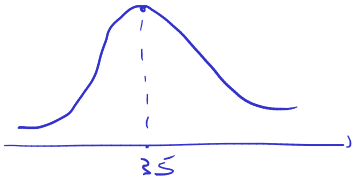
Seminar 7

(1) $r_1 = 40 = \underbrace{\theta}_{s_\theta(t)} + \text{noise}$

$w(\theta) = \mathcal{N}(\mu = 35, \sigma^2 = 2)$



a) ML: $\hat{\theta}_{ML} = \arg\max_\theta \; w(r|\theta)$

$= \arg\min_\theta \; d(r, s_\theta)^2$

$s_\theta = [\ \theta\ ]$

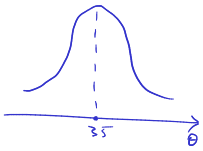
$r = [\ 40\ ]$

$d(r, s_\theta)^2 = (40-\theta)^2$

$\hat{\theta}_{ML} = 40$

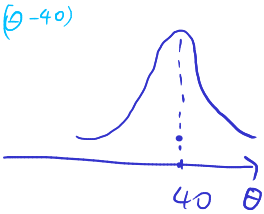
b). MAP: $\hat{\theta}_{ML} = \arg\max_\theta \; w(r|\theta) \cdot w(\theta)$

$w(\theta) = \mathcal{N}(\mu = 35, \sigma^2 = 2) = \frac{1}{\sqrt{2}\sqrt{2\pi}} \cdot e^{-\frac{(\theta-35)^2}{2\cdot 2}}$



$w(r|\theta) = \mathcal{N}(\mu = \theta, \sigma^2 = 2) = \frac{1}{\sqrt{2}\cdot\sqrt{2\pi}} \cdot e^{-\frac{(r-\theta)^2}{2\cdot 2}}$ (40)

$= \frac{1}{\sqrt{2}\cdot\sqrt{2\pi}} \cdot e^{-\frac{(40-\theta)^2}{2\cdot 2}} \Leftarrow (\theta-40)^2$



$w(\theta|r)$

$w(r|\theta)\cdot w(\theta) = \left(\frac{1}{\sqrt{2}\sqrt{2\pi}}\right)^2 \cdot e^{-\frac{(\theta-35)^2 + (40-\theta)^2}{4}}$ ← Want maximum

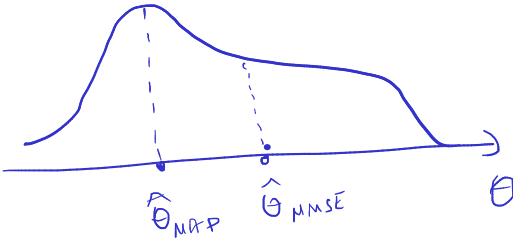
D ← Want minimum

$e^{-x} = \frac{1}{e^x}$

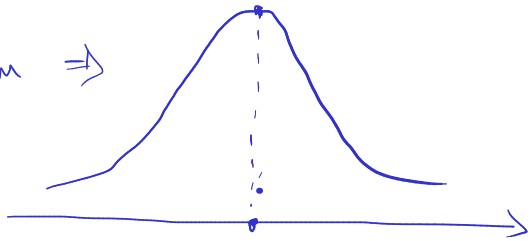
$\frac{\partial D}{\partial \theta} = 0 \Leftrightarrow 2\cdot(\theta-35) + 2(40-\theta)(-1) = 0$

$\theta - 35 + \theta - 40 = 0 \Rightarrow \hat{\theta}_{MAP} = \frac{35+40}{2} = 37.5$

c) MMSE : $\hat{\theta}_{MMSE} = E\{w(\theta|r)\}$



$w(\theta|r)$ = gaussian $\Rightarrow$



$\mu = \hat{\theta}_{MMSE} = \hat{\theta}_{MAP} = 37.5$

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d). $w(\theta) = \mathcal{N}(\mu = 35, \sigma^2 = 20) = \frac{1}{\sqrt{20}\cdot\sqrt{2\pi}} e^{-\frac{(\theta-35)^2}{2\cdot 20}}$

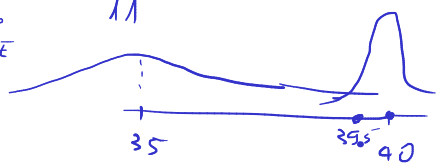
$w(\theta|r)$

$w(r|\theta)\cdot w(\theta) = \frac{1}{\sqrt{40}\cdot 2\pi} \cdot e^{-\frac{(\theta-40)^2}{4} - \frac{(\theta-35)^2}{40}} = \frac{1}{\sqrt{40}\cdot 2\pi} \cdot e^{-\frac{10(\theta-40)^2 + (\theta-35)^2}{40}}$

Want minimum

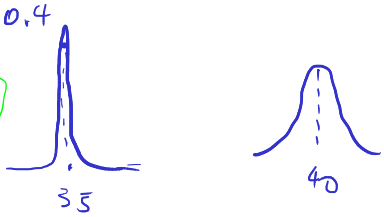
$\frac{\partial D}{\partial \theta} = \frac{20}{10}(\theta-40) + 1(\theta-35) = 0$

$\Leftrightarrow 10\theta - 400 + \theta - 35 = 0 \Leftrightarrow \hat{\theta}_{\substack{MAP \\ MMSE}} = \frac{435}{11} = 39.5$



e). ..... Same

$w(r|\theta)\cdot w(\theta) = \frac{1}{\sqrt{2}\sqrt{2\pi}} \cdot \frac{1}{\sqrt{0.2}\cdot\sqrt{2\pi}} \cdot e^{-\frac{(\theta-40)^2}{4} - \frac{(\theta-35)^2}{0.4}} = \ldots\ldots \cdot e^{-\frac{(\theta-40)^2 + 10(\theta-35)^2}{4}}$



$\frac{\partial D}{\partial \theta} = 1(\theta-40) - \frac{20}{10}(\theta-35) = 0$

$\Rightarrow \hat{\theta}_{MMSE} = \frac{40+350}{11} = 35.45$

(2) $r_1 = 40$

$r_2 = 38.1$

$r_3 = 39.2$

$r = [40 \quad 38.1 \quad 39.2]$

$$w(r|\theta) = w(r_1|\theta) \cdot w(r_2|\theta) \cdot w(r_3|\theta)$$

$$= \frac{1}{\sqrt{2}\sqrt{2\pi}} e^{-\frac{(40-\theta)^2}{4}} \cdot \frac{1}{\sqrt{2}\sqrt{2\pi}} e^{-\frac{(38.1-\theta)^2}{4}} \cdot \frac{1}{\sqrt{2}\sqrt{2\pi}} e^{-\frac{(39.2-\theta)^2}{4}}$$

$$= \left(\frac{1}{\sqrt{2}\cdot\sqrt{2\pi}}\right)^3 \cdot e^{-\frac{(\theta-40)^2 + (\theta-38.1)^2 + (\theta-39.2)^2}{4}} \quad (1)$$

a) ML: $\frac{\partial D}{\partial \theta} = 2(\theta-40) + 2(\theta-38.1) + 2(\theta-39.2) = 0$

$$\Leftrightarrow \hat{\theta}_{ML} = \frac{40 + 38.1 + 39.2}{3}$$

b). MAP MMSE

$$w(r|\theta) \cdot w(\theta) = \left(\frac{1}{\sqrt{2}\cdot\sqrt{2\pi}}\right)^3 \cdot e^{-\frac{(\theta-40)^2 + (\theta-38.1)^2 + (\theta-39.2)^2}{4}} \cdot \frac{1}{\sqrt{2}\sqrt{2\pi}} \cdot e^{-\frac{(\theta-35)^2}{4}}$$

$$= \left(\frac{1}{\sqrt{2}\sqrt{2\pi}}\right)^4 \cdot e^{-\frac{(\theta-40)^2 + (\theta-38.1)^2 + (\theta-39.2)^2 + (\theta-35)^2}{4}} \quad (1)$$

$$\frac{\partial D}{\partial \theta} = 0 \Leftrightarrow 2(\theta-40) + 2(\theta-38.1) + 2(\theta-39.2) + 2(\theta-35) = 0$$

$$\Rightarrow \hat{\theta}_{MAP\ MMSE} = \frac{40 + 38.1 + 39.2 + 35}{4} =$$