ISMT S-136 Time Series Analysis with Python

Harvard Summer School

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Lecture 7

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Yule-Walker Equations

Let x_t be a causal AR(p) process:

$$x_t = \sum_{j=1}^{p} \phi_j x_{t-j} + w_t, \quad \text{where} \quad w_t \sim wn(0, \sigma_w^2),$$

then the following system of equations (called *Yule-Walker equations*) holds:

$$\begin{aligned} \gamma(h) =& \phi_1 \gamma(h-1) + \ldots + \phi_p \gamma(h-p), \;\; h = 1, 2, \ldots, p, \\ \sigma_w^2 =& \gamma(0) - \phi_1 \gamma(1) - \ldots - \phi_p \gamma(p). \end{aligned}$$

Yule-Walker Estimation

Consider the *Yule-Walker equations*
for a causal AR(p) process $x_t = \sum_{j=1}^{p} \phi_j x_{t-j} + w_t$:

$$\begin{aligned}\gamma(h) =& \phi_1 \gamma(h-1) + \ldots + \phi_p \gamma(h-p), \ h = 1, 2, \ldots, p, \\ \sigma_w^2 =& \gamma(0) - \phi_1 \gamma(1) - \ldots - \phi_p \gamma(p).\end{aligned}$$

If we replace $\gamma(h)$ by the sample autocovariance function $\hat{\gamma}(h)$ (i.e. use method of moments),
we get the estimates $\hat{\phi}_1, \ldots, \hat{\phi}_p$ and $\hat{\sigma}_w^2$ (namely Yule-Walker estimators) for unknown parameters $\phi_1, \ldots, \phi_p$ and σ_w^2, respectively, as a solution to the following linear system of equations:

$$\begin{aligned}\hat{\gamma}(h) =& \hat{\phi}_1 \hat{\gamma}(h-1) + \ldots + \hat{\phi}_p \hat{\gamma}(h-p), \ h = 1, 2, \ldots, p, \\ \hat{\sigma}_w^2 =& \hat{\gamma}(0) - \hat{\phi}_1 \hat{\gamma}(1) - \ldots - \hat{\phi}_p \hat{\gamma}(p).\end{aligned}$$

Yule-Walker Estimation for ARMA?

Can we do same for MA(q) and general ARMA(p, q)?
Yes, but the Yule-Walker equations are more complex and as a result the Yule-Walker estimators are not efficient (not "optimal").

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Maximum Likelihood and Least Squares Estimation

Consider the following causal AR(1) process

$$x_t = \mu + \phi(x_{t-1} - \mu) + w_t, \text{ where } w_t \overset{\text{iid}}{\sim} \mathcal{N}(0, \sigma_w^2).$$

Given observations $x_1, x_2, \ldots, x_n$, the likelihood function is then

$$\begin{aligned} &L(\mu, \phi, \sigma_w^2) \\ &\quad = f(x_1, x_2, \ldots, x_n \,|\, \mu, \phi, \sigma_w^2) \\ &\quad = f(x_1 \,|\, \mu, \phi, \sigma_w^2) f(x_2 \,|\, x_1, \mu, \phi, \sigma_w^2) \cdots f(x_n \,|\, x_{n-1}, \mu, \phi, \sigma_w^2) \\ &\quad = f(x_1 \,|\, \mu, \phi, \sigma_w^2) f_w((x_2 - \mu) - \phi(x_1 - \mu)) \cdots f_w((x_n - \mu) - \phi(x_{n-1} - \mu)) \end{aligned}$$

and since $x_t \,|\, x_{t-1}, \mu, \phi, \sigma_w^2 \sim \mathcal{N}(\mu + \phi(x_{t-1} - \mu), \sigma_w^2)$, we get

$$L(\mu, \phi, \sigma_w^2) = f(x_1 \,|\, \mu, \phi, \sigma_w^2) \prod_{t=2}^{n} f_w((x_t - \mu) - \phi(x_{t-1} - \mu)),$$

where $f_w(z) = \frac{1}{\sqrt{2\pi}\sigma_w} e^{-\frac{z^2}{2\sigma_w^2}}$ is the pdf of w_t.

Maximum Likelihood and Least Squares Estimation

Because x_t is causal, we can write:

$$x_1 = \mu + \sum_{j=0}^{\infty} \phi^j w_{1-j}, \quad \text{i.e. } x_1 \sim \mathcal{N}(\mu, \frac{\sigma_w^2}{1-\phi^2}).$$

Then

$$\begin{aligned} L(\mu, \phi, \sigma_w^2) =& f(x_1 \mid \mu, \phi, \sigma_w^2) \prod_{t=2}^{n} f_w((x_t - \mu) - \phi(x_{t-1} - \mu)) \\ =& \left(\frac{1}{\sqrt{2\pi}\sigma_w} \right)^n \left(1 - \phi^2\right)^{\frac{1}{2}} \; e^{-\frac{S(\mu,\phi)}{2\sigma_w^2}}, \end{aligned}$$

where

$$\underbrace{S(\mu, \phi)}_{\substack{\text{unconditional} \\ \text{sum} \\ \text{of squares}}} = (1 - \phi^2)(x_1 - \mu) + \sum_{t=2}^{n} \left[(x_t - \mu) - \phi(x_{t-1} - \mu) \right]^2.$$

Maximum Likelihood and Least Squares Estimation

Alternatively, we could drop x_1 (i.e. condition on x_1) and write

$$L(\mu, \phi, \sigma_w^2 \mid x_1) = \prod_{t=2}^{n} f_w((x_t - \mu) - \phi(x_{t-1} - \mu))$$
$$= \left(\frac{1}{\sqrt{2\pi}\sigma_w} \right)^{n-1} e^{-\frac{S_c(\mu,\phi)}{2\sigma_w^2}},$$

where

$$\underbrace{S_c(\mu, \phi)}_{\text{conditional sum of squares}} = \sum_{t=2}^{n} \left[(x_t - \mu) - \phi(x_{t-1} - \mu) \right]^2.$$

Maximum Likelihood and Least Squares Estimation

If we have a general ARMA(p, q), which is also Gaussian and causal, one can similarly write the likelihood function.

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Definition of Autoregressive Integrated Moving Average (ARIMA)

Def.

If $\nabla^d x_t$ is ARMA(p,q), then x_t is called
Autoregressive Integrated Moving Average, ARIMA(p, d, q).

Definition of Autoregressive Integrated Moving Average (ARIMA)

Note:
If x_t is ARIMA(p, d, q) with $\mathrm{E}\left[\nabla^d x_t\right] = \mu$, then one can write

$$\phi(B) \underbrace{(1-B)^d x_t}_{\nabla^d x_t} = \delta + \theta(B) w_t,$$

where $\delta = \mu(1 - \phi_1 - \ldots - \phi_p)$.

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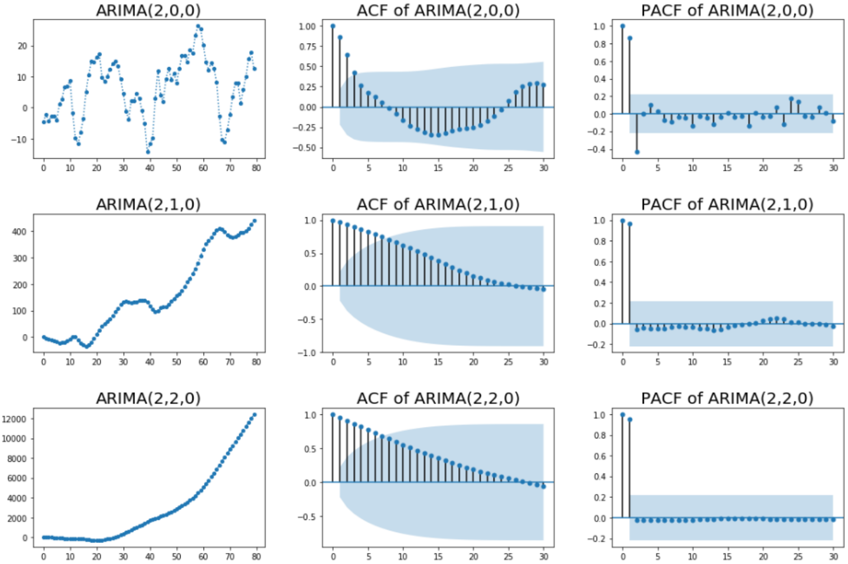
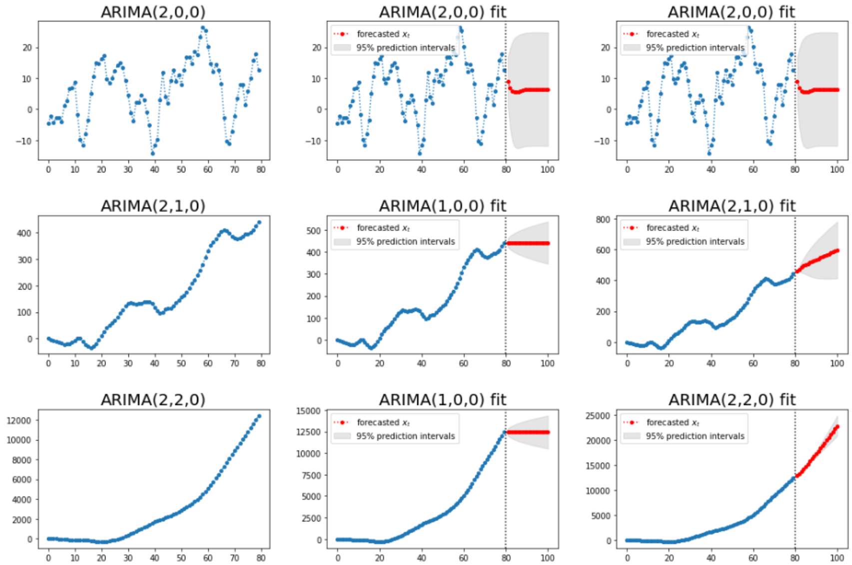
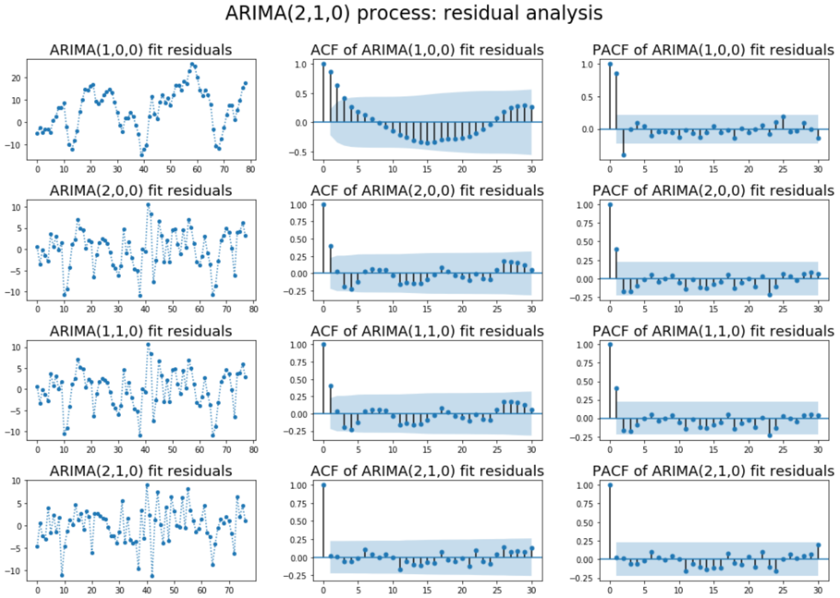
ARIMA Example



ARIMA Example (continued)



ARIMA Example (continued)



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Definition of *Pure* Seasonal ARMA

Def.

x_t is said to follow *pure* Seasonal ARMA, denoted by ARMA$(P,Q)_s$, if

$$\underbrace{\Phi_P(B^s)}_{\substack{\text{seasonal}\\ \text{AR}\\ \text{operator}}} x_t = \underbrace{\Theta_Q(B^s)}_{\substack{\text{seasonal}\\ \text{MA}\\ \text{operator}}} w_t,$$

where

$$\begin{aligned}\Phi_P(B^s) =& 1 - \Phi_1 B^s - \Phi_2 B^{2s} - \ldots - \Phi_P B^{Ps},\\ \Theta_Q(B^s) =& 1 + \Theta_1 B^s + \Theta_2 B^{2s} + \ldots + \Theta_Q B^{Qs}.\end{aligned}$$

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Definition of *Mixed* Seasonal ARMA

Def.

x_t is said to follow *mixed* Seasonal ARMA, denoted by ARMA$(p, q) \times (P, Q)_s$, if

$$\underbrace{\Phi_P(B^s)}_{\substack{\text{seasonal}\\\text{AR}\\\text{operator}}}\phi(B)x_t = \underbrace{\Theta_Q(B^s)}_{\substack{\text{seasonal}\\\text{MA}\\\text{operator}}}\theta(B)w_t,$$

where

$$\Phi_P(B^s) = 1 - \Phi_1 B^s - \Phi_2 B^{2s} - \ldots - \Phi_P B^{Ps},$$
$$\Theta_Q(B^s) = 1 + \Theta_1 B^s + \Theta_2 B^{2s} + \ldots + \Theta_Q B^{Qs}$$

and

$$\phi(B) = 1 - \phi_1 B - \phi_2 B^2 - \ldots - \phi_p B^p,$$
$$\theta(B) = 1 + \theta_1 B + \theta_2 B^2 + \ldots + \theta_q B^q.$$

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Definition of *Multiplicative* Seasonal ARIMA (SARIMA)

Def.

x_t is said to follow *multiplicative* Seasonal ARIMA (SARIMA), denoted by ARIMA$(p, d, q) \times (P, D, Q)_s$, if

$$\underbrace{\Phi_P(B^s)}_{\text{seasonal AR operator}} \phi(B) \underbrace{(1 - B^s)^D}_{\nabla_s^D} \underbrace{(1 - B)^d}_{\nabla^d} x_t = \delta + \underbrace{\Theta_Q(B^s)}_{\text{seasonal MA operator}} \theta(B) w_t,$$

where

$$\begin{aligned} \Phi_P(B^s) =& 1 - \Phi_1 B^s - \Phi_2 B^{2s} - \ldots - \Phi_P B^{Ps}, \\ \Theta_Q(B^s) =& 1 + \Theta_1 B^s + \Theta_2 B^{2s} + \ldots + \Theta_Q B^{Qs} \end{aligned}$$

and

$$\begin{aligned} \phi(B) =& 1 - \phi_1 B - \phi_2 B^2 - \ldots - \phi_p B^p, \\ \theta(B) =& 1 + \theta_1 B + \theta_2 B^2 + \ldots + \theta_q B^q. \end{aligned}$$