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SMT S-136 Time Series Analysis with Python

Part I of Assignment 6

Let x_t be a causal AR(2) process:

$$x_t = \phi_1 x_{t-1} + \phi_2 x_{t-2} + w_t, \quad \text{where } w_t \overset{iid}{\sim} \mathcal{N}(0, \sigma_w^2).$$

Assume we know ϕ_1, ϕ_2, and σ_w. Suppose we observe x_1, x_2, and x_3, i.e. $n = 3$.

(a) Find the minimum mean square error predictor x_4^3 (the superscript "3" here indicates that the predictor is based on x_1, x_2, x_3) of x_4 using the conditional expectation:

$$x_4^3 = \mathrm{E}\left[x_4 \,|\, x_1, x_2, x_3\right].$$

(b) Find the linear minimum mean square error predictor x_4^3 of x_4. You need to justify your answer here.

(c) For the predictors you obtained in (a) and (b), find the one-step-ahead prediction error P_4^3.

(d) In both cases (a) and (b), construct the 95% prediction interval for x_4^3, that is, an interval (L, R), such that

$$P(L \le x_4 \le R \,|\, x_1, x_2, x_3) = 0.95$$

SOLUTION:

x_3 observable

x_2 observable

a) $x_4^3 = E[x_4 | x_1, x_2, x_3] = E[\phi_1 x_3 + \phi_2 x_2 + w_t \,|\, x_1, x_2, x_3]$

$= \underline{\underline{\phi_1 x_3 + \phi_2 x_2}}$ as x_t is a causal AR(2) process

b) $x_4^3 = \underline{\underline{\phi_1 x_3 + \phi_2 x_2}}$

$\Rightarrow$ As $E[x_4 | x_1, x_2, x_3] = E[\phi_1 x_3 + \phi_2 x_2 + w_t \,|\, x_1, x_2, x_3]$ is already a linear prediction of x_4, the predictors in a) and b) are the same

c) $P_4^3 = E[(x_4 - x_4^3)^2] = E[(\phi_1 x_3 + \phi_2 x_2 + w_4 - \phi_1 x_3 - \phi_2 x_2 \quad] = E[\sigma_w^2] = \underline{\underline{\sigma_w^2}}$

$\Rightarrow$ As AR(2) is a causal process and x_1, x_2, x_3, so all x_t with $t < n+t-4$, is observable, the prediction error is σ_w^2

d) Interval bounds : $x_{n+1}^n \pm 1{,}96 \cdot \sqrt{P_{n+1}^n} = x_4^3 \pm 1{,}96 \cdot \sqrt{P_4^3}$

$\Rightarrow$ Positive bound: $\phi_1 x_3 + \phi_2 x_2 + 1{,}96 \cdot \sqrt{\sigma_w^2} = \phi_1 x_3 + \phi_2 x_2 + 1{,}96\sigma_w$

$\Rightarrow$ Negative bound: $\phi_1 x_3 + \phi_2 x_2 - 1{,}96 \cdot \sqrt{\sigma_w^2} = \phi_1 x_3 + \phi_2 x_2 - 1{,}96\sigma_w$

$\Rightarrow$ $\underline{\underline{[\phi_1 x_3 + \phi_2 x_2 + 1{,}96\sigma_w,\ \phi_1 x_3 + \phi_2 x_2 - 1{,}96\sigma_w]}}$