ISMT S-136 Time Series Analysis with Python

Harvard Summer School

Dmitry Kurochkin

Summer 2021
Lecture 5

Contents

Contents

Example: Causality of an MA(q) Process

Let x_t be MA(q):

$$x_t = w_t + \theta_1 w_{t-1} + \theta_2 w_{t-2} + \ldots + \theta_q w_{t-q}, \quad \text{where} \quad w_t \sim wn(0, \sigma_w^2),$$

which can be written as:

$$\underbrace{1}_{\phi(B)} x_t = (1 + \theta_1 B + \theta_2 B^2 + \ldots + \theta_q B^q) w_t,$$

The causality condition for an MA(q) then is

$$\underbrace{1}_{\phi(z)} \neq 0 \quad \text{for all} \quad |z| \leq 1,$$

i.e. MA(q) is always causal!

Contents

Example: Causality of an AR(1) Process

Let x_t be AR(1):

$$x_t = \phi x_{t-1} + w_t, \quad \text{where} \quad w_t \sim wn(0, \sigma_w^2),$$

which can be equivalently rewritten as:

$$\underbrace{(1 - \phi B)}_{\phi(B)} x_t = w_t.$$

The causality condition for an AR(1) then is

$$\underbrace{1 - \phi z}_{\phi(z)} \neq 0 \quad \text{for all} \quad |z| \leq 1,$$

or

$$\left| \frac{1}{\phi} \right| > 1.$$

Contents

Example 3.8 Parameter Redundancy, Causality, Invertibility

Let

$$x_t = 0.4x_{t-1} + 0.45x_{t-2} + w_t + w_{t-1} + 0.25w_{t-2},$$

which is equivalent to

$$\underbrace{(1 - 0.4B - 0.45B^2)}_{=\phi(B)} x_t = \underbrace{(1 + B + 0.25B^2)}_{=\theta(B)} w_t.$$

Is this process ARMA(2,2)?

We notice that

$$\underbrace{(1 + 0.5B)(1 - 0.9B)}_{=\phi(B)} x_t = \underbrace{(1 + 0.5B)(1 + 0.5B)}_{=\theta(B)} w_t,$$

i.e. the ARMA(1,1) model!

Example 3.8 Parameter Redundancy, Causality, Invertibility

We have the following ARMA(1,1) process:

$$\underbrace{(1-0.9B)}_{=\phi(B)}x_t = \underbrace{(1+0.5B)}_{=\theta(B)}w_t,$$

Root of $\phi(z)=0$ is $z=\frac{10}{9}$, which is outside of unit circle: $|z|>1$. Therefore, this process is causal.

To get the coefficients ψ_j, $j=0,1,2,\ldots$, we can write $\psi(z)=\frac{\theta(z)}{\phi(z)}$, or

$$\underbrace{(1-0.9z)}_{\phi(z)}\underbrace{(\psi_0+\psi_1 z+\psi_2 z^2+\ldots)}_{\psi(z)} = \underbrace{1+0.5z}_{\theta(z)},$$

which is equivalent to

$$\psi_0+(\psi_1-0.9\psi_0)z+(\psi_2-0.9\psi_1)z^2+\ldots = 1+0.5z.$$

$\Rightarrow x_t = w_t + 1.4\sum_{j=1}^{\infty} 0.9^{j-1}w_{t-j}$ ("MA representation").

Example 3.8 Parameter Redundancy, Causality, Invertibility

We have the following ARMA(1,1) process:

$$\underbrace{(1-0.9B)}_{=\phi(B)} x_t = \underbrace{(1+0.5B)}_{=\theta(B)} w_t,$$

Root of $\theta(z) = 0$ is $z = -2$, which is outside of unit circle: $|z| > 1$. Therefore, this process is invertible.

To get the coefficients π_j, $j = 0, 1, 2, \ldots$, we can write $\pi(z) = \frac{\phi(z)}{\theta(z)}$, or

$$\underbrace{(1+0.5z)}_{\theta(z)} \underbrace{(\pi_0 + \pi_1 z + \pi_2 z^2 + \ldots)}_{\pi(z)} = \underbrace{1-0.9z}_{\phi(z)},$$

which is equivalent to

$$\pi_0 + (\pi_1 + 0.5\pi_0)z + (\pi_2 + 0.5\pi_1)z^2 + \ldots = 1 - 0.9z.$$

$\Rightarrow w_t = -1.4 \sum_{j=0}^{\infty} (-0.5)^{j-1} x_{t-j}$ ("AR representation").

Example 3.8 Parameter Redundancy, Causality, Invertibility

We have the following ARMA(1,1) process:

$$\underbrace{(1-0.9B)}_{=\phi(B)}x_t = \underbrace{(1+0.5B)}_{=\theta(B)}w_t,$$

which is

$$x_t = 0.9x_{t-1} + 0.5w_{t-1} + w_t,$$

and obtained

$$w_t = -1.4\sum_{j=0}^{\infty}(-0.5)^{j-1}x_{t-j},$$

then

$$x_t = 1.4\sum_{j=1}^{\infty}(-0.5)^{j-1}x_{t-j} + w_t.$$

Contents

Example 3.9 Causal Conditions for an AR(2) Process

Let x_t be AR(2):

$$x_t = \phi_1 x_{t-1} + \phi_2 x_{t-2} + w_t, \quad \text{where} \quad w_t \sim wn(0, \sigma_w^2),$$

which can be equivalently rewritten as:

$$\underbrace{\left(1 - \phi_1 B - \phi_2 B^2\right)}_{\phi(B)} x_t = w_t.$$

The causality condition then is

$$\underbrace{1 - \phi_1 z - \phi_2 z^2}_{\phi(z)} \neq 0 \quad \text{for all} \quad |z| \leq 1,$$

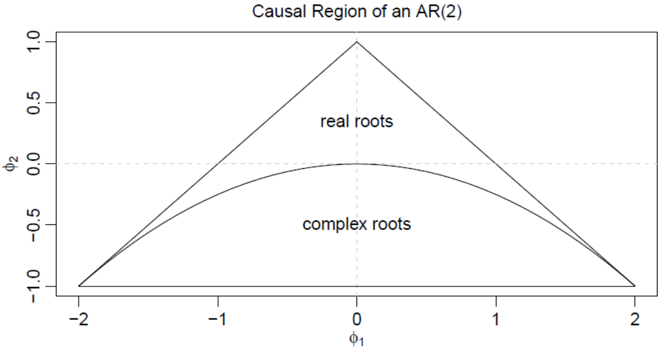
or

$$\left| \frac{-\phi_1 \pm \sqrt{\phi_1^2 + 4\phi_2}}{2\phi_2} \right| > 1.$$

Example 3.9 Causal Conditions for an AR(2) Process

We notice that the causality condition $\left|\frac{-\phi_1 \pm \sqrt{\phi_1^2 + 4\phi_2}}{2\phi_2}\right| > 1$ is equivalent to

$$\phi_1 + \phi_2 < 1, \phi_2 - \phi_1 < 1, \quad \text{and} \quad |\phi_2| < 1.$$



Source: *Time Series Analysis and Its Applications: With R Examples* by R. Shumway and D. Stoffer

Contents

Definition of Partial Autocorrelation Function (PACF)

Assume x_t is stationary. Let's denote regression of

x_{t+h} on $\{x_{t+h-1}, x_{t+h-2}, \ldots, x_{t+1}\}$ by $\hat{x}_{t+h}$:

$$\begin{aligned}\hat{x}_{t+h} =& \sum_{j=1}^{h-1} \beta_j x_{t+h-j} \\ =& \beta_1 x_{t+h-1} + \beta_2 x_{t+h-2} + \ldots + \beta_{h-1} x_{t+1},\end{aligned}$$

where coefficients β_j, $j = 1, 2, \ldots, h-1$, minimize

$$\mathrm{E}\left[\left(x_{t+h} - \hat{x}_{t+h}\right)^2\right].$$

Definition of Partial Autocorrelation Function (PACF)

Similarly, let's denote regression of

x_t on $\{x_{t+1}, x_{t+2}, \ldots, x_{t+h-1}\}$ by $\hat{x}_t$:

$$\begin{aligned} \hat{x}_t =& \sum_{j=1}^{h-1} \beta_j x_{t+j} \\ =& \beta_1 x_{t+1} + \beta_2 x_{t+2} + \ldots + \beta_{h-1} x_{t+h-1}, \end{aligned}$$

where coefficients β_j, $j = 1, 2, \ldots, h-1$, minimize

$$\mathrm{E}\left[(x_t - \hat{x}_t)^2 \right].$$

Definition of Partial Autocorrelation Function (PACF)

Remark

Because x_t is stationary, the set of coefficients β_j in

$$\begin{aligned}\hat{x}_{t+h} =& \sum_{j=1}^{h-1} \beta_j x_{t+h-j} \\ =& \beta_1 x_{t+h-1} + \beta_2 x_{t+h-2} + \ldots + \beta_{h-1} x_{t+1},\end{aligned}$$

is, in fact, same as in

$$\begin{aligned}\hat{x}_{t} =& \sum_{j=1}^{h-1} \beta_j x_{t+j} \\ =& \beta_1 x_{t+1} + \beta_2 x_{t+2} + \ldots + \beta_{h-1} x_{t+h-1},\end{aligned}$$

Definition of Partial Autocorrelation Function (PACF)

Def.

Let x_t be a stationary process, then ϕ_{hh} $(h = 0, 1, 2, \ldots)$, defined as:

$$\begin{aligned}
\phi_{00} &= \mathrm{Corr}(x_t, x_t) = \rho(0) = 1, \\
\phi_{11} &= \mathrm{Corr}(x_{t+1}, x_t) = \rho(1), \\
\phi_{hh} &= \mathrm{Corr}(x_{t+h} - \hat{x}_{t+h}, x_t - \hat{x}_t), h \geq 2,
\end{aligned}$$

is called *partial autocorrelation function* (PACF).

Here, $\hat{x}_{t+h}$ and $\hat{x}_t$ are as defined above:

$$\begin{aligned}
\hat{x}_{t+h} &= \sum_{j=1}^{h-1} \beta_j x_{t+h-j} = \beta_1 x_{t+h-1} + \beta_2 x_{t+h-2} + \ldots + \beta_{h-1} x_{t+1}, \\
\hat{x}_t &= \sum_{j=1}^{h-1} \beta_j x_{t+j} = \beta_1 x_{t+1} + \beta_2 x_{t+2} + \ldots + \beta_{h-1} x_{t+h-1}.
\end{aligned}$$

Example 3.15 The PACF of an AR(1)

Let x_t be AR(1):

$$x_t = \phi x_{t-1} + w_t, \quad \text{where} \quad w_t \sim wn(0, \sigma_w^2),$$

where $|\phi| < 1$ (i.e. x_t is causal) then

$$\begin{aligned}
\phi_{00} =& 1, \\
\phi_{11} =& \rho(1) = \phi, \\
\phi_{22} =& \mathrm{Corr}(x_{t+2} - \hat{x}_{t+2}, x_t - \hat{x}_t) \\
=& \mathrm{Corr}(\underbrace{x_{t+2} - \phi x_{t+1}}_{w_{t+2}}, \underbrace{x_t - \phi x_{t+1}}_{\text{depends on past}}) = 0
\end{aligned}$$

For causal AR(1),

$$\phi_{hh} = 0 \quad \text{for all} \quad h > 1,$$

Example 3.16 The PACF of an AR(p)

Let x_t be AR(p),

$$x_t = \phi_1 x_{t-1} + \phi_2 x_{t-2} + \ldots + \phi_p x_{t-p} + w_t, \quad \text{where} \quad w_t \sim wn(0, \sigma_w^2),$$

be causal then

$$\phi_{hh} = 0 \quad \text{for all} \quad h > p,$$

because $\hat{x}_{t+1} = \phi_1 x_{t-1} + \phi_2 x_{t-2} + \ldots + \phi_p x_{t-p}$ for $h > p$.

Example 3.17 The PACF of an Invertible MA(q)

Let x_t be MA(q),

$$x_t = w_t + \theta_1 w_{t-1} + \theta_2 w_{t-2} + \ldots + \theta_q x_{t-q} + w_t, \quad \text{where} \quad w_t \sim wn(0, \sigma_w^2),$$

be invertible then

$$x_t = -\sum_{j=1}^{\infty} \pi_j x_{t-j} + w_t$$

and PACF will never cut off.

Behavior of the ACF and PACF for ARMA Models

	AR(p)	MA(q)	ARMA(p, q)
ACF	Tails off	Cuts off after lag q	Tails off
PACF	Cuts off after lag p	Tails off	Tails off

Source: *Time Series Analysis and Its Applications: With R Examples* by R. Shumway and D. Stoffer