

Decision and Estimation in Data Processing

Chapter II. Elements of Signal Detection Theory

II.1 Introduction

Introduction

- ▶ Signal detection = the problem of deciding which signal is present from 2 or more possibilities
 - ▶ one possibility may be that there is no signal
- ▶ Based on **noisy** observations
 - ▶ signals are affected by noise

The model for signal detection

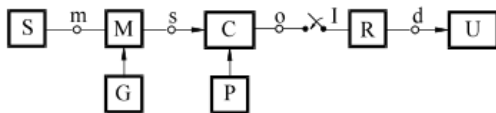


Figure 1: Signal detection model

► Contents:

- Information source: generates messages a_n with probabilities $p(a_n)$
- Modulator: transmits a signal $s_n(t)$ for message a_n
- Channel: adds random noise
- Sampler: takes samples from the signal $s_n(t)$
- Receiver: **decides** what message a_n has been transmitted

Practical scenarios

► Data transmission

- constant voltage levels (e.g. $s_n(t) = \text{constant}$)
- PSK modulation (Phase Shift Keying): $s_n(t) = \text{cosine}$ with same frequency but various initial phase
- FSK modulation (Frequency Shift Keying): $s_n(t) = \text{cosines}$ with different frequencies
- OFDM modulation (Orthogonal Frequency Division Multiplexing): particular case of FSK

► Radar

- a signal is emitted; if there is an obstacle, the signal gets reflected back
- the receiver waits for possible reflections of the signal and must decide
 - no reflection is present -> no object
 - reflected signal is present -> object detected

Generalizations

- ▶ Decide between more than two signals
- ▶ Number of observations:
 - ▶ use only one sample
 - ▶ use multiple samples
 - ▶ observe the whole continuous signal for some time T

II.2 Detection of constant signals based on 1 sample

Detection of a constant signal, 1 sample

- ▶ Simplest case: detection of a constant signal contaminated with white normal noise, using 1 sample
 - ▶ two messages a_0 and a_1
 - ▶ messages are encoded as constant signals
 - ▶ for a_0 : send $s_0(t) = 0$
 - ▶ for a_1 : send $s_1(t) = A$
 - ▶ over the signals there is additive noise
 - ▶ receiver takes just 1 sample
 - ▶ decision: compare sample with a threshold

Threshold-based decision

- ▶ The value of the sample taken is $r = s + n$
 - ▶ s is the true underlying signal ($s_0 = 0$ or $s_1 = A$)
 - ▶ n is a sample of the noise
- ▶ n is a (continuous) random variable
- ▶ r is a random variable also
 - ▶ what distribution does r have compared to n ?
- ▶ Decision is taken by comparing with a threshold T :
 - ▶ if $r < T$, take decision D_0 : decide the true signal is s_0
 - ▶ if $r \geq T$, take decision D_1 : decide the true signal is s_1

Hypotheses

- ▶ Receiver chooses between **two hypotheses**:
 - ▶ H_0 : true signal is s_0 (a_0 has been transmitted)
 - ▶ H_1 : true signal is s_1 (a_1 has been transmitted)
- ▶ Possible results
 1. Correct rejection: no signal present, no signal detected.
 - ▶ Decision D_0 when hypothesis is H_0
 - ▶ Probability is $P_n = P(D_0 \cap H_0)$
 2. **False alarm**: no signal present, signal detected (error)
 - ▶ Decision D_1 when hypothesis is H_0
 - ▶ Probability is $P_{fa}P(D_1 \cap H_0)$
 3. **Miss**: signal present, no signal detected (error)
 - ▶ Decision D_0 when hypothesis is H_1
 - ▶ Probability is $P_m = P(D_0 \cap H_1)$
 4. **Hit**: signal present, signal detected
 - ▶ Decision D_1 when hypothesis is H_1
 - ▶ Probability is $P_d = P(D_1 \cap H_1)$

Maximum likelihood criterion

- ▶ Choose the hypothesis that **seems most likely** given the observed sample r
- ▶ The **likelihood** of an observation r = the probability density of r given a hypothesis H_0 or H_1
- ▶ Likelihood in case of hypothesis H_0 : $w(r|H_0)$
 - ▶ r is only noise, so value is taken from the noise distribution
- ▶ Likelihood in case of hypothesis H_1 : $w(r|H_1)$
 - ▶ r is $A + \text{noise}$, so value is taken from the distribution of $(A + \text{noise})$
- ▶ **Likelihood ratio** test:

$$\frac{w(r|H_1)}{w(r|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} 1$$

Graphical interpretation

- ▶ Consider noise having a normal distribution
- ▶ Plot the two density functions for H_0 , H_1

Decision via threshold

- ▶ Likelihood ratio test for ML = comparing r with a threshold T
- ▶ The threshold = the cross-over point of the two distributions

Normal noise

- ▶ Particular case: the noise has normal distribution $\mathcal{N}(0, \sigma^2)$
- ▶ Likelihood ratio is $\frac{w(r|H_1)}{r|H_0} = \frac{e^{-\frac{(r-A)^2}{2\sigma^2}}}{e^{-\frac{r^2}{2\sigma^2}}} \underset{H_0}{\overset{H_1}{\geq}} 1$
- ▶ For normal distribution, it is easier to apply *natural logarithm* to the terms
 - ▶ logarithm is a monotonic increasing function, so it won't change the comparison
 - ▶ if $A < B$, then $\log(A) < \log(B)$
- ▶ The **log-likelihood** of an observation = the logarithm of the likelihood value
 - ▶ usually the natural logarithm, but any one can be used

Log-likelihood test for ML

- ▶ For normal noise, the ML decision means the log-likelihood test

$$\frac{(r - A)^2}{r^2} \underset{H_0}{\overset{H_1}{\gtrless}} 1$$

- ▶ Applying square root

$$\frac{|r - A|}{|r|} \underset{H_0}{\overset{H_1}{\gtrless}} 1$$

- ▶ $|r - A|$ = distance from r to A , $|r|$ = distance from r to 0
- ▶ ML decision with normal noise: choose the value 0 or A which is **nearest** to r
 - ▶ very general principle, encountered in many other scenarios
 - ▶ also known as **nearest neighbor** principle / decision
 - ▶ ML receiver is also known as **minimum distance receiver**
 - ▶ equivalent with setting a threshold $T = \frac{A}{2}$

Generalizations

- ▶ What if the noise has another distribution?
 - ▶ Threshold T is still the cross-over point, whatever that is
 - ▶ There can be more cross-overs, so multiple thresholds
 - ▶ Can think that \mathbb{R} axis is split into **decision regions** R_0 and R_1
- ▶ What if the noise distributions are different for H_0 and H_1 ?
 - ▶ Threshold T is the cross-over point, whatever that is
- ▶ What if the signal $s_0(t)$ (for H_0) is not 0, but another constant value B ?
 - ▶ T is the crossover point, the distributions are centered on B and A
 - ▶ In case of normal noise, choose B or A whichever is nearest (threshold is at middle point)

Generalizations

- ▶ What if we have more than two signal levels?
 - ▶ e.g. 4 possible signals: -6, -2, 2, 6
 - ▶ Just choose the most likely hypothesis, out of 4 likelihood functions
 - ▶ Not a single threshold value, now there are more

Exercises

- ▶ A signal can have two possible values, 0 or 5. The receiver takes one sample with value $r = 2.25$
 1. Considering that the noise is white gaussian noise, what signal is decided based on the Maximum Likelihood criterion?
 2. What if the signal 0 is affected by gaussian noise $\mathcal{N}(0, 0.5)$, while the signal 5 is affected by uniform noise $\mathcal{U}[-4, 4]$?
 3. Repeat a. and b. assuming the value 0 is replaced by -1
- ▶ A signal can have four possible values: -6, -2, 2, 6. Each value lasts for 1 second. The signal is affected by white noise with normal distribution. The receiver takes 1 sample per second. Using ML criterion, decide what signal has been transmitted, if the received samples are:

4, 6.6, -5.2, 1.1, 0.3, -1.5, 7, -7, 4.4

Computing conditional error probabilities

- ▶ We can compute the conditional probabilities of errors
- ▶ Consider the decision regions:
 - ▶ R_0 : when $r \in R_0$, decision is D_0 , i.e. $(-\infty, T)$ for gaussian noise
 - ▶ R_1 : when $r \in R_1$, decision is D_1 , i.e. $[T, \infty)$ for gaussian noise
- ▶ Probability of false alarm **if** original signal is $s_0(t)$

$$P(D_1|H_0) = \int_{R_1} w(r|H_0)dx$$

- ▶ Probability of miss **if** original signal is $s_1(t)$

$$P(D_0|H_1) = \int_{R_0} w(r|H_1)dx$$

- ▶ These probabilities do not account for the probability that the signal actually is $s_0(t)$ or $s_1(t)$
 - ▶ they are **conditional** (“if”)

Computing conditional error probabilities

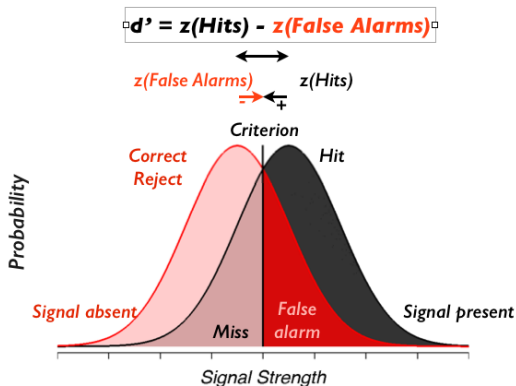


Figure 2: Conditional error probabilities

[image from <http://gru.stanford.edu/doku.php/tutorials/sdt>]

Reminder: the Bayes rule

- ▶ Reminder: the Bayes rule

$$P(A \cap B) = P(B|A) \cdot P(A)$$

- ▶ Interpretation
 - ▶ The probability $P(A)$ is taken out from $P(B|A)$
 - ▶ $P(B|A)$ gives no information on $P(A)$, the chances of A actually happening
 - ▶ Example: $P(\text{score} \mid \text{shoot}) = \frac{1}{2}$. How many goals are scored?

Exercise

- ▶ A signal can have two possible values, 0 or 5. The signal 0 is affected by gaussian noise $\mathcal{N}(0, 0.5)$, while the signal 5 is affected by uniform noise $\mathcal{U}[-4, 4]$. The receiver performs ML decision based on a single sample.
 1. Compute the probability of a wrong decision when the original signal is $s_0(t)$
 2. Compute the probability of a wrong decision when the original signal is $s_1(t)$

Pitfalls of ML decision criterion

- ▶ The ML is based on comparing **conditional** probability density functions
 - ▶ conditioned by H_0 or by H_1
- ▶ Conditioning by H_0 and H_1 ignores the probability of H_0 or H_1 actually happening
 - ▶ We don't know how $p(H_0)$ or $P(H_1)$
- ▶ If $p(H_0) > p(H_1)$, we may want to move the threshold towards H_1 , and vice-versa
 - ▶ because it is more likely that the signal is $s_0(t)$
 - ▶ and thus we want to “encourage” decision D_0

The minimum error probability criterion

- ▶ Takes into account the probabilities $P(H_0)$ and $P(H_1)$
- ▶ Goal is to **minimize the total probability of error** P_e
 - ▶ errors = false alarms and misses
- ▶ We need to find the decision regions R_0 and R_1

Probability of error

- ▶ Probability of false alarm

$$\begin{aligned}P(D_1 \cap H_0) &= P(D_1|H_0) \cdot P(H_0) \\&= \int_{R_1} w(r|H_0) dx \cdot P(H_0) \\&= (1 - \int_{R_0} w(r|H_0) dx) \cdot P(H_0)\end{aligned}$$

- ▶ Probability of miss

$$\begin{aligned}P(D_0 \cap H_1) &= P(D_0|H_1) \cdot P(H_1) \\&= \int_{R_0} w(r|H_1) dx \cdot P(H_1)\end{aligned}$$

- ▶ The sum is

$$P_e = P(H_0) + \int_{R_0} [w(r|H_1) \cdot P(H_1) - w(r|H_0) \cdot P(H_0)] dx$$

Minimum probability of error

- ▶ We want to minimize P_e , i.e. to minimize the integral
- ▶ To minimize the integral, we choose R_0 such that for all $r \in R_0$, the term inside the integral is **negative**
 - ▶ because integrating over all the interval where the function is negative ensures minimum value of integral
- ▶ So, when $w(r|H_1) \cdot P(H_1) - w(r|H_0) \cdot P(H_0) < 0$ we have $r \in R_0$, i.e. decision D_0
- ▶ Conversely, when $w(r|H_1) \cdot P(H_1) - w(r|H_0) \cdot P(H_0) > 0$ we have $r \in R_1$, i.e. decision D_1
- ▶ Therefore

$$w(r|H_1) \cdot P(H_1) - w(r|H_0) \cdot P(H_0) \underset{H_0}{\overset{H_1}{\gtrless}} 0$$

$$\frac{w(r|H_1)}{w(r|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{P(H_0)}{P(H_1)}$$

Interpretation

- ▶ Similar to ML, but threshold depends on probabilities of the two hypotheses
 - ▶ When one hypotheses is more likely than the other, the threshold is pushed in its favor, towards the other
- ▶ Also based on a **likelihood ratio** test, just like ML

Minimum probability of error - gaussian noise

- Assuming the noise is gaussian (normal), $\mathcal{N}(0, \sigma^2)$

$$w(r|H_1) = e^{-\frac{(r-A)^2}{2\sigma^2}}$$

$$w(r|H_0) = e^{-\frac{r^2}{2\sigma^2}}$$

- Apply natural logarithm

$$-\frac{(r-A)^2}{2\sigma^2} + \frac{r^2}{2\sigma^2} \underset{H_0}{\overset{H_1}{\geq}} \ln \left(\frac{P(H_0)}{P(H_1)} \right)$$

- Equivalently

$$2rA - A^2 \underset{H_0}{\overset{H_1}{\geq}} 2\sigma^2 \cdot \ln \left(\frac{P(H_0)}{P(H_1)} \right)$$

$$r \underset{H_0}{\overset{H_1}{\geq}} \underbrace{\frac{A^2 + 2\sigma^2 \cdot \ln \left(\frac{P(H_0)}{P(H_1)} \right)}{2A}}_T$$

Decision regions

- ▶ We still compare with a threshold T , but its value is shifted towards the less probable hypothesis
 - ▶ T depends on the ratio $\frac{P(H_0)}{P(H_1)}$
- ▶ Decision regions
 - ▶ $R_0 = (-\infty, T]$
 - ▶ $R_1 = [T, \infty)$
 - ▶ will be different for other noise distributions (non-gaussian)

- An information source provides two messages with probabilities $p(a_0) = \frac{2}{3}$ and $p(a_1) = \frac{1}{3}$. The messages are encoded as constant signals with values -5 (a_0) and 5 (a_1). The signals are affected by gaussian noise $\mathcal{N}(0, \sigma^2 = 1)$. The receiver takes one sample r . Decision is done by comparing r with a threshold value T , as follows: if $r < T$ it is decided that the transmitted message is a_0 , otherwise it is a_1 .
1. Find the threshold value T according to the minimum probability of error criterion
 2. What if the signal 5 is affected by uniform noise $\mathcal{U}[-4, 4]$?
 3. What are the probabilities of false alarm and of miss?

Minimum risk (cost) criterion

- ▶ What if we care more about one type of errors (e.g. false alarms) than other kind (e.g. miss)?
- ▶ Minimum risk (cost) criterion: assign costs to decisions, minimize average cost
 - ▶ C_{ij} = cost of decision D_i when true hypothesis was H_j
 - ▶ C_{00} = cost for good detection D_0 in case of H_0
 - ▶ C_{10} = cost for false alarm (detection D_1 in case of H_0)
 - ▶ C_{01} = cost for miss (detection D_0 in case of H_1)
 - ▶ C_{11} = cost for good detection D_1 in case of H_1
- ▶ The risk = the average cost

$$R = C_{00}P(D_0 \cap H_0) + C_{10}P(D_1 \cap H_0) + C_{01}P(D_0 \cap H_1) + C_{11}P(D_1 \cap H_1)$$

- ▶ Minimum risk criterion: **minimize the risk R**

Computations

- ▶ Proof on table:
 - ▶ Use Bayes rule
 - ▶ Notations: $w(r|H_j)$ (*likelihood*)
 - ▶ Probabilities: $\int_{R_i} w(r|H_j) dV$
- ▶ Conclusion, **decision rule is**

$$\frac{w(r|H_1)}{w(r|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{(C_{10} - C_{00})p(H_0)}{(C_{01} - C_{11})p(H_1)}$$

Interpretation

- ▶ Similar to ML and to minimum probability of error criteria
 - ▶ also uses a **likelihood ratio** test
- ▶ Both probabilities and the assigned costs can move threshold towards one side or the other
- ▶ If $C_{10} - C_{00} = C_{01} - C_{11}$, reduces to previous criterion (minimum probability of error)
 - ▶ e.g. if $C_{00} = C_{11} = 0$, and $C_{10} = C_{01}$

In gaussian noise

- ▶ If the noise is gaussian (normal), then similar to other criteria, apply logarithm
- ▶ Equivalently

$$-(r - A)^2 + r^2 \underset{H_0}{\overset{H_1}{\geq}} \underbrace{2\sigma^2 \cdot \ln \left(\frac{(C_{10} - C_{00})p(H_0)}{(C_{01} - C_{11})p(H_1)} \right)}_C$$

$$r \underset{H_0}{\overset{H_1}{\geq}} \underbrace{\frac{A^2 + 2\sigma^2 \cdot \ln \left(\frac{(C_{10} - C_{00})p(H_0)}{(C_{01} - C_{11})p(H_1)} \right)}{2A}}_T$$

In gaussian noise

- ▶ In general, for likelihood ratio test $\frac{w(r|H_1)}{w(r|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} K$, the threshold is

$$T = \frac{A^2 + 2\sigma^2 \cdot \ln K}{2A}$$

Example

- ▶ Example at blackboard: 0 / 5, random noise with $N(0, \sigma^2)$, one sample

Neymar-Pearson criterion

- ▶ Neymar-Pearson criterion: maximize probability of a hit ($P(D_1 \cap H_1)$) while keeping probability of false alarms smaller than a limit ($P(D_1 \cap H_0) \leq \lambda$)
- ▶ Deduce the threshold T from the limit condition $P(D_1 \cap H_0) = \lambda$

Exercise

- ▶ An information source provides two messages with probabilities $p(a_0) = \frac{2}{3}$ and $p(a_1) = \frac{1}{3}$.
- ▶ The messages are encoded as constant signals with values -5 (a_0) and 5 (a_1).
- ▶ The signals are affected by noise with triangular distribution $[-5, 5]$.
- ▶ The receiver takes one sample r .
- ▶ Decision is done by comparing r with a threshold value T , as follows: if $r < T$ it is decided that the transmitted message is a_0 , otherwise it is a_1 .
 1. Find the threshold value T according to the Neyman-Pearson criterion, considering $P_{fa} \leq 10^{-2}$
 2. What is the probability of hit?

Two non-zero levels

- ▶ What if the s_0 signal is not 0, but another constant signal $s_0 = B$?
- ▶ Noise distribution $w(r|H_0)$ is centered on B , not 0
- ▶ Otherwise, everything else stays the same
- ▶ Performance is defined by the gap between the two levels ($A - B$)
 - ▶ same performance if $s_0 = 0$, $s_1 = A$ or if $s_0 = -\frac{A}{2}$ and $s_1 = \frac{A}{2}$
- ▶ Valid for all decision criteria

Differential vs single-ended signalling

- ▶ Single-ended signaling: one signal is 0, other is non-zero
 - ▶ $s_0 = 0, s_1 = A$
- ▶ Differential signaling: use two non-zero levels with different sign, same absolute value
 - ▶ $s_0 = -\frac{A}{2}, s_1 = \frac{A}{2}$
- ▶ Which is better?

Differential vs single-ended signalling

- ▶ If gap difference between levels is the same, performance is the same
- ▶ Average power of a signal = average squared value
- ▶ For differential signal: $P = \left(\pm \frac{A}{2}\right)^2 = \frac{A^2}{4}$
- ▶ For signal ended signal: $P = P(H_0) \cdot 0 + P(H_1) (A)^2 = \frac{A^2}{2}$
 - ▶ assuming equal probabilities $P(H_0) = P(H_1) = \frac{1}{2}$
- ▶ Differential uses half the power of single-ended (i.e. better)

Summary of criteria

- ▶ We have seen decision based on 1 sample r , between 2 constant levels
- ▶ All decisions are based on a likelihood-ratio test

$$\frac{w(r|H_1)}{w(r|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} K$$

- ▶ Different criteria differ in the chosen value of K (likelihood threshold)
- ▶ Depending on the noise distributions, the real axis is partitioned into regions
 - ▶ region R_0 : if r is in here, decide D_0
 - ▶ region R_1 : if r is in here, decide D_1
 - ▶ e.g. $R_0 = (-\infty, \frac{A+B}{2}]$, $R_1 = (\frac{A+B}{2}, \infty)$ (ML)

Summary of criteria

- ▶ For gaussian noise,

Receiver Operating Characteristic

- ▶ The receiver performance is usually represented with “**Receiver Operating Characteristic**” (ROC) graph
- ▶ It is a graph of hit probability $P_d = P(D_1 \cap H_1)$ (correct detection) as a function of false alarm probability $P_{fa} = P(D_1 \cap H_0)$

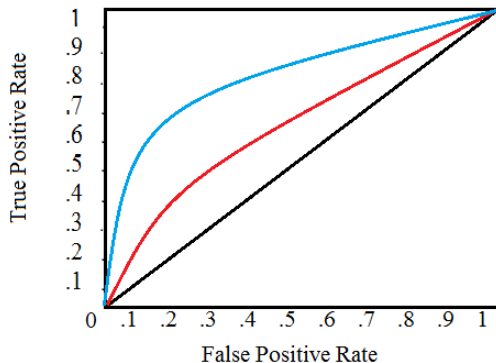


Figure 3: Sample ROC curves

Receiver Operating Characteristic

- ▶ There is always a **tradeoff** between good P_d and bad P_{fa}
 - ▶ to increase P_d one must also increase P_{fa}
 - ▶ if we want to make sure we don't miss any real detections (increase P_d), we pay by increasing the chances of false alarms
- ▶ Different criteria = different likelihood thresholds K = different points on the graph = different tradeoffs
 - ▶ but the tradeoff cannot be avoided
- ▶ How to improve the receiver?
 - ▶ i.e. increase P_D while keeping P_{fa} the same

Performance of likelihood-ratio decoding in WGN

- ▶ WGN = “White Gaussian Noise”
- ▶ Assume equal probabilities $P(H_0) = P(H_1) = \frac{1}{2}$
- ▶ All decisions are based on a likelihood-ratio test

$$\frac{w(r|H_1)}{w(r|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} K$$

- ▶ Hit probability is

$$\begin{aligned} P_{hit} &= P(D_1|H_1)P(H_1) \\ &= P(H_1) \int_T^\infty w(r|H_1) \\ &= P(H_1)(F(\infty) - F(T)) \\ &= \frac{1}{4} \left(1 - \operatorname{erf} \left(\frac{T - A}{\sqrt{2}\sigma} \right) \right) \\ &= Q \left(\frac{T - A}{\sqrt{2}\sigma} \right) \end{aligned}$$

Performance of likelihood-ratio decoding in WGN

- False alarm probability is

$$\begin{aligned}P_{fa} &= P(D_1|H_0)P(H_0) \\&= P(H_0) \int_T^\infty w(r|H_0) \\&= P(H_0)(F(\infty) - F(T)) \\&= \frac{1}{4} \left(1 - \operatorname{erf} \left(\frac{T - 0}{\sqrt{2}\sigma} \right) \right) \\&= Q \left(\frac{T}{\sqrt{2}\sigma} \right)\end{aligned}$$

- Therefore $\frac{T}{\sqrt{2}\sigma} = Q^{-1}(P_{fa})$
- Replacing in P_{hit} yields

$$P_{hit} = Q \left(\underbrace{Q^{-1}(P_{fa})}_{\text{constant}} - \frac{A}{\sqrt{2}\sigma} \right)$$

Signal-to-noise ratio

- ▶ **Signal-to-noise ratio (SNR)** = $\frac{\text{power of original signal}}{\text{power of noise}}$
- ▶ Average power of a signal = average squared value = $\overline{X^2}$
 - ▶ Original signal power is $\frac{A^2}{2}$
 - ▶ Noise power is $\overline{X^2} = \sigma^2$ (when noise mean value $\mu = 0$)
- ▶ In our case, $\text{SNR} = \frac{A^2}{2\sigma^2}$

$$P_{hit} = Q \left(\underbrace{Q^{-1}(P_{fa})}_{\text{constant}} - \sqrt{\text{SNR}} \right)$$

- ▶ For a fixed P_{fa} , P_{hit} increases with SNR
 - ▶ Q is a monotonic decreasing function

Performance depends on SNR

- ▶ Receiver performance increases with SNR increase
 - ▶ high SNR: good performance
 - ▶ poor SNR: bad performance

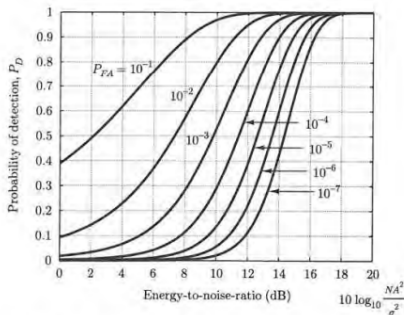


Figure 4: Detection performance depends on SNR

[source: *Fundamentals of Statistical Signal Processing*, Steven Kay]

Decision between hypotheses

- ▶ Statistical decision is not useful merely for detecting signals
- ▶ We are in fact deciding between two different probability distributions
 - ▶ regardless of what the two distributions mean
- ▶ For detection of constant signals, we choose between two distributions with **different average value**, generally
 - ▶ one distribution has average value 0, the other one A
- ▶ But we can choose between distributions that differ in other parameters
 - ▶ average value, or
 - ▶ variance, or
 - ▶ shape, etc

Decision between hypotheses

- ▶ Example: We have a sample with value $r = 2.5$. It can come from a distribution $\mathcal{N}(0, \sigma^2 = 1)$ (hypothesis H_0) or from $\mathcal{N}(0, \sigma^2 = 2)$ (hypothesis H_1). Which hypothesis do we think is true?
 - ▶ It is the variance that differs, not the average value
- ▶ We can use the exact same criteria as before
 - ▶ Draw the two distributions
 - ▶ Compute the likelihoods $w(r|H_0)$ and $w(r|H_1)$ for r
 - ▶ Decide based on likelihood ratio using some criterion

II.3 Detection of constant signals with multiple samples

Multiple samples from a constant signal

- ▶ Suppose we have multiple samples, not just 1
- ▶ The samples are arranged in a **sample vector**

$$\mathbf{r} = [r_1, r_2, \dots, r_N]$$

- ▶ In each hypotheses, the signal is a **random process**
 - ▶ H_0 : random process with average value 0
 - ▶ H_1 : random process with average value A
- ▶ Assuming the noise is stationary and ergodic, the signal random process is stationary and ergodic (signal = constant + noise)
- ▶ The values of \mathbf{r} are described by the **distribution of order N** of the random processes, $w_N(\mathbf{r}) = w_N(r_1, r_2, \dots, r_N)$
- ▶ Assuming the noise is white noise, the sample times don't matter

Likelihood-based of vector samples

- ▶ We can apply **the same criteria** based on likelihood ratio as for 1 sample

$$\frac{w_N(\mathbf{r}|H_0)}{w_N(\mathbf{r}|H_1)} \underset{H_0}{\overset{H_1}{\gtrless}} K$$

- ▶ Notes
 - ▶ \mathbf{r} is a vector; we consider the likelihood of all the samples
 - ▶ the hypotheses H_0 and H_1 are the same as for 1 sample
 - ▶ $w_N(\mathbf{r}|H_0)$ = likelihood of the whole vector \mathbf{r} being obtained in hypothesis H_0
 - ▶ $w_N(\mathbf{r}|H_1)$ = likelihood of the whole vector \mathbf{r} being obtained in hypothesis H_1
 - ▶ the value of K is given by the actual decision criterion used
- ▶ Interpretation: we choose the hypothesis that is most likely to have produced the observed data
 - ▶ the same, but now the data = multiple samples

- ▶ Assuming the noise is white noise, the samples r_i are **multiple independent realizations of the same distribution**
- ▶ In that case the joint distributions $w_N(\mathbf{r}|H_j)$ can be decomposed as a product

$$w_N(\mathbf{r}|H_j) = w(r_1|H_j) \cdot w(r_2|H_j) \cdot \dots \cdot w(r_N|H_j)$$

- ▶ The $w(r_i|H_j)$ are just the likelihoods of each individual sample
 - ▶ e.g. the likelihood of obtaining $[5.1, 4.7, 4.9]$ = likelihood of obtaining 5.1 \times likelihood of getting 4.7 \times likelihood of getting 4.9

- ▶ Then all likelihood ratio criteria can be written as

$$\frac{w_N(\mathbf{r}|H_1)}{w_N(\mathbf{r}|H_0)} = \frac{w(r_1|H_1)}{w(r_1|H_0)} \cdot \frac{w(r_2|H_1)}{w(r_2|H_0)} \cdots \frac{w(r_N|H_1)}{w(r_N|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} K$$

- ▶ The likelihood ratio of a vector of samples = product of likelihood ratio for each sample

Particular case: AWGN

- ▶ AWGN = “Additive White Gaussian Noise”
- ▶ In hypothesis H_1 : $w(r_i|H_1) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(r_i-A)^2}{2\sigma^2}}$
- ▶ In hypothesis H_0 : $w(r_i|H_1) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{r_i^2}{2\sigma^2}}$
- ▶ Likelihood ratio for vector \mathbf{r}

$$\frac{w_N(\mathbf{r}|H_1)}{w_N(\mathbf{r}|H_0)} = \frac{e^{-\frac{\sum (r_i-A)^2}{2\sigma^2}}}{e^{-\frac{\sum (r_i)^2}{2\sigma^2}}}$$

- ▶ We can interpret this likelihood ratio in two ways

Interpretation 1: average value of samples

- Interpretation 1: average value of samples

$$\begin{aligned}\frac{w_N(\mathbf{r}|H_1)}{w_N(\mathbf{r}|H_0)} &= \frac{e^{-\frac{\sum (r_i - A)^2}{2\sigma^2}}}{e^{-\frac{\sum (r_i)^2}{2\sigma^2}}} \\ &= e^{-\frac{\sum (r_i - A)^2 - \sum (r_i)^2}{2\sigma^2}} \\ &= e^{-\frac{\sum (r_i^2 - 2r_i A + A^2) - \sum (r_i)^2}{2\sigma^2}} \\ &= e^{-\frac{\sum (-2r_i A + A^2)}{2\sigma^2}} \\ &= e^{-\frac{-2A \sum (r_i) + N A^2}{2\sigma^2}} \\ &= e^{-\frac{-2A \frac{\sum (r_i)}{N} + A^2}{2 \frac{\sigma^2}{N}}}\end{aligned}$$

Average value of N gaussian random variables

- ▶ Let U_r = average value of the samples r_i

$$U_r = \frac{1}{N} \sum r_i$$

- ▶ What distribution does it have?
- ▶ Consider the sum $S_r = \sum r_i$ of the N samples r_i
 - ▶ From chapter 1: the sum of normal r.v. $\mathcal{N}(\mu, \sigma^2)$ has:
 - ▶ normal distribution $\mathcal{N}(\mu_S, \sigma_S^2)$ with
 - ▶ average value: $\mu_S = N \cdot \mu$
 - ▶ variance: $\sigma_S^2 = N \cdot \sigma^2$
- ▶ Then $U_r = \frac{1}{N} S_r$, and from the properties of average values we have
 - ▶ U_r has normal distribution with:
 - ▶ average value $= \frac{1}{N} \mu_S = \frac{1}{N} N \mu = \mu$
 - ▶ variance $= \left(\frac{1}{N}\right)^2 \sigma_S^2 = \left(\frac{1}{N}\right)^2 N \sigma^2 = \frac{1}{N} \sigma^2$

Average value of N gaussian random variables

- ▶ The mean value of N realizations of a normal distribution has a normal distribution with
 - ▶ same average value
 - ▶ variance N times smaller
- ▶ If N gets very large, the mean value is a very good **estimator** of the distribution's average value
 - ▶ its distribution gets very narrow around the average value

Interpretation 1: average value of samples

$$\begin{aligned}\frac{w_N(\mathbf{r}|H_1)}{w_N(\mathbf{r}|H_0)} &= e^{\frac{-2AU_r + A^2}{2\frac{\sigma^2}{N}}} \\ &= \frac{e^{\frac{U_r^2 - 2AU_r + A^2}{2\frac{\sigma^2}{N}}}}{e^{\frac{U_r^2}{2\frac{\sigma^2}{N}}}} \\ &= \frac{e^{\frac{(U_r - A)^2}{2\frac{\sigma^2}{N}}}}{e^{\frac{U_r^2}{2\frac{\sigma^2}{N}}}} \\ &= \frac{w(U_r|H_1)}{w(U_r|H_0)}\end{aligned}$$

- The likelihood ratio of N gaussian samples = the likelihood ratio of **the mean of the samples**

Interpretation 1: average value of samples

- ▶ The likelihood ratio of N gaussian samples = the likelihood ratio of **the mean of the samples**
 - ▶ the mean has smaller variance $\frac{1}{N}\sigma^2$, so is more accurate
 - ▶ it is like the noise distribution gets N times narrower (due to averaging)
- ▶ Detection of constant signals with N samples is the same as detection with 1 sample, but:
 - ▶ use the average value of the samples r_i
 - ▶ its distributions are N times narrower (variance is N times smaller)
- ▶ As N increases, the probability of errors decrease \Rightarrow better performance

Exercise

Exercise:

- ▶ A signal can have two values, 0 (hypothesis H_0) or 6 (hypothesis H_1). The signal is affected by AWGN $\mathcal{N}(0, \sigma^2 = 1)$. The receiver takes 5 samples with values $\{1.1, 4.4, 3.7, 4.1, 3.8\}$.
 1. What is decision according to Maximum Likelihood criterion?
 2. What is decision according to minimum probability of error criterion, assuming $P(H_0) = 2/3$ and $P(H_1) = 1/3$?

Interpretation 2: geometrical

- ▶ Useful mainly for Maximum Likelihood criterion
- ▶ Likelihood ratio for vector \mathbf{r}

$$\frac{w_N(\mathbf{r}|H_1)}{w_N(\mathbf{r}|H_0)} = \frac{e^{-\frac{\sum (r_i - A)^2}{2\sigma^2}}}{e^{-\frac{\sum (r_i)^2}{2\sigma^2}}} \underset{H_0}{\overset{H_1}{\geq}} K$$

- ▶ For Maximum Likelihood we compare to 1

$$\frac{e^{-\frac{\sum (r_i - A)^2}{2\sigma^2}}}{e^{-\frac{\sum (r_i)^2}{2\sigma^2}}} \underset{H_0}{\overset{H_1}{\geq}} 1$$

$$e^{-\frac{\sum (r_i - A)^2}{2\sigma^2} + \frac{\sum (r_i)^2}{2\sigma^2}} \underset{H_0}{\overset{H_1}{\geq}} 1$$

$$-\sum (r_i - A)^2 + \sum (r_i)^2 \underset{H_0}{\overset{H_1}{\geq}} 0$$

$$\sum (r_i)^2 \underset{H_0}{\overset{H_1}{\geq}} \sum (r_i - A)^2$$

Interpretation 2: geometrical

- ▶ $\sqrt{\sum (r_i)^2}$ is the geometrical (Euclidian) distance between point $\mathbf{r} = [r_1, r_2, \dots, r_N]$ and point $\mathbf{0} = [0, 0, \dots, 0]$
- ▶ $\sqrt{\sum (r_i - A)^2}$ is the geometrical (Euclidian) distance between point $\mathbf{r} = [r_1, r_2, \dots, r_N]$ and point $\mathbf{A} = [A, A, \dots, A]$
- ▶ ML decision chooses **the closest signal vector (point)** to the received vector (point), in a N-dimensional space
 - ▶ it is known as “minimum distance receiver”
 - ▶ same interpretation as in the 1-D case
- ▶ Question: what is the geometrical interpretation for the other criteria?

Exercise

Exercise:

- ▶ A signal can have two values, 0 (hypothesis H_0) or 6 (hypothesis H_1). The signal is affected by AWGN $\mathcal{N}(0, \sigma^2 = 1)$. The receiver takes 2 samples with values $\{1.1, 4.4\}$.
 1. What is decision according to Maximum Likelihood criterion? Use the geometrical interpretation.

Interpretation 3: correlation value

- Likelihood ratio for vector \mathbf{r}

$$\frac{w_N(\mathbf{r}|H_1)}{w_N(\mathbf{r}|H_0)} = \frac{e^{-\frac{\sum (r_i - A)^2}{2\sigma^2}}}{e^{-\frac{\sum (r_i)^2}{2\sigma^2}}} \underset{H_0}{\overset{H_1}{\gtrless}} K$$

$$e^{-\frac{\sum (r_i - A)^2}{2\sigma^2} + \frac{\sum (r_i)^2}{2\sigma^2}} \underset{H_0}{\overset{H_1}{\gtrless}} K$$

$$-\sum (r_i - A)^2 + \sum (r_i)^2 \underset{H_0}{\overset{H_1}{\gtrless}} 2\sigma^2 \ln K$$

$$2\sum r_i A - NA^2 \underset{H_0}{\overset{H_1}{\gtrless}} 2\sigma^2 \ln K$$

$$\frac{1}{N}\sum r_i A \underset{H_0}{\overset{H_1}{\gtrless}} \underbrace{\frac{A^2}{2} + \frac{1}{N}\sigma^2 \ln K}_{L=\text{const}}$$

Interpretation 3: correlation value

- ▶ The **correlation value** (sometimes just “the correlation”) of two signals x and y is

$$C_{x,y} = \frac{1}{N} \sum x[n]y[n]$$

- ▶ It is the value of the correlation function in 0

$$C_{x,y} = R_{xy}[0] = \overline{x[n]y[n+0]}$$

- ▶ For continuous signals

$$C_{x,y} = \frac{1}{T} \int_{T/2}^{T/2} x(t)y(t)dt$$

- ▶ $\frac{1}{N} \sum r_i A$ is the correlation value of the received samples $\mathbf{r} = [r_1, r_2, \dots, r_N]$ with the **target** samples $\mathbf{A} = [A, A, \dots, A]$

Interpretation 3: correlation value

- ▶ If the correlation value of the received samples with the target samples $\mathbf{A} = [A, A, \dots A]$ is greater than a certain threshold L , we decide that signal is detected.
 - ▶ otherwise, the signal is rejected
- ▶ This is **similar to signal detection based on 1 sample**, with the sample value being $C_{x,y}$

