Seminar 7

(1)

noise $= \mathcal{N}(\mu = 0, \sigma^2 = 2)$

$$\boxed{r_1 = 40 = \theta + \text{noise}}$$

$$w(r|\theta) = \mathcal{N}(\mu = \theta, \sigma^2 = 2)$$

prior: $\mathcal{N}(\mu = 35, \sigma^2 = 2)$

a) ML:

$$\hat{\theta}_{ML} = \underset{\theta}{\operatorname{argmax}}\ w(r|\theta) = \underset{\theta}{\operatorname{argmin}}\ d(r, \Lambda_\theta)^2$$

$r = 40 \Rightarrow \Lambda_\theta = \theta \Rightarrow d(r, \Lambda_\theta)^2 = (40-\theta)^2$

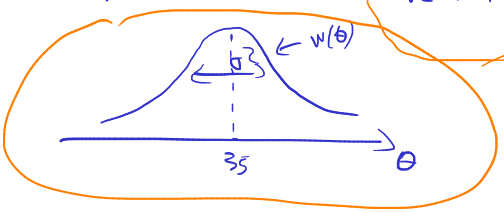
$$\hat{\theta}_{ML} = \underset{\theta}{\operatorname{argmin}}\ (40-\theta)^2 \Rightarrow \hat{\theta}_{ML} = 40$$

$\left(2(40-\theta)(-1) = 0 \Rightarrow \hat{\theta}_{ML} = 40\right)$

b). MAP: likelihood · prior distrib.

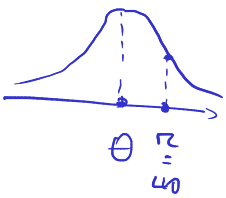
$$\hat{\theta}_{MAP} = \underset{\theta}{\operatorname{argmax}}\ w(r|\theta) \cdot w(\theta)$$

$$w(\theta) = \mathcal{N}(\mu = 35, \sigma^2 = 2) = \frac{1}{\sqrt{2}\cdot\sqrt{2\pi}}\, e^{-\frac{(\theta-35)^2}{2\cdot 2}}$$

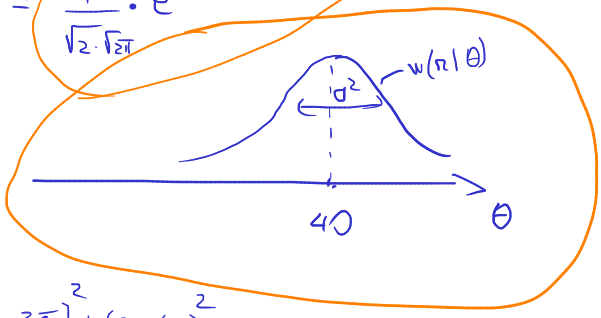


$$w(r|\theta) = \mathcal{N}(\mu = \theta, \sigma^2 = 2) = \frac{1}{\sqrt{2}\cdot\sqrt{2\pi}} \cdot e^{-\frac{(40-\theta)^2}{2\cdot 2}}$$

$r = 40$



$$= \frac{1}{\sqrt{2}\cdot\sqrt{2\pi}} \cdot e^{-\frac{(\theta-40)^2}{2\cdot 2}}$$



$$w(r|\theta)\cdot w(\theta) = \frac{1}{2\cdot\sqrt{2\pi}} \cdot e^{-\frac{(\theta-35)^2 + (\theta-40)^2}{4}}$$

$$\hat{\theta}_{MAP} = \underset{\theta}{\operatorname{argmax}} \frac{1}{2\cdot 2\pi} \cdot e^{-\frac{(\theta-35)^2+(\theta-40)^2}{4}}$$

$$= \underset{\theta}{\operatorname{argmin}} \underbrace{(\theta-35)^2+(\theta-40)^2}_{D}$$

$e^{-x} = \frac{1}{e^{x}}$; x small $\Rightarrow$ big

D = minimal

$$\frac{\partial D}{\partial \theta} = 0 \iff 2(\theta-35) + 2(\theta-40) = \emptyset$$

$$\iff 2\cdot\theta = 35+40 \Rightarrow \hat{\theta}_{MAP} = \frac{35+40}{2} = 37.5$$

c). $\hat{\theta}_{MMSE} = E\{ w(\theta|r) \}$

Property:

$$w(r|\theta)\cdot w(\theta) = \mathcal{N}\left(\mu = \frac{\mu_1\cdot\sigma_2^2 + \mu_2\cdot\sigma_1^2}{\sigma_1^2+\sigma_2^2}, \sigma^2 = \frac{\sigma_1^2\sigma_2^2}{\sigma_1^2+\sigma_2^2}\right)$$

$\mu_1 = 40$ $\quad \mu_2 = 35$

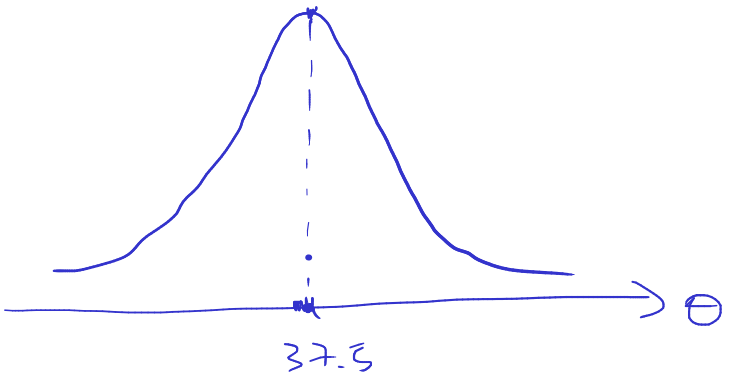
$\sigma_1^2 = 2$ $\quad \sigma_2 = 2$

$= \frac{40\cdot 2 + 35\cdot 2}{4}$ $\quad = \frac{2\cdot 2}{2+2}$

$= 20 + 17.5$ $\quad = 1$

$= 37.5$

$$= \mathcal{N}(\mu = 37.5, \sigma^2 = 1)$$



$\hat{\theta}_{MMSE}$ = mean of this function = $\mu = 37.5$

d) $$w(\theta) = \frac{1}{\sqrt{20}\cdot\sqrt{2\pi}} \cdot e^{-\frac{(\theta-35)^2}{2\cdot 20}}$$

$$w(x|\theta)\cdot w(\theta) = \frac{1}{\sqrt{2}\cdot\sqrt{2\pi}} \cdot \frac{1}{\sqrt{20}\cdot\sqrt{2\pi}} \cdot e^{-\frac{(\theta-35)^2}{40} - \frac{(\theta-40)^2}{4}}$$

$$= \frac{1}{\sqrt{40}\cdot 2\pi} \cdot e^{-\frac{(\theta-35)^2 + 10(\theta-40)^2}{40}}$$

minimum !

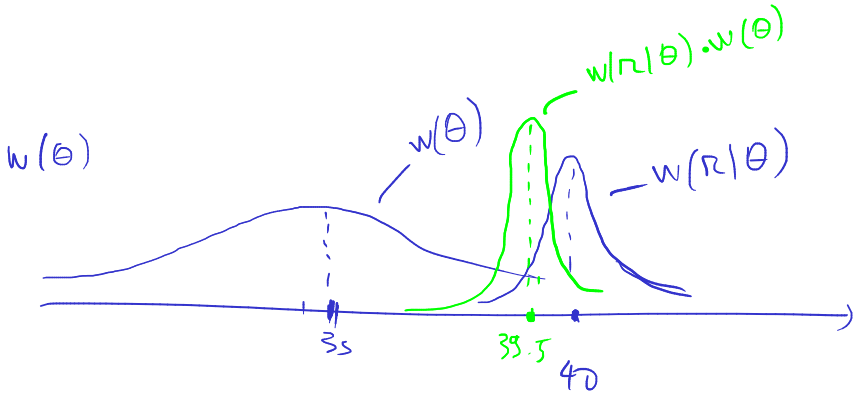
$$\Delta = (\theta-35)^2 + 10(\theta-40)^2$$

$$\frac{\partial \Delta}{\partial \theta} = 2(\theta-35) + 20(\theta-40) = 0$$

$$\Leftrightarrow \theta + 10\theta = 35 + 10\cdot 40$$

$$11\theta = 435 \Rightarrow \hat{\theta}_{MAP} = \frac{435}{11} = 39.5$$

$$\hat{\theta}_{MAP} = \theta_{MMSE}$$



(2)

$$\Lambda_\theta = \begin{bmatrix} \theta & \theta & \theta \end{bmatrix}$$

$$r = \begin{bmatrix} 40 & 38.1 & 39.2 \end{bmatrix}$$

$r_i = \underbrace{\theta}_{\Lambda_\theta(t)} + \text{noise}$

a) ML:

$$\hat{\theta}_{ML} = \underset{\theta}{\operatorname{argmax}}\ w(r|\theta)$$

$$= \underset{\theta}{\operatorname{argmin}}\ d(r, \Lambda_\theta)^2$$

$$\Delta = d(r, \Lambda_\theta)^2 = (\theta - 40)^2 + (\theta - 38.1)^2 + (\theta - 39.2)^2$$

$$\frac{\partial \Delta}{\partial \theta} = 2(\theta - 40) + 2(\theta - 38.1) + 2(\theta - 39.2) = 0$$

$$\Rightarrow \hat{\theta}_{ML} = \frac{40 + 38.1 + 39.2}{3}$$

b). c).

$$\hat{\theta}_{MAP} = \underset{\theta}{\operatorname{argmax}}\ w(r|\theta) \cdot w(\theta)$$

$$w(\theta) = \frac{1}{\sqrt{2}\cdot\sqrt{2\pi}} \cdot e^{-\frac{(\theta-35)^2}{2\cdot 2}}$$

$$w(r|\theta) = \frac{1}{\sqrt{2}\cdot\sqrt{2\pi}} \cdot e^{-\frac{(\theta-40)^2}{2\cdot 2}} \cdot \frac{1}{\sqrt{2}\sqrt{2\pi}} \cdot e^{-\frac{(\theta-38.1)^2}{2\cdot 2}} \cdot \frac{1}{\sqrt{2}\sqrt{2\pi}} \cdot e^{-\frac{(\theta-39.2)^2}{2\cdot 2}}$$

$$= \left(\frac{1}{\sqrt{2}\cdot\sqrt{2\pi}}\right)^3 \cdot e^{-\frac{(\theta-40)^2 + (\theta-38.1)^2 + (\theta-39.2)^2}{2\cdot 2}}$$

Want minimum

$$w(r|\theta)\cdot w(\theta) = \left(\frac{1}{\sqrt{2}\cdot\sqrt{2\pi}}\right)^4 \cdot e^{-\frac{(\theta-40)^2 + (\theta-38.1)^2 + (\theta-39.2)^2 + (\theta-35)^2}{2\cdot 2}}$$

$$\frac{\partial \Delta}{\partial \theta} = 0 \Rightarrow 2(\theta-40) + 2(\theta-38.1) + 2(\theta-39.2) + 2(\theta-35) = 0$$

$$\hat{\theta}_{MAP} = \frac{40 + 38.1 + 39.2 + 35}{4}$$

$\hat{\theta}_{MAP} = \hat{\theta}_{MMSE}$