Decision and Estimation in Data Processing

Chapter II. Elements of Signal Detection Theory

II.1 Detection based on a single sample

The model for signal detection

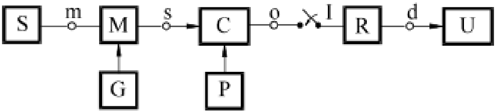

Figure 1: Signal detection model

- Contents:
  - Information source: generates messages a_n with probabilities $p(a_n)$
  - Modulator: transmits a signal $s_n(t)$ for message a_n
  - Channel: adds random noise
  - Sampler: takes samples from the signal $s_n(t)$
  - Receiver: **decides** what message a_n has been transmitted

Example

- A simple case (binary):
  - two messages a_0 and a_1
  - signals are constants (i.e. 0 for a_0, 5 for a_1)
  - take just 1 sample
  - decide: compare with a threshold
- General case: many messages, various signals, more samples (or continuous)

Detection for the binary case

- Receiver guesses between two hypotheses:
  - H_0: a_0 has been transmitted
  - H_1: a_1 has been transmitted
- The sample $r = s + n$
  - if more samples, then they are vectors $\overrightarrow{r} = \overrightarrow{s} + \overrightarrow{n}$
- Decision based on regions:
  - if r in region R_0, then decide D_0: was a_0
  - if r in region R_1, then decide D_1: was a_1
  - for single sample, regions are intervals: below/above the threshold
  - for 2 samples: regions are areas in a 2D plane, etc.
- Possible errors:
  - **false alarm**: was a_0, but decided D_1
  - probability is $P(D_1 \cap a_0)$
  - **miss**: was a_1, but decided D_0
  - probability is $P(D_0 \cap a_1)$

Minimum risk (cost) criterion

- How to choose the threshold? We need criteria
  - In general: how to delimit regions R_i?
- Minimum risk (cost) criterion: assign costs to decisions, minimize average cost
  - C_{ij} = cost of decision D_i when symbol was a_j
  - C_{00} = cost for good a_0 detection
  - C_{10} = cost for false alarm
  - C_{01} = cost for miss
  - C_{11} = cost for good a_1 detection
- The risk = the average cost

$$R = C_{00}P(D_0 \cap a_0) + C_{10}P(D_1 \cap a_0) + C_{01}P(D_0 \cap a_1) + C_{11}P(D_1 \cap a_1)$$

- Minimum risk criterion: **minimize the risk R**

Computations

- Proof on table:
  - Use Bayes rule
  - Notations: $w(r|a_j)$ (*likelihood*)
  - Probabilities: $\int_{R_i} w(r|a_j)dV$
- Conclusion, **decision rule is**

$$\frac{w(r|a_1)}{w(r|a_0)} \gtrless \frac{(C_{10} - C_{00})p(a_0)}{(C_{01} - C_{11})p(a_1)}$$

$$\Lambda(r) \gtrless K$$

- Interpretation: effect of costs, probabilities (move threshold)
- Can also apply logarithm (useful for normal distribution)

$$\ln \Lambda(r) \gtrless \ln K$$

- Example at blackboard: random noise with $N(0, \sigma^2)$, one sample

Ideal observer criterion

- Minimize the probability of decision error P_e
  - definition of P_e
- Particular case of minimum risk, with
  - $C_{00} = C_{11} = 0$ (good decisions bear no cost)
  - $C_{10} = C_{01}$ (pay the same in case of bad decisions)

$$\frac{w(r|a_1)}{w(r|a_0)} \gtrless \frac{p(a_0)}{p(a_1)}$$

Maximum likelihood criterion

- Particular case of above, with equal probability of messages

$$\frac{w(r|a_1)}{w(r|a_0)} \gtrless 1$$

$$\ln \frac{w(r|a_1)}{w(r|a_0)} \gtrless 0$$

- Example at blackboard: random noise with $N(0, \sigma^2)$, one sample
- Example at blackboard: random noise with $N(0, \sigma^2)$, **two** samples