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ISMT S-136 Time Series Analysis with Python

Part I of Assignment 8

Let

$$x_t = w_t,$$
$$y_t = w_t + \theta w_{t-1} + u_t,$$

where w_t and u_t are independent white noise series with variances σ_w^2 and σ_u^2, respectively.

(a) Find autocorrelation functions (ACFs) of x_t and y_t.

(b) Show that x_t and y_t are jointly stationary and find the cross-correlation function (CCF), $\rho_{xy}(h)$, in terms of σ_w, σ_u, and θ.

SOLUTION:

a) $\rho_x(h) = \operatorname{Corr}(x_{t+h}, x_t) = \dfrac{\operatorname{Cov}(x_{t+h}, x_t)}{\sqrt{\operatorname{Var}(x_{t+h}) \cdot \operatorname{Var}(x_t)}} = \dfrac{\gamma_x(t+h,t)}{\sqrt{\gamma_x(h,h) \cdot \gamma_x(t,t)}} \overset{\text{ind.}}{=} \underline{\underline{0}}$

$\Rightarrow \gamma_x(t+h,t) = \operatorname{Cov}(x_{t+h}, x_t) = \operatorname{Cov}(w_{t+h}, w_t) \overset{\text{ind.}}{=} 0$

$\rho_y(h) = \operatorname{Corr}(y_{t+h}, y_t) = \dfrac{\operatorname{Cov}(y_{t+h}, y_t)}{\sqrt{\operatorname{Var}(y_{t+h}) \cdot \operatorname{Var}(y_t)}} = \dfrac{\gamma_y(t+h,t)}{\sqrt{\gamma_y(h,h) \cdot \gamma_y(t,t)}} \overset{\text{ind.}}{=} \underline{\underline{0}}$

$\Rightarrow \gamma_y(t+h,t) = \operatorname{Cov}(y_{t+h}, y_t) = \operatorname{Cov}(w_{t+h} + \theta \cdot w_{t+h-1} + u_{t+h},\ w_t + \theta \cdot w_{t-1} + u_t) \overset{\text{ind.}}{=} 0$

b) $\rho_{xy}(s,t) = \operatorname{Corr}(x_s, y_t) = \dfrac{\operatorname{Cov}(x_s, y_t)}{\sqrt{\operatorname{Var}(x_s) \cdot \operatorname{Var}(y_t)}} = \dfrac{\gamma_{xy}(s,t)}{\sqrt{\gamma_x(s,s) \cdot \gamma_y(t,t)}}$

$\Rightarrow \gamma_{xy}(h) = \operatorname{Cov}(x_{t+h}, y_t) = \operatorname{Cov}(w_{t+h},\ w_t + \theta \cdot w_{t-1} + u_t)$

$$= \begin{cases} \theta \cdot \sigma_w^2 & \text{if } h = -1 \\ \sigma_w^2 & \text{if } h = 0 \\ 0 & \text{if } h = 1 \\ 0 & \text{if } |h| \geq 2 \end{cases}$$

$\Rightarrow \gamma_x(s,s) = \operatorname{Var}(x_s) = \operatorname{Var}(w_t) = \sigma_w^2$

$\Rightarrow \gamma_y(t,t) = \operatorname{Var}(y_t) = \operatorname{Var}(w_t + \theta \cdot w_{t-1} + u_t) \overset{\text{ind.}}{=} \sigma_w^2 + \theta^2 \sigma_w^2 + \sigma_u^2 = (1+\theta^2)\sigma_w^2 + \sigma_u^2$

$\Rightarrow$ Lag $= -1$: $\dfrac{\theta \cdot \sigma_w^2}{\sqrt{\sigma_w^4 + \theta^2 \sigma_w^4 + \sigma_w^2 \sigma_u^2}} = \dfrac{\theta^2 \cdot \sigma_w^4}{\sigma_w^4 + \theta^2 \sigma_w^4 + \sigma_u^2 \sigma_w^2} = \dfrac{\theta^2 \sigma_w^2}{\sigma_w^2 + \theta^2 \sigma_w^2 + \sigma_u^2}$

Lag $= 0$: $\dfrac{\sigma_w^2}{\sqrt{\sigma_w^4 + \theta^2 \sigma_w^4 + \sigma_w^2 \sigma_u^2}} = \dfrac{\sigma_w^4}{\sigma_w^4 + \theta^2 \sigma_w^4 + \sigma_w^2 \sigma_u^2} = \dfrac{\sigma_w^2}{\sigma_w^2 + \theta^2 \sigma_w^2 + \sigma_u^2}$

$$\rho_{xy}(s,t) = \begin{cases} \dfrac{\theta^2 \sigma_w^2}{\sigma_w^2 + \theta^2 \sigma_w^2 + \sigma_u^2} & \text{if } h = -1 \\ \dfrac{\sigma_w^2}{\sigma_w^2 + \theta^2 \sigma_w^2 + \sigma_u^2} & \text{if } h = 0 \\ 0 & \text{else} \end{cases}$$

$\Rightarrow$ x_t and y_t jointly stationary