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ISMT S-136 Time Series Analysis with Python

Part I of Assignment 5

Let x_t be a causal AR(1) process:

$$x_t = \phi x_{t-1} + w_t.$$

(a) Find β_1 and β_2 that minimize $\mathrm{E}\left[(x_{t+3} - \hat{x}_{t+3})^2\right]$, where $\hat{x}_{t+3} = \beta_1 x_{t+2} + \beta_2 x_{t+1}$. Show your work.

(b) Similarly, find β_1 and β_2 that minimize $\mathrm{E}\left[(x_t - \hat{x}_t)^2\right]$, where $\hat{x}_t = \beta_1 x_{t+1} + \beta_2 x_{t+2}$.

(c) Using the results you obtained in (a) and (b), find ϕ_{33}, the partial autocorrelation function (PACF) of x_t at lag $h = 3$.
Hint: Because of causality, $\mathrm{Corr}(w_{t+3}, x_t - \phi x_{t+1}) = 0$.

SOLUTION:

a) $E[(x_{t+3} - \hat{x}_{t+3})^2] = E[(x_{t+3} - \beta_1 x_{t+2} - \beta_2 x_{t+1})^2] =: g(\beta)$

$= Var(x_{t+3} - \beta_1 x_{t+2} - \beta_2 x_{t+1}) + \underbrace{E[x_{t+3} - \beta_1 x_{t+2} - \beta_2 x_{t+1}]^2}_{=0} = Var(x_{t+3} - \beta_1 x_{t+2} - \beta_2 x_{t+1})$

$= Cov(x_{t+3} - \beta_1 x_{t+2} - \beta_2 x_{t+1}, x_{t+3} - \beta_1 x_{t+2} - \beta_2 x_{t+1})$

$= Cov(x_{t+3}, x_{t+3}) - \beta_1 Cov(x_{t+3}, x_{t+2}) - \beta_2 Cov(x_{t+3}, x_{t+1}) - \beta_1 Cov(x_{t+2}, x_{t+3}) + \beta_1^2 Cov(x_{t+2}, x_{t+2})$
$+ \beta_1\beta_2 Cov(x_{t+2}, x_{t+1}) - \beta_2 Cov(x_{t+1}, x_{t+3}) + \beta_1\beta_2 Cov(x_{t+1}, x_{t+2}) + \beta_2^2 Cov(x_{t+1}, x_{t+1})$

$= \gamma(0) - \beta_1\gamma(1) - \beta_2\gamma(2) - \beta_1\gamma(1) + \beta_1^2\gamma(0) + \beta_1\beta_2\gamma(1) - \beta_2\gamma(2) + \beta_1\beta_2\gamma(1) + \beta_2^2\gamma(0)$

$\frac{\partial g(\beta)}{\partial \beta_1}$: $0 = -\gamma(1) - \gamma(1) + 2\beta_1\gamma(0) + \beta_2\gamma(1) + \beta_2\gamma(1) = -2\gamma(1) + 2\beta_1\gamma(0) + 2\beta_2\gamma(1)$

$0 = -\gamma(1) + \beta_1\gamma(0) + \beta_2\gamma(1) \Rightarrow \beta_2 = \frac{\gamma(1) - \beta_1\gamma(0)}{\gamma(1)}$

$\frac{\partial g(\beta)}{\partial \beta_2}$: $0 = -\gamma(2) + \beta_1\gamma(1) - \gamma(2) + \beta_1(\gamma(1) + 2\beta_2\gamma(0) = -2\gamma(2) + 2\beta_1\gamma(1) + 2\beta_2\gamma(0)$

$0 = -\gamma(2) + \beta_1\gamma(1) + \beta_2\gamma(0) \Rightarrow \beta_2 = \frac{\gamma(2) - \beta_1\gamma(1)}{\gamma(0)}$

$\Rightarrow \frac{\gamma(1) - \beta_1\gamma(0)}{\gamma(1)} = \frac{\gamma(2) - \beta_1\gamma(1)}{\gamma(0)} \Rightarrow \underline{\underline{\beta_2 = \beta_1 = \phi}}$

b) $\hat{x}_{t+3} = \beta_1 \cdot x_{t+2} + \beta_2 \cdot x_{t+1} \xleftrightarrow{\text{same}} \hat{x}_t = \beta_1 x_{t+1} + \beta_2 x_{t+2}$

$\Rightarrow$ <u>same coefficients</u> β_1 and β_2 as in question a)

c) $\phi_{33} = Corr(x_{t+3} - \hat{x}_{t+3}, x_t - \hat{x}_t)$

$= Corr(x_{t+3} - \beta_1 x_{t+2} - \beta_2 x_{t+1}, x_t - \beta_1 x_{t+1} + \beta_2 x_{t+2})$

$\Rightarrow$ for all $h > p$: $\phi_{hh} = 0$

$\rightarrow$ as $p = 1$ (AR(1) process) and $h = 3$ (3 lags): $\underline{\underline{\phi_{33} = 0}}$