ISMT S-136 Time Series Analysis with Python

Harvard Summer School

Dmitry Kurochkin

Summer 2021
Lecture 4

Contents

Contents

Example 3.5 The MA(1) Process

Consider the following process, namely *moving average* of order 1:

$$x_t = w_t + \theta w_{t-1}, \quad \text{where} \quad w_t \sim wn(0, \sigma_w^2).$$

Show that

(a) $E(x_t) = 0$ for all t

(b)

$$\gamma(h) = \begin{cases} (1+\theta^2)\sigma_w^2 & \text{if} \quad h = 0 \\ \theta\sigma_w^2 & \text{if} \quad h = \pm 1 \\ 0 & \text{otherwise} \end{cases}$$

(c)

$$\rho(h) = \begin{cases} 1 & \text{if} \quad h = 0 \\ \frac{\theta}{1+\theta^2} & \text{if} \quad h = \pm 1 \\ 0 & \text{otherwise} \end{cases}$$

Contents

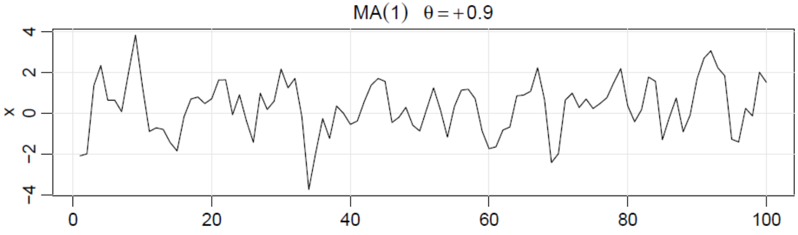
Example 3.5 The Sample Path of an MA(1) Process

Consider the following process, namely *moving average* of order 1:

$$x_t = w_t + \theta w_{t-1}, \quad \text{where} \quad w_t \sim wn(0, \sigma_w^2).$$

Let $\theta = 0.9$, i.e. $\rho(1) = 0.497$ in this case.

Notice that: $\rho(h) = \begin{cases} \frac{\theta}{1+\theta^2} & \text{if } \; h = \pm 1 \\ 0 & \text{if } \; |h| \geq 2 \end{cases}$



Source: *Time Series Analysis and Its Applications: With R Examples*
by R. Shumway and D. Stoffer

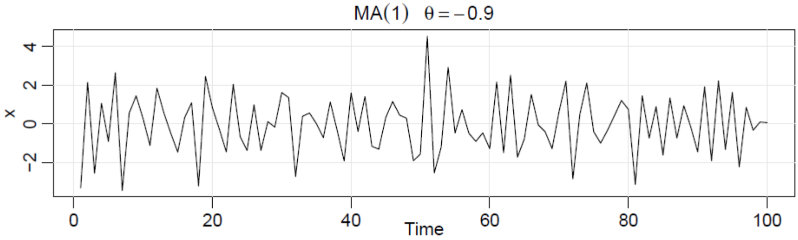
Example 3.5 The Sample Path of an MA(1) Process

Consider the following process, namely *moving average* of order 1:

$$x_t = w_t + \theta w_{t-1}, \quad \text{where} \quad w_t \sim wn(0, \sigma_w^2).$$

Let $\theta = -0.9$, i.e. $\rho(1) = -0.497$ in this case.

Notice that: $\rho(h) = \begin{cases} \frac{\theta}{1+\theta^2} & \text{if} \quad h = \pm 1 \\ 0 & \text{if} \quad |h| \geq 2 \end{cases}$



Source: *Time Series Analysis and Its Applications: With R Examples* by R. Shumway and D. Stoffer

Contents

Example 3.6 Non-uniqueness of MA Models and Invertibility

Consider the following MA(1) process:

$$x_t = w_t + \theta w_{t-1}, \quad \text{where} \quad w_t \overset{\text{iid}}{\sim} \mathcal{N}(0, \sigma_w^2).$$

Then one can see non-uniqueness of the MA model in the following sense:

1. Autocorrelation function: Notice that for any $a \neq 0$, $\rho(h)$ of an MA(1) is same for $\theta = a$ and $\theta = \frac{1}{a}$.
2. Autocovariance function: For any $a \neq 0$ and $c > 0$, $\gamma(h)$ of an MA(1) is same for $\theta = a, \sigma_w = c$ and $\theta = \frac{1}{a}, \sigma_w = ac$.

Example 3.6 Non-uniqueness of MA Models and Invertibility

Therefore, given $a \neq 0$, the MA(1) processes

$$x_t = w_t + a w_{t-1}, \quad \text{where} \quad w_t \overset{\text{iid}}{\sim} \mathcal{N}(0, \sigma_w^2),$$

and

$$x_t = w_t + \frac{1}{a} w_{t-1}, \quad \text{where} \quad w_t \overset{\text{iid}}{\sim} \mathcal{N}(0, a^2 \sigma_w^2),$$

represent the same time series model.

Which choice is preferable?

Example 3.6 Non-uniqueness of MA Models and Invertibility

We notice that the following MA(1) process

$$x_t = w_t + \theta w_{t-1}, \quad \text{where} \quad w_t \overset{\text{iid}}{\sim} \mathcal{N}(0, \sigma_w^2),$$

can be rewritten as follows:

$$w_t = -\theta w_{t-1} + x_t$$

and therefore

$$w_t = \sum_{j=0}^{\infty} (-\theta)^j x_{t-j}, \quad \text{if } |\theta| < 1.$$

If such infinite AR representation is possible (i.e. $|\theta| < 1$ in the case of MA(1)), the process is called an *invertible* process

Contents

Definition of Moving Average Model

Def.

Assume that

$$x_t = w_t + \theta_1 w_{t-1} + \theta_2 w_{t-2} + \ldots \theta_q w_{t-q},$$

where $w_t \sim wn(0, \sigma_w^2)$ and $\theta_1, \theta_2, \ldots, \theta_q$ are constants ($\theta_q \neq 0$), then the process x_t is called *moving average model* of order q, abbreviated MA(q).

Contents

Definition of Moving Average Model Model

Def.

The moving average operator is defined to be

$$\theta(B) = 1 + \theta_1 B + \theta_2 B^2 + \ldots + \theta_q B^q.$$

Note:

Then the moving average model of order q can be written as follows:

$$x_t = \theta(B) w_t.$$

Contents

Definition of Autoregressive Moving Average Model

Def.

Assume that

$$x_t = \alpha + \phi_1 x_{t-1} + \phi_2 x_{t-2} + \dots \phi_p x_{t-p} \\ + w_t + \theta_1 w_{t-1} + \theta_2 w_{t-2} + \dots \theta_q w_{t-q}$$

is a stationary process, where

$w_t \sim wn(0, \sigma_w^2)$,
$\phi_1, \phi_2, \dots, \phi_p$ are constants ($\phi_p \neq 0$),
$\theta_1, \theta_2, \dots, \theta_q$ are constants ($\theta_q \neq 0$), and
there is no parameter redundancy,

then the process x_t is called *autoregressive moving average model* of order p,q, abbreviated ARMA(p,q).

Definition Autoregressive Moving Average Model

Note:

Assumig $\alpha = 0$, the autoregressive moving average model of order p,q can be written as follows:

$$\phi(B)x_t = \theta(B)w_t,$$

where

$$\phi(B) = 1 - \phi_1 B - \phi_2 B^2 - \ldots - \phi_p B^p$$

and

$$\theta(B) = 1 + \theta_1 B + \theta_2 B^2 + \ldots + \theta_q B^q.$$

Contents

Parameter Redundancy

Remark:

We notice that if both sides of

$$\phi(B)x_t = \theta(B)w_t$$

are left-multiplied by some operator $\eta(B)$, we get an equivalent model:

$$\eta(B)\phi(B)x_t = \eta(B)\theta(B)w_t$$

with possibly more parameters.

Example 3.7 Parameter Redundancy

Let

$$x_t = w_t.$$

This process, however, is equivalent to

$$\underbrace{(1 - 0.5B)}_{\eta(B)} x_t = \underbrace{(1 - 0.5B)}_{\eta(B)} w_t,$$

i.e. the following seemingly ARMA(1,1):

$$x_t = 0.5x_{t-1} - 0.5w_{t-1} + w_t.$$

For example, if we fit ARMA(1,1) to a realization of $x_t = w_t$, we can get significant results for all parameters - over-parametrized model.

Requirements on AR and MA polynomials

We will require that the AR and MA polynomials of ARMA(p,q),

$$\phi(z) = 1 - \phi_1 z - \ldots - \phi_p z^p, \phi_p \neq 0,$$

and

$$\theta(z) = 1 + \theta_1 z + \ldots + \theta_q z^q, \theta_q \neq 0,$$

have no common factors.

Contents

Definition of Causality

Def.

ARMA(p, q) model is called *causal* if it can be written as:

$$x_t = \sum_{j=0}^{\infty} \psi_j w_{t-j}$$
$$= \psi_0 w_t + \psi_1 w_{t-1} + \psi_2 w_{t-2} + \dots,$$

where $\sum_{j=0}^{\infty} |\psi_j| < \infty$.

Definition of Causality

Note

Let

$$\psi(B) = \sum_{j=0}^{\infty} \psi_j B^j$$
$$= \psi_0 + \psi_1 B + \psi_2 B^2 + \dots,$$

then a causal x_t can be written as follows ("MA representation"):

$$x_t = \psi(B) w_t.$$

Contents

Roots of AR Polynomial of Causal ARMA

Claim

ARMA(p, q) model is causal if and only if

$$\phi(z) \neq 0 \quad \text{for all} \quad |z| \leq 1.$$

Note that one can rewrite the ARMA model

$$\phi(B)x_t = \theta(B)w_t$$

as follows:

$$x_t = \underbrace{\frac{\theta(B)}{\phi(B)}}_{\psi(B)} w_t.$$

Contents

Definition of Invertibility

Def.

ARMA(p, q) model is called *invertible* if it can be written as:

$$
\begin{aligned}
w_t =& \sum_{j=0}^{\infty} \pi_j x_{t-j} \\
=& \pi_0 x_t + \pi_1 x_{t-1} + \pi_2 x_{t-2} + \dots,
\end{aligned}
$$

where $\sum_{j=0}^{\infty} |\pi_j| < \infty$.

Definition of Invertibility

Note

Let

$$\pi(B) = \sum_{j=0}^{\infty} \pi_j B^j$$
$$= \pi_0 + \pi_1 B + \pi_2 B^2 + \dots,$$

then an invertible x_t can be written as follows ("AR representation"):

$$\pi(B) x_t = w_t.$$

Contents

Roots of MA Polynomial of Invertibile ARMA

Claim

ARMA(p, q) model is invertible if and only if

$$\theta(z) \neq 0 \quad \text{for all} \quad |z| \leq 1.$$

Note that one can rewrite the ARMA model

$$\phi(B)x_t = \theta(B)w_t$$

as follows:

$$\underbrace{\frac{\psi(B)}{\theta(B)}}_{\pi(B)} x_t = w_t.$$