

Chapter III: The Z Transform

III.1 Introducing the Z transform

Preliminaries: complex numbers

- ▶ real and imaginary part
- ▶ **modulus and phase**
- ▶ graphical interpretation
- ▶ Euler formula
- ▶ modulus and phase of e^{jx}

Definition of Z transform

- ▶ The Z Transform of a signal $x[n]$, called $X(z)$, is defined as:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

- ▶ It is defined only for the values of z where the sum is finite (called *region of convergence*)
- ▶ Notation:

$$\mathcal{Z}(x[n]) = X(Z)$$

$$x[n] \xleftrightarrow{\mathcal{Z}} X(Z)$$

- ▶ Similar to the Laplace transform for analog signals
- ▶ The Z transform associates a **polynomial** to a signal (think Decision and Estimation class)
- ▶ Why?
 - ▶ Convolution of two signals = multiplication of polynomials
 - ▶ Short descriptions of complicated signals (i.e. exponential signals)

Examples

$x[n] = 1, 2, 5, 7, 0$, (with time origin in 1 or in 5)

$\delta[n]$, $\delta[n - k]$, $\delta[n + k]$

$\left(\frac{1}{2}\right)^n$

$x[n] = a^n u[n]$

$x[n] = -a^n u[-n - 1]$

Region of convergence

- ▶ For finite-support signals, the CR is the whole Z plane, possibly except 0 or ∞
- ▶ For causal signals, the CR is *the outside of a circle*:

$$|z| > r_1$$

- ▶ For anti-causal signals, the CR is *the inside of a circle*:

$$|z| < r_2$$

- ▶ For bilateral signals, both the causal and the anti-causal terms of the sum must converge \longrightarrow the CR is the area between two circles:

$$r_1 < |z| < r_2$$

- ▶ For finite-support signals, the two “circles” are 0 and ∞
- ▶ Two different signals can have the same expression of $X(z)$, but with different RC!
 - ▶ RC is an essential part in specifying a Z transform
 - ▶ should never be omitted

The Inverse Z Transform

- ▶ From a purely mathematical point of view, $X(z)$ is a complex function
- ▶ Proper definition of inverse transform is based on the theory of complex functions

$$X(z) = \sum_{-\infty}^{\infty} x[k]z^{-k}$$

- ▶ Multiply with z^{n-1} and integrate along a contour C inside the Convergence Region:

$$\oint_C X(z)z^{n-1}dz = \oint_C \sum_{-\infty}^{\infty} x[k]z^{n-k-1}dz = \sum_{-\infty}^{\infty} x[k] \oint_C z^{n-k-1}dz$$

The Inverse Z Transform

- ▶ The Cauchy integral theorem says that:

$$\frac{1}{2\pi j} \oint_C z^{n-k-1} dz = \begin{cases} 1, & \text{if } k = n \\ 0, & \text{if } k \neq n \end{cases}$$

- ▶ And therefore:

$$x[n] = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz$$

- ▶ We will not use this relation in practice, but instead will rely on **partial fraction decomposition**

Properties of Z transform

1. Linearity

If $x_1[n] \xleftrightarrow{Z} X_1(z)$ with CR1, and $x_2[n] \xleftrightarrow{Z} X_2(z)$ with CR2, then:

$$ax_1[n] + bx_2[n] \xleftrightarrow{Z} aX_1(z) + bX_2(z)$$

and CR is at least the intersection of CR1 and CR2.

Proof: use definition

Properties of Z transform

2. Shifting in time

If $x[n] \xleftrightarrow{\mathcal{Z}} X(z)$ with CR, then:

$$x[n - k] \xleftrightarrow{\mathcal{Z}} z^{-k} X(z)$$

with same RC, possibly except 0 and ∞ .

Proof: by definition

- ▶ valid for all k , also for $k < 0$
- ▶ delay of 1 sample $= z^{-1}$

Properties of Z transform

3. Modulation in time

If $x[n] \xleftrightarrow{\mathcal{Z}} X(z)$ with CR, then:

$$e^{j\omega_0 n} x[n] \xleftrightarrow{\mathcal{Z}} X(e^{-j\omega_0} z)$$

with same CR

Proof: by definition

Properties of Z transform

4. Reflected signal

If $x[n] \xrightarrow{Z} X(z)$ with CR $r_1 < |z| < r_2$, then:

$$x[-n] \xrightarrow{Z} X(z^{-1})$$

with CR $\frac{1}{r_2} < |z| < \frac{1}{r_1}$

Proof: by definition

Properties of Z transform

5. Derivative of Z transform

If $x[n] \xleftrightarrow{Z} X(z)$ with CR, then:

$$nx[n] \xleftrightarrow{Z} \frac{dX(z)}{dz}$$

with same CR

Proof: by derivating the difference

Properties of Z transform

6. Transform of difference

If $x[n] \xrightarrow{Z} X(z)$ with CR, then:

$$x[n] - x[n-1] \xrightarrow{Z} (1 - z^{-1})X(z)$$

with same CR except $z = 0$.

Proof: using linearity and time-shift property

Properties of Z transform

7. Accumulation in time

If $x[n] \xleftrightarrow{Z} X(z)$ with CR, then:

$$y[n] = \sum_{k=-\infty}^n x[k] \xleftrightarrow{Z} \frac{X(z)}{(1 - z^{-1})}$$

with same CR except $z = 1$.

Proof: $x[n] = y[n] - y[n - 1]$, apply previous property

Properties of Z transform

8. Complex conjugation

If $x[n] \xrightarrow{Z} X(z)$ with CR, and $x[n]$ is a complex signal, then:

$$x^*[n] \xrightarrow{Z} X^*(z^*)$$

with same CR except $z = 0$.

Proof: apply definition

Properties of Z transform

9. Convolution in time

If $x_1[n] \xleftrightarrow{Z} X_1(z)$ with CR1, and $x_2[n] \xleftrightarrow{Z} X_2(z)$ with CR2, then:

$$x[n] = x_1[n] * x_2[n] \xleftrightarrow{Z} X(z) = X_1(z) \cdot X_2(z)$$

and CR the intersection of CR1 and CR2.

Proof: use definition

- ▶ **Very important property!**
- ▶ Can compute the convolution of two signals via the Z transform

Properties of Z transform

10. Correlation in time

If $x_1[n] \xleftrightarrow{Z} X_1(z)$ with CR1, and $x_2[n] \xleftrightarrow{Z} X_2(z)$ with CR2, then:

$$r_{x_1 x_2}[l] = \sum_{n=-\infty}^{\infty} x_1[n] x_2[n-l] \xleftrightarrow{Z} R_{x_1 x_2}(z) = X_1(z) \cdot X_2(z^{-1})$$

and CR the intersection of CR1 and with the CR of $X_2(z^{-1})$ (see reflection property)

Proof: correlation = convolution with second signal reflected, use convolution and reflection properties

11. Initial value theorem

If $x[n]$ is a causal signal, then

$$x[0] = \lim_{z \rightarrow \infty} X(z)$$

Proof:

$$X(z) = \sum_{n=0}^{\infty} x[n]z^{-n} = x[0] + x[1]z^{-1} + x[2]z^{-2} + \dots$$

When $z \rightarrow \infty$, all terms z^{-k} vanish.

Common Z transform pairs

- Easily found all over the Internet

Sequence	Transform	ROC
$\delta[n]$	1	All z
$u[n]$	$\frac{1}{1-z^{-1}}$	$ z > 1$
$-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$ z < 1$
$\delta[n-m]$	z^{-m}	All z except 0 or ∞
$a^n u[n]$	$\frac{1}{1-az^{-1}}$	$ z > a $
$-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$ z < a $
$na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$ z > a $
$-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$ z < a $
$\begin{cases} a^n & 0 \leq n \leq N-1, \\ 0 & \text{otherwise} \end{cases}$	$\frac{1-a^N z^{-N}}{1-az^{-1}}$	$ z > 0$
$\cos(\omega_0 n) u[n]$	$\frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$ z > 1$
$r^n \cos(\omega_0 n) u[n]$	$\frac{1-r\cos(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$ z > r$

III.2. Z transforms which are Rational Functions

Rational functions

- ▶ Many Z transforms are in the form of a rational function, i.e. a fraction where
 - ▶ numerator = polynomial in z^{-1} or z
 - ▶ denominator = polynomial in z^{-1} or z

$$X(z) = \frac{B(z)}{A(z)} = \frac{\sum_{k=0}^M b_k z^{-k}}{\sum_{k=0}^N a_k z^{-k}} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \dots + b_M z^{-M}}{a_0 + a_1 z^{-1} + a_2 z^{-2} + \dots + a_N z^{-N}}$$

- ▶ Example:

$$X(z) = \frac{3 + 2z^{-1} + 4z^{-2}}{1 - 5z^{-2} + 7z^{-4}}$$

Poles and zeros

- ▶ A polynomial is completely determined by its roots and a scaling factor
- ▶ **Definition:** the **zeros** of $X(z)$ are the roots of the numerator $B(z)$
- ▶ **Definition:** the **poles** of $X(z)$ are the roots of the denominator $A(z)$
- ▶ The zeros are usually named z_1, z_2, \dots, z_M , and the poles p_1, p_2, \dots, p_N .

The transform $X(z)$ can be rewritten as:

$$X(z) = \frac{b_0}{a_0} \cdot z^{N-M} \cdot \frac{(z - z_1) \dots (z - z_M)}{(z - p_1) \dots (z - p_N)} = \frac{b_0}{a_0} \cdot \frac{(1 - z_1 z^{-1}) \dots (1 - z_M z^{-1})}{(1 - p_1 z^{-1}) \dots (1 - p_N z^{-1})}$$

It has:

- ▶ M zeros with finite values
- ▶ N poles with finite values
- ▶ and either N-M zeros in 0, if $N > M$, or N-M poles in 0, if $N < M$ (trivial poles/zeros)

Graphical representation

- ▶ The graphical representation of poles and zeros in the complex plane is called **the pole-zero plot**
- ▶ Graphical: poles = “x”, zeros = “0”
- ▶ CR cannot contain poles
- ▶ Example: at whiteboard