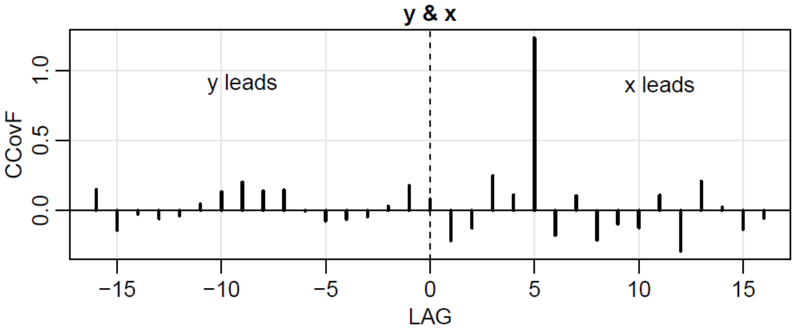
i.e. $\gamma_{yx}(h)$ is "shifted" $\gamma_x(h-5)$.

Notice that x_t and y_t are jointly stationary.

Example 1.24 Prediction Using Cross-Correlation (continued)

Stationary x_t and $y_t = a x_{t-5} + w_t$, where $w_t \overset{\text{iid}}{\sim} \mathcal{N}(0, \sigma_w^2)$.
Then $\gamma_{yx}(h) = a\gamma_x(h-5)$.

Example:



Source: *Time Series Analysis and Its Applications: With R Examples*
by R. Shumway and D. Stoffer

Sample Cross-Covariance Function

Def.

Given a realization $(x_1, y_t), (x_2, y_2), \ldots, (x_n, y_n)$ of jointly stationary time series x_t and y_t, the sample cross-covariance function is defined as follows:

$$\hat{\gamma}_{xy}(h) = \frac{1}{n} \sum_{t=1}^{n-h} (x_{t+h} - \bar{x})(y_t - \bar{y})$$

for $h = 0, 1, \ldots, n-1$, and

$$\hat{\gamma}_{xy}(-h) = \hat{\gamma}_{yx}(h).$$

Sample Cross-Correlation Functions (Sample CCF)

Def.

Given $\hat{\gamma}_{xy}(h)$ for $h = 0, \pm 1, \ldots, \pm(n-1)$ for a realization $(x_1, y_t), (x_2, y_2), \ldots, (x_n, y_n)$ of jointly stationary time series x_t and y_t, the corresponding *sample CCF* is:

$$\hat{\rho}_{xy}(h) = \frac{\hat{\gamma}_{xy}(h)}{\sqrt{\hat{\gamma}_x(0)\hat{\gamma}_y(0)}},$$

where $\hat{\gamma}_x(\cdot)$ and $\hat{\gamma}_y(\cdot)$ are ACFs of x_t and y_t, respectively.

Example 1.29 Prewhitening and Cross Correlation Analysis

Let

$$
\begin{aligned}
x_t =& 2\cos(2\pi t \frac{1}{12}) + w_{t,1},\\
y_t =& 2\cos(2\pi [t+5] \frac{1}{12}) + w_{t,2},
\end{aligned}
$$

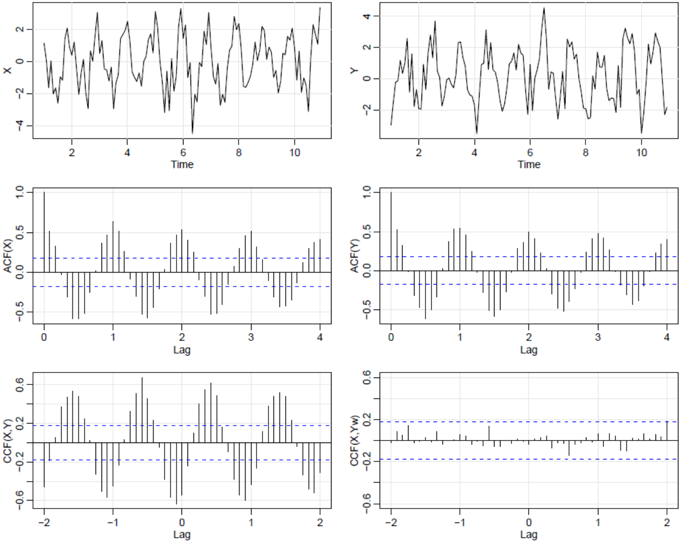
where $w_{t,1}, w_{t,2} \overset{\text{iid}}{\sim} \mathcal{N}(0,1)$.

Then

$$
\begin{aligned}
\gamma_{xy}(h) =& \mathrm{Cov}(x_{t+h}, y_t)\\
=& \mathrm{Cov}(2\cos(2\pi[t+h]\frac{1}{12}) + w_{t+h,1}, 2\cos(2\pi[t+5]\frac{1}{12}) + w_{t,2})\\
=& \mathrm{Cov}(w_{t+h,1}, w_{t,2})\\
=& 0,
\end{aligned}
$$

for all h.

Example 1.29 Prewhitening and Cross Correlation Analysis (continued)



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Vector-valued Time Series

Def.
We define a *vector-valued* time series x_t as the following collection $x_{t,1}, x_{t,2}, \ldots, x_{t,k}$ of k time series:

$$x_t = (x_{t,1}, x_{t,2}, \ldots, x_{t,k})^T = \begin{bmatrix} x_{t,1} \\ x_{t,2} \\ \vdots \\ x_{t,k} \end{bmatrix}.$$

Stationarity of Vector-valued Time Series

Def.

A vector-valued time series $x_t = (x_{t,1}, x_{t,2}, \ldots, x_{t,k})^T$ is called *stationary* if

$$x_i \text{ and } x_j \text{ are jointly stationary}$$

for all $i \neq j$, $i, j \in \{1, 2, \ldots, k\}$.

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Autocovariance Matrix

Notation:
Let $x_t = (x_{t,1}, x_{t,2}, \dots, x_{t,k})^T$ be a stationary vector-valued time series.
We introduce an expectation $\mu = (\mu_1, \mu_2, \dots, \mu_k)^T$ (i.e. $k \times 1$ matrix) of x_t as follows:

$$\mu = \mathrm{E}[x_t] = \begin{bmatrix} \mathrm{E}[x_{t,1}] \\ \mathrm{E}[x_{t,2}] \\ \vdots \\ \mathrm{E}[x_{t,k}] \end{bmatrix}$$

Autocovariance Matrix

Def.
Let $x_t = (x_{t,1}, x_{t,2}, \dots, x_{t,k})^T$ be a stationary vector-valued time series. The $k \times k$ matrix

$$\Gamma(h) = \mathrm{E}[(x_{t+h} - \mu)(x_t - \mu)^T]$$

$$= \mathrm{E}\left[\begin{bmatrix} x_{t+h,1} - \mu_1 \\ x_{t+h,2} - \mu_2 \\ \vdots \\ x_{t+h,k} - \mu_p \end{bmatrix} \begin{bmatrix} x_{t,1} - \mu_1 & x_{t,2} - \mu_2 & \dots & x_{t,k} - \mu_p \end{bmatrix}\right]$$

$$= \mathrm{E}\begin{bmatrix} (x_{t+h,1} - \mu_1)(x_{t,1} - \mu_1) & (x_{t+h,1} - \mu_1)(x_{t,2} - \mu_2) & \dots & (x_{t+h,1} - \mu_1)(x_{t,k} - \mu_k) \\ (x_{t+h,2} - \mu_2)(x_{t,1} - \mu_1) & (x_{t+h,2} - \mu_2)(x_{t,2} - \mu_2) & \dots & (x_{t+h,2} - \mu_2)(x_{t,k} - \mu_k) \\ \vdots & & \ddots & \\ (x_{t+h,k} - \mu_k)(x_{t,1} - \mu_1) & (x_{t+h,k} - \mu_k)(x_{t,2} - \mu_2) & \dots & (x_{t+h,k} - \mu_k)(x_{t,k} - \mu_k) \end{bmatrix}$$

is called *autocovariance matrix*.

Notice:

1. Off-diagonal elements, $\gamma_{ij}(h) = \mathrm{E}[(x_{t+h,i} - \mu_i)(x_{t,j} - \mu_j)]$, $i \neq j$, of $\Gamma(h)$ are the cross-covariance functions between $x_{t,i}$ and $x_{t,j}$.
2. Diagonal elements, $\gamma_{ij}(h) = \mathrm{E}[(x_{t+h,i} - \mu_i)(x_{t,i} - \mu_i)]$, of $\Gamma(h)$ are the autocovariance functions of $x_{t,i}$.

Sample Autocovariance Matrix

Def.

The *sample autocovariance matrix* is defined as

$$\hat{\Gamma}(h) = \frac{1}{n} \sum_{t=1}^{n-h} (x_{t+h} - \bar{x})(x_t - \bar{x})^T,$$

for $h = 0, 1, \ldots, n-1$, and

$$\hat{\Gamma}(-h) = \hat{\Gamma}^T(h).$$

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Vector Autoregressive Model of First Order (VAR(1))

Def.

We say that a vector-valued time series $x_t = (x_{t,1}, x_{t,2}, \dots, x_{t,k})^T$ follows *Vector Autoregressive* model of first order, abbreviated VAR(1), if

$$x_t = \Phi x_{t-1} + w_t,$$

where

$$\Phi = \begin{bmatrix} \Phi_{11} & \Phi_{12} & \dots & \Phi_{1k} \\ \Phi_{21} & \Phi_{22} & \dots & \Phi_{2k} \\ \vdots & & \ddots & \\ \Phi_{k1} & \Phi_{k2} & \dots & \Phi_{kk} \end{bmatrix}$$

and w_t is the *vector Gaussian white noise*, that is, multivariate normal with zero mean, $\mathrm{E}[w_s w_t^T] = 0$ for all $s \neq t$, and some covariance matrix,

$$\Sigma_w = \mathrm{E}[w_t w_t^T],$$

which is generally not diagonal.

Vector Autoregressive Model of Order p (VAR(p))

Def.

We say that a vector-valued time series $x_t = (x_{t,1}, x_{t,2}, \ldots, x_{t,k})^T$ follows *Vector Autoregressive* model of first order, abbreviated VAR(p), if

$$x_t = \Phi_1 \, x_{t-1} + \ldots + \Phi_p \, x_{t-p} + w_t,$$

where for each $j \in \{1, 2, \ldots, p\}$

$$\Phi_j = \begin{bmatrix} \Phi_{j,11} & \Phi_{j,12} & \ldots & \Phi_{j,1k} \\ \Phi_{j,21} & \Phi_{j,22} & \ldots & \Phi_{j,2k} \\ \vdots & & \ddots & \\ \Phi_{j,k1} & \Phi_{j,k2} & \ldots & \Phi_{j,kk} \end{bmatrix}$$

and w_t is the *vector Gaussian white noise*, that is, multivariate normal with zero mean, $\mathrm{E}[w_s w_t^T] = 0$ for all $s \neq t$, and some covariance matrix,

$$\Sigma_w = \mathrm{E}[w_t w_t^T],$$

which is generally not diagonal.

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Parameter Estimation: Yule–Walker Equations for VAR

If we have a VAR(p) process:

$$x_t = \Phi_1 \, x_{t-1} + \ldots + \Phi_p \, x_{t-p} + w_t,$$

where $x_t = (x_{t,1}, x_{t,2}, \ldots, x_{t,k})^T$, Φ_p are $k \times k$ matrices, and w_t is the vector Gaussian white noise.

Then similarly to the univariate AR(p) case, one show that the following Yule-Walker equations hold:

$$\begin{aligned} \Gamma(h) &= \Phi_1 \, \Gamma(h-1) + \ldots + \Phi_p \, \Gamma(h-p), \quad \text{for } h = 1, 2, \ldots \\ \Gamma(0) &= \Phi_1 \, \Gamma(-1) + \ldots + \Phi_p \, \Gamma(-p) + \Sigma_w, \end{aligned}$$

where

$$\Sigma_w = \mathrm{E}[w_t w_t^T].$$

Parameter Estimation: Yule–Walker Equations for VAR

If we replace $\Gamma(h)$ by the sample autocovariance matrix $\hat{\Gamma}(h)$ (i.e. use method of moments),
we get the estimates (namely Yule-Walker estimators) for unknown parameters as a solution to the following linear system of equations:

$$\hat{\Gamma}(h) = \Phi_1 \hat{\Gamma}(h-1) + \ldots + \Phi_p \hat{\Gamma}(h-p), \quad \text{for } h = 1, 2, \ldots$$
$$\hat{\Gamma}(0) = \Phi_1 \hat{\Gamma}(-1) + \ldots + \Phi_p \hat{\Gamma}(-p) + \Sigma_w.$$

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Parameter Estimation: Likelihood for VAR(p)

If we have a VAR(p) process:

$$x_t = \Phi_1 x_{t-1} + \ldots + \Phi_p x_{t-p} + w_t,$$

where $x_t = (x_{t,1}, x_{t,2}, \ldots, x_{t,k})^T$, Φ_j are $k \times k$ matrices, and w_t is the vector Gaussian white noise.

Then similarly to the univariate AR(p) case, one can write the likelihood function explicitly.

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Vector Autoregressive Moving Average (VARMA)

Def.

We say that a vector-valued time series $x_t = (x_{t,1}, x_{t,2}, \ldots, x_{t,k})^T$ follows *Vector Autoregressive Moving Average* model of order p, q, abbreviated VARMA(p, q), if

$$x_t = \sum_{j=1}^{p} \Phi_j \, x_{t-j} + \sum_{i=1}^{q} \Theta_i w_{t-i} + w_t,$$

where for each $j \in \{1, 2, \ldots, p\}$ and $i \in \{1, 2, \ldots, q\}$ we have

$$\Phi_j = \begin{bmatrix} \Phi_{j,11} & \Phi_{j,12} & \ldots & \Phi_{j,1k} \\ \Phi_{j,21} & \Phi_{j,22} & \ldots & \Phi_{j,2k} \\ \vdots & & \ddots & \\ \Phi_{j,k1} & \Phi_{j,k2} & \ldots & \Phi_{j,kk} \end{bmatrix}, \; \Theta_i = \begin{bmatrix} \Theta_{i,11} & \Theta_{i,12} & \ldots & \Theta_{i,1k} \\ \Theta_{i,21} & \Theta_{i,22} & \ldots & \Theta_{i,2k} \\ \vdots & & \ddots & \\ \Theta_{i,k1} & \Theta_{i,k2} & \ldots & \Theta_{i,kk} \end{bmatrix},$$

respectively, and w_t is the *vector Gaussian white noise*, that is, multivariate normal with zero mean, $\mathrm{E}[w_s w_t^T] = 0$ for all $s \neq t$, and some covariance matrix,

$$\Sigma_w = \mathrm{E}[w_t w_t^T],$$

which is generally not diagonal.

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Parameter Estimation: Likelihood for VARMA(p, q)

If we have a VARMA(p, q) process:

$$x_t = \sum_{j=1}^{p} \Phi_j \, x_{t-j} + \sum_{i=1}^{q} \Theta_i w_{t-i} + w_t,$$

where $x_t = (x_{t,1}, x_{t,2}, \ldots, x_{t,k})^T$, Φ_j and Θ_i are $k \times k$ matrices, and w_t is the vector Gaussian white noise.

Then similarly to the univariate ARMA(p) case, one can write the likelihood function explicitly.

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VAR with Exogenous Inputs (VARX)

Def.

Assume there is a vector $u_t = (u_{t,1}, u_{t,2}, \ldots, u_{t,r})^T$ of r pre-determined inputs.

We say that a vector-valued time series $x_t = (x_{t,1}, x_{t,2}, \ldots, x_{t,k})^T$ follows *Vector Autoregressive* model of order p with *exogenous inputs*, abbreviated VARX(p), if

$$x_t = A u_t + \Phi_1 \, x_{t-1} + \Phi_2 \, x_{t-2} + \ldots + \Phi_p \, x_{t-p} + w_t,$$

where for each $j \in \{1, 2, \ldots, p\}$

$$\Phi_j = \begin{bmatrix} \Phi_{j,11} & \Phi_{j,12} & \ldots & \Phi_{j,1k} \\ \Phi_{j,21} & \Phi_{j,22} & \ldots & \Phi_{j,2k} \\ \vdots & & \ddots & \\ \Phi_{j,k1} & \Phi_{j,k2} & \ldots & \Phi_{j,kk} \end{bmatrix}, \; A = \begin{bmatrix} A_{11} & A_{12} & \ldots & A_{1r} \\ A_{21} & A_{22} & \ldots & A_{2r} \\ \vdots & & \ddots & \\ A_{k1} & A_{k2} & \ldots & A_{kr} \end{bmatrix},$$

and w_t is the *vector Gaussian white noise*, that is, multivariate normal with zero mean, $\mathrm{E}[w_s w_t^T] = 0$ for all $s \neq t$, and some covariance matrix,

$$\Sigma_w = \mathrm{E}[w_t w_t^T],$$

which is generally not diagonal.

VARMA with Exogenous Inputs (VARMAX)

Def.

Assume there is a vector $u_t = (u_{t,1}, u_{t,2}, \ldots, u_{t,r})$ of r pre-determined inputs. We say that a vector-valued time series $x_t = (x_{t,1}, x_{t,2}, \ldots, x_{t,k})^T$ follows *Vector Autoregressive Moving Average* model of order p, q with *exogenous inputs*, abbreviated VARMAX(p, q), if

$$x_t = Au_t + \sum_{j=1}^{p} \Phi_j \, x_{t-j} + \sum_{i=1}^{q} \Theta_i w_{t-i} + w_t,$$

where for each $j \in \{1, 2, \ldots, p\}$ and $i \in \{1, 2, \ldots, q\}$ we have

$$\Phi_j = \begin{bmatrix} \Phi_{j,11} & \Phi_{j,12} & \ldots & \Phi_{j,1k} \\ \Phi_{j,21} & \Phi_{j,22} & \ldots & \Phi_{j,2k} \\ \vdots & & \ddots & \\ \Phi_{j,k1} & \Phi_{j,k2} & \ldots & \Phi_{j,kk} \end{bmatrix}, \quad \Theta_i = \begin{bmatrix} \Theta_{i,11} & \Theta_{i,12} & \ldots & \Theta_{i,1k} \\ \Theta_{i,21} & \Theta_{i,22} & \ldots & \Theta_{i,2k} \\ \vdots & & \ddots & \\ \Theta_{i,k1} & \Theta_{i,k2} & \ldots & \Theta_{i,kk} \end{bmatrix},$$

respectively, A is an $k \times r$ matrix, and w_t is the *vector white noise*, that is, multivariate normal with zero mean, $\mathrm{E}[w_s w_t^T] = 0$ for all $s \neq t$, and some covariance matrix,

$$\Sigma_w = \mathrm{E}[w_t w_t^T],$$

which is generally not diagonal.