

Decision and Estimation in Data Processing

Chapter II. Elements of Signal Detection Theory

II.1 Introduction

Introduction

- ▶ Signal detection = the problem of deciding which signal is present from 2 or more possibilities
 - ▶ one possibility may be that there is no signal
- ▶ Based on **noisy** observations
 - ▶ signals are affected by noise

The model for signal detection

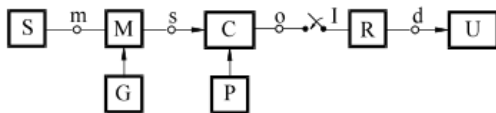


Figure 1: Signal detection model

► Contents:

- Information source: generates messages a_n with probabilities $p(a_n)$
- Modulator: transmits a signal $s_n(t)$ for message a_n
- Channel: adds random noise
- Sampler: takes samples from the signal $s_n(t)$
- Receiver: **decides** what message a_n has been transmitted

- ▶ Data transmission

- ▶ binary voltage levels (e.g. $s_n(t) = \text{constant}$)
- ▶ PSK modulation (Phase Shift Keying): $s_n(t) = \text{cosine}$ with same frequency but various initial phase
- ▶ FSK modulation (Frequency Shift Keying): $s_n(t) = \text{cosines}$ with different frequencies

- ▶ Radar

- ▶ a signal is emitted; if there is an obstacle, the signal gets reflected back
- ▶ the receiver waits for possible reflections of the signal and must decide
 - ▶ no reflection is present -> no object
 - ▶ reflected signal is present -> object detected

Generalizations

- ▶ Decide between more than two signals
- ▶ Number of observations:
 - ▶ use only one sample
 - ▶ use multiple samples
 - ▶ observe the whole continuous signal for some time T

II.2 Detection of constant signals

Detection of a constant signal, white normal noise, 1 sample

- ▶ Simplest case: detection of a constant signal contaminated with white normal noise, using 1 sample
 - ▶ two messages a_0 and a_1
 - ▶ messages are encoded as constant signals
 - ▶ for a_0 : send $s_0(t) = 0$
 - ▶ for a_1 : send $s_1(t) = A$
 - ▶ over the signals there is white noise, normal distribution $\mathcal{N}(0, \sigma^2)$
 - ▶ receiver takes just 1 sample
 - ▶ decision: compare sample with a threshold

Decision

- ▶ The value of the sample taken is $r = s + n$
 - ▶ s is the true underlying signal ($s_0 = 0$ or $s_1 = A$)
 - ▶ n is a sample of the noise
- ▶ n is a (continuous) random variable, with normal distribution
- ▶ r is a random variable also
 - ▶ what distribution does it have?
- ▶ Decision is taken by comparing with a threshold T :
 - ▶ if $r < T$, take decision D_0 : decide the true signal is s_0
 - ▶ if $r \geq T$, take decision D_1 : decide the true signal is s_1

Hypotheses

- ▶ Receiver chooses between **two hypotheses**:
 - ▶ H_0 : true signal is s_0 (a_0 has been transmitted)
 - ▶ H_1 : true signal is s_1 (a_1 has been transmitted)
- ▶ Possible results
 1. No signal present, no signal detected.
 - ▶ Decision D_0 when hypothesis is H_0
 - ▶ Probability is $P(D_0 \cap H_0)$
 2. **False alarm**: no signal present, signal detected (error)
 - ▶ Decision S_1 when hypothesis is H_0
 - ▶ Probability is $P(D_1 \cap H_0)$
 3. **Miss**: signal present, no signal detected (error)
 - ▶ Decision D_0 when hypothesis is H_1
 - ▶ Probability is $P(D_0 \cap H_1)$
 4. Signal detected correctly: signal present, signal detected
 - ▶ Decision D_1 when hypothesis is H_1
 - ▶ Probability is $P(D_1 \cap H_1)$

Maximum likelihood criterion

- ▶ Choose the hypothesis that **seems most likely** given the observed sample r
- ▶ The **likelihood** of an observation r = the probability density of r given a hypothesis H_0 or H_1
- ▶ Likelihood in case of hypothesis H_0 : $w(r|H_0)$
 - ▶ r is only noise, so value is taken from the noise distribution
- ▶ Likelihood in case of hypothesis H_1 : $w(r|H_1)$
 - ▶ r is $A + \text{noise}$, so value is taken from the distribution of $(A + \text{noise})$
- ▶ Likelihood test:

$$\frac{w(r|H_1)}{w(r|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} 1$$

Graphical interpretation

- ▶ Consider noise having a normal distribution
- ▶ Plot the two density functions for H_0 , H_1

Decision via threshold

- ▶ Decision via ML = comparing r with a threshold T
- ▶ The threshold = the cross-over point of the two distributions

Normal noise

- ▶ Particular case: the noise has normal distribution $\mathcal{N}(0, \sigma^2)$
- ▶ Likelihood test is $\frac{w(r|H_1)}{r|H_0} = \frac{e^{-\frac{(r-A)^2}{2\sigma^2}}}{e^{-\frac{r^2}{2\sigma^2}}} \underset{H_0}{\geq} 1$
- ▶ For normal distribution, it is easier to apply *natural logarithm* to the terms
 - ▶ logarithm is a monotonic increasing function, so it won't change the comparison
 - ▶ if $A < B$, then $\log(A) < \log(B)$
- ▶ The **log-likelihood** of an observation = the logarithm of the likelihood value
 - ▶ usually the natural logarithm, but any one can be used

Log-likelihood test for ML

- ▶ For normal noise, the ML decision means the log-likelihood test

$$\frac{(r - A)^2}{r^2} \underset{H_0}{\overset{H_1}{\gtrless}} 1$$

- ▶ Applying square root

$$\frac{|r - A|}{|r|} \underset{H_0}{\overset{H_1}{\gtrless}} 1$$

- ▶ $|r - A|$ = distance from r to A , $|r|$ = distance from r to 0
- ▶ ML decision with normal noise: choose the value 0 or A which is **nearest** to r
 - ▶ very general principle, encountered in many other scenarios
 - ▶ also known as **nearest neighbor** principle / decision
 - ▶ equivalent with setting a threshold $T = \frac{A}{2}$

Generalizations

- ▶ What if the noise has another distribution?
 - ▶ Threshold T is still the cross-over point, whatever that is
- ▶ What if the noise distributions are different for H_0 and H_1 ?
 - ▶ Threshold T is the cross-over point, whatever that is
- ▶ What if the signal $s_0(t)$ (for H_0) is not 0, but another constant value B ?
 - ▶ T is the crossover point, the distributions are centered on B and A
 - ▶ In case of normal noise, choose B or A whichever is nearest (threshold is at middle point)

Generalizations

- ▶ What if we have more than two signal levels?
 - ▶ e.g. 4 possible signals: -6, -2, 2, 6
 - ▶ Just choose the most likely hypothesis, out of 4 likelihood functions
 - ▶ Not a single threshold value, now there are more

Pitfalls of ML decision

- ▶ The ML is based on comparing **conditional** probability density functions
 - ▶ conditioned by H_0 or by H_1
- ▶ Conditioning by H_0 and H_1 ignores the probability of H_0 or H_1 actually happening
- ▶ Reminder: the Bayes rule

$$p(A \cap B) = p(B|A) \cdot P(A))$$

- ▶ Interpretation
 - ▶ The probability $p(A)$ is taken out from $p(B|A)$
 - ▶ $p(B|A)$ gives no information on $p(A)$, the chances of A actually happening
 - ▶ Example: $P(\text{goal} \mid \text{if shoot}) = 0.5$. How many goals do you score?
 - ▶ Answer: that depends on how many times you actually shot for goal, which is not given here

The minimum error criterion

- ▶ The ML decision

Minimum risk (cost) criterion

- ▶ How to choose the threshold? We need criteria
 - ▶ In general: how to delimit regions R_i ?
- ▶ Minimum risk (cost) criterion: assign costs to decisions, minimize average cost
 - ▶ C_{ij} = cost of decision D_i when symbol was a_j
 - ▶ C_{00} = cost for good a_0 detection
 - ▶ C_{10} = cost for false alarm
 - ▶ C_{01} = cost for miss
- ▶ The risk = the average cost

$$R = C_{00}P(D_0 \cap a_0) + C_{10}P(D_1 \cap a_0) + C_{01}P(D_0 \cap a_1) + C_{11}P(D_1 \cap a_1)$$

- ▶ Minimum risk criterion: **minimize the risk R**

Computations

- ▶ Proof on table:
 - ▶ Use Bayes rule
 - ▶ Notations: $w(r|a_j)$ (*likelihood*)
 - ▶ Probabilities: $\int_{R_i} w(r|a_j) dV$
- ▶ Conclusion, **decision rule is**

$$\frac{w(r|a_1)}{w(r|a_0)} \geq \frac{(C_{10} - C_{00})p(a_0)}{(C_{01} - C_{11})p(a_1)}$$

$$\Lambda(r) \geq K$$

- ▶ Interpretation: effect of costs, probabilities (move threshold)
- ▶ Can also apply logarithm (useful for normal distribution)

$$\ln \Lambda(r) \geq \ln K$$

- ▶ Example at blackboard: random noise with $N(0, \sigma^2)$, one sample

Ideal observer criterion

- ▶ Minimize the probability of decision error P_e
 - ▶ definition of P_e
- ▶ Particular case of minimum risk, with
 - ▶ $C_{00} = C_{11} = 0$ (good decisions bear no cost)
 - ▶ $C_{10} = C_{01}$ (pay the same in case of bad decisions)

$$\frac{w(r|a_1)}{w(r|a_0)} \geq \frac{p(a_0)}{p(a_1)}$$

Maximum likelihood criterion

- ▶ Particular case of above, with equal probability of messages

$$\frac{w(r|a_1)}{w(r|a_0)} \geq 1$$

$$\ln \frac{w(r|a_1)}{w(r|a_0)} \geq 0$$

- ▶ Example at blackboard: random noise with $N(0, \sigma^2)$, one sample
- ▶ Example at blackboard: random noise with $N(0, \sigma^2)$, **two** samples