Decision and Estimation in Data Processing

Chapter II. Elements of Signal Detection Theory

II.1 Introduction

Introduction

- Signal detection = the problem of deciding which signal is present from 2 or more possibilities
  - one possibility may be that there is no signal
- Based on **noisy** observations
  - signals are affected by noise

The model for signal detection

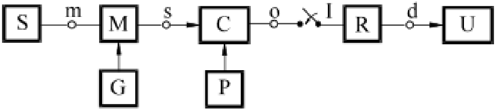

Figure 1: Signal detection model

- Contents:
  - Information source: generates messages a_n with probabilities $p(a_n)$
  - Modulator: transmits a signal $s_n(t)$ for message a_n
  - Channel: adds random noise
  - Sampler: takes samples from the signal $s_n(t)$
  - Receiver: **decides** what message a_n has been transmitted

Practical scenarios

- Data transmission
  - binary voltage levels (e.g. $s_n(t) =$ constant)
  - PSK modulation (Phase Shift Keying): $s_n(t) =$ cosine with same frequency but various initial phase
  - FSK modulation (Frequency Shift Keying): $s_n(t) =$ cosines with different frequencies
- Radar
  - a signal is emitted; if there is an obstacle, the signal gets reflected back
  - the receiver waits for possible reflections of the signal and must decide
    - no reflection is present -> no object
    - reflected signal is present -> object detected

Generalizations

- Decide between more than two signals
- Number of observations:
  - use only one sample
  - use multiple samples
  - observe the whole continuous signal for some time T

II.2 Detection of constant signals

Detection of a constant signal, white normal noise, 1 sample

- Simplest case: detection of a constant signal contaminated with white normal noise, using 1 sample
  - two messages a_0 and a_1
  - messages are encoded as constant signals
    - for a_0: send $s_0(t) = 0$
    - for a_1: send $s_1(t) = A$
  - over the signals there is white noise, normal distribution $\mathcal{N}(0, \sigma^2)$
  - receiver takes just 1 sample
  - decision: compare sample with a threshold

Decision

- The value of the sample taken is $r = s + n$
  - s is the true underlying signal ($s_0 = 0$ or $s_1 = A$)
  - n is a sample of the noise
- n is a (continuous) random variable, with normal distribution
- r is a random variable also
  - what distribution does it have?
- Decision is taken by comparing with a threshold T:
  - if $r < T$, take decision D_0: decide the true signal is s_0
  - if $r \geq T$, take decision D_1: decide the true signal is s_1

Hypotheses

- Receiver chooses between **two hypotheses**:
  - H_0: true signal is s_0 (a_0 has been transmitted)
  - H_1: true signal is s_1 (a_1 has been transmitted)
- Possible results
  1. No signal present, no signal detected.
     - Decision D_0 when hypothesis is H_0
     - Probability is $P(D_0 \cap H_0)$
  2. **False alarm**: no signal present, signal detected (error)
     - Decision S_1 when hypothesis is H_0
     - Probability is $P(D_1 \cap H_0)$
  3. **Miss**: signal present, no signal detected (error)
     - Decision D_0 when hypothesis is H_1
     - Probability is $P(D_0 \cap H_1)$
  4. Signal detected correctly: signal present, signal detected
     - Decision D_1 when hypothesis is H_1
     - Probability is $P(D_1 \cap H_1)$

Maximum likelihood criterion

- Choose the hypothesis that **seems most likely** given the observed sample r
- The **likelihood** $=$ the probability density of r given

Minimum risk (cost) criterion

- How to choose the threshold? We need criteria
  - In general: how to delimit regions R_i?
- Minimum risk (cost) criterion: assign costs to decisions, minimize average cost
  - C_{ij} = cost of decision D_i when symbol was a_j
  - C_{00} = cost for good a_0 detection
  - C_{10} = cost for false alarm
  - C_{01} = cost for miss
  - C_{11} = cost for good a_1 detection
- The risk = the average cost

$$R = C_{00}P(D_0 \cap a_0) + C_{10}P(D_1 \cap a_0) + C_{01}P(D_0 \cap a_1) + C_{11}P(D_1 \cap a_1)$$

- Minimum risk criterion: **minimize the risk R**

Computations

- Proof on table:
  - Use Bayes rule
  - Notations: $w(r|a_j)$ (*likelihood*)
  - Probabilities: $\int_{R_i} w(r|a_j)dV$
- Conclusion, **decision rule is**

$$\frac{w(r|a_1)}{w(r|a_0)} \gtrless \frac{(C_{10} - C_{00})p(a_0)}{(C_{01} - C_{11})p(a_1)}$$

$$\Lambda(r) \gtrless K$$

- Interpretation: effect of costs, probabilities (move threshold)
- Can also apply logarithm (useful for normal distribution)

$$\ln \Lambda(r) \gtrless \ln K$$

- Example at blackboard: random noise with $N(0, \sigma^2)$, one sample

Ideal observer criterion

- Minimize the probability of decision error P_e
  - definition of P_e
- Particular case of minimum risk, with
  - $C_{00} = C_{11} = 0$ (good decisions bear no cost)
  - $C_{10} = C_{01}$ (pay the same in case of bad decisions)

$$\frac{w(r|a_1)}{w(r|a_0)} \gtrless \frac{p(a_0)}{p(a_1)}$$

Maximum likelihood criterion

- Particular case of above, with equal probability of messages

$$\frac{w(r|a_1)}{w(r|a_0)} \gtrless 1$$

$$\ln \frac{w(r|a_1)}{w(r|a_0)} \gtrless 0$$

- Example at blackboard: random noise with $N(0, \sigma^2)$, one sample
- Example at blackboard: random noise with $N(0, \sigma^2)$, **two** samples