ISMT S-136 Time Series Analysis with Python

Harvard Summer School

Dmitry Kurochkin

Summer 2021
Lecture 3

Contents

Contents

Linear Process: Definition

<u>Def.</u>

The time series defined as

$$x_t = \mu + \sum_{j=-\infty}^{\infty} \psi_j w_{t-j}, \quad \text{where} \quad w_t \sim wn(0, \sigma_w^2)$$

and $\sum_{j=-\infty}^{\infty} |\psi_j| < \infty$, is called *linear process*.

Contents

Stationarity of Linear Process

Claim

If $\sum_{j=-\infty}^{\infty} \psi_j^2 < \infty$, the linear process

$$x_t = \mu + \sum_{j=-\infty}^{\infty} \psi_j w_{t-j}, \quad \text{where} \quad w_t \sim wn(0, \sigma_w^2)$$

is stationary and its autocovariance function is given by

$$\gamma(h) = \sigma_w^2 \sum_{j=-\infty}^{\infty} \psi_{j+h} \psi_j \quad \text{for} \quad h \geq 0.$$

Contents

Example 3.1 The AR(1) Model

Consider the following process, namely *autoregressive model* of order 1:

$$x_t = \phi x_{t-1} + w_t, \quad \text{where} \quad w_t \sim wn(0, \sigma_w^2).$$

Assume $|\phi| < 1$.

Given $\mathrm{Var}(x_t) \leq c$ for some constant $c \in \mathbb{R}$, show that

(a) $x_t = \sum_{j=0}^{+\infty} \phi^j w_{t-j}$ (called the stationary solution)

(b) $E(x_t) = 0$ for all t

(c) $\gamma(h) = \frac{\sigma_w^2 \phi^h}{1-\phi^2}$, $h \geq 0$

(d) $\rho(h) = \phi^h$, $h \geq 0$

(e) $\rho(h) = \phi\rho(h-1)$, $h = 1, 2, \ldots$

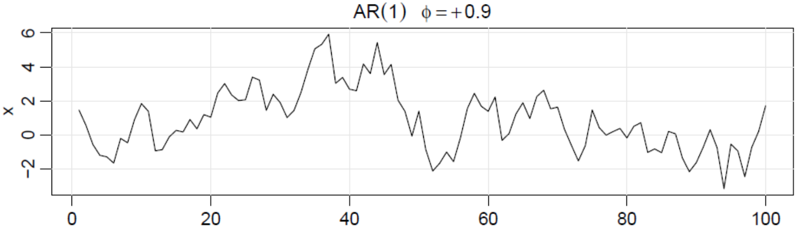
Contents

Example 3.2 The Sample Path of an AR(1) Process

Consider the following process, namely *autoregressive model* of order 1:

$$x_t = \phi x_{t-1} + w_t, \quad \text{where} \quad w_t \sim wn(0, \sigma_w^2).$$

Let $\phi = 0.9$.
Notice that: $\rho(h) = \phi^h$, $h \geq 0$.



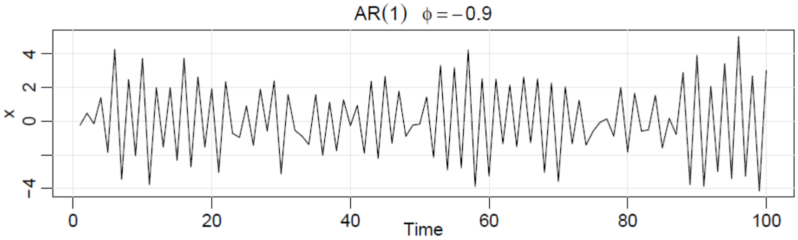
Source: *Time Series Analysis and Its Applications: With R Examples*
by R. Shumway and D. Stoffer

Example 3.2 The Sample Path of an AR(1) Process

Consider the following process, namely *autoregressive model* of order 1:

$$x_t = \phi x_{t-1} + w_t, \quad \text{where} \quad w_t \sim wn(0, \sigma_w^2).$$

Let $\phi = -0.9$.
Notice that: $\rho(h) = \phi^h$, $h \geq 0$.



Source: *Time Series Analysis and Its Applications: With R Examples*
by R. Shumway and D. Stoffer

Contents

Example 3.3 Explosive AR Models and Causality

Consider the following process, namely *autoregressive model* of order 1:

$$x_t = \phi x_{t-1} + w_t, \quad \text{where} \quad w_t \sim wn(0, \sigma_w^2).$$

Let's see what happens if $|\phi| > 1$ ("explosive" model):

Formally, we can write

$$x_t = \left(\frac{1}{\phi}\right) x_{t+1} + \left(\frac{-1}{\phi}\right) w_{t+1},$$

where $|\frac{1}{\phi}| < 1$ and therefore

$$x_t = -\sum_{j=1}^{\infty} \left(\frac{1}{\phi}\right)^j w_{t+j}.$$

The process is not *causal* (the process depends on future)
- in this form, the model is useless!

Example 3.3 Explosive AR Models and Causality

Using the ACF of a linear process, [1]
we can conclude that

$$x_t = -\sum_{j=1}^{\infty} \left(\frac{1}{\phi}\right)^j w_{t+j}$$

has

$$\gamma(h) = \frac{\sigma_w^2 \phi^{-2} \phi^{-h}}{1 - \phi^{-2}},\ h \geq 0.$$

[1] Recall that the linear process $x_t = \mu + \sum_{j=-\infty}^{\infty} \psi_j w_{t-j}$ has $\gamma(h) = \sigma_w^2 \sum_{j=-\infty}^{\infty} \psi_{j+h} \psi_j$ for $h \geq 0$

Example 3.3 Explosive AR Models and Causality

Let's now consider AR(1) with Gaussian white noise:

$x_t = \phi x_{t-1} + w_t,$ where $w_t \overset{\text{iid}}{\sim} \mathcal{N}(0, \sigma_w^2)$.

We demonstrated that

1. if $|\phi| < 1$, $x_t = \sum_{j=0}^{+\infty} \phi^j w_{t-j}$ and $\gamma(h) = \frac{\sigma_w^2 \phi^h}{1-\phi^2},\ h \geq 0$,
2. if $|\phi| > 1$, $x_t = -\sum_{j=1}^{\infty} \left(\frac{1}{\phi}\right)^j w_{t+j}$ and $\gamma(h) = \frac{\sigma_w^2 \phi^{-2} \phi^{-h}}{1-\phi^{-2}},\ h \geq 0$,
3. if $|\phi| = 1$, x_t is called *random walk* and is not stationary.

Therefore,

$$x_t = \phi x_{t-1} + w_t, \quad \text{where} \quad w_t \overset{\text{iid}}{\sim} \mathcal{N}(0, \sigma_w^2)$$

is equivalent to

$$y_t = \frac{1}{\phi} y_{t-1} + v_t, \quad \text{where} \quad v_t \overset{\text{iid}}{\sim} \mathcal{N}(0, \sigma_w^2 \phi^{-2}).$$

Contents

Definition of Autoregressive Model

Def.

Assume that

$$x_t = \alpha + \phi_1 x_{t-1} + \phi_2 x_{t-2} + \dots \phi_p x_{t-p} + w_t,$$

where $w_t \sim wn(0, \sigma_w^2)$ and $\phi_1, \phi_2, \dots, \phi_p$ are constants ($\phi_p \neq 0$), then the process x_t is called *autoregressive model* of order p, abbreviated AR(p).

Contents

Definition of Autoregressive Model

Def.
The autoregressive operator is defined to be

$$\phi(B) = 1 - \phi_1 B - \phi_2 B^2 - \ldots - \phi_p B^p.$$

Note:
Assuming $\mathrm{E}(x_t) = 0$ (i.e. $\alpha = 0$), the autoregressive model of order p can be written as follows:

$$\phi(B) x_t = w_t.$$

Contents

Example 3.5 The MA(1) Process

Consider the following process, namely *moving average* of order 1:

$$x_t = w_t + \theta w_{t-1}, \quad \text{where} \quad w_t \sim wn(0, \sigma_w^2).$$

Show that

(a) $E(x_t) = 0$ for all t

(b)

$$\gamma(h) = \begin{cases} (1+\theta^2)\sigma_w^2 & \text{if} \quad h = 0 \\ \theta\sigma_w^2 & \text{if} \quad h = \pm 1 \\ 0 & \text{otherwise} \end{cases}$$

(c)

$$\rho(h) = \begin{cases} 1 & \text{if} \quad h = 0 \\ \frac{\theta}{1+\theta^2} & \text{if} \quad h = \pm 1 \\ 0 & \text{otherwise} \end{cases}$$

Contents

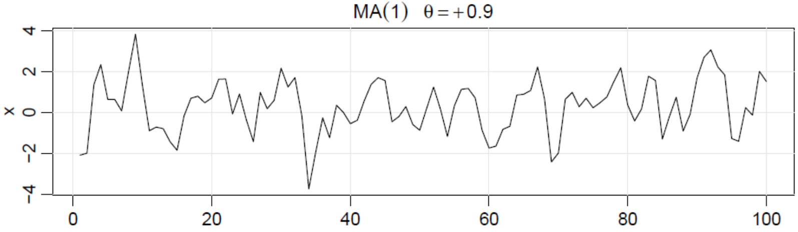
Example 3.5 The Sample Path of an MA(1) Process

Consider the following process, namely *moving average* of order 1:

$$x_t = w_t + \theta w_{t-1}, \quad \text{where} \quad w_t \sim wn(0, \sigma_w^2).$$

Let $\theta = 0.9$, i.e. $\rho(1) = 0.497$ in this case.

Notice that: $\rho(h) = \begin{cases} \frac{\theta}{1+\theta^2} & \text{if} \quad h = \pm 1 \\ 0 & \text{if} \quad |h| \geq 2 \end{cases}$



Source: *Time Series Analysis and Its Applications: With R Examples* by R. Shumway and D. Stoffer

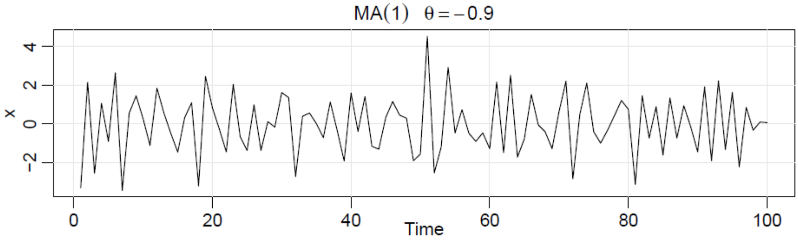
Example 3.5 The Sample Path of an MA(1) Process

Consider the following process, namely *moving average* of order 1:

$$x_t = w_t + \theta w_{t-1}, \quad \text{where} \quad w_t \sim wn(0, \sigma_w^2).$$

Let $\theta = -0.9$, i.e. $\rho(1) = -0.497$ in this case.

Notice that: $\rho(h) = \begin{cases} \frac{\theta}{1+\theta^2} & \text{if } h = \pm 1 \\ 0 & \text{if } |h| \geq 2 \end{cases}$



Source: *Time Series Analysis and Its Applications: With R Examples*
by R. Shumway and D. Stoffer

Contents

Example 3.6 Non-uniqueness of MA Models and Invertibility

Consider the following MA(1) process:

$$x_t = w_t + \theta w_{t-1}, \quad \text{where} \quad w_t \overset{\text{iid}}{\sim} \mathcal{N}(0, \sigma_w^2).$$

Then one can see non-uniqueness of the MA model in the following sense:

1. Autocorrelation function: Notice that for any $a \neq 0$, $\rho(h)$ of an MA(1) is same for $\theta = a$ and $\theta = \frac{1}{a}$.

2. Autocovariance function: For any $a \neq 0$ and $c > 0$, $\gamma(h)$ of an MA(1) is same for $\theta = a, \sigma_w = c$ and $\theta = \frac{1}{a}, \sigma_w = ac$.

Example 3.6 Non-uniqueness of MA Models and Invertibility

Therefore, given $a \neq 0$, the MA(1) processes

$$x_t = w_t + a w_{t-1}, \quad \text{where} \quad w_t \overset{\text{iid}}{\sim} \mathcal{N}(0, \sigma_w^2),$$

and

$$x_t = w_t + \frac{1}{a} w_{t-1}, \quad \text{where} \quad w_t \overset{\text{iid}}{\sim} \mathcal{N}(0, a^2 \sigma_w^2),$$

represent the same time series model.

Which choice is preferable?

Example 3.6 Non-uniqueness of MA Models and Invertibility

We notice that the following MA(1) process

$$x_t = w_t + \theta w_{t-1}, \quad \text{where} \quad w_t \overset{\text{iid}}{\sim} \mathcal{N}(0, \sigma_w^2),$$

can be rewritten as follows:

$$w_t = -\theta w_{t-1} + x_t$$

and therefore

$$w_t = \sum_{j=0}^{\infty} (-\theta)^j x_{t-j}, \quad \text{if } |\theta| < 1.$$

If such infinite AR representation is possible (i.e. $|\theta| < 1$ in the case of MA(1)), the process is called an *invertible* process

Contents

Definition of Moving Average Model

Def.

Assume that

$$x_t = w_t + \theta_1 w_{t-1} + \theta_2 w_{t-2} + \dots \theta_q w_{t-q},$$

where $w_t \sim wn(0, \sigma_w^2)$ and $\theta_1, \theta_2, \dots, \theta_q$ are constants ($\theta_q \neq 0$), then the process x_t is called *moving average model* of order q, abbreviated MA(q).

Contents

Definition of Moving Average Model Model

Def.

The moving average operator is defined to be

$$\theta(B) = 1 + \theta_1 B + \theta_2 B^2 + \ldots + \theta_q B^q.$$

Note:

Then the moving average model of order q can be written as follows:

$$x_t = \theta(B) w_t.$$