RIEMANN HYPOTHESIS AND ITS RELATION TO COMPUTER SCIENCE

NIKAN KADKHODAZADEH

Abstract. This paper looks at the Riemann hypothesis and the connection between the distribution of prime numbers and the zeros of the Riemann zeta function. Despite being purely mathematical in conjecture, it has a profound connection to computer science. We will explore a couple of these connections and will explore the consequences of proving the hypothesis.

Contents

1. Introduction

In 1859, Bernhard Riemann gave a talk on a paper he recently published. This paper focused on a function called the zeta function, defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

He mentions a conjecture about the zeros of the zeta function known as the **Riemann hypothesis**, which is stated below.

Riemann Hypothesis. The Riemann hypothesis states that the zeros of the zeta function that lie between $0 \leq s \leq 1$ have $\Re(s) = \frac{1}{2}$.

2. Preliminaries

This section will define important functions that will be referenced later on.

2.1. RIEMANN ZETA FUNCTION.

Definition 2.1. The **Riemann Zeta Function** is defined by the analytic continuation of the series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

where $s \in \mathbb{C}$.

We write s as $a + bi$, where $a, b \in \mathbb{R}$, so that the real part of s is $\mathrm{Re}(s) = a$ and the imaginary part is $\mathrm{Im}(s) = b$.

2.2. GAMMA FUNCTION.

Definition 2.2. The **Gamma Function**, denoted as $\Gamma(s)$, is defined as

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} \, dt.$$

For example, $\Gamma(s)$ for $s = 1$ is

$$\Gamma(1) = \int_0^\infty e^{-t} t^{1-1} \, dt = \int_0^\infty e^{-t} \, dt = \lim_{t \to \infty} (-e^{-t}) + e^0 = 1.$$

One very interesting property of the gamma function is that it generalizes the factorial function. Specifically, $\Gamma(n) = (n-1)!$. The proof is not shown in this paper but can easily be found. This is an example of analytic continuation.

2.3. PRIME COUNTING FUNCTION.

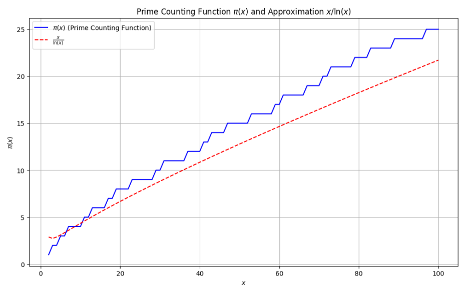
Definition 2.3. $\pi(x)$ will be **the prime counting function** which will count the number of primes less than or equal to x where $x \in \mathbb{R}^+$.

$$\pi(x) = \#\{p \leq x : p \; is \; prime\}$$

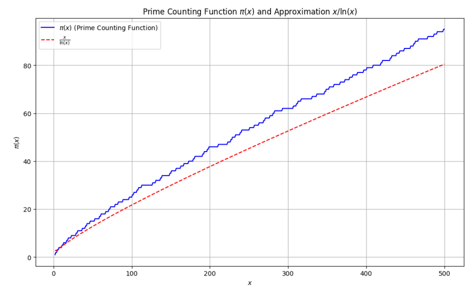
In the 18th century, Gauss and Legendre conjectured a way to approximate it which is

$$\pi(x) \approx \frac{x}{ln(x)}$$

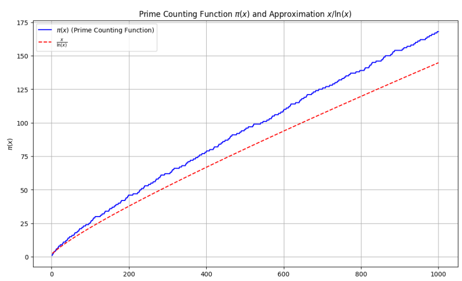
We can use code and software to plot $\pi(x)$ and $\frac{x}{lnx}$. By running some code [1] we can get these plots:



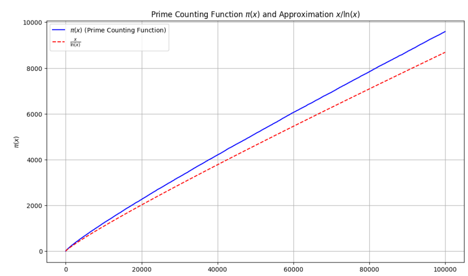
(A) $x = 100$



(B) $x = 500$



(C) $x = 1000$



(D) $x = 100000$

FIGURE 1. Plots of different values of $\pi(x)$ and $\frac{x}{ln(x)}$

2.4. **The prime number theorem.**

$$\lim_{x\to\infty} \frac{\pi(x)}{\frac{x}{\ln(x)}} = 1$$

This means that as x approaches infinity, the error between the 2 functions approaches to 0. However, The blue and red lines seem to be diverging, not converging as x gets larger, does this mean the **the prime number theorem** is wrong? No, The theorem does not say that the difference $\pi(x) - \frac{x}{ln(x)}$ gets smaller as x gets larger, instead what it does say is that the **ratio** between the approximation and the actual value gets closer to 1 as n gets larger and larger.

x	$\pi(x)$	$\frac{x}{\ln(x)}$	Ratio $\left(\frac{\pi(x)}{x/\ln(x)}\right)$
10^2	25	22	88.00%
10^3	168	145	86.31%
10^4	1229	1086	88.36%
10^5	9592	8686	90.55%
10^6	78498	72382	92.21%
10^7	664579	620421	93.36%
10^8	5761455	5428681	94.22%
10^9	50847534	48254942	94.90%
10^{10}	455052511	434294482	95.44%

TABLE 1. Prime Counting Function, Approximation, and Ratio

3. Main Content

In his 1859 paper, Bernhard Riemman wrote a short paper titled "On the Number of Primes Less Than A Given Magnitude". Where he introduced the connection between the prime counting function $\pi(x)$ and the Riemann Zeta function $\zeta(s)$. He discovered a hidden relation between the distribution of primes and the zeros of his zeta function.

3.1. **Analytic continuation.**

Riemann used a advanced technique called analytic continuation to extend the domain of a function, He used this to extend the domain of Euler's zeta function.

When Riemann extended the domain, in the new territory, the $\zeta(s)$ function can be seen crossing through the origin, what this means s that for some values, the $\zeta(s)$ function evaluates to 0, we call these **zeta zeros**. Some zeros are easy to find, when you input an even

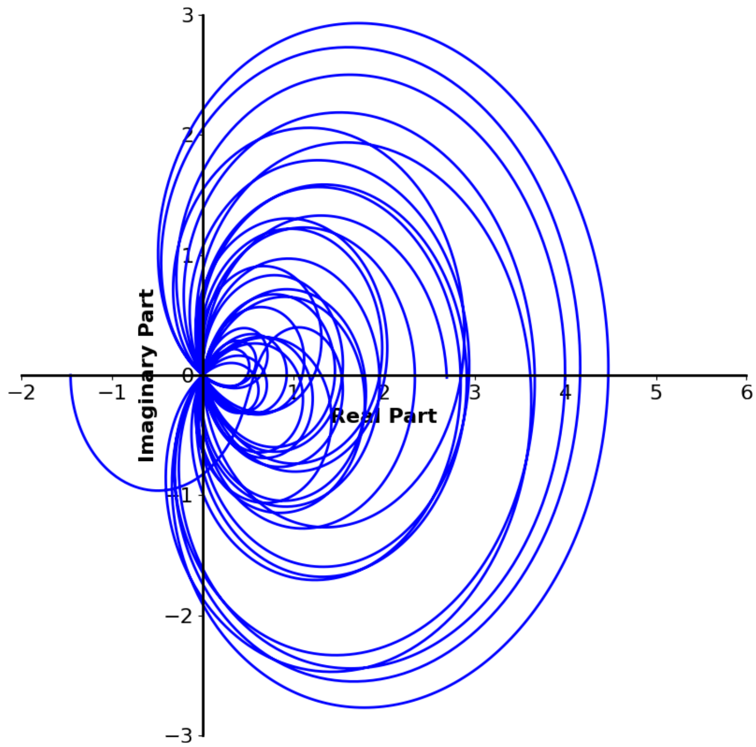


FIGURE 2. Zeta Function

negative integer, you will get a zero. We call these the "trivial zeros". We do not care about these trivial zeros, what we are interested in are the non trivial zeros.

3.2. **The Riemann Hypothesis.** These other zeta zeros exhibit a very compelling pattern, all the non trivial zeros lie within a single region called the **critical strip**, where $0 \leq s \leq 1$. Riemann proved that there are infinitely many zeros in the critical strip. Riemann hypothesized that all the non trivial zeros are not just anywhere in the strip but on a single vertical line, called the **critical line** which is where the $\Re(s) = \frac{1}{2}$. More formally.

The Riemann hypothesis states that the zeros of the zeta function that lie between $0 \leq s \leq 1$ have $\Re(s) = \frac{1}{2}$.

This is the million dollar problem. At this point, you might be asking yourself "what does this have to do with prime numbers?"

3.3. **Riemann's Hypothegsis shows the distribution of prime numbers can be predicted.**

4. Conclusion

References

[1] Nikan Kadkhodazadeh. Riemanncode. https://github.com/nikankad/RiemannCode, 2023. Accessed: 2024-11-16.

Department of Computer Science, University of Southern Maine, Portland, ME

Email address: nikan.kadkhodazadeh@maine.edu