Chapter III. Elements of Estimation Theory

II.1 Introduction

What means "Estimation"?

- A sender transmits a signal $s_\Theta(t)$ which depends on an **unknown** parameter Θ
- The signal is affected by noise, we receive $r(t) = s_\Theta(t) + noise$
- We want to **find out** the correct value of the parameter
  - based on samples from the received signal, or the full continuous signal
  - available data is noisy $=>$ we "estimate" the parameter
- The found value is $\hat{\Theta}$, **the estimate** of Θ ("estimatul", rom)
  - there will always be some estimation error $\epsilon = \hat{\Theta} - \Theta$
- Examples:
  - Unknown amplitude of constant signal: $r(t) = A + noise$, estimate A
  - Unknown phase of sine signal: $r(t) = \cos(2\pi f t + \phi)$, estimate ϕ
  - Record speech signal, estimate/decide what word is pronounced

Estimation vs Decision

- Consider the following estimation: $r(t) = A + noise$, estimate A
- For detection, we have to choose between **two known values** of A:
  - i.e. A can be 0 or 5 (hypotheses H_0 and H_1)
- For estimation, A can be anything $=>$ we choose between **infinite number of options** for A:
  - A might be any value in $\mathbb{R}$, in general

Estimation vs Decision

- Detection = Estimation constrained to only a few discrete options
- Estimation = Detection with an infinite number of options available
- The statistical methods used are quite similar
  - In practice, distinction between Estimation and Detections is somewhat blurred
  - (e.g. when choosing between 1000 hypotheses, do we call it "Detection" or "Estimation"?)

Available data

- The available data is the received signal $r(t)$
  - affected by noise, and depending on the unknown Θ
- We consider **N samples** from $r(t)$, taken at some sample times t_i

$$\mathbf{r} = [r_1, r_2, ... r_N]$$

- Each sample r_i is a random variable that depends on Θ (and the noise)
  - Each sample has a distribution that depends on Θ

$$w_i(r_i; \Theta)$$

- The whole sample vector $\mathbf{r}$ is a N-dimensional random variable that depends on Θ (and the noise)
  - It has a N-dimensional distribution that depends on Θ

$$w(\mathbf{r}; \Theta)$$

Types of estimation

- We consider estimating a parameter Θ under two circumstances:

1. No distribution is known about the parameter, except maybe some allowed range (e.g. $\Theta > 0$)
   - The parameter can be any value in the allowed range, equally likely
2. We know a distribution $p(\Theta)$ for Θ, which tells us the values of Θ that are more likely than others
   - this is known as *a priori* (or *prior*) distribution (i.e. "known beforehand")

II.2 Maximum Likelihood estimation

Maximum Likelihood definition

- When no distribution is known about the parameter, we use a method known as Maximum Likelihood Estimation (MLE)
- The distribution of the received data, $w(\mathbf{r}; \Theta)$, is known as the **likelihood function**
  - we know the vector $\mathbf{r}$ we received, so this is a constant
  - the unknown variable in this function is Θ

$$L(\Theta) = w(\mathbf{r}; \Theta)$$

- Maximum Likelihood Estimation: The estimate $\hat{\Theta}$ is **the value that maximizes the likelihood of the observed data**
  - i.e. the value Θ that maximizes $w(r; \Theta)$

$$\hat{\Theta} = \arg\max_{\Theta} L(\Theta) = \arg\max_{\Theta} w(r; \Theta)$$

- If Θ is allowed to live only in a certain range, restrict the maximization only to that range.

Computations

- Find maximum by setting derivative to 0

$$\frac{dL(\Theta)}{d\Theta} = 0$$

- We can also maximize **natural logarithm** of the likelihood function ("log-likelihood function")

$$\frac{d \ln \left(L(\Theta) \right)}{d\Theta} = 0$$

Computations

Method:

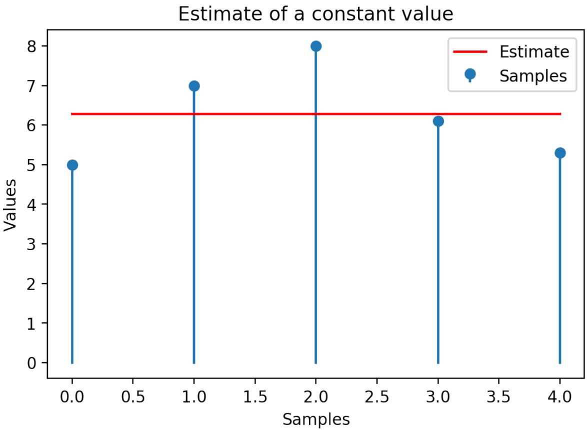
1. Find the function
$$L(\Theta) = w(\mathbf{r};\Theta)$$
2. Set the condition that derivative of $L(\Theta)$ or $\ln((L(\Theta))$ is 0
$$\frac{dL(\Theta)}{d\Theta} = 0, \text{ or } \frac{d\ln\left(L(\Theta)\right)}{d\Theta} = 0$$
3. Solve and find the value $\hat{\Theta}$
4. Check that second derivative at point $\hat{\Theta}$ is negative, to check that point is a maximum
   - because derivative $= 0$ for both maximum and minimum points

Examples:

Estimating a constant signal in gaussian noise:

- Find the Maximum Likelihood estimate of a constant value A from 5 noisy measurements $r_i = A + noise$ with values $[5, 7, 8, 6.1, 5.3]$. The noise is AWGN $\mathcal{N}(\mu = 0, \sigma^2)$.
- Solution: at whiteboard.
- The estimate $\hat{A}$ is the average value of the samples (not surprisingly)

Numerical simulation



General signal in AWGN

- Consider that the true underlying signal is $s_\Theta(t)$
- Consider AWGN noise $\mathcal{N}(\mu = 0, \sigma^2)$.
- The samples r_i are taken at sample moments t_i
- The samples r_i have normal distribution with average $s_\Theta(t_i)$ and variance σ^2
- Overall likelihood function = product of likelihoods for each sample r_i

$$\begin{aligned} L(\Theta) &= \prod_{i=1}^{N} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(r_i - s_\Theta(t_i))^2}{2\sigma^2}} \\ &= \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{\sum (r_i - s_\Theta(t_i))^2}{2\sigma^2}} \end{aligned}$$

- The log-likelihood is

$$\ln\left(L(\Theta)\right) = \underbrace{\ln\left(\frac{1}{\sigma\sqrt{2\pi}}\right)}_{constant} - \frac{\sum (r_i - s_\Theta(t_i))^2}{2\sigma^2}$$

General signal in AWGN

- The maximum of the function = the minimum of the exponent

$$\hat{\Theta} = \arg\max_{\Theta} w(r; \Theta) = \arg\min \sum (r_i - s_{\Theta}(t_i))^2$$

- The term $\sum (r_i - s_{\Theta}(t_i))^2$ is the **squared distance** $d(\mathbf{r}, s_{\Theta})$

$$d(\mathbf{r}, s_{\Theta}) = \sqrt{\sum (r_i - s_{\Theta}(t_i))^2}$$

$$(d(\mathbf{r}, s_{\Theta}))^2 = \sum (r_i - s_{\Theta}(t_i))^2$$

- Maximum Likelihood estimate $\hat{\Theta}$ = the value that makes $s_{\Theta}(t_i)$ **closest to the received values r**
    - closer = more likely
    - closest = most likely = maximum likelihood
- For continuous signals? Same, but use distance function for continuous signals

General signal in AWGN

- Find maximum by setting derivative to 0

$$\frac{d \ln (L(\Theta))}{d\Theta} = 0$$

means

$$\sum (r_i - s_\Theta(t_i)) \frac{ds_\Theta(t_i)}{d\Theta} = 0$$

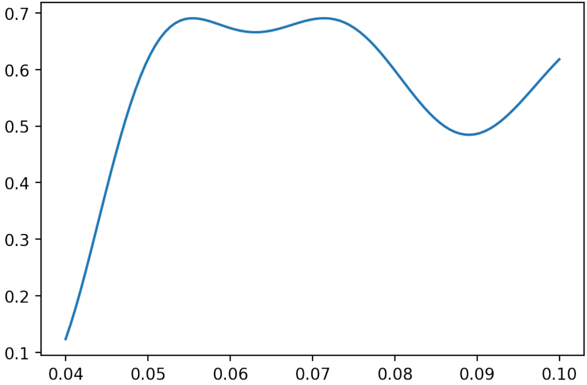
Numerical simulation

Estimating the frequency f of a cosine signal

- Find the Maximum Likelihood estimate of the frequency f of a cosine signal, from 10 noisy measurements $r_i = cos(2\pi f t_i) + noise$ with values [...]. The noise is AWGN $\mathcal{N}(\mu = 0, \sigma^2)$. The sample times $t_i = [0, 1, 2, 3, 4, 5, 6, 7, 8, 9]$
- Solution: at whiteboard.

Numerical simulation

The likelihood function is:



Numerical simulation

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True frequency = 0.070000, Estimate = 0.071515
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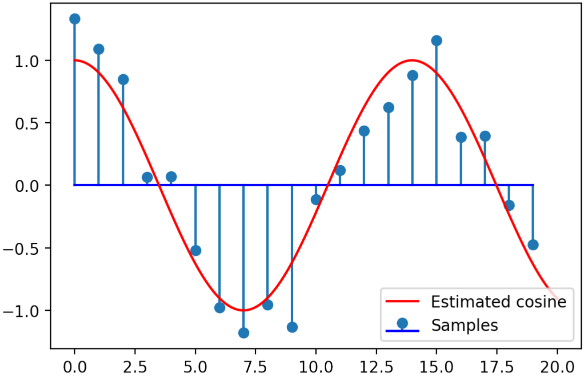

ML Estimation and ML Detection

- In ML Estimation, the estimate $\hat{\Theta}$ is the value that maximizes the likelihood function
- In ML Detection, the decision criterion $\frac{w(r|H_1)}{w(r|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} 1$ means "choose the hypothesis that maximizes the likelihood function".
- Therefore it is the same principle, merely in a different context:
  - in Detection we are restricted to a few predefined options
  - in Estimation we are unrestricted $=>$ choose the maximizing value

II.3 Bayesian estimation

Prior distribution

- Suppose we know beforehand a distribution of Θ, $w(\Theta)$
  - we know beforehand how likely it is to have a certain value
  - known as *a priori* distribution or *prior* distribution
- The estimation must take it into account
  - the estimate will be slightly "moved" towards more likely values
- Known as "Bayesian estimation"
  - Thomas Bayes = discovered the Bayes rule
  - Stuff related to Bayes rule are often named "Bayesian"

Cost function

- The **estimation error** is the difference between the estimate $\hat{\Theta}$ and the true value Θ
$$\epsilon = \hat{\Theta} = \Theta$$
- The **cost function** $C(\epsilon)$ assigns a cost to each possible estimation error
  - when $\epsilon = 0$, the cost $C(0) = 0$
  - small errors ϵ have small costs
  - large errors ϵ have large costs
- Usual types of cost functions:
  - Quadratic: $C(\epsilon) = \epsilon^2 = \left(\hat{\Theta} - \Theta\right)^2$
  - Uniform (“hit or miss”): $C(\epsilon) = \begin{cases} 0, & \text{if } |\epsilon| = |\hat{\Theta} - \Theta| \leq E \\ 1, & \text{if } |\epsilon| = |\hat{\Theta} - \Theta| > E \end{cases}$
  - Linear: $C(\epsilon) = |\epsilon| = |\hat{\Theta} - \Theta|$
  - draw them at whiteboard

The Bayesian risk

- For each pair of values $\mathbf{r}$ and Θ, $w(\mathbf{r};\Theta)$ tells us how likely it is to have them
- Multiplying with $C(\epsilon$ gives us the cost, for each $\mathbf{r}$ and Θ

$$C(\epsilon)w(\mathbf{r};\Theta)$$

- Integrating over Θ gives the cost for a certain $\mathbf{r}$

$$\int_{-\infty}^{\infty} C(\epsilon)w(\mathbf{r};\Theta)d\Theta$$

- Further integrating also over $\mathbf{r}$ gives the global cost for all $\mathbf{r}$ and all Θ

$$R = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} C(\epsilon)w(\mathbf{r};\Theta)d\Theta d\mathbf{r}$$

Minimizing the risk

- We want to minimize the risk R
- Bayes rule: $w(\mathbf{r};\Theta) = w(\Theta|\mathbf{r})w(\mathbf{r})$
- Replacing in R, we obtain

$$\begin{aligned} R &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} C(\epsilon)w(\Theta|\mathbf{r})w(\mathbf{r})d\Theta d\mathbf{r} \\ &= \int_{-\infty}^{\infty} w(\mathbf{r})\left[\int_{-\infty}^{\infty} C(\epsilon)w(\Theta|\mathbf{r})d\Theta\right] d\mathbf{r} \end{aligned}$$

- Since $w(\mathbf{r}) \geq 0$, minimizing the inner integral will minimize R

$$I = \int_{-\infty}^{\infty} C(\epsilon)w(\Theta|\mathbf{r})d\Theta$$

- Next, we'll replace $C(\epsilon)$ with its definition and derivate over $\hat{\Theta}$
  - Attention: $\hat{\Theta}$, not Θ!

MMSE estimator

- When the cost function is quadratic $C(\epsilon) = \epsilon^2 = \left(\hat{\Theta} - \Theta\right)^2$

$$I = \int_{-\infty}^{\infty} (\hat{\Theta} - \Theta)^2 w(\Theta|\mathbf{r}) d\Theta$$

- We want the $\hat{\Theta}$ that minimizes I, so we derivate

$$\frac{dI}{d\hat{\Theta}} = 2 \int_{-\infty}^{\infty} (\hat{\Theta} - \Theta) w(\Theta|\mathbf{r}) d\Theta = 0$$

- Equivalent to

$$\hat{\Theta} \underbrace{\int_{-\infty}^{\infty} w(\Theta|\mathbf{r})}_{1} d\Theta = \int_{-\infty}^{\infty} \Theta w(\Theta|\mathbf{r}) d\Theta$$

- The **Minimum Mean Squared Error (MMSE)** estimator is

$$\hat{\Theta} = \int_{-\infty}^{\infty} \Theta \cdot w(\Theta|\mathbf{r}) d\Theta$$

Interpretation

- $w(\Theta|\mathbf{r})$ is the **posterior** (or **a posteriori**) distribution
  - it is the distribution of Θ after we know the data we received
  - the prior distribution $w(\Theta)$ is the one before knowing any data
- The MMSE estimation is the **average value** of the posterior distribution

The MAP estimator

- When the cost function is uniform
$$C(\epsilon) = \begin{cases} 0, & \text{if } |\epsilon| = |\hat{\Theta} - \Theta| \leq E \\ 1, & \text{if } |\epsilon| = |\hat{\Theta} - \Theta| > E \end{cases}$$ $$\begin{split}
- Keep in mind that $\Theta = \hat{\Theta} - \epsilon$
- We obtain

$$I = \int_{-\infty}^{\hat{\Theta}-E} w(\Theta|\mathbf{r}) d\Theta + \int_{T\hat{h}eta+E}^{\infty} w(\Theta|\mathbf{r}) d\Theta$$
$$I = 1 - \int_{\hat{\Theta}-E}^{\hat{\Theta}+E} w(\Theta|\mathbf{r}) d\Theta$$

The MAP estimator

- To minimize I, we must maximize $\int_{\hat{\Theta}-E}^{\hat{\Theta}+E} w(\Theta|\mathbf{r}) d\Theta$, the integral around point $\hat{\Theta}$
- For E a very small, the function $w(\Theta|\mathbf{r})$ is approximately constant, so we pick the point where the function is maximum
- The **Maximum A Posteriori (MAP)** estimator is

$$\hat{\Theta} = \arg\max w(\Theta|\mathbf{r})$$

- $\arg\max$ = "the value which maximizes the function"
  - $\max f(x)$ = the maximum value of a function
  - $\arg\max f(x)$ = the x for which the function reaches its maximum

Interpretation

- The MAP estimator chooses Θ as the value where the posterior distribution is maximum
- The MMSE estimator chooses Θ as average value of the posterior distribution

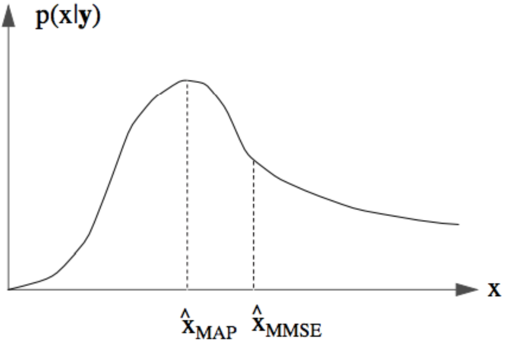

Figure 1: MAP vs MMSE estimators

Finding the posterior distribution

- That's cool, but how do we find this posterior distribution $w(\Theta|\mathbf{r})$?
- Use the Bayes rule

$$w(\Theta|\mathbf{r}) = \frac{w(\mathbf{r};\Theta)}{w(\mathbf{r})} = \frac{w(\mathbf{r}|\Theta) \cdot w(\Theta)}{w(\mathbf{r})}$$

- Since $w(\mathbf{r})$ is constant for a given $\mathbf{r}$ the MAP estimator is

$$\hat{\Theta} = \arg\max w(\Theta|\mathbf{r}) = \arg\max w(\mathbf{r}|\Theta) w(\Theta)$$

- The MAP estimator is the one which **maximizes** the likelihood of the observed data, **but multiplying with the prior distribution** $w(\Theta)$
- The MMSE estimator is the **average** of the same thing

Relation with Maximum Likelihood Estimator

- The MLE estimator was just $\arg\max w(\mathbf{r}|\Theta)$
- The MAP estimator = like the MLE estimator but with the prior distribution $w(\Theta)$
- If $w(\Theta)$ is a constant, the MAP estimator reduces to MLE
  - $w(\Theta) =$ constant means all values Θ are equally likely
  - i.e. we don't have a clue where the real Θ might be

Relation with Detection

- The minimum probability of error criterion $\frac{w(r|H_1)}{w(r|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{P(H_0)}{P(H_1)}$
- It can be rewritten as $w(r|H_1) \cdot P(H_1) \underset{H_0}{\overset{H_1}{\gtrless}} w(r|H_0)P(H_0)$
  - i.e. choose the hypothesis where $w(r|H) \cdot P(H)$ is maximum
  - $w(r|H_1)$, $w(r|H_0)$ are the likelihood of observed data
  - $P(H_1)$, $P(H_0)$ are the prior probabilities (known beforehand)
- The MAP estimator is where $w(\mathbf{r}|\Theta)w(\Theta)$ is maximum
  - $w(\mathbf{r}|\Theta)$ is the likelihood of observed data
  - $w(\Theta)$ is the prior distribution (known beforehand)
- Therefore it is the same principle, merely in a different context:
  - in Detection we are restricted to a few predefined options
  - in Estimation we are unrestricted $=>$ choose the maximizing value of the whole function

Exercise

Exercise: constant value, 3 measurement, Gaussian same σ

- We want to estimate today's temperature in Sahara
- Our thermometer reads 40 degrees, but the value was affected by Gaussian noise $\mathcal{N}(0, \sigma^2 = 2)$ (crappy thermometer)
- We know that this time of the year, the temperature is around 35 degrees, with a Gaussian distribution $\mathcal{N}(35, \sigma^2 = 2)$.
- Estimate the true temperature using MLE, MAP and MLE estimators

Exercise

Exercise: constant value, 3 measurements, Gaussian same σ

- What if he have three thermometers, showing 40, 38, 41 degrees

Exercise: constant value, 3 measurements, Gaussian different σ

- What if the temperature this time of the year has Gaussian distribution $\mathcal{N}(35, \sigma_2^2 = 3)$
  - different variance, $\sigma_2 \neq \sigma$

General signal in AWGN

- Consider that the true underlying signal is $s_\Theta(t)$
- Consider AWGN noise $\mathcal{N}(\mu = 0, \sigma^2)$.
- As in Maximum Likelihood function, overall likelihood function

$$w(\mathbf{r}|\Theta) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{\sum (r_i - s_\Theta(t_i))^2}{2\sigma^2}}$$

- But now this function is also **multiplied with** $w(\Theta)$

$$w(\mathbf{r}|\Theta) \cdot w(\Theta)$$

General signal in AWGN

- MAP estimator is the argument that maximizes this product

$$\hat{\Theta}_{MAP} = \arg\max w(\mathbf{r}|\Theta) w(\Theta)$$

- Taking logarithm

$$\begin{aligned}\hat{\Theta}_{MAP} &= \arg\max \ln\left(w(\mathbf{r}|\Theta)\right) + \ln\left(w(\Theta)\right) \\ &= \arg\max -\frac{\sum (r_i - s_\Theta(t_i))^2}{2\sigma^2} + \ln\left(w(\Theta)\right)\end{aligned}$$

Gaussian prior

- If the prior distribution is also Gaussian $\mathcal{N}(\mu_\Theta, \sigma_\Theta^2)$

$$\ln\left(w(\Theta)\right) = -\frac{\sum(\Theta - \mu_\Theta)^2}{2\sigma_\Theta^2}$$

- MAP estimation becomes

$$\hat{\Theta}_{MAP} = \arg\min \frac{\sum(r_i - s_\Theta(t_i))^2}{2\sigma^2} + \frac{\sum(\Theta - \mu_\Theta)^2}{2\sigma_\Theta^2}$$

- Can be rewritten as

$$\hat{\Theta}_{MAP} = \arg\min d(\mathbf{r}, s_\Theta)^2 + \underbrace{\frac{\sigma^2}{\sigma_\Theta^2}}_{\lambda} \cdot d(\Theta, \mu_\Theta)^2$$

Interpretation

- MAP estimator with Gaussian noise and Gaussian prior

$$\hat{\Theta}_{MAP} = \arg\min d(\mathbf{r}, s_\Theta)^2 + \underbrace{\frac{\sigma^2}{\sigma_\Theta^2}}_{\lambda} \cdot d(\Theta, \mu_\Theta)^2$$

- $\hat{\Theta}_{MAP}$ is close to its expected value μ_Θ and it makes the true signal close to received data $\mathbf{r}$
  - Example: "search for a house that is close to job and close to the Mall"
  - λ controls the relative importance of the two terms
- Particular cases
  - σ_Θ very small = the prior is very specific (narrow) = λ large = second term very important = $\hat{\Theta}_{MAP}$ close to μ_Θ
  - σ_Θ very large = the prior is very unspecific = λ small = first term very important = $\hat{\Theta}_{MAP}$ close to ML estimation

Applications

- In general, practical applications:
  - can use various prior distributions
  - estimate **multiple parameters** (a vector of parameters)
- Applications
  - denoising of signals
  - signal restoration
  - signal compression

Estimator bias

- How good is an estimator?
  - Many ways to characterize
- An estimator $\hat{\Theta}$ is a **random variable**
  - can have different values, because it is computed based on the received samples, which depend on noise
  - example: in lab, try on multiple computers $=>$ slightly different results
- As a random variable, it has:
  - an average value (expected value): $E\left\{\hat{\Theta}\right\}$
  - a variance: $E\left\{(\hat{\Theta} - \Theta)^2\right\}$

Estimator bias

- **Unbiased** estimator = if the average value of the estimator is the true value of Θ

$$E\left\{\hat{\Theta}\right\} = \Theta$$

- **Biased** estimator = if the average value of the estimator is different from the true value Θ
  - the difference $E\left\{\hat{\Theta}\right\} - \Theta$ is called **the bias** of the estimator

Estimator bias

- Example: for constant signal A with AWGN noise (zero-mean), ML estimator is $\hat{A}_{ML} = \frac{1}{N}\sum_i r_i$
- Then:

$$
\begin{aligned}
E\left\{\hat{A}_{ML}\right\} &= \frac{1}{N} E\left\{\sum_i r_i\right\} \\
&= \frac{1}{N}\sum_{i=1}^{N} E\left\{r_i\right\} \\
&= \frac{1}{N}\sum_{i=1}^{N} E\left\{A + noise\right\} \\
&= \frac{1}{N}\sum_{i=1}^{N} A \\
&= A
\end{aligned}
$$

- This estimator in unbiased

Estimator variance

- Unbiased estimators are good, but if the **variance** of the estimator is large, then estimated values can be far from the true value
- We prefer estimators with **small variance**, even if maybe slightly biased