Decision and Estimation in Data Processing

Chapter I. Random Signals

I.1 Random variables

Random variables

- A **random variable** is a variable that holds a value produced by a (partially) random phenomenon
  - basically it is *a name* attached to an arbitrary value
  - short notation: r.v.
- Typically denoted as X, Y etc..
- Examples:
  - $X =$ The value of a dice
  - $V_{in} =$ The value of the voltage in one point of a circuit

Off-topic: Glossary

- "*i.e.*" = *id est* = "that is" = "adică"
- "*e.g.*" = *exampli gratia* = "for example" = "de exemplu"

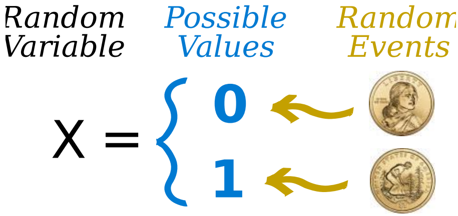
Realizations

- **A realization** of a random variable = one possible value it can take
  - e.g. the value 3 of a dice
  - at different times, one may get different realizations
- **Sample space** Ω = the set of all values that can be taken by a random variable X
  - i.e. the set of all possible realizations
- Example: rolling a dice
  - The r.v. is denoted as X
  - We might get a realization $X = 6$
  - But we could have got any value from the sample space

$$\Omega = \{1, 2, 3, 4, 5, 6\}$$

Rolling a die

- Random variable X = "the face obtained by throwing a coin"



(image from *https://www.mathsisfun.com/data/random-variables.html*)

Discrete and continuous random variables

- **Discrete** random variable: if Ω is a discrete set
  - Example: value of a dice
- **Continuous** random variable: if Ω is a continuous set
  - Example: a voltage value

Why random variables?

- Random variables are a great model for **noise**
- Examples:
  - Measure a voltage in a circuit
  - Measure several times, the value is never precisely the same. The values always *varies* a little.
  - i.e. it is affected by noise

Probability Mass Function

- Consider a **discrete** r.v. A
- The **probability mass function (PMF)** $=$ the probability that A has value x

$$w_A(x) = P\{A = x\}$$

- Also known as the **distribution** of A
- Example: what is the PMF of a dice? Plot on board.

Computing probability based on PMF

- Probability that A is equal to some value v

$$P\{A = v\} = w_A(v)$$

- Probability that A is between a and b (including):

$$P\{a \leq A \leq b\} = \sum_{x=a}^{b} w_A(x)$$

Cumulative Distribution Function

- The **cumulative distribution function (CDF)** = the probability that the value of A is smaller or equal than x

$$F_A(x) = P\left\{A \leq x\right\}$$

- In Romanian: *"funcție de repartitie"*
- Example: what is the CDF of a dice? Plot on board.
- For discrete r.v., the CDF is "stairwise"

Computing probability based on CDF

- Probability that A is equal to some value v

$$P\left\{A = v\right\} = F_A(v) - F_A(v-1)$$

- Probability that A is between a and b (including):

$$P\left\{a \leq A \leq b\right\} = F_A(b) - F_A(a-1)$$

Relation between PMF and CDF

- CDF is the *cumulative sum* (i.e. the integral) of PMF

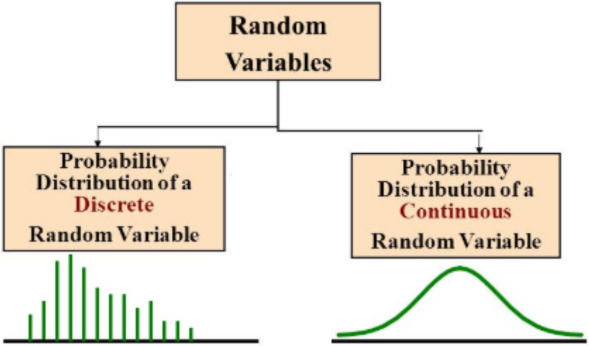
$$F_A(x) = \sum_{t=-\infty}^{t=x} w_A(t)$$

- Example for dice: easy to notice graphically

Probability Density Function

- Consider a **continuous** r.v. A
  - assume it takes values in some interval $[a, b]$
- The **Probability Density Function (PDF)** of A = probability that the value of A is in a small vicinity *epsilon* around x, divided by *epsilon*
- Denoted as $w_A(x)$, also known as **the distribution** of A
- Informally, the PDF gives the probability that the value of A is **close to** x

Continuous and discrete random variables



(image from "Probability Distributions: Discrete and Continuous", Seema Singh, *https://towardsdatascience.com/probability-distributions-discrete-and-continuous-7a94ede66dc0*)

Probability of an exact value

- The probability that a continuous r.v. A is **exactly** equal to a value x is **zero**
  - because there are an infinity of possibilities (continuous)
  - That's why we can't define a probability mass function like for discrete r.v.
- That's why the PDF says **in a small vicinity** around some value x, and not precisely equal to x

Computing probability based on PDF

- Probability that A is equal to some value v is always 0

$$P\{A = v\} = 0$$

- Probability that A is between a and b = integral of PDF from a to b:

$$P\{a \leq A \leq b\} = \int_a^b w_A(x)dx$$

Cumulative Distribution Function

- The **cumulative distribution function (CDF)** $=$ the probability that the value of A is smaller or equal than x

$$F_A(x) = P\{A \leq x\}$$

- In Romanian: *"funcție de repartiție"*
- Same definition as for discrete r.v.

Computing probability based on CDF

- Probability that A is between a and b:

$$P\{a \leq A \leq b\} = F_A(b) - F_A(a)$$

- Doesn't matter if we consider closed or open interval
  - $[a, b]$ or (a, b)
  - why?

Relation between PDF and CDF

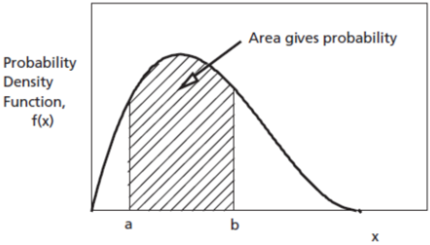
- CDF is **the integral** of PMF
- PDF is **the derivative** of CDF

$$F_A(x) = \int_{-\infty}^{x} w_A(x)\mathrm{d}x$$

$$\begin{aligned} w_A(x) &= \frac{\mathrm{d}F_A(x)}{\mathrm{d}x} \\ &= \lim_{\epsilon \to 0} \frac{F_A(x+\epsilon) - F_A(x-\epsilon)}{2\epsilon} \\ &= \lim_{\epsilon \to 0} \frac{P(A \in [x-\epsilon, x+\epsilon])}{2\epsilon} \end{aligned}$$

Graphical interpretation

- Probability that a continuous r.v. A is between a and b is **the area below the PDF**
  - i.e. the integral from a to b
- Probability that A is exactly equal to a certain value is zero
  - the area below a single point is zero



(image from "https://intellipaat.com/blog/tutorial/statistics-and-probability-tutorial/probability-distributions-of-continuous-variables/*)

Discrete vs continuous r.v.

Comparison of discrete vs continous random variables:

- The CDF $F_A(x)$ is defined identically, means same thing
- The PDF/PMF $w_A(x)$ is the derivative of CDF
  - for continuous r.v.:
    - it is a proper derivative
    - it means probability to be "around" x
  - for discrete r.v:
    - sort of "discrete derivative"
    - it means probability to be exactly equal to x

Properties of random variables

CDF:

- The CDF is always ≥ 0
- The CDF is always monotonously increasing (non-decreasing)
- The CDF starts from 0 and goes up to 1

$$F_A(-\infty) = 0 \quad F_A(\infty) = 1$$

PDF/PFM:

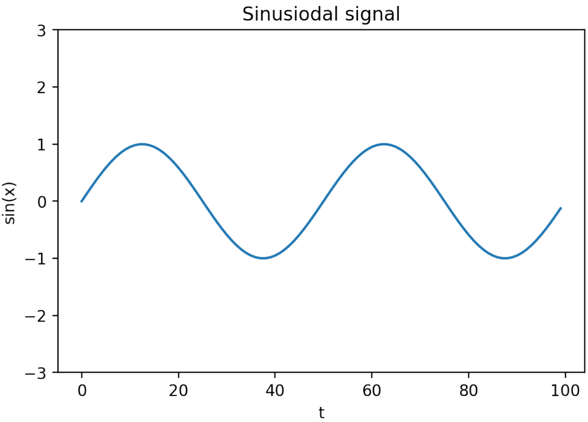
- The PDF/PMF are always ≥ 0
- Integral/sum over all of the PDF/PMF $= 1$

$$\int_{-\infty}^{\infty} w_A(x)\mathrm{d}x = 1$$

$$\sum_{x=-\infty}^{\infty} w_A(x) = 1$$

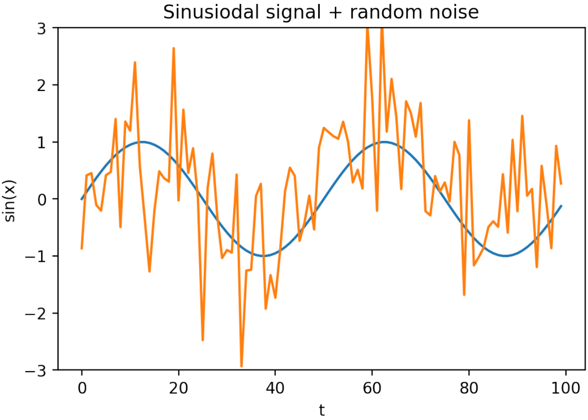
Different distributions

- Normal sine signal



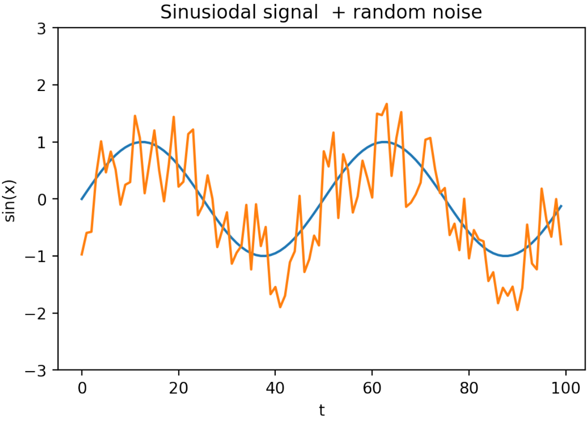
Different distributions

- Sine + noise 1 (normal, $\mu = 0, \sigma^2 = 1$)



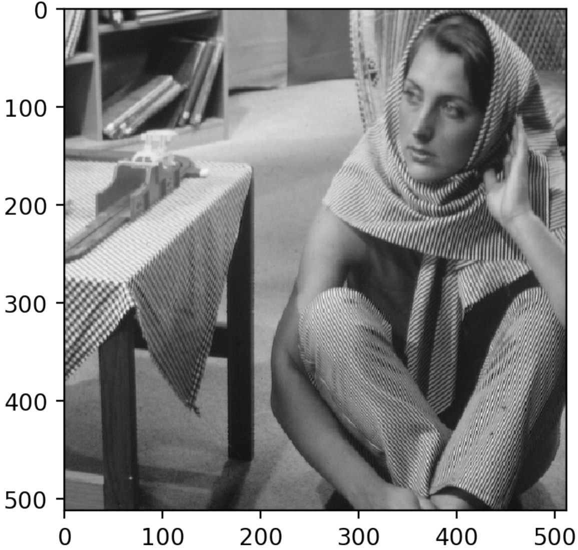
Different distributions

- Sine + noise 2 (uniform $\mathcal{U}[-1, 1]$)
- What's different? The distribution type



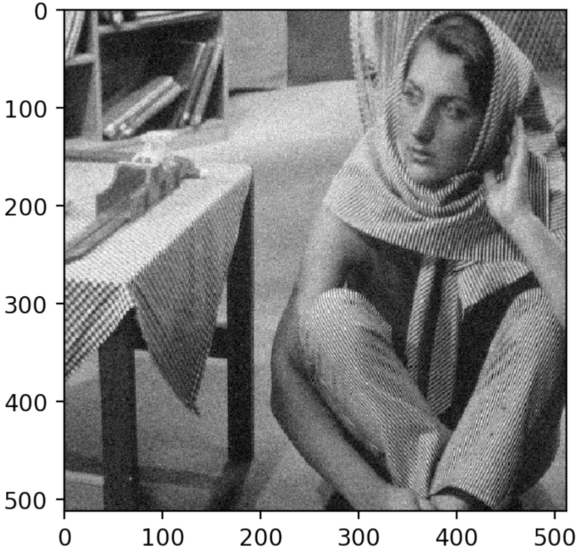
Different distributions

- Clean Image



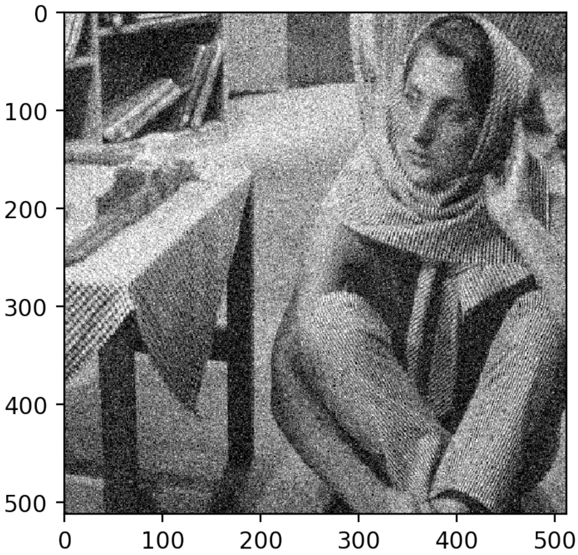
Different distributions

- Image + noise (normal, $\mu = 0, \sigma^2 = 1$)



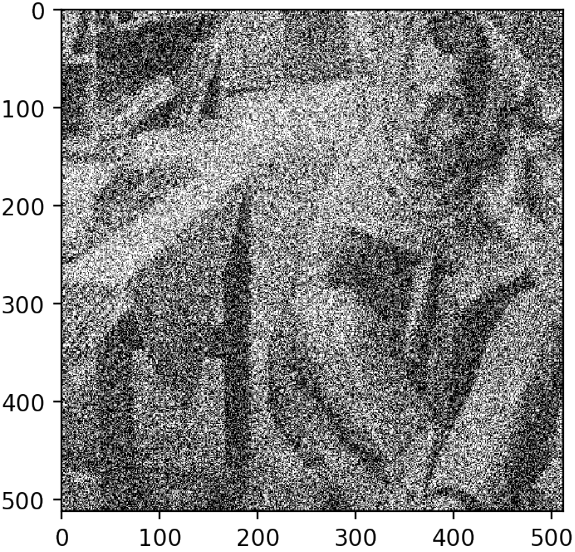
Different distributions

- Image + larger noise (normal, $\mu = 0, \sigma^2 = 10$)



Different distributions

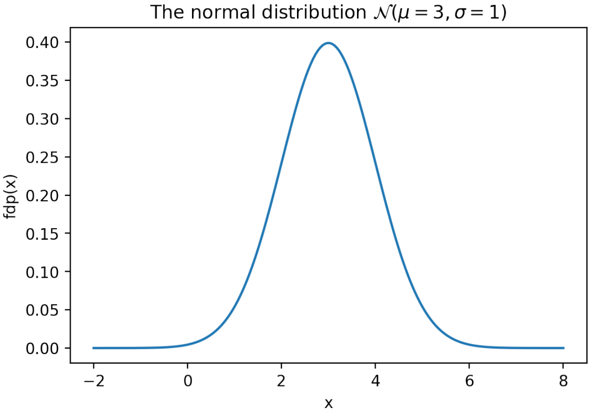
- Image + noise (uniform, $\mathcal{U}[-5, 5]$)



The normal distribution

- Probability density function

$$w_A(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$



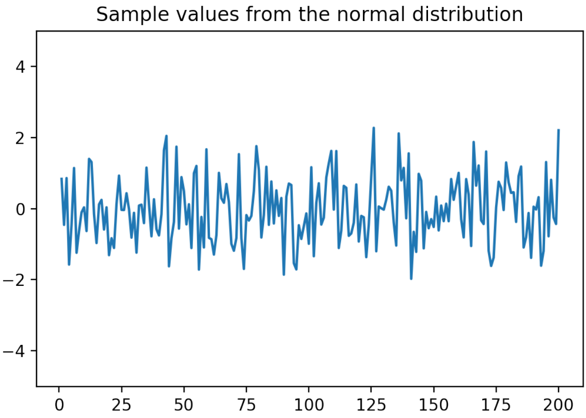
The normal distribution

- Has two parameters:
  - **Average value** $\mu =$ "center" of the function
  - **Standard deviation** $\sigma =$ "width" of the function
    - Small $\sigma =$ narrow and tall
    - Big $\sigma =$ wide and low
- The front constant is just for normalization (ensures that integral $= 1$)
- Extremely often encountered in real life
- Any real value is possible ($w_A(x) > 0, \forall x \in \mathbb{R}$)
- Usually denoted as $\mathcal{N}(\mu, \sigma^2)$

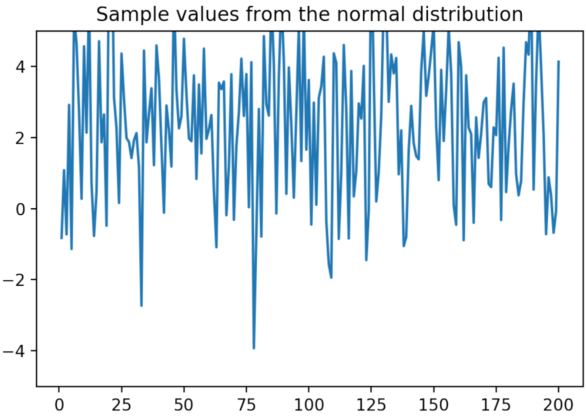
The normal distribution

- The distribution decreases as x gets farther from μ
  - Because of the term $-(x-\mu)^2$ at the exponent
  - Most likely values: around μ $(x-\mu=0)$
  - Values closer to μ are more likely, values farther from μ are less likely
- The function describes a preference for values around μ, with decreasing preference when getting farther from μ

Example of values from the normal distribution (mu=0, sigma=1)



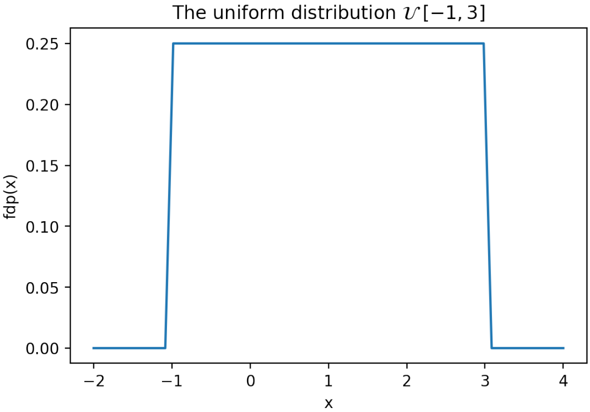
Example of values from the normal distribution (mu=2, sigma=4)



The uniform distribution

- The probability density function = a constant, between two endpoints

$$w_A(x) = \begin{cases} \frac{1}{b-a}, & x \in [a,b] \\ 0, & elsewhere \end{cases}$$



The uniform distribution

- Has two parameters: the limits a and b of the interval
- The "height" of the function is $\frac{1}{b-a}$
  - in order for the integral to be 1
- Only values from the interval $[a, b]$ are possible
  - value cannot be outside interval (probability is 0)
- Denoted as $\mathcal{U}\left[a, b\right]$

Other distributions

- Many other distributions exist, relevant for particular applications

Computing probabilities for the normal distribution

- How to compute $\int_a^b$ for a normal distribution?
  - Can't be done with algebraic formula, non-elementary function
- Use *the error function*:

$$erf(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt$$

- The CDF of a normal distribution $\mathcal{N}(\mu, \sigma^2)$

$$F_A(X) = \frac{1}{2}(1 + erf(\frac{x - \mu}{\sigma\sqrt{2}}))$$

- The values of *erf()* are available / are computed numerically
  - e.g. on Google, search for $erf(0.5)$
  - Other useful values:
    - $erf(-\infty) = -1$
    - $erf(\infty) = 1$

Exercise

Exercise:

- Let X be a r.v. with distribution $\mathcal{N}(3, 2)$. Compute the probability that $X \in [2, 4]$

Sum of constant + random variable

- Consider a random variable A
- What is $B = 5 + A$?

Answer:

- B is also a random variable
- B has same type of distribution, but the function is “shifted” by 5 to the right

Example:

- A is normal variable with $w_A(x) = \mathcal{N}(\mu = 3, \sigma^2 = 2)$
- What is the distribution of B = 5 + A?
- Answer: $w_B(x) = \mathcal{N}(\mu = 8, \sigma^2 = 2)$

R.v. as functions of other r.v.

- A function applied to a r.v. produces another r.v.
- Examples: if B is a r.v. with distribution $\mathcal{U}\,[0, 10]$, then
  - $C = 5 + A$ is another r.v., with distribution $\mathcal{U}\,[5, 15]$
  - $D = A^2$ is also another r.v.
  - $E = cos(A)$ is also another r.v.
- Reason: since A is random, the values B, C, D are also random
- A, B, C, D are *not independent*
  - A certain value of one of them automatically implies the value of the others

Multiple random variables

- Consider a system with two continuous r.v. A and B
- What is the probability that the pair (A, B) has values around (x, y)?
- Distribution of the values of (A, B) is described by:
  - joint probability density function $w_{AB}(x, y)$
  - joint cumulative density function $F_{AB}(x, y)$

Multiple random variables

- Joint cumulative distribution function:

$$F_{AB}(x, y) = P\{A \leq x \cap B \leq y\}$$

- Joint probability density function:

$$w_{AB}(x, y) = \frac{\partial^2 P_{AB}(x, y)}{\partial x \partial y}$$

- The joint PDF gives the probability that the value of the pair (A, B) is in a vicinity of (x, y)
- Similar for discrete random variables

$$w_{AB}(x, y) = P\{A = x \cap B = y\}$$

Independent random variables

- Two v.a. A and B are **independent** if the value of one of them does not influence in any way the value of the other
- For independent r.v., the probability that A is around x and B is around y is **the product** of the two probabilities

$$w_{AB}(x, y) = w_A(x) \cdot w_B(y)$$

- Relation holds for CDF / PDF / PMF, continuous or discrete r.v.
- Same for more than two r.v.

Independent random variables

Exercise:

- Compute the probability that three r.v. X, Y and Z i.i.d. $\mathcal{N}(-1, 1)$ are all positive simultaneously
  - ***i.i.d*** $=$ "independent and identically distributed"

Multiple normal variables

- Consider a set of N normal r.v. $(A_1, ...A_N)$, with different μ_i, but same σ
- Then probability that $(A_1, ...A_N)$ is around $(x_1, ...x_N)$ is

$$w_{A_1,...A_N}(x_1, ...x_N) = \frac{1}{(\sigma\sqrt{2\pi})^N} e^{\frac{(x_1-\mu_1)^2+...+(x_N-\mu_N)^2}{2\sigma^2}}$$

- The probability depends on the **Euclidean distance** between $\mathbf{x} = (x_1, ...x_N)$ and $\mu = (\mu_1, ...\mu_N)$

Euclidean distance

- **Euclidean (geometric) distance** between two N-dimensional vectors:

$$d(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\| = \sqrt{(u_1 - v_1)^2 + ... + (u_N - v_N)^2}$$

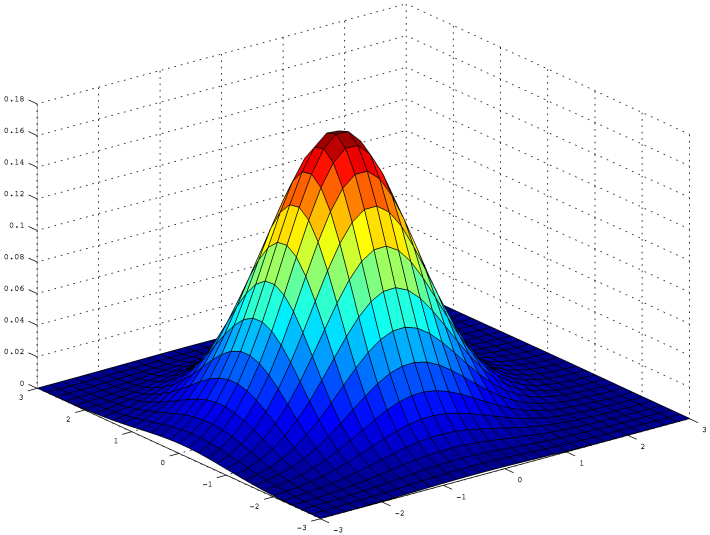
- One-dimensional: $\|\mathbf{u} - \mathbf{v}\| = |u - v|$
- 2D: $\|\mathbf{u} - \mathbf{v}\| = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2}$
- 3D: $\|\mathbf{u} - \mathbf{v}\| = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + (u_3 - v_3)^2}$
- ...
- N-dimensional: $\|\mathbf{u} - \mathbf{v}\| = \sqrt{\sum_{i=1}^{N} (u_i - v_i)^2}$
- ...
- Continuous signals: $\|\mathbf{u} - \mathbf{v}\| = \sqrt{\int_{-\infty}^{\infty} (u(t) - v(t))^2 dt}$

Multiple normal variables

- Probability of N normal random variables, independent, with same σ but possibly different μ_i depends on the **squared Euclidean distance to the mean vector** $\mu = (\mu_1, ...\mu_N)$
  - Close to μ: higher probability
  - Far from μ: lower probability
  - Two points at same distance from μ have same probability

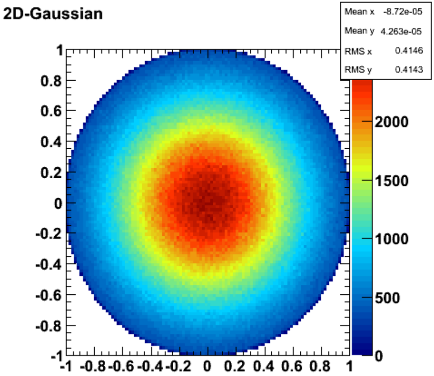
2D normal distribution

- Distribution of 2 normal random variables (2D normal distribution)



2D normal distribution - top view

- View from top
- Here, $\mu = (0, 0)$
- Probability density decreases as distance from center increases, in circles (symmetrically)



Statistical averages

- R.v. are described by statistical averages ("*moments*")
- **The average value** (moment of order 1)
- Continuous r.v.:

$$\overline{A} = E\{A\} = \int_{-\infty}^{\infty} x \cdot w_A(x) dx$$

- Discrete r.v.:

$$\overline{A} = E\{A\} = \sum_{x=-\infty}^{\infty} x \cdot w_A(x)$$

- (Example: the entropy of H(X) = the average value of the information)
- Usual notation: μ

Properties of the average value

- Computing the average value is a **linear** operation
  - because the underlying integral / sum is a linear operation
- Linearity

$$E\{c_1 A + c_2 B\} = c_1 E\{A\} + c_2 E\{B\}$$

- Or:

$$E\{cA\} = cE\{A\}, \forall c \in \mathbb{R}$$

$$E\{A + B\} = E\{A\} + E\{B\}$$

- No proof given here

Average squared value

- **Average squared value** = average value of the squared values
- Moment of order 2
- Continuous r.v.:

$$\overline{A^2} = E\{A^2\} = \int_{-\infty}^{\infty} x^2 \cdot w_A(x) dx$$

- Discrete r.v.:

$$\overline{A^2} = E\{A^2\} = \sum_{-\infty}^{\infty} x^2 \cdot w_A(x)$$

- Interpretation: average of squared values = average power of a signal

Variance

- **Variance**= average squared value of the difference to the average value
- Continuous r.v.:

$$\sigma^2 = \overline{\{A-\mu\}^2} = \int_{-\infty}^{\infty} (x-\mu)^2 \cdot w_A(x) dx$$

- Discrete r.v.:

$$\sigma^2 = \overline{\{A-\mu\}^2} = \sum_{-\infty}^{\infty} (x-\mu)^2 \cdot w_A(x)$$

- Interpretation: how much do the values vary around the average value
  - $\sigma^2 =$ large: large spread around the average value
  - $\sigma^2 =$ small: values are concentrated around the average value

Relation between the three values

- Relation between the average value, the average squared value, and the variance:

$$
\begin{aligned}
\sigma^2 &= \overline{\{A-\mu\}^2} \\
&= \overline{A^2 - 2 \cdot A \cdot \mu + \mu^2} \\
&= \overline{A^2} - 2\mu\overline{A} + \mu^2 \\
&= \overline{A^2} - \mu^2
\end{aligned}
$$

Sum of random variables

- Sum of two or more **independent** r.v. is also a r.v.
- Its distribution = the **convolution** of the distributions of the two r.v.
- If $C = A + B$

$$w_C(x) = w_A(x) \star w_B(x)$$

- Particular case: if A and B are normal r.v., with $\mathcal{N}(\mu_A, \sigma_A^2)$ and $\mathcal{N}(\mu_B, \sigma_B^2)$, then:
  - C is also a normal r.v., with $\mathcal{N}(\mu_C, \sigma_C^2)$, having:
  - average = sum of the two averages: $\mu_C = \mu_A + \mu_B$
  - variance = sum of the two variances: $\sigma_C^2 = \sigma_A^2 + \sigma_B^2$

I.2 Random processes

Random process

- A **random process** = a sequence of random variables indexed in time
- **Discrete-time** random process $f[n]$ = a sequence of random variables at discrete moments of time
  - e.g.: a sequence 50 of throws of a dice, the daily price on the stock market
- **Continuous-time** random process $f(t)$ = a continuous sequence of random variables at every moment
  - e.g.: a noise voltage signal, a speech signal
- Every sample from a random process is a (different) random variable!
  - e.g. $f(t_0)$ = value at time t_0 is a r.v.

Realizations of random processes

- A **realization** of the random process = a particular sequence of realizations of the underlying r.v.
  - e.g. we see a given noise signal on the oscilloscope, but *we could have seen any other realization just as well*
- When we consider a random process = we consider the set of all possible realizations

Distributions of order 1 of random processes

- Every sample $f(t_1)$ from a random process is a random variable
  - it is described by a **distribution of order 1**
  - has a CDF $F_1(x;t_1)$
  - has a PDF $w_1(x;t_1) = \frac{dF_1(x;t_1)}{dx}$
  - everything depends on the time moment t_1
- The sample at time t_2 is a different random variable with **possibly different** functions
  - has a different CDF $F_1(x;t_2)$
  - has a different PDF $w_1(x;t_2) = \frac{dF_1(x;t_2)}{dx}$
- These functions specify how the value of one sample is distributed
- The index w_1 indicates we consider a single random variable (distribution of order 1)
- Same for discrete-time random processes

Distributions of order 2

- A pair of random variables $f(t_1)$ and $f(t_2)$ form a system of 2 r.v.
  - they are described by a **distribution of order 2**
  - have a joint CDF $F_2(x_i, x_j; t_1, t_2)$
  - have a joint PDF $w_2(x_i, x_j; t_1, t_2) = \frac{\partial^2 F_2(x_i, x_j; t_1, t_2)}{\partial x_i \partial x_j}$
  - depend on time moments t_1 and t_2
- These functions specify how the pair of values is distributed
- Same for discrete-time random processes

Distributions of order n

- Generalize to n samples of the random process
- A set of n random variables $f(t_1), ...f(t_n)$ from the random process $f(t)$
  - are described by **distribution of order n**
  - have joint CDF $F_n(x_1, ...x_n; t_1, ...t_n)$
  - have joint PDF $w_n(x_1, ...x_n; t_1, ...t_n) = \frac{\partial^2 F_n(x_1,...x_n;t_1,...t_n)}{\partial x_1 ... \partial x_n}$
  - depend on time moments t_1, t_2, ... t_n
- These functions specify how the whole set of n values is distributed
- Same for discrete-time random processes

Statistical averages

Random processes are characterized using statistical and temporal averages (*moments*)

For continuous random processes:

1. Average value

$$\overline{f(t_1)} = \mu(t_1) = \int_{-\infty}^{\infty} x \cdot w_1(x; t_1) dx$$

2. Average squared value (*valoarea patratica medie*)

$$\overline{f^2(t_1)} = \int_{-\infty}^{\infty} x^2 \cdot w_1(x; t_1) dx$$

Statistical averages - variance

3. Variance (= *varianța*)

$$\sigma^2(t_1) = \overline{\{f(t_1) - \mu(t_1)\}^2} = \int_{-\infty}^{\infty} (x - \mu(t_1)^2 \cdot w_1(x; t_1) dx$$

- The variance can be computed as:

$$\begin{aligned}\sigma^2(t_1) =& \overline{\{f(t_1) - \mu(t_1)\}^2} \\ =& \overline{f(t_1)^2 - 2f(t_1)\mu(t_1) + \mu(t_1)^2} \\ =& \overline{f^2(t_1)} - \mu(t_1)^2\end{aligned}$$

- Note:
  - these three values are calculated across all realizations, at time t_1
  - they characterize only the sample at time t_1
  - at a different time t_2, the r.v. $f(t_2)$ is different so all average values might be different

Statistical averages - autocorrelation

4. The autocorrelation function

$$R_{ff}(t_1, t_2) = \overline{f(t_1)f(t_2)} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 w_2(x_1, x_2; t_1, t_2) dx_1 dx_2$$

5. The correlation function (for different random processes $f(t)$ and $g(t)$)

$$R_{fg}(t_1, t_2) = \overline{f(t_1)g(t_2)} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 y_2 w_2(x_1, y_2; t_1, t_2) dx_1 dy_2$$

- Note:
  - these functions may have different values for a different pair of values (t_1, t_2)

Discrete random processes

For **discrete random processes**, nothing changes (except notation from $f(t)$ to $f[t]$):

Temporal averages

- What to do when we only have access to a single realization $f^{(k)}(t)$?
- Compute values **for a single realization $f^{(k)}(t)$, across all time moments**
- For continuous random processes:

1. Temporal average value

$$\overline{f^{(k)}(t)} = \mu^{(k)} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f^{(k)}(t)dt$$

2. Temporal average squared value

$$\overline{[f^{(k)}(t)]^2} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} [f^{(k)}(t)]^2 dt$$

Temporal variance

3. Temporal variance

$$\sigma^2 = \overline{\left\{f^{(k)}(t) - \mu^{(k)}\right\}^2} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} (f^{(k)}(t) - \mu^{(k)})^2 dt$$

- The variance can be computed as:

$$\sigma^2 = \overline{[f^{(k)}(t)]^2} - [\mu^{(k)}]^2$$

- Note:
  - these values do not depend anymore on time t (integrated)

Temporal autocorrelation

4. The temporal autocorrelation function

$$
\begin{aligned}
R_{ff}(t_1, t_2) =& \overline{f^{(k)}(t_1 + t)f^{(k)}(t_2 + t)} \\
&= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f^{(k)}(t_1 + t)f^{(k)}(t_2 + t)dt
\end{aligned}
$$

5. The temporal correlation function (for different random processes $f(t)$ and $g(t)$)

$$
\begin{aligned}
R_{fg}(t_1, t_2) =& \overline{f^{(k)}(t_1 + t)g^{(k)}(t_2 + t)} \\
&= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f^{(k)}(t_1 + t)g^{(k)}(t_2 + t)dt
\end{aligned}
$$

Discrete random processes

For **discrete random processes**, replace $\int$ with $\sum$, T with N, and divide to $2N+1$ instead of $2T$

1. $\overline{f^{(k)}[t]} = \mu^{(k)} = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{t=-N}^{N} f^{(k)}[t]$
2. $\overline{[f^{(k)}[t]]^2} = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{t=-N}^{N} (f^{(k)}[t])^2$
3. $\sigma^2 = \overline{\left\{f^{(k)}[t] - \mu^{[k]}\right\}^2} = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{t=-N}^{N} (f^{(k)}[t] - \mu^{(k)})^2$

Discrete random processes

4. Temporal autocorrelation:

$$
\begin{aligned}
R_{ff}(t_1, t_2) &= \overline{f^{(k)}[t_1 + t]f^{(k)}[t_2 + t]} \\
&= \lim_{N \to \infty} \frac{1}{2N+1} \sum_{t=-N}^{N} f^{(k)}[t_1 + t]f^{(k)}[t_2 + t]
\end{aligned}
$$

5. Temporal correlation:

$$
\begin{aligned}
R_{fg}(t_1, t_2) &= \overline{f^{(k)}[t_1 + t]g^{(k)}[t_2 + t]} \\
&= \lim_{N \to \infty} \frac{1}{2N+1} \sum_{t=-N}^{N} f^{(k)}[t_1 + t]g^{(k)}[t_2 + t]
\end{aligned}
$$

Finite length realizations

If the realization is not from time $-\infty$ to ∞, but only from a t_{min} to t_{max}, just use $\int_{t_{min}}^{t_{max}}$ or $\sum_{t_{min}}^{t_{max}}$ for the temporal averages

- Example: Compute the temporal averages for the finite-length realization

$$\{1, -1, 2, -2, 3, -3, 4, -4, 5, -5\}$$

Statistical and temporal averages

- Statistical averages are usually the relevant values
  - but they require to know the distributions
- In real life, with unknown signals, we can only measure one realization
  - so we can only compute the temporal values for one realization
- Fortunately, in many cases they are the same (ergodicity, see later)

Stationary random processes

- All the statistical averages are dependent on the time
  - i.e. they might be different for a sample at t_2
- **Stationary** random process = when all statistical averages are **identical if we shift the time origin** (e.g. delay the signal)
- Equivalent definition: if all the PDF are identical when shifting the time origin

$$w_n(x_1, ...x_n; t_1, ...t_n) = w_n(x_1, ...x_n; t_1 + \tau, ...t_n + \tau)$$

- Basically, nothing should depend on the time t

Strict-sense and wide-sense stationary

- Strictly stationary / strongly stationary / strict-sense stationary:
  - relation holds for every n
- Weakly stationary / wide-sense stationary:
  - relation holds only for $n = 1$ and $n = 2$ (the most used)

Consequences of stationarity

- For $n = 1$:
$$w_1(x_i; t_1) = w_1(x_i; t_2) = w_1(x_i)$$

- The average value, average squared value, variance of a sample are all **identical** for any time t

$$\overline{f(t)} = constant, \forall t$$

$$\overline{f^2(t)} = constant, \forall t$$

$$\sigma^2(t) = constant, \forall t$$

Consequences of stationarity

- For $n = 2$:

$$w_2(x_i, x_j; t_1, t_2) = w_2(x_i, x_j; 0, t_2 - t_1) = w_2(x_i, x_j; t_2 - t_1)$$

- The autocorrelation function depends only on the **time difference** $\tau = t_2 - t_1$ between the samples

$$R_{ff}(t_1, t_2) = R_{ff}(0, t_2 - t_1) = R_{ff}(\tau) = \overline{f(t)f(t+\tau)}$$

- Depends on a single value $\tau =$ time difference of the two samples

Consequences of stationarity

- Definition of autocorrelation function for **stationary** r.p:
  - the function now depends on $\tau = t_2 - t_1$, instead of t_1 and t_2
- Statistical autocorrelation: no change
- Temporal autocorrelation:
  - for continuous r.p.

$$R_{ff}(\tau) = \overline{f(t)f(t+\tau)}$$
$$= \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} f^{(k)}(t)f^{(k)}(t+\tau)dt$$

  - for discrete r.p.

$$R_{ff}(\tau) = \overline{f(t)f(t+\tau)}$$
$$= \lim_{N\to\infty} \frac{1}{2N+1}\sum_{t=-N}^{N} f^{(k)}[t]f^{(k)}[t+\tau]$$

  - finite length: limit the integrals / sums to the length of the signal,

Consequences of stationarity

- Same for correlation function between two different r.p
- Depends only on the **time difference** $\tau = t_2 - t_1$ between the samples

$$R_{fg}(t_1, t_2) = R_{fg}(0, t_2 - t_1) = R_{fg}(\tau) = \overline{f(t)g(t+\tau)}$$

- Definition is similar to the autocorrelation definition on the previous slide

Interpretation of autocorrelation

- $R_{ff}(\tau) =$ the average value of the product of two samples which are time τ apart
  - e.g. tells us if the two samples vary in same direction or not
- Same for correlation, but the samples are taken from different r.p f and g
- Example:
  - $R_{ff}(0.5) > 0$ means two samples separated by 0.5 seconds tend to vary in same direction (both positive, both negative $=>$ their product is mostly positive)
  - $R_{ff}(1) < 0$ means two samples separated by 1 second tend to vary in opposite directions (when one is positive, the other is negative $=>$ their product is mostly negative)
  - $R_{ff}(2) = 0$ means two samples separated by 2 seconds are uncorrelated (their product is 0 on average, so equally positive and negative)

Ergodic random processes

- In practice, we have access to a single realization
- **Ergodic** random process = the temporal averages on any realization are equal to the statistical averages
- Ergodicity means:
  - We can compute / estimate all averages from a single realization (any)
    - but the realization must be very long (length $\to \infty$) for precise results
  - Realizations are all similar to the others, statistically
    - so a single realization is characteristic of the whole process

Ergodic random processes

- Most random processes we care about are ergodic and stationary
  - e.g. voltage noises
- Example of non-ergodic process:
  - throw a dice, then the next 50 values are identical to the first
  - a single realization is not characteristic

I.3 More on autocorrelation

The Power Spectral Density of a random process

- The Power Spectral Density (PSD) $S_{ff}(\omega)$ is the power of the random process at every frequency f $(\omega = 2\pi f)$
- The PSD describes how the power of a signal is distributed in frequency
  - e.g. some random processes have more power at low frequency, others at high frequency etc.
- The power in the frequency band $[f_1, f_2]$ is equal to $\int_{f_1}^{f_2} S_{ff}(\omega) d\omega$
- The whole power of the signal is $P = \int_{-\infty}^{\infty} S_{ff}(\omega) d\omega$
- The PSD is a measurable quantity
  - it can be determined experimentally
  - it is important in practical (engineering) applications

The Wiener-Khinchin theorem

- *Rom: teorema Wiener-Hincin*

Theorem:

- **The Power Spectral Density = the Fourier transform of the autocorrelation function**

$$S_{ff}(\omega) = \int_{-\infty}^{\infty} R_{ff}(\tau) e^{-j\omega\tau} d\tau$$

$$R_{ff}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{ff}(\omega) e^{j\omega\tau} d\omega$$

- No proof
  - Makes a relation between two rather different domains
  - autocorrelation function: a *statistical* property
  - PSD function: a *physical* property (relevant for engineering purposes)

White noise

- **White noise** = a random process with autocorrelation function equal to a Dirac function
$$R_{ff}(\tau) = \delta(\tau)$$
  - is a random process: every sample of white noise is a random variable
  - autocorrelation is a Dirac: autocorrelation is 0 for any $\tau \neq 0$
  - any two different samples ($\tau \neq 0$) have zero correlation (are uncorrelated)
    - values of any two different samples are not related
- Power spectral density of white noise = Fourier transform of a Dirac = a constant $\forall \omega$
  - equal distribution of power at all frequencies up to ∞
- White noise can have any distribution (normal, uniform etc.)
  - the term "white noise" doesn't refer to the distribution of sample values, but to the fact that all samples are unrelated to each other

Band-limited white noise

- In real life, power spectral density goes to 0 at very high frequencies
  - because total power $P = \int_{-\infty}^{\infty} S_{ff} \omega$ cannot be infinite
  - known as "*band-limited white noise*"
- In this case, autocorrelation = approximately a Dirac, but not infinitely thin
  - samples which are very close are necessarily a bit correlated
  - e.g. due to small parasitic capacities

AWGN

- **AWGN** = Additive White Gaussian Noise
  - is the usual type of noise considered in applications
- It means:
  - additive: the noise is added to the original signal (e.g. not multiplied with it)
  - gaussian: the samples have normal distribution
  - white: the samples are uncorrelated (unrelated) with each other

2018-2019 Exam

- Chapter 1 ends here for 2018-2019 exam. Following slides not needed.

Properties of the autocorrelation function

1. Is even

$$R_{ff}(\tau) = R_{ff}(-\tau)$$

   - Proof: change variable in definition

2. At infinite it goes to a constant

$$R_{ff}(\infty) = \overline{f(t)}^2 = const$$

   - Proof: two samples separated by ∞ are independent

3. Is maximum in 0

$$R_{ff}(0) \geq R_{ff}(\tau)$$

   - Proof: start from $\overline{(f(t) - f(t+\tau))^2} \geq 0$
   - Interpretation: different samples might vary differently, but a sample always varies identically with itself

Properties of the autocorrelation function

4. Value in 0 = the power of the random process

$$R_{ff}(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{ff}(\omega)d\omega$$

   - Proof: Put $\tau = 0$ in inverse Fourier transform of Wiener-Khinchin theorem

5. Variance = difference between values at 0 and ∞

$$\sigma^2 = R_{ff}(0) - R_{ff}(\infty)$$

   - Proof: $R_{ff}(0) = \overline{f(t)^2}$, $R_{ff}(\infty) = \overline{f(t)}^2$

Autocorrelation of filtered random processes

- Consider a stationary random process applied as input to a LTI system
  - either continuous-time: input $x(t)$, system $H(s)$, output $y(t)$
  - or discrete-time: input $x[n]$, system $H(z)$, output $y[n]$
- How does the autocorrelation of y depend on that of the input x?
  - y is the convolution between x and the impulse response h

Computations

- For discrete-time processes

$$
\begin{aligned}
R_{yy}(\tau) =& \overline{y[n]y[n+\tau]} \\
=& \overline{\sum_{k_1=-\infty}^{\infty} h[k_1]x[n-k_1] \sum_{k_2=-\infty}^{\infty} h[k_2]x[n+\tau-k_2]} \\
=& \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} h[k_1]h[k_2]\overline{x[n-k_1]x[n+\tau-k_2]} \\
=& \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} h[k_1]h[k_2]R_{xx}[\tau-k_1+k_2]
\end{aligned}
$$

- From Wiener-Hincin theorem:

$$
S_{ff}(\omega) = \sum_{\tau=-\infty}^{\infty} R_{ff}(\tau)e^{-j\omega\tau}
$$

Computations

- Therefore

$$S_{yy}(\omega) = \sum_{\tau=-\infty}^{\infty} \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} h[k_1]h[k_2]R_{xx}[\tau - k_1 + k_2]e^{-j\omega\tau}$$

- Change of variable: $\tau - k_1 + k_2 = u$
  - then $\tau = u + k_1 - k_2$

$$\begin{aligned}
S_{yy}(\omega) = &\sum_{u=-\infty}^{\infty} \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} h[k_1]h[k_2]R_{xx}[u]e^{-j\omega(u+k_1+k_2)} \\
= &\sum_{u=-\infty}^{\infty} R_{xx}[u]e^{-j\omega u} \sum_{k_1=-\infty}^{\infty} h[k_1]e^{-j\omega k_1} \sum_{k_2=-\infty}^{\infty} h[k_2]e^{j\omega k_2} \\
= &S_{xx}(\omega) \cdot H(\omega) \cdot H*^(\omega) \\
= &S_{xx}(\omega) \cdot |H(\omega)|^2
\end{aligned}$$

Result

$$S_{yy}(\omega) = S_{xx}(\omega) \cdot |H(\omega)|^2$$

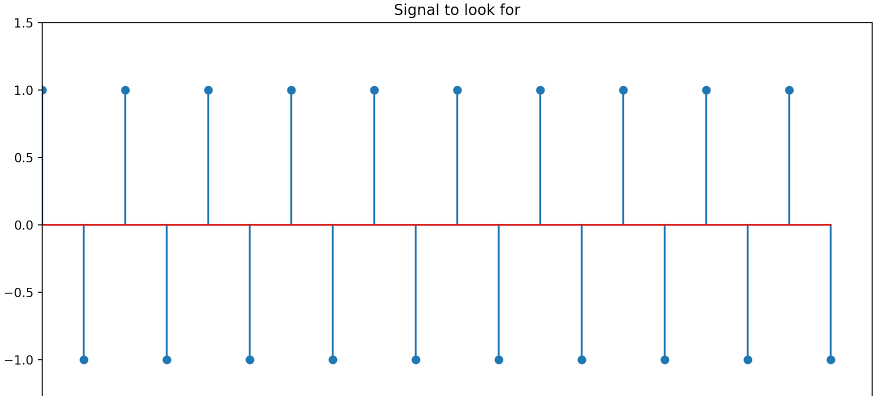
- The PSD of $y =$ the PSD of x multiplied with the squared amplitude response of the filter
- Same relation is valid for continuous processes as well

Applications of (auto)correlation

- Searching for a certain part in a large signal
- Correlation of two signals = measure of **similarity** of the two signals
  - The correlation function measures the similarity of a signal with all the shifted versions of the other
  - Example at blackboard
- Correlation can be used to locate data
  - The (auto)correlation function has large values when the two signals match
  - Large value when both positive and negative areas match,
  - Small values when they don't match

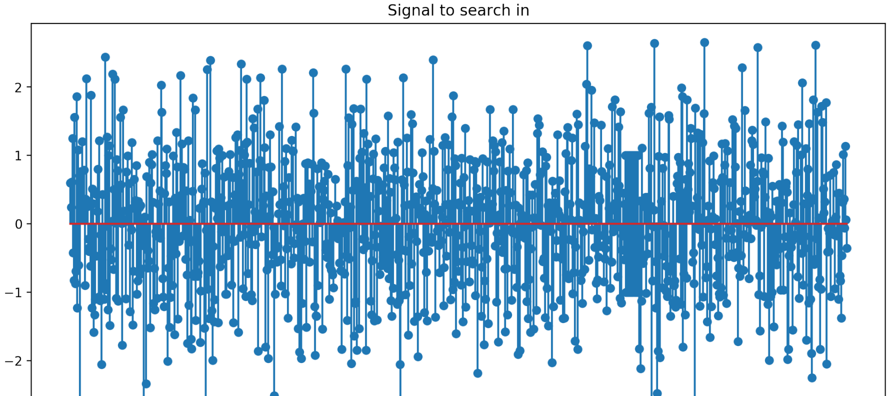
The signal to look for

```
/home/ncleju/.local/bin/pweave:6: UserWarning: In Matplotlib
individual lines on a stem plot will be added as a LineCollec
instead of individual lines. This significantly improves the
performance of a stem plot. To remove this warning and switch
new behaviour, set the "use_line_collection" keyword argument
  from pweave.scripts import weave
```



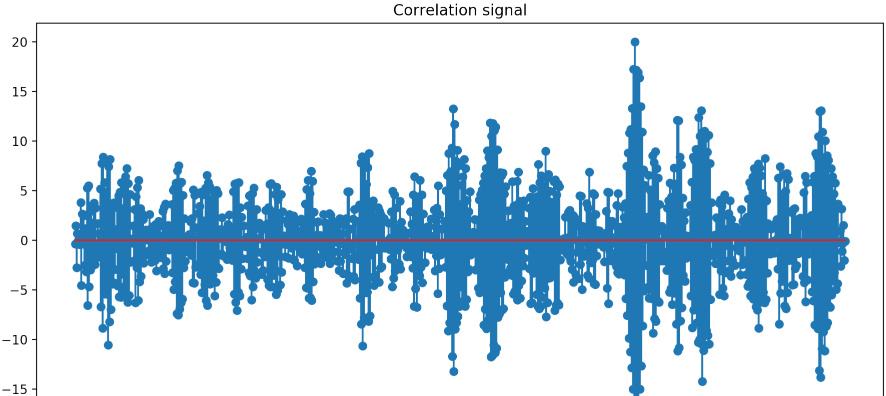
The complete signal

```
/home/ncleju/.local/bin/pweave:6: UserWarning: In Matplotlib
individual lines on a stem plot will be added as a LineCollec
instead of individual lines. This significantly improves the
performance of a stem plot. To remove this warning and switch
new behaviour, set the "use_line_collection" keyword argument
  from pweave.scripts import weave
```



Correlation result

```
/home/ncleju/.local/bin/pweave:6: UserWarning: In Matplotlib
individual lines on a stem plot will be added as a LineCollec
instead of individual lines. This significantly improves the
performance of a stem plot. To remove this warning and switch
new behaviour, set the "use_line_collection" keyword argument
  from pweave.scripts import weave
```



System identification

- Determining the impulse response of an unknown LTI system
- Based on correlation between input and output of the system

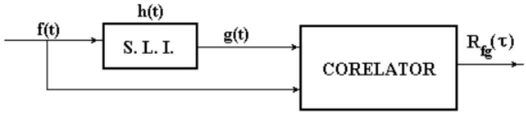


Figure 1: System identification setup

System identification

$$
\begin{aligned}
R_{fg}(\tau) =& \overline{f[n]g[n+\tau]} \\
=& \overline{f[n] \sum_{k=-\infty}^{\infty} h[k] f[n+\tau-k]} \\
=& \sum_{k=-\infty}^{\infty} h[k] \overline{f[n] f[n+\tau-k]} \\
=& \sum_{k=-\infty}^{\infty} h[k] R_{ff}[\tau-k] \\
=& h[\tau] \star R_{ff}[\tau]
\end{aligned}
$$

- If the input f is **white noise** with power A, $R_{ff}[n] = A \cdot \delta[n]$, and

$$R_{fg}(\tau) = h[\tau] \star R_{ff}[\tau] = A \cdot h[\tau] \star \delta[\tau] = A \cdot h[\tau]$$

- Then the correlation is proportional with the impulse response of the unknown system