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ISMT S-136 Time Series Analysis with Python

Part I of Assignment 7

Let x_t be an AR(2)-ARCH(1) process:

$$x_t = \phi_1 x_{t-1} + \phi_2 x_{t-2} + r_t,$$
$$r_t = \sigma_t \varepsilon_t, \quad \text{where } \varepsilon_t \overset{\text{iid}}{\sim} \mathcal{N}(0,1),$$
$$\sigma_t^2 = \alpha_0 + \alpha_1 r_{t-1}^2.$$

Suppose we observe x_1, x_2, and x_3, i.e. $n = 3$.

(a) Here, we assume that we know ϕ_1, ϕ_2, α_0, and α_1. Assume r_t is independent of $x_{t-1}, x_{t-2}, \dots$ for all t. Find the minimum mean square error predictor x_4^3 (the superscript "3" here indicates that the predictor is based on x_1, x_2, x_3) of x_4, i.e. the predictor that minimizes $\mathrm{E}\left[(x_4 - x_4^3)^2\right]$.

(b) Assume the process x_t follows AR(2)-ARCH(1) with unknown parameters. Does fitting AR(2)-ARCH(1) provide any advantage comparable to just using AR(2) model for this process x_t? Please be specific.
Hint: In practice, one needs to estimate the parameters first.

SOLUTION:

a)
$$x_4^3 = E[x_4 \mid x_3, x_2, x_1] = E[\phi_1 x_3 + \phi_2 x_2 + r_4 \mid x_3, x_2, x_1]$$
$$= E[\phi_1 x_3 + \phi_2 x_2 + \sigma_4 + \varepsilon_4 \mid x_3, x_2, x_1]$$
$$= E[\phi_1 x_3 + \phi_2 x_2 + \sqrt{\alpha_0 + \alpha_1 r_3^2} \cdot \varepsilon_4 \mid x_3, x_2, x_1]$$
$$= \phi_1 x_3 + \phi_2 x_2 + E[\sqrt{\alpha_0 + \alpha_1 r_3^2} \cdot \varepsilon_4 \mid x_3, x_2, x_1]$$
$$\overset{\text{ind.}}{=} \phi_1 x_3 + \phi_2 x_2 + E[\sqrt{\alpha_0 + \alpha_1 r_3^2} \mid x_3, x_2, x_1] \cdot \underbrace{E[\varepsilon_4 \mid x_3, x_2, x_1]}_{0}$$
$$= \underline{\underline{\phi_1 x_3 + \phi_2 x_2}}$$

b) The main advantage of fitting an AR(2)-ARCH(1) model against fitting an AR(2) model is the advantage of being able to correctly model the non-constant variance (dependent on past realizations / variances in time). Using an AR(2) model always implies a constant variance which is not appropriate for the process x_t. The ARCH(1) extension therefore allows to count for the autoregressive error terms.