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Riemann's Zeta Function and Prime Numbers

an approximation using von Mangoldt's explicit formula

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Abstract

This essay aims to explain the connection between Riemann's zeta function and prime numbers. It relies heavily on Riemann's 1859 manuscript, in which he approximates the prime-counting function using the non-trivial zeros of the zeta function. The main result of this essay is von Mangoldt's explicit formula, which is a modified version of Riemann's approximation.

Sammanfattning

Syftet med denna uppsats är att klargöra kopplingen mellan Riemanns zeta funktion och primtalen. Den är till stor del baserad på Riemanns manuskript från 1859, där han approximerar primtalsfunktionen med hjälp av zeta funktionens icke-triviala nollställen. Huvudresultatet i uppsatsen är von Mangoldts explicita formel, vilken är en modifierad version av Riemanns approximation.

Contents

1. Introduction

Prime numbers. The building blocks of all numbers. The atoms of mathematics. A concept simple enough to teach to our children, yet it holds mysteries of such complexity that not even the greatest mathematical minds have been able to solve them.

Prime numbers have attracted mathematicians for thousands of years. The proof of infinite prime numbers originates from Euclid's collected work Elements, written around 300 BC [12]. About 2000 years later, Gauss studied the distribution of primes. He tried to find a formula describing the number of primes less than a given magnitude using the so-called prime-counting function $\pi(x)$. Let $n \in \mathbb{N}$, then we define the *prime-counting function* as

$$\pi(n) = \begin{cases} 1 & if \quad n = p \\ 0 & if \quad n \neq p. \end{cases}$$

By adding all $\pi(n)$ for all n in some interval $[0, x]$, where $x \in \mathbb{R}$, we obtain a step function, $\pi(x)$, that jumps by one each time x is a prime number. The problem with this function is that it requires former knowledge about the prime numbers and does not tell us much about their actual distribution. In 1792, Gauss conjectured that the prime-counting function is *asymptotically equal* to $x/\ln(x)$ [6], which is was proven independently by Hadamard [9] and Poussin [13] in 1896. This result was a great step towards understanding the distribution of the primes, yet, for finite numbers, $x/\ln(x)$ will have a significant error. Gauss was never able to solve this problem, but one of his students was determined to, Bernhard Riemann. He found that the key to reducing the error in Gauss's estimate of the prime-counting function lies in the *non-trivial zeros* of the *zeta function*.

To understand Riemann's approach, we must go back further. In around 1730, Euler studied the convergence of infinite series, more specifically,

$$\sum_{n=1}^{\infty} \frac{1}{n^s}.$$

It can be assumed that Euler knew that the series would converge for all real $s > 1$, but it was rigorously proved by Dirichlet [1, 4]. As a function of s, this series is known as the zeta function, denoted by ζ. Euler also found the first know connection between the zeta function and the prime numbers, stated in the following lemma.

Lemma 4.2. *Let $p = 2, 3, 5, 7, 11, ...$ be all prime numbers and let $s > 1$. Then the zeta function can be expressed as*

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \frac{1}{1 - \frac{1}{p^s}},$$

which is known as the Euler product formula.

In 1859, Riemann was determined to finish Gauss's quest of understanding prime numbers, and Euler's product formula was his starting point. In his manuscript "Ueber die anzahl der primzahlen unter einer gegebenen grösse" (On the number of primes less than a given magnitude) [14], he provides an approximation of $\pi(x)$ dependent on the non-trivial zeros of ζ, with the Euler product as a starting point. Another crucial part is the analytic continuation of ζ, which is non-zero in its original domain. As Riemann continued ζ to the complex plane, the zeros of the function were revealed. To a great extent, this essay is based on the ideas presented in his manuscript, and the reader is encouraged to read it. The original and the German transcription can be found at [10], and the English translation in [4, 10].

Riemann's approximation, similar to Fourier series, is an oscillating function that approximates the prime-counting function, and the error decreases for each non-trivial zero added. This approximation formulated explicitly reads

$$\pi(x) = R(x) - \sum_{\rho} R(x^{\rho}) - \sum_{m} R(x^{-2m}), \quad R(x) = 1 + \sum_{k=1}^{\infty} \frac{(\ln(x))^k}{k!k\zeta(k+1)}, \tag{1.1}$$

where ρ is a non-trivial root of ζ. Compared to the original step function, this representation is analytic, and it does not require any former knowledge about the prime numbers.

This essay will derive a slightly modified version of Riemann's formula. After Riemann's manuscript was published, Hans von Mangoldt proved that a more elegant explicit formula could be obtained by modifying the prime-counting function [4]. This modified version of the prime-counting function is known as Chebyshev's second function and is denoted by ψ. Instead of taking a step 1 for each prime p, it takes a step of $\ln(p)$ for each prime power p^k, $k \in \mathbb{N}$. Von Mangoldt preserves the essence of the prime-counting function in the sense that π can be stated in terms of ψ. The explicit formula for ψ is known as *von Mangoldt's explicit formula* and is the main result of this essay, stated below.

Theorem 7.1. *Let $x > 1$ be a real number and let ρ be a non-trivial root to the zeta function, then ψ can be expressed as*

$$\psi(x) = x - \sum_{\rho} \frac{x^{\rho}}{\rho} + \sum_{n=1}^{\infty} \frac{x^{-2n}}{2n} - \frac{\zeta'(0)}{\zeta(0)}.$$

Compared to (1.1), von Mangoldt's explicit formula is relatively compact, yet, it provides us with the same conclusion, that we can predict the distribution of the prime numbers using the non-trivial zeros of ζ.

Nevertheless, the fact remains it is still an approximation but far better than the asymptotical equality. Furthermore, if we could find all the non-trivial zeros of ζ, this approximation would converge to ψ. Thus, we have a function that predicts the location of prime numbers without any prior knowledge of them. If the Riemann hypothesis is proven to be true, the location of the zeros would be revealed, and the hunt for the prime distribution would be history.

A flowchart of the historical development can be seen in Fig. 1.1. It is important to mention that even though the main result is von Mangoldt's explicit formula, most of the underlying theory originates from Riemann's manuscript.

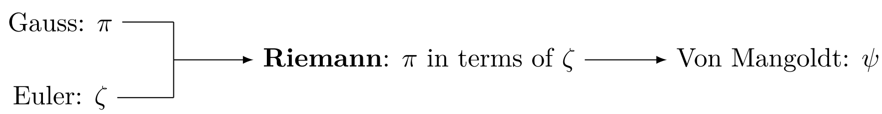

Figure 1.1 – Flowchart of the historical course of the functions, beginning with Gauss and Euler studying the prime-counting function and the zeta function separately. Later, Riemann finds that the key to reducing the error in Gauss's approximation of $\pi(x)$ lies in the non-trivial zeros of ζ. Finally, von Mangoldt constructs a more elegant formula using Riemann's result and ψ.

The primary source of this essay is the book *Riemann's Zeta Function* by H. M. Edwards [4]. Chapter one of the book is based on the manuscript written by Riemann in 1859. Edwards spends the remaining eleven chapters answering the questions arising in chapter one and proving the less rigorous parts of the manuscript, except for the Riemann hypothesis.

This essay is divided into eight sections. In Section 2, some of the essential underlying theory that is required to understand the essay is stated. The reader is expected to be well familiar with calculus and have a basic understanding of real and complex analysis. Section 3 provides a short introduction to the prime-counting functions, J, and ψ. Section 4 is devoted to Riemann's zeta function. By modifying the factorial function, ζ will be continued to the complex plane, and trivial zeros will be introduced. Section 5 introduces the xi function, denoted by ξ, which is used to find the non-trivial zeros of ζ. Furthermore, Theorem 5.5 provides a way to express ξ in terms of the non-trivial zeros. In Section 6, the main goal is to formulate Chebyshev's function in terms of ζ, which, together with Theorem 5.5, is essential for the derivation of von Mangoldt explicit formula. In Section 7, von Mangoldt's explicit formula is stated and proven, using the results of Section 5 and 6. The main goal of the last Section 8, is to show how von Mangoldt's formula improves as the number of non-trivial zeros increases.

In Fig. 1.2, a flowchart of the overall structure can be seen with the most important definitions, lemmas, and theorems.

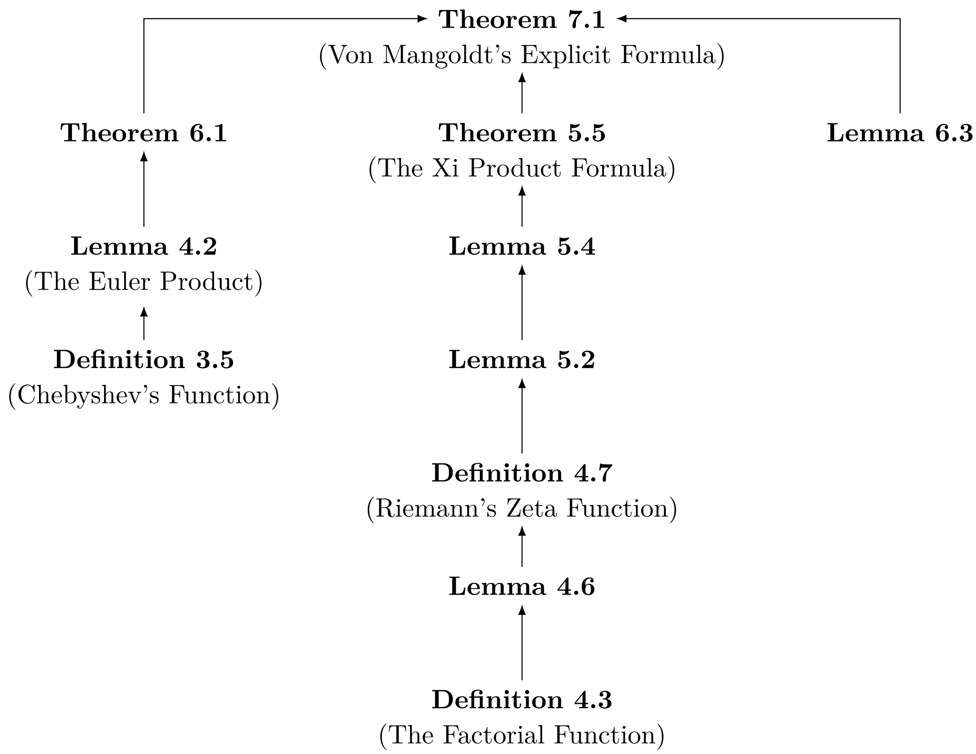

Figure 1.2 – The left side of the chart describes the construction of Theorem 6.1, which lets us state Chebyshev's function ψ in terms of ζ, using Euler's product formula. The expression for ψ is in integral form, and to evaluate the integral, we will need Lemma 6.3 (right side). In the middle, we see the course of development to arrive at the xi product formula. We are beginning with the continuation of ζ and then defining the xi function to locate the non-trivial zeros of ζ.

2. Preliminaries

This section will introduce some concepts that the reader is assumed to be familiar with. The concepts in question have their primary origin in real and complex analysis. Throughout the paper, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{C}$ will be used to denote the sets of natural numbers, integers, real numbers, and complex numbers. Each result in this section is stated without proof, but the reader can consult the indicated references if needed.

Definition 2.1. Let f be a function on the open set $\Omega \subseteq \mathbb{C}$ and consider the difference quotient

$$\frac{f(z) - f(z_0)}{z - z_0}$$

for all $z_0 \neq z \in \Omega$. If

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists, f has a *complex derivative* at z_0. Furthermore, if f has a complex derivative at all $z_0 \in \Omega$, f is said to be *holomorphic* on Ω. A function f that is holomorphic on $\mathbb{C}$ is said to be *entire*.

Definition 2.2. A point z_0 is called a *zero of order* m for the function f if f is holomorphic on some neighbourhood of z_0 and f and its first $m - 1$ derivatives vanish at z_0 but $f^{(m)}(z_0) \neq 0$.

Theorem 2.3. *Let f and g be holomorphic functions on some neighbourhood of z_0. Then f has a zero of order m at z_0 if and only if f can be written as*

$$f(z) = (z - z_0)^m g(z),$$

where $g(z_0) \neq 0$.

Proof. See e.g. [16]. □

A zero of order one is called a *simple zero*. For simplicity, we will not define poles by Laurent series. To understand this essay, it will be sufficient to know that if a function f has a *pole* in at z_0, then $f(z_0)$ is undefined. The following theorem and corollary will cover their most important properties and how they relate to zeros.

Theorem 2.4. *f has a pole of order k at z_0 if and only if f can be written as*

$$f(z) = \frac{g(z)}{(z - z_0)^k},$$

where g is holomorphic on some neighbourhood of z_0 and $g(z_0) \neq 0$.

Proof. See e.g. page 281 in [16]. □

If some function were to have a zero and a pole simultaneously, we would get what is called a *removable singularity*.

Corollary 2.5. *If f has a zero of order m and a pole of order k at z_0, f can be written as*

$$f(z) = \frac{(z-z_0)^m g(z)}{(z-z_0)^k},$$

where g is holomorphic on some neighbourhood of z_0 and $g(z_0) \neq 0$. Furthermore,

(i) if $m = k$, f has a removable singularity,

(ii) if $m > k$, f has a removable singularity and a zero of order $m - k$,

(iii) if $m < k$, f has a removable singularity and a pole of order $k - m$.

The following definition describes a type function that is holomorphic almost everywhere, except for a finite number of discrete points.

Definition 2.6. A *meromorphic function* f on Ω with singular set S is a function $f : \Omega \setminus S \to \mathbb{C}$ such that

(i) the set S is closed in Ω and discrete,

(ii) the function f is holomorphic on $\Omega \setminus S$,

(iii) for each $z_0 \in S$ and $r > 0$ such that $D(z_0, r) \subseteq \Omega$ and $S \cap D(z_0, r) = \{z_0\}$, the function $f|_{D(z_0,r)\setminus\{z_0\}}$ has a finite order pole at z_0.

Another important concept concerning complex functions is *multiple-valued functions*, which are commonly obtain by taking the inverse of a single-valued function that is not bijective. For instance, let $z, w \in \mathbb{C}$ and $k \in \mathbb{Z}$, then we have that $z = e^w = e^{w+2\pi ki}$, resulting in its inverse, $\log(z)$, being multiple-valued. Before we define the complex logarithm note that, $\ln|z|$ is the ordinary real-valued logarithm defined for all $|z| \neq 0$, and $\arg(z)$ is the complex argument of z.

Definition 2.7. Let $z \in \mathbb{C} \setminus \{0\}$. Then we define the complex logarithm as

$$\log(z) \coloneqq \ln|z| + i\arg(z).$$

By definition, $\log(z)$ will have infinitely many values at any given z since $\arg(z)$ is 2π-periodic. It is often convenient to divide it into single-valued *branches* only considering some argument $(\theta, \theta + 2\pi]$. The line form zero to infinity with $\arg(z) = \theta$, is called the *branch cut*, and will be discontinuous. Usually the negative real axis is chosen as the branch cut of $\log(z)$. The branch cut is important to consider during integration since it can not be integrated along or across. With the logarithm defined we may continue to the following definition.

Definition 2.8. Let $\alpha \in \mathbb{C}$ be a constant and $z \neq 0$, we define

$$z^\alpha \coloneqq e^{\alpha \log(z)}.$$

We will now turn our attention to integrals and convergence for the remaining part of this section. We will define the *Stieltje integral* and the *complex line integral* with their most relevant properties. We will also define *uniform convergence* and a few properties related to sums and integrals. The reader is assumed to understand the concepts of the Riemann integral, supremum, and infimum. If not, they are all defined in [15].

Definition 2.9. Let $[a, b]$ be a given intervall. By a *partition* P of $[a, b]$ we mean a finite set of points $x_0, x_1, ..., x_n$, where

$$a = x_0 \leq x_1 \leq ... \leq x_{n-1} \leq x_n = b.$$

We write

$$\Delta x_i = x_i - x_{i-1} \quad (i = 1, ..., n).$$

Definition 2.10. Let α be a monotonically increasing function on the interval $[a, b]$. Corresponding to each partition P of $[a, b]$, we write

$$\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1}).$$

where $\Delta\alpha_i \geq 0$. For any real function f which is bounded on $[a, b]$ we put

$$U(P, f, \alpha) = \sum_{i=1}^{n} M_i \Delta\alpha_i, \qquad L(P, f, \alpha) = \sum_{i=1}^{n} m_i \Delta\alpha_i$$

where $M_i = \sup f(x)$ and $m_i = \inf f(x)$ are corresponding to each partition P of $[a, b]$ and $(x_{i-1} \leq x \leq x_i)$. We define

$$\overline{\int_a^b} f(x) d\alpha = \inf U(P, f, \alpha), \qquad \underline{\int_a^b} f(x) d\alpha = \sup L(P, f, \alpha)$$

where inf and sup are taken over all partitions P of $[a, b]$. If the upper and lower Riemann integrals are equal we denote their common value by

$$\int_a^b f(x) d\alpha(x)$$

and is known as the *Riemann-Stieltje integral* or the *Stieltje integral* of f with respect to α over $[a, b]$.

Further reading about the proof of the existence of Stieltjes integral can be found in [11]. In many cases, we can use a more convenient form if the integrand and α' are Riemann integrable. In other words, the upper and lower Riemann integrals are equal. If a function f is Riemann integrable, we use the notation $f \in \mathscr{R}$.

Theorem 2.11. *Assume α increases monotonically and $\alpha' \in \mathscr{R}$ on $[a, b]$. Let f be a bounded function on $[a, b]$. Then $f \in \mathscr{R}$ if and only if $f\alpha' \in \mathscr{R}$. In this case the following equality holds*

$$\int_a^b f(x) d\alpha(x) = \int_a^b f(x)\alpha'(x) dx. \tag{2.1}$$

Proof. See e.g. page 131 in [15]. $\square$

From (2.1), it is clear that the Riemann integral is a special case of Stieltje's integral. We obtain the standard Riemann integral by setting $\alpha(x) = x$. The complex line integral is a special case as well, which we will see later. As with the Riemann integral, Stieltje's integral, can be integrated by parts, which brings us to the following theorem.

Theorem 2.12. *Suppose α increases monotonically on $[a, b]$, and f, f' are continuous on $[a, b]$, then*

$$\int_a^b f(x)d\alpha(x) = f(x)\alpha(x)\Big|_a^b - \int_a^b \alpha(x)f'(x)dx$$

Proof. See e.g. page 141 [15]. □

The following two definitions states the criteria for uniform convergence of a sequence of functions and a series of functions.

Definition 2.13. Let $\Omega \subseteq \mathbb{C}$ be an open set. We say that a sequence of functions $\{f_n\}$, $n = 1, 2, 3, ...$ converges *uniformly* on Ω if to a function f if for every $\epsilon > 0$ there is an integer N such that $n \geq N$ implies

$$|f_n(z) - f(z)| \leq \epsilon$$

for all $z \in \Omega$.

Definition 2.14. Let $\Omega \subseteq \mathbb{C}$ be an open set. A series of functions $\sum_{k=1}^{\infty} f_k(z)$ converges *uniformly* on E to the function $f(z)$ if for every $\epsilon > 0$ there is an integer N such that $n \geq N$ implies

$$\left| \sum_{k=1}^{n} f_k(z) - f(z) \right| < \epsilon$$

for all $z \in \Omega$.

In practice, Definition 2.14 is not a very useful way of determining the convergence of a series, unlike the following theorem, known as *Weierstrass M-test*.

Theorem 2.15. *Suppose $\{f_n\}$ is a sequence of functions defined on a set $\Omega \subseteq \mathbb{C}$, and suppose*

$$|f_n(z)| \leq M_n \quad (z \in E, n = 1, 2, 3...).$$

Then $\sum_{n=1}^{\infty} f_n$ converges uniformly on Ω if $\sum_{n=1}^{\infty} M_n$ converges.

Proof. See e.g. page 148 in [15]. □

Another important property following from uniform convergence is the possibility to move a limit inside an integral.

Theorem 2.16. *Let α be monotonically increasineg on $[a, b]$. Suppose $\{f_n\}$ is a sequence of continuous functions defined on $[a, b]$, and suppose f_n converge uniformly to a limit function f. Then f is continuous and*

$$\lim_{n\to\infty} \int_a^b f_n(x)d\alpha = \int_a^b \lim_{n\to\infty} f_n(x)d\alpha = \int_a^b f(x)d\alpha.$$

Proof. See e.g page 151 in [15]. □

Considering integrals in the area of complex analysis, one of the fundamental concepts is integration along curves in the complex plane. As an extension of the real line, the complex plane unfolds the possibility of integrating along various paths, not being restricted to the line of the real numbers. Consequently, we need to define what kind of curve we are allowed to integrate along.

Definition 2.17. A *contour* Γ is either a single point or a finite sequence of directed smooth curves $(\gamma_1, \gamma_2, ..., \gamma_n)$ where the endpoint of γ_k coincides with the initial point of γ_{k+1} for $k = 1, 2, ..., n-1$.

Theorem 2.18. *Let f be continuous on the directed smooth curve γ. Then if $z(t)$, $a \leq t \leq b$, is a parametrization of γ consistent with its direction, we have*

$$\int_\gamma f(z)dz = \int_a^b f(z(t))z'(t)dt.$$

As mentioned before, this is a special case of Stieltjes integral and by comparison with Theorem 2.11, the resemblance is quite apparent. To clarify, we rewrite the integral as follows

$$\int_a^b f(z(t))z'(t)dt = \int_a^b (f \circ z)(t)z'(t)dt = \int_a^b (f \circ z)(t)dz(t).$$

If we instead of a curve integrate over a contour, we integrate each curve on its own and add them together.

Definition 2.19. Let Γ be a contour and let f be continuous on Γ. Then the *contour integral* along Γ is defined by

$$\int_\Gamma f(z)dz := \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz + ... + \int_{\gamma_n} f(z)dz.$$

In some cases it will be sufficient to approximate the value of an integral rather than an exact evaluation. If we consider a contour integral it seems natural that its value must be less or equal to the maximum of the function times the length of the contour.

Theorem 2.20. *Suppose f is continuous and bounded on a contour Γ such that $|f(z)| \leq M$, for some real $M \geq 0$. Then*

$$\left| \int_\Gamma f(z)dz \right| \leq M\ell(\Gamma)$$

where $\ell(\Gamma)$ *is the length of* Γ*. In particular, we have*

$$\left| \int_\Gamma f(z)dz \right| \leq \max_z |f(z)|\ell(\Gamma).$$

Proof. See e.g [16] or page 33 in [7]. □

The remaining definitions and theorem is concerning *Mellin inversion*, which will be used to transform a real-valued integral to a complex line integral, provided a certain relation between the integrand and the function defined by the integral.

Definition 2.21. Let f be a real-valued function on $x \in [0, \infty)$ and let $s \in \mathbb{C}$. The *Mellin transformation* is the operation mapping of the function f, into the complex function ϕ, by the relation

$$\phi(s) = \int_0^\infty f(x)x^{s-1}dx.$$

Definition 2.22. Let f be piecewise continuous on $[0, \infty)$ and assume f satisfies

$$f(x) = O(x^a) \quad (x \to 0) \quad \text{and} \quad f(x) = O(x^b) \quad (x \to \infty).$$

Then $\phi(s)$ is defined in the strip $-a < \operatorname{Re} s < -b$, which is called the *fundamental strip*.

Theorem 2.23. *Let* f *and* ϕ *be defined as above. Then for any real* c *inside the fundamental strip the following equality holds*

$$f(x) = \lim_{R\to\infty} \frac{1}{2\pi i} \int_{c-iR}^{c+iR} \phi(s)x^{-s}ds$$

This is the *Mellin inverse* of ϕ.

Proof. See e.g. [5] or [2]. □

3. Prime-counting Functions

This section introduces the essential prime counting functions. The goal is to define Chebyshev's function, which is later approximated by von Mangoldt's explicit formula in Section 7. We will also establish the relation between Chebyshev's function and Riemann's prime counting function. First and foremost, we define the function $J_0(n)$ for counting prime powers.

Definition 3.1. Let n and k be integers and p be any prime number, then

$$J_0(n) = \begin{cases} 1/k & if \quad n = p^k \\ 0 & if \quad n \neq p^k \end{cases} \tag{3.1}$$

If we where to sum all $J_0(n)$ in some interval $n \in [0, x]$, $x \in \mathbb{R}$ we would get a value describing the amount of primes and powers of prime from 0 to x.

Definition 3.2. Let n be an integer and x an real number. Then for all $x \geq 0$ we define Riemann's prime counting function as

$$J(x) = \frac{1}{2}\left(\sum_{n<x} J_0(n) + \sum_{n\leq x} J_0(n)\right)$$

$J(x)$ is a step-function that jumps by 1 for every prime number and $\frac{1}{k}$ for every prime with power k. The function is defined to be halfway between the steps for every prime power and is constant in every interval between.

To define Chebyshev's function, we need to define a function that assigns $\ln(p)$ to any $n = p^k$. This function is known as von Mangoldt's function.

Definition 3.3. Let n and k be integers and p be any prime number, then

$$\Lambda(n) = \begin{cases} \ln p & if \quad n = p^k \\ 0 & if \quad n \neq p^k \end{cases}$$

Lemma 3.4. $\Lambda(n) = \ln(n) J_0(n)$ *for all* $n \in \mathbb{N}$.

Proof.

$$\ln(n) J_0(n) = \begin{cases} \frac{\ln(n)}{k} & if \quad n = p^k \\ 0 & if \quad n \neq p^k \end{cases}$$

and since

$$n = p^k \implies \frac{\ln\left(p^k\right)}{k} = \frac{k \ln(p)}{k} = \ln(p)$$

we can see that $\ln(n) J_0(n) = \Lambda(n)$. □

We have shown that von Mangoldt's function $\Lambda(n)$ is equal to $J_0(n)$ scaled by the natural logarithm of n, and can now define Chebyshev's function.

Definition 3.5. Let n be an integer and x an real number. Then for all $x \geq 0$ we define Chebyshev's psi function as

$$\psi(x) = \frac{1}{2}\left(\sum_{n<x} \Lambda(n) + \sum_{n \leq x} \Lambda(n)\right)$$

By the same logic as of Lemma 3.4 $\psi(x)$ is related to $J(x)$ by a factor of $\ln(x)$. With this modified version of the prime-counting function, we can, as with the original, look at its behaviour as x goes to infinity, leading us to the following lemma.

Lemma 3.6. *Let $x \geq 0$, then $\psi(x)$ is asymptotically equal to x.*

Proof. See page 72 in [4]. $\square$

4. Riemann's Zeta Function

As the introduction suggests, von Mangoldt's explicit formula depends on the non-trivial zeros of Riemann's zeta function. This section introduces Riemann's zeta function and its most relevant properties. To do so, we begin with the original definition and the so-called Euler product formula. Then the zeta function will be analytically continued to the whole complex plane by modifying the factorial function.

Definition 4.1. For all real $s > 1$ we define the *zeta function* as the convergent series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

An important property of this function is that it can be expressed in terms of prime numbers. This is the first indication of its importance in approximating the distribution of prime numbers.

Lemma 4.2. *Let $p = 2, 3, 5, 7, 11, ...$ be all prime numbers and let $s > 1$. Then the zeta function can be expressed as*

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \frac{1}{1 - \frac{1}{p^s}},$$

which is known as the Euler product formula.

Proof. The absolute convergence of $\sum_{n=1}^{\infty} \frac{1}{n^s}$ for $\operatorname{Re} s > 1$, implies the convergence of

$$\prod_p \left(1 - \frac{1}{p^s}\right).$$

Let s be fixed, $\operatorname{Re} s > 1$, $\epsilon > 0$, and choose some large N such that

$$\sum_{n=N+1}^{\infty} \left|\frac{1}{n^s}\right| < \epsilon.$$

Since

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + ...$$

then

$$\zeta(s)\left(1 - \frac{1}{2^s}\right) = \left(1 - \frac{1}{2^s}\right) + \left(\frac{1}{2^s} - \frac{1}{4^s}\right) + \left(\frac{1}{3^s} - \frac{1}{6^s}\right) + ... = 1 + \frac{1}{3^s} + \frac{1}{5^s} + ... \ .$$

By multiplying with $1 - \frac{1}{2^s}$, we remove all fractions containing multiples of two. If we continue this process for each prime number, we will, step by step, remove all fractions containing a multiple of any prime number. Thus,

$$\zeta(s)\left(1 - \frac{1}{2^s}\right)...\left(1 - \frac{1}{(p_{N-1})^s}\right)\left(1 - \frac{1}{(p_N)^s}\right) = 1 + \frac{1}{(p_{N+1})^s} + ...$$

Form our choice N, if $n \geq N$, it follows that

$$\left| \zeta(s) \prod_{k=1}^{n} \Big(1 - \frac{1}{(p_k)^s}\Big) - 1 \right| < \epsilon.$$

Hence, $\zeta(s) \prod_p \left(1 - \frac{1}{p^s}\right) = 1$, and thus we have arrived at our desired expression

$$\zeta(s) = \prod_p \frac{1}{1 - \frac{1}{p^s}}.$$

□

Lemma 4.2 does not reveal anything about the actual distribution of primes. We seek a function that generates the prime numbers, but this expression requires us to know them already.

As we want to find the zeros of ζ, we now encounter some difficulties. ζ is non-zero for all $s > 1$, thus we have to look elsewhere. The goal is to continue the function so that it is defined on the whole complex plane, which requires some work, but both sides of Euler's product formula will converge for all complex numbers s where $\operatorname{Re} s > 1$, without modification. The problem of ζ being non-zero, unfortunately, remains. The remaining part of this section aims to continue the zeta function to the whole complex plane. The consequences of this continuation will be of great importance, and to this day, it holds one of the greatest mysteries in modern mathematics. The continuation will be performed by modifying the following function.

Definition 4.3. We define the *factorial function* as

$$\Pi(s) = \int_0^\infty \mathrm{e}^{-x} x^s \, dx$$

which hold for all $s \in \mathbb{C} \setminus \{-n\}$, where $n \in \mathbb{N}$.

As the name suggests, $\Pi(s)$ is an extension of $n!$, such that $\Pi(n) = n!$ for $n \in \mathbb{N}$. In the continuation of ζ we will use Definition 4.3 but we will also need some of the following properties.

Lemma 4.4. *The factorial function has the following properties*

$$\Pi(s) = \prod_{n=1}^{\infty} \frac{n^{1-s}(n+1)^s}{s+n} = \prod_{n=1}^{\infty} \Big(1 + \frac{s}{n}\Big)^{-1} \Big(1 + \frac{1}{n}\Big)^s \tag{4.1}$$

$$\Pi(s) = s\Pi(s-1) \tag{4.2}$$

$$\frac{\pi s}{\Pi(s)\Pi(-s)} = \sin(\pi s) \tag{4.3}$$

$$\Pi(s) = 2^s \Pi\Big(\frac{s}{2}\Big) \Pi\Big(\frac{s-1}{2}\Big) \pi^{-1/2} \tag{4.4}$$

which hold for all $s \in \mathbb{C} \setminus \{-n\}$, *where* $n \in \mathbb{N}$.

The proof of these properties will be left as an exercise for the reader. If needed, the reader can consult [3]. Equation (4.1) and (4.4) will not be used in the continuation but is used later in Section 5 and 7.

Lemma 4.5. *The factorial function has a pole of order 1 for all $s = -k$ where $k = 1, 2, 3, \ldots$*

Proof. By equation (4.1) and $s = -k$ we get

$$\Pi(-k) = \prod_{n=1}^{\infty} \frac{n^{1+k}(n+1)^{-k}}{-k+n} = \prod_{n=1}^{\infty} \frac{n^{k+1}}{(n-k)(n+1)^k}.$$

Both n^{k+1} and $(n+1)^k$ are non zero for all n and k but the term but the $n - k = 0$ whenever $n = k$, therefore any $k = 1, 2, 3, \ldots$ will result in a first order pole. $\square$

Before moving on to the actual continuation of ζ the following lemma is required.

Lemma 4.6. *Let s and z be complex and let Γ be the contour displayed in Fig. 4.1. Then the following equality holds*

$$\int_\Gamma \frac{(-z)^s}{e^z - 1} \frac{dz}{z} = 2i \sin(\pi s) \int_0^\infty \frac{t^{s-1}}{e^t - 1} dt.$$

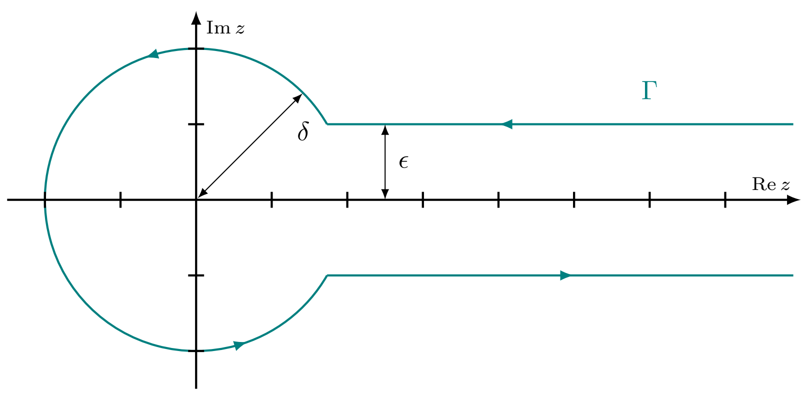

Figure 4.1 – The contour Γ, a straight line from infinity, around the origin in a circle of radius $\delta > 0$, and back to infinity, where the straight lines are shifted $\epsilon > 0$ above/below the positive x-axis.

Proof. To evaluate this contour integral it is important to notice that $(-z)^s = e^{s \log(-z)}$. By definition $\log(z)$ is not defined on the real negative axis due to the branch cut, hence $(-z)^s$ will not be defined on the positive real axis. Therefore the path of integration must be taken slightly above and below the x-axis. First, let us rewrite

$$\int_\Gamma \frac{(-z)^s}{e^z - 1} \frac{dz}{z} = -\int_\Gamma \frac{(-z)^{s-1}}{e^z - 1} dz = -\int_\Gamma \frac{(-z)^{s-1} e^{-z}}{1 - e^{-z}} dz \tag{4.5}$$

to make the parameterization and evaluation more convenient. We will split the integral into three separate curve integrals in the most natural way, the two linear segments and the circle sector. For the linear parts we will use $z = t \pm i\epsilon$, and for the circle sector $z = \delta e^{i\theta}$, giving us

$$\int_\Gamma \frac{(-z)^{s-1}e^{-z}}{1-e^{-z}}dz = \int_\infty^{\epsilon_0} \frac{(-(t+i\epsilon))^{s-1}e^{-(t+i\epsilon)}}{1-e^{-(t+i\epsilon)}}dt + \int_{\delta_0}^{2\pi+\delta_0} \frac{(-\delta e^{i\theta})^{s-1}e^{-\delta e^{i\theta}}}{1-e^{-\delta e^{i\theta}}} i\delta e^{i\theta}d\theta$$
$$+ \int_{\overline{\epsilon}_0}^{\infty} \frac{(-(t-i\epsilon))^{s-1}e^{-(t-i\epsilon)}}{1-e^{-(t-i\epsilon)}}dt \equiv \mathrm{I} + \mathrm{II} + \mathrm{III}.$$

Here ϵ_0 and $\overline{\epsilon}_0$ represents the point where the curves meet, and δ_0 the radian measure of the initial point of the circle sector. Starting with II we can see that for small δ we have

$$|1-e^{-\delta e^{i\theta}}| \geq |1-e^{-\delta}| \geq \tfrac{\delta}{2}.$$

Thus,

$$\begin{aligned} |\mathrm{II}| &\leq 2\pi \max_\theta \frac{\left|(-\delta e^{i\theta})^{s-1}\right|\left|e^{-\delta e^{i\theta}}\right|}{\delta/2}\delta \\ &\leq 4\pi\delta^{\operatorname{Re} s-1}e^{-\theta \operatorname{Im} s}e^{\delta}. \end{aligned} \tag{4.6}$$

If we let $\delta \to 0$, (4.6) goes to zeros, thus, II goes to zero. Now we rewrite I and III as

$$\mathrm{I} + \mathrm{III} = \int_\infty^{\epsilon_0} \frac{e^{(s-1)\log(-t-i\epsilon)}e^{-t-i\epsilon}}{1-e^{-t-i\epsilon}}dt + \int_{\overline{\epsilon}_0}^{\infty} \frac{e^{(s-1)\log(-t+i\epsilon)}e^{-t+i\epsilon}}{1-e^{-t+i\epsilon}}dt$$
$$= \int_\infty^{\epsilon_0} \frac{e^{(s-1)(\log\left(\sqrt{t^2+\epsilon^2}\right)+i(-\pi+\epsilon'))}e^{-t-i\epsilon}}{1-e^{-t-i\epsilon}}dt + \int_{\overline{\epsilon}_0}^{\infty} \frac{e^{(s-1)(\log\left(\sqrt{t^2+\epsilon^2}\right)+i(\pi-\epsilon''))}e^{-t+i\epsilon}}{1-e^{-t+i\epsilon}}dt$$

where ϵ' and ϵ'' are chosen so that $(-\pi+\epsilon')$ and $(\pi-\epsilon'')$ are the initial and terminal argument of the linear segments. The value of the integrals is independent of ϵ as long as ϵ is sufficiently small. By uniform convergence we may let $\epsilon \to 0^+$, giving us

$$\begin{aligned} &= \int_\infty^{\epsilon_0} \frac{e^{(s-1)(\log(t)-i\pi)}e^{-t}}{1-e^{-t}}dt + \int_{\overline{\epsilon}_0}^{\infty} \frac{e^{(s-1)(\log(t)+i\pi)}e^{-t}}{1-e^{-t}}dt \\ &= -\int_\infty^{\epsilon_0} \frac{t^{s-1}e^{-i\pi s}e^{-t}}{1-e^{-t}}dt - \int_{\overline{\epsilon}_0}^{\infty} \frac{t^{s-1}e^{i\pi s}e^{-t}}{1-e^{-t}}dt \\ &= e^{-i\pi s}\int_{\epsilon_0}^{\infty} \frac{t^{s-1}e^{-t}}{1-e^{-t}}dt - e^{i\pi s}\int_{\overline{\epsilon}_0}^{\infty} \frac{t^{s-1}e^{-t}}{1-e^{-t}}dt. \end{aligned}$$

The exponential terms will give us $-2i\sin(\pi s)$, and since δ and ϵ has gone to zero, so will ϵ_0, thus

$$= -2i\sin(\pi s)\int_0^\infty \frac{t^{s-1}e^{-t}}{1-e^{-t}}dt.$$

By (4.5) we have that

$$\int_\Gamma \frac{(-z)^s}{e^z-1}\frac{dz}{z} = 2i\sin(\pi s)\int_0^\infty \frac{t^{s-1}}{e^t-1}dt.$$

$\square$

Equipped with Lemma 4.6 we can begin the continuation of ζ. We begin with the integral

$$\Pi(s-1) = \int_0^\infty \mathrm{e}^{-x} x^{s-1} dx$$

and then by substitution of x to nx

$$\Pi(s-1) = \int_0^\infty \mathrm{e}^{-nx} (nx)^{s-1} n dx = n^s \int_0^\infty \mathrm{e}^{-nx} x^{s-1} dx.$$

Thus,

$$\int_0^\infty \mathrm{e}^{-nx} x^{s-1} dx = \frac{\Pi(s-1)}{n^s}$$

which is valid for $s > 0$ and $n \in \mathbb{N}$. Next step is to sum both sides over n and using the fact that $\sum_{n=1}^\infty (\frac{1}{r})^n = \frac{1}{r-1}$ if $r > 1$. Since the exponential term $e^{-nx} = (\frac{1}{e^x})^n$ is the only term dependent on n we evaluate the sum

$$\sum_{n=1}^\infty \left(\frac{1}{e^x}\right)^n = \frac{1}{e^x - 1}$$

for $x > 0$. Since the sum converges on the interval of the integral the evaluation is justified, which gives us

$$\int_0^\infty \frac{x^{s-1}}{e^x - 1} dx = \Pi(s-1) \sum_{n=1}^\infty \frac{1}{n^s}.$$

By Lemma 4.6 we have

$$\sum_{n=1}^\infty \frac{1}{n^s} = \frac{1}{\Pi(s-1) 2i \sin(\pi s)} \int_\Gamma \frac{(-z)^s}{e^z - 1} \frac{dz}{z}$$

and using (4.2) and (4.3) of Lemma 4.4 the factorial function and $\sin(\pi s)$ can be rewritten so that

$$\sum_{n=1}^\infty \frac{1}{n^s} = \frac{\Pi(-s)}{2\pi i} \int_\Gamma \frac{(-z)^s}{e^z - 1} \frac{dz}{z}.$$

Definition 4.7. Let s be a complex number and let $\Pi(s)$ be the factorial function. Then we define the *Riemann zeta function* as

$$\zeta(s) = \frac{\Pi(-s)}{2\pi i} \int_\Gamma \frac{(-z)^s}{e^z - 1} \frac{dz}{z} \tag{4.7}$$

which is valid for all s in $\mathbb{C} \setminus \{1\}$.

From now on, all use of $\zeta(s)$ will be Riemann's zeta function. Therefore, it is crucial to notice that Riemann's zeta function is not strictly equal to the zeta function. They are identical on the domain $\operatorname{Re} s > 1$, but Riemann's zeta function is its own function for the remaining part of the complex plane. The new function is, as its precursor, analytic, and its range agrees with the original where it is defined. A common misconception is putting these equal and evaluating $\zeta(-1) = -\frac{1}{12}$ as $\sum_{n=1}^\infty n = 1+2+3+...$, but assigning this value to the diverging sum of the natural numbers is incorrect.

Even though the function (4.7) satisfies the goal of extending the zeta function, it could be more convenient. By evaluating the integral, we obtain the *functional equation* of ζ.

Lemma 4.8. *Let s be a complex number and let $\Pi(s)$ be the factorial function. Then*

$$\zeta(s) = \Pi(-s)(2\pi)^{s-1} 2\sin\left(\frac{s\pi}{2}\right)\zeta(1-s), \tag{4.8}$$

for all $s \in \mathbb{C} \setminus \{1\}$. This is the functional equation of $\zeta(s)$.

Proof. See page 12 in [4]. $\square$

Lemma 4.9. *Let $n \in \mathbb{N}$ and $\zeta(s)$ be defined as in (4.7). Then $\zeta(-2n) = 0$ for all $n \in \mathbb{N}$. We refer to these zeros as the trivial zeros of the zeta function.*

Proof. By observation of (4.8) the only possible zero terms are $\sin\left(\frac{s\pi}{2}\right)$ and $\zeta(1-s)$. The first term, $\sin\left(\frac{s\pi}{2}\right) = 0$ for all $s = 2k$ where $k \in \mathbb{Z}$ while the second term is, by definition, non zero for all $s < 0$. Therefore all zeros in the half-plane $\operatorname{Re} s < 0$ originate from the sin-term, i.e $s = -2n$. For $\operatorname{Re} s > 0$, all zeros of sin will correspond with the first order poles of the factorial function by Lemma 4.5, thus being removable. In conclusion, $\zeta(-2n) = 0$ are the only trivial zeros of ζ.

$\square$

Since $\zeta(s)$ is non-zero in its original domain $\operatorname{Re} s > 1$, the term $\zeta(1-s)$ must be non-zero for all $\operatorname{Re} s < 0$. By the relation between $\zeta(s)$ and $\zeta(1-s)$ in (4.8), there is a region from $\operatorname{Re} s = 0$ to $\operatorname{Re} s = 1$ where neither of the terms has to converge to a non zero value. This fact opens up a possibility for other zeros, which will be examined further in the following section.

5. The Xi Function

In this section, the xi function is defined and used to find the non-trivial zeros of ζ. Furthermore, one of the most important theorems of this essay, namely Theorem 5.5, is derived. Theorem 5.5 lets us express the xi function in terms of the non-trivial zeros of ζ, which is essential for deriving von Mangoldt's explicit formula.

Definition 5.1. Let s be a complex number. Then we define the *xi function* as

$$\xi(s) = \Pi(s/2)(s-1)\pi^{-s/2}\zeta(s), \qquad \forall s \in \mathbb{C}. \tag{5.1}$$

Unlike the meromorphic functions of which it is composed, the ξ is an entire function. All poles of the factorial function will correspond with the trivial zeros of ζ and the pole $\zeta(1)$ with $s-1$.

Lemma 5.2. *Let s be a complex number. Then*

$$\xi(s) = \xi(1-s),$$

for all $s \in \mathbb{C}$. This is the functional equation of ξ.

This follows from inserting $1-s$ in Definition 5.1 and applying equation (4.3) and (4.4), which will give us $\xi(s)$.

As suggested before, the zeta function potentially has more zeros. To examine this further, we will look at the possibility of zeros such that $\xi(\rho) = 0$. By Definition 5.1, ξ is non-zero for all $\operatorname{Re} s > 1$ since ζ will converge to a non-zero value in that domain. The right-hand side of (5.1) has a zero at $s = 1$ but is removable due to the pole of $\zeta(1)$. Hence all zeros of ξ must lie in the half plane $\operatorname{Re} s < 1$.

By Lemma 5.2, we have that $\xi(\rho) = 0$ if and only if $\xi(1-\rho) = 0$, which implies that $\operatorname{Re}\rho > 0$ since the contrary would imply $1-\rho$ to have a real part greater than 1.

Lemma 5.3. *All zeros $\xi(\rho) = 0$ lie in the strip $0 \le \operatorname{Re} s \le 1$. This strip is known as the critical strip.*

Note that by Definition 5.1 all zeros ρ must be zeros of ζ since all other terms are non zero on in the strip.

Lemma 5.4. *The zeros $\xi(\rho) = 0$ are the non-trivial zeros of ζ.*

Both the non-trivial zeros ρ and the trivial zeros are visualized in Fig. 5.1. One can see that all non-trivial zeros lie within the critical strip, but all confirmed zeros lie on the so-called *critical line*. It is known that infinitely many zeros lie on the critical line, and so far, no zeros have been found outside it. The famous *Riemann hypothesis* conjectures that all zeros lie on the critical line, but it is not yet proven.

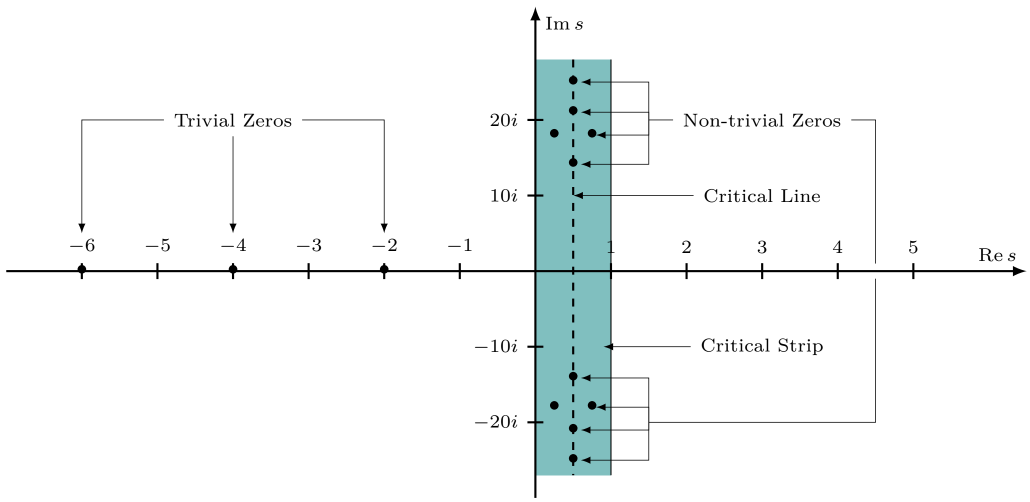

Figure 5.1 – The zeros $\zeta(s) = 0$, visualized in the complex plane. The trivial zeros are located on the negative even integers, and the non-trivial zeros inside the critical strip. The zeros on the critical line are actual known zeros. The zeros outside the critical line are not real zeros, and they visualize the other, theoretically possible zeros that are allowed by Lemma 5.2.

To summarize, we have found that the non-trivial zeros of ζ are the zeros of ξ. The following theorem provides a formula for ξ in terms of these zeros. This formula is of great importance since it will let us express Chebyshev's function in terms of the non-trivial zeros in Section 7.

Theorem 5.5. *Let s be a complex number and let ρ be a non-trival zero of ζ. Then $\xi(s)$ can be expanded as the infinite product*

$$\xi(s) = \xi(0) \prod_{\rho} \left(1 - \frac{s}{\rho}\right)$$

for all $s \in \mathbb{C}$.

Proof. The complete proof of Theorem 5.5 is, more or less, an essay on its own; see Chapter 2 in [4] or [8]. The reader is encouraged to read the complete proof if interested. However, for the purpose of this essay, the following derivation will be sufficient to understand its relation to the non-trivial zeros, which is the goal. First, notice that $\log \xi(s)$ has a logarithmic singularity for all $s = \rho$,

$$s = \rho \iff 1 - \frac{s}{\rho} = 0.$$

This indicates that the function defined by

$$\sum_{\rho} \log\left(1 - \frac{s}{\rho}\right),$$

if convergent, will behave in a similar way as $\log \xi(s)$ near infinity and will only differ by an additive constant. Hence,

$$\log \xi(s) = c + \sum_{\rho} \log\left(1 - \frac{s}{\rho}\right).$$

With $s = 0$, the sum is zero, thus $\log \xi(0) = c$. Thus,

$$\log \xi(s) = \log \xi(0) + \sum_{\rho} \log\left(1 - \frac{s}{\rho}\right) \iff \xi(s) = \xi(0) \prod_{\rho} \left(1 - \frac{s}{\rho}\right).$$

□

We now have all needed results concerning the xi function. Before moving on we will need this last theorem justify the convergence of a series in Section 7.

Theorem 5.6. *For any given $\epsilon > 0$ the series*

$$\sum_{\rho} \frac{1}{|\rho - \frac{1}{2}|^{1+\epsilon}}$$

converges, where ρ ranges over all roots ρ of $\xi(\rho) = 0$.

Proof. See page 12 in [4]. □

6. Chebyshev's Function in Terms of Zeta

This section aims express Chebyshev's function in terms of the zeta function. The integral formula for Chebychev's function stated in Theorem 6.1 is, together with the product formula of Theorem 5.5, the most central part in von Mangoldt's derivation of the explicit formula. Furthermore, Lemma 6.3 is stated and proven. It will be used to evaluate Chebyshev's integral formula in the following section.

Theorem 6.1. *Let s be a complex number and let a and x be real numbers. Then Chebyshev's function can be expressed in terms of ζ as follows*

$$\psi(x) = \lim_{R\to\infty} \frac{1}{2\pi i} \int_{a-iR}^{a+iR} \left[-\frac{\zeta'(s)}{\zeta(s)} \right] x^s \frac{ds}{s}$$

for $x > 1$, $\operatorname{Re} s > 1$ and $a > 1$.

Proof. We begin with the Euler product of Lemma 4.2 and our goal is to modify it in a way that relates it to Chebyshvs's function.

$$\zeta(s) = \prod_p \frac{1}{1 - \frac{1}{p^s}} \qquad (\operatorname{Re} s > 1).$$

By taking the logarithm on both sides we get

$$\log \zeta(s) = \log \left(\prod_p \frac{1}{1 - \frac{1}{p^s}} \right) = \sum_p \log \left(\frac{1}{1 - \frac{1}{p^s}} \right) = \sum_p -\log\bigl(1 - p^{-s}\bigr)$$

and using series representation of $\log(1-x) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - ...$ gives us

$$-\log\bigl(1 - p^{-s}\bigr) = p^{-s} + \frac{1}{2}p^{-2s} + \frac{1}{3}p^{-3s} + ... = \sum_n \left(\frac{1}{n}\right) p^{-ns}$$

hence,

$$\log \zeta(s) = \sum_p \left[\sum_n \left(\frac{1}{n}\right) p^{-ns} \right] \qquad (\operatorname{Re} s > 1).$$

The double series is absolutely convergent which implies that the order of summation does not matter.

$$\log \zeta(s) = \sum_p \sum_n \left(\frac{1}{n}\right) p^{-ns} \qquad (\operatorname{Re} s > 1)$$

then it follows form Stieltje integration that

$$\log \zeta(s) = \int_0^\infty x^{-s} dJ(x). \qquad (\operatorname{Re} s > 1)$$

Since the logarithm of ζ will have a logarithmic singularity at the roots of ζ, it will behave poorly outside the plane $\operatorname{Re} s > 1$. In this proof, it will be sufficient to stay in the $\operatorname{Re} s > 1$ plane, but later, it will be useful to have a more well-behaved function on the whole complex plane. We take the derivative with respect to s on both sides. $\frac{d}{ds} \log \zeta(s) = \frac{\zeta'(s)}{\zeta(s)}$, which is analytic in the complex plane, with the exception of the poles that occur at the zeros and poles of the zeta function. On the right-hand side we only have one term dependent on s,

$$\frac{d}{ds}x^{-s} = \frac{d}{ds}\left(\frac{1}{x}\right)^s = \left(\frac{1}{x}\right)^s \log\left(\frac{1}{x}\right) = -x^{-s}\log(x)$$

combining these gives us

$$\frac{\zeta'(s)}{\zeta(s)} = -\int_0^\infty x^{-s}\log(x)dJ(x).$$

Since $\psi(x)$ is the same as $J(x)$ scaled by $\log(x)$ we can change the element of integration so that

$$\int_0^\infty x^{-s}\log(x)dJ(x) = \int_0^\infty x^{-s}d\psi(x).$$

Thus,

$$-\frac{\zeta'(s)}{\zeta(s)} = \int_0^\infty x^{-s}d\psi(x).$$

Since we want ψ expressed in terms of ζ we need to separate the ψ-term. By Stieltje integration by parts we rewrite the right hand side as

$$\int_0^\infty x^{-s}d\psi(x) = x^{-s}\psi(x)\Big|_0^\infty - \int_0^\infty \psi(x)\frac{d}{dx}(x^{-s})dx.$$

In the second term, all we have to do is differentiate but in the first term we have to consider $x^{-s}\psi(x)$ as x goes to zero and infinity. As $x \to 0^+$ it follows that $x^{-s}\psi(x) = 0$, since $\psi(x) \equiv 0$ for all $x < 2$. As $x \to \infty$ we recall Lemma 3.6 which states that $\psi(x) \sim x$, thus $x^{-s}\psi(x) \to 0$ for all $\operatorname{Re} s > 1$. Therefore the first term vanishes completely, which leaves us with

$$-\frac{\zeta'(s)}{\zeta(s)} = s\int_0^\infty \psi(x)x^{-s-1}dx.$$

Then, by Mellin inversion, we get the desired expression

$$\psi(x) = \lim_{R\to\infty}\frac{1}{2\pi i}\int_{a-iR}^{a+iR}\left[-\frac{\zeta'(s)}{\zeta(s)}\right]x^s\frac{ds}{s}.$$

□

In the next section, we will evaluate this integral to gain an explicit formula for ψ. For this to be possible, we need the following lemmas.

Lemma 6.2. *Let s be a complex number and let a and x be real numbers. Then the integral*

$$\lim_{R\to\infty}\frac{1}{2\pi i}\int_{a-iR}^{a+iR} x^s\frac{ds}{s} = 1$$

provided $x > 1$, $\operatorname{Re} s > 1$ and $a > 0$.

Proof. See page 55 in [4]. □

Lemma 6.3. *Let s an β be complex numbers and let a and x be real numbers. Then for all β, the integral*

$$\lim_{R\to\infty} \frac{1}{2\pi i} \int_{a-iR}^{a+iR} x^s \frac{ds}{s-\beta} = x^\beta$$

provided $x > 1$, $\operatorname{Re} s > 1$ and $a > 0$.

Proof. By substitution of the variable $s = t + \beta$ the we get the expression

$$\lim_{R\to\infty} \frac{1}{2\pi i} \int_{a-iR}^{a+iR} x^s \frac{ds}{s-\beta} = \lim_{R\to\infty} \frac{1}{2\pi i} \int_{(a-\beta)-iR}^{(a-\beta)+iR} x^\beta x^t \frac{dt}{t} = x^\beta \lim_{R\to\infty} \frac{1}{2\pi i} \int_{(a-\beta)-iR}^{(a-\beta)+iR} x^t \frac{dt}{t}.$$

Other than the limits of integration, we have an expression in the same form as Lemma 6.2. Thus, we need to consider the term $a-\beta$. Considering the real part, a must be greater than $\operatorname{Re}\beta$ to fulfill Lemma 6.2. If we switch the limits of integration from $(a-\beta) \pm iR$ to $\operatorname{Re}(a-\beta) \pm iR$ the difference between the integrals will be negligible. Thus, provided $a > \operatorname{Re}\beta$ and $x > 1$

$$x^\beta \lim_{R\to\infty} \frac{1}{2\pi i} \int_{(a-\beta)-iR}^{(a-\beta)+iR} x^t \frac{dt}{t} = x^\beta \lim_{R\to\infty} \frac{1}{2\pi i} \int_{\operatorname{Re}(a-\beta)-iR}^{\operatorname{Re}(a-\beta)+iR} x^t \frac{dt}{t} = x^\beta.$$

□

7. Von Mangoldt's Explicit Formula

This section provides proof of von Mangoldt's explicit formula stated in Theorem 7.1. The main results from sections 5 and 6, namely Theorem 5.5 and Theorem 6.1, together with Lemma 6.3, will be central for this proof.

As seen in the previous section, the integral formula of Theorem 6.1 contains a ζ-term. We will modify the product formula of Theorem 5.5 to obtain an expression for the ζ-term dependent on the non-trivial zeros of ζ. Then we use the new expression and Lemma 6.3 to evaluate the integral, resulting in von Mangoldt's explicit formula.

Theorem 7.1. *Let $x > 1$ be a real number and let ρ be a non trivial root to the zeta function, then ψ can be expressed as*

$$\psi(x) = x - \sum_{\rho} \frac{x^{\rho}}{\rho} + \sum_{n=1}^{\infty} \frac{x^{-2n}}{2n} - \frac{\zeta'(0)}{\zeta(0)}.$$

Proof. We use Theorem 5.5 to obtain an expression for $-\frac{\zeta'(s)}{\zeta(s)}$. By the definition of ξ we have that

$$\Pi\Big(\frac{s}{2}\Big)\pi^{-s/2}(s-1)\zeta(s) = \xi(0)\prod_{\rho}\left(1 - \frac{s}{\rho}\right)$$

Just like in section 6 we can use the logarithmic derivative to get the desired expression of ζ. The logarithmic derivative of the previous equation becomes

$$\frac{d}{ds}\left(\log \Pi\Big(\frac{s}{2}\Big) - \frac{s}{2}\log(\pi) + \log(s-1) + \log\zeta(s)\right) = \frac{d}{ds}\left(\log\xi(0) + \sum_{\rho}\log\left(1 - \frac{s}{\rho}\right)\right)$$

$$\frac{d}{ds}\log\Pi\Big(\frac{s}{2}\Big) - \frac{1}{2}\log(\pi) + \frac{1}{s-1} + \frac{\zeta'(s)}{\zeta(s)} = \frac{d}{ds}\sum_{\rho}\log\left(1 - \frac{s}{\rho}\right) \tag{7.1}$$

where

$$\frac{d}{ds}\sum_{\rho}\log\left(1 - \frac{s}{\rho}\right) = \sum_{\rho}\frac{1}{1 - \frac{s}{\rho}}\Big(-\frac{1}{\rho}\Big) = \sum_{\rho}\frac{1}{s-\rho}. \tag{7.2}$$

The logarithmic derivative of the factorial function can be evaluated with the product formula of (4.1) as follows

$$\begin{aligned}
\frac{d}{ds}\log\Pi\Big(\frac{s}{2}\Big) &= \frac{d}{ds}\log\left(\prod_{n=1}^{\infty}\Big(1 + \frac{s}{2n}\Big)^{-1}\left(1 + \frac{1}{n}\right)^{s/2}\right) \\
&= \frac{d}{ds}\sum_{n=1}^{\infty}\log\left(\Big(1 + \frac{s}{2n}\Big)^{-1}\left(1 + \frac{1}{n}\right)^{s/2}\right) \\
&= \frac{d}{ds}\sum_{n=1}^{\infty}\left[\frac{s}{2}\log\Big(1 + \frac{1}{n}\Big) - \log\Big(1 + \frac{s}{2n}\Big)\right] \\
&= \sum_{n=1}^{\infty}\left[\frac{1}{2}\log\Big(1 + \frac{1}{n}\Big) - \frac{1}{s+2n}\right].
\end{aligned} \tag{7.3}$$

Using (7.2) and (7.3) in (7.1) gives us the expression

$$-\frac{\zeta'(s)}{\zeta(s)} = \frac{1}{s-1} - \sum_{\rho} \frac{1}{s-\rho} + \sum_{n=1}^{\infty} \left[\frac{1}{2}\log\left(1+\frac{1}{n}\right) - \frac{1}{s+2n}\right] - \frac{1}{2}\log(\pi). \tag{7.4}$$

Next we evaluate with $s = 0$ which gives

$$-\frac{\zeta'(0)}{\zeta(0)} = -1 - \sum_{\rho} \frac{1}{\rho} - \sum_{n=1}^{\infty} \left[\frac{1}{2}\log\left(1+\frac{1}{n}\right) - \frac{1}{2n}\right] - \frac{1}{2}\log(\pi)$$

and by subtraction

$$-\frac{\zeta'(s)}{\zeta(s)} + \frac{\zeta'(0)}{\zeta(0)} = \frac{1}{s-1} + 1 - \sum_{\rho} \left[\frac{1}{s-\rho} + \frac{1}{\rho}\right] - \sum_{n=1}^{\infty} \left[\frac{1}{s+2n} - \frac{1}{2n}\right]$$

then

$$-\frac{\zeta'(s)}{\zeta(s)} = \frac{s}{s-1} - \sum_{\rho} \frac{s}{\rho(s-\rho)} + \sum_{n=1}^{\infty} \frac{s}{2n(s+2n)} - \frac{\zeta'(0)}{\zeta(0)}. \tag{7.5}$$

Before moving forth, we need to address the convergence of the series of (7.4) to justify the differentiation. We will show that they converge uniformly in any disk $|s| \leq R$. On the sum over n the absolute value

$$\left|\frac{1}{s+2n} - \frac{1}{2}\log\left(1+\frac{1}{n}\right)\right|$$

can be rewritten by adding zero with $1/2n$ terms and expanding the logarithm. We get

$$\left|\frac{1}{s+2n} - \frac{1}{2n} + \frac{1}{2n} - \frac{1}{2}\left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \ldots\right)\right|$$
$$\leq \left|\frac{s}{2n(s+2n)}\right| + \left|\frac{1}{4n^2} - \frac{1}{6n^3} + \ldots\right| \leq \frac{R}{(2n)^2} + \frac{1}{n^2} \leq C\frac{1}{n^2}$$

for all sufficiently large n the series converge uniformly by Weierstrass M-test. The series over ρ converges uniformly by Lemma 5.6 because when ρ and $1-\rho$ are paired

$$\left|\frac{1}{s-\rho} + \frac{1}{s-(1-\rho)}\right| = \left|\frac{1}{(s-\frac{1}{2}) - (\rho-\frac{1}{2})} + \frac{1}{(s-\frac{1}{2}) + (\rho-\frac{1}{2})}\right|$$
$$= \left|\frac{2(s-\frac{1}{2})}{(s-\frac{1}{2})^2 - (\rho-\frac{1}{2})^2}\right| \leq C\frac{1}{|\rho-\frac{1}{2}|^2}$$

for all sufficiently large ρ once R is fixed. Thereby we conclude that the differentiation is valid. Then it follows that (7.5) is valid, except at the zeros and poles of ζ, that is, 1, ρ and $-2n$. Since the terms converge uniformly, we can integrate the expression termwise on a finite interval.

With all details sorted out we can put our new expression for $-\frac{\zeta'(s)}{\zeta(s)}$ into the integral of Theorem 6.1. On a finite interval, we get

$$\psi(x) = \lim_{h\to\infty} \frac{1}{2\pi i} \int_{a-ih}^{a+ih} \left[\frac{s}{s-1} - \sum_{\rho} \frac{s}{\rho(s-\rho)} + \sum_{n=1}^{\infty} \frac{s}{2n(s+2n)} - \frac{\zeta'(0)}{\zeta(0)}\right] x^s \frac{ds}{s}$$

$$= x - \lim_{h\to\infty} \sum_{\rho} \frac{1}{2\pi i} \int_{a-ih}^{a+ih} \frac{x^s ds}{\rho(s-\rho)} + \lim_{h\to\infty} \sum_{n=1}^{\infty} \frac{1}{2\pi i} \int_{a-ih}^{a+ih} \frac{x^s ds}{2n(s+2n)} - \frac{\zeta'(0)}{\zeta(0)}$$

by Lemma 6.3. The terms containing sums will be valid on infinite intervals to (See e.g. page 58 [4]). Thus by Lemma 6.3 we obtain the desired formula

$$\psi(x) = x - \sum_{\rho} \frac{x^{\rho}}{\rho} + \sum_{n=1}^{\infty} \frac{x^{-2n}}{2n} - \frac{\zeta'(0)}{\zeta(0)} \quad (x > 1).$$

$\square$

8. Numerical Approximations of $\psi(x)$

In this section, von Mangoldt's explicit formula is reformulated to be more conveniently used in a numerical approximation. The approximation of $\psi(x)$ with different numbers of non-trivial zeros is visualized in Fig. 8.1.

We begin with the constant term $\frac{\zeta'(0)}{\zeta(0)}$. It has the numerical value $\log(2\pi)$, see e.g. page 66 in [4]. Furthermore, we use the fact that $\log(1-x) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - ...$, and we see that

$$\sum_{n=1}^{\infty} \frac{x^{-2n}}{2n} = \frac{1}{2}x^{-2} + \frac{1}{4}x^{-4} + \frac{1}{6}x^{-6}... = -\frac{1}{2}\left(-x^{-2} - \frac{1}{2}x^{-4} - \frac{1}{3}x^{-6}...\right)$$

$$= -\frac{1}{2}\log\left(1 - x^{-2}\right).$$

Hence, an alternative formulation of von Mangoldt's explicit formula is

$$\psi(x) = x - \sum_{\rho} \frac{x^{\rho}}{\rho} - \frac{1}{2}\log\left(1 - x^{-2}\right) - \log(2\pi) \quad (x > 1). \tag{8.1}$$

Remark 8.1. $\psi(x)$ is strictly real-valued, even though we sum over the complex roots of ζ. It is because all terms containing some ρ will cancel their imaginary parts when paired with the term containing $\overline{\rho}$.

Proof. Let ρ be a non-trivial zero of ζ and $x > 1$. For each term of the sum

$$\sum_{\rho} \frac{x^{\rho}}{\rho}$$

we want to show that if we add it together with the term containing $\overline{\rho}$, the imaginary part is zero. Let $\rho = a + bi$, then

$$\frac{x^{\rho}}{\rho} + \frac{x^{\overline{\rho}}}{\overline{\rho}} = \frac{x^{a+bi}}{a+bi} + \frac{x^{a-bi}}{a-bi} = \frac{x^a}{a^2+b^2}\left(ax^{bi} + ax^{-bi} - bx^{bi} + bx^{-bi}\right)$$

$$= \frac{x^a}{a^2+b^2}\left[2a\left(\frac{e^{bi\log(x)} + e^{-bi\log(x)}}{2}\right) + 2b\left(\frac{e^{bi\log(x)} - e^{-bi\log(x)}}{2i}\right)\right]$$

$$= \frac{x^a}{a^2+b^2}\Big[2a\cos(b\log(x)) + 2b\sin(b\log(x))\Big]. \tag{8.2}$$

Since $a, b, x \in \mathbb{R}$ and $x > 1$, (8.2) must be real. $\square$

In Fig. 8.1 we can see (8.1) beside Chebyshev's function with four different numbers of non-trivial zeros added. Each zero is added in pairs with its complex conjugate.

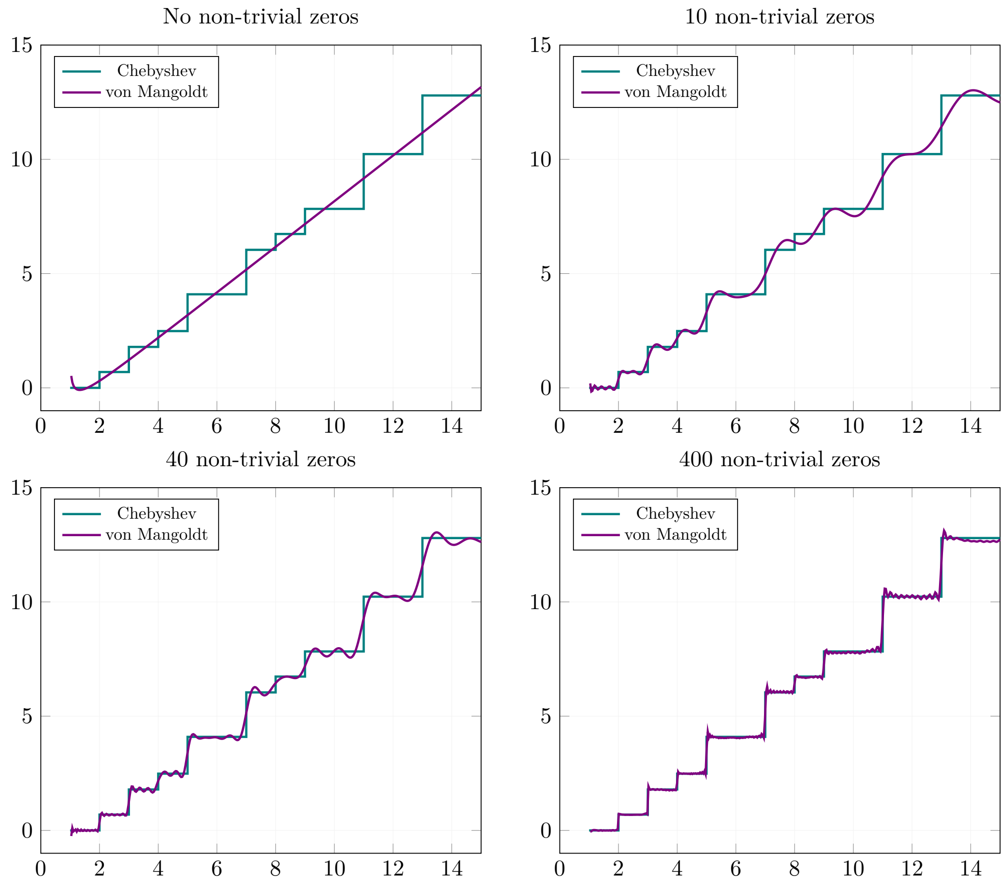

Figure 8.1 – The four figures display Chebyshev's function beside its approximation with von Mangoldt's explicit formula. The figures show how the approximation improves for each number of added zeros, 0, 10, 40, and 400.

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