Chapter III: The Z Transform

III.1 Introducing the Z transform

Preliminaries: complex numbers

- real and imaginary part
- **modulus and phase**
- graphical interpretation
- Euler formula
- modulus and phase of e^{jx}

Definition of Z transform

- The Z Transform of a signal $x[n]$, called $X(z)$, is defined as:

$$X(z) = \sum_{-\infty}^{\infty} x[n] z^{-n}$$

- It is defined only for the values of z where the sum is finite (called *region of convergence)*
- Notation:

$$\mathcal{Z}\left(x[n]\right) = X(Z)$$

$$x[n] \overset{\mathrm{Z}}{\longleftrightarrow} X(Z)$$

- Similar to the Laplace transform for analog signals
- The Z transform associates **a polynomial** to a signal (think Decision and Estimation class)
- Why?
  - Convolution of two signals = multiplication of polynomials
  - Short descriptions of complicated signals (i.e. exponential signals)

Examples

$x[n] = 1, 2, 5, 7, 0,$ (with time origin in 1 or in 5)
$\delta[n]$, $\delta[n-k]$, $\delta[n+k]$
$\left(\frac{1}{2}\right)^n$
$x[n] = a^n u[n]$
$x[n] = -a^n u[-n-1]$

Region of convergence

- For finite-support signals, the CR is the whole Z plane, possibly except 0 or ∞
- For causal signals, the CR is *the outside of a circle*:

$$|z| > r_1$$

- For anti-causal signals, the CR is *the inside of a circle*:

$$|z| < r_2$$

- For bilateral signals, both the causal and the anti-causal terms of the sum must converges —> the CR is the area between two circles:

$$r_1 < |z| < r_2$$

- For finite-support signals, the two "circles" are 0 and ∞
- Two different signals can have the same expression of $X(z)$, but with different RC!
    - RC is an essential part in specifying a Z transform
    - should never be omitted

The Inverse Z Transform

- From a purely mathematical point of view, $X(z)$ is a complex function
- Proper definition of inverse transform is based on the theory of complex functions

$$X(z) = \sum_{-\infty}^{\infty} x[k] z^{-k}$$

- Multiply with z^{n-1} and integrate along a contour C inside the Convergence Region:

$$\oint_C X(z) z^{n-1} dz = \oint_C \sum_{-\infty}^{\infty} x[k] z^{n-k-1} dz = \sum_{-\infty}^{\infty} x[k] \oint_C z^{n-k-1} dz$$

The Inverse Z Transform

- The Cauchy integral theorem says that:

$$\frac{1}{2\pi j}\oint_C z^{n-k-1}dz = \begin{cases} 1, & \text{if } k = n \\ 0, & \text{if } k \neq n \end{cases}$$

- And therefore:

$$x[n] = \frac{1}{2\pi j}\oint_C X(z)z^{n-1}dz$$

- We will not use this relation in practice, but instead will rely on **partial fraction decomposition**

Properties of Z transform

1. Linearity

If $x_1[n] \overset{Z}{\longleftrightarrow} X_1(z)$ with CR1, and $x_2[n] \overset{Z}{\longleftrightarrow} X_2(z)$ with CR2, then:

$$ax_1[n] + bx_2[n] \overset{Z}{\longleftrightarrow} aX_1(z) + bX_2(z)$$

and CR is at least the intersection of CR1 and CR2.
Proof: use definition

Properties of Z transform

2. Shifting in time

If $x[n] \overset{\mathrm{Z}}{\longleftrightarrow} X(z)$ with CR, then:

$$x[n-k] \overset{\mathrm{Z}}{\longleftrightarrow} z^{-k}X(z)$$

with same RC, possibly except 0 and ∞.
Proof: by definition

- valid for all k, also for $k < 0$
- delay of 1 sample $= z^{-1}$

Properties of Z transform

3. Modulation in time

If $x[n] \overset{\mathrm{Z}}{\longleftrightarrow} X(z)$ with CR, then:

$$e^{j\omega_0 n}x[n] \overset{\mathrm{Z}}{\longleftrightarrow} X\left(e^{-j\omega_0}z\right)$$

with same CR
Proof: by definition

Properties of Z transform

4. Reflected signal

If $x[n] \overset{\mathrm{Z}}{\longleftrightarrow} X(z)$ with CR $r_1 < |z| < r_2$, then:

$$x[-n] \overset{\mathrm{Z}}{\longleftrightarrow} X\left(z^{-1}\right)$$

with CR $\frac{1}{r_2} < |z| \frac{1}{r_1}$
Proof: by definition

Properties of Z transform

5. Derivative of Z transform

If $x[n] \overset{\mathrm{Z}}{\longleftrightarrow} X(z)$ with CR, then:

$$nx[n] \overset{\mathrm{Z}}{\longleftrightarrow} \frac{dX(z)}{dz}$$

with same CR

Proof: by derivating the difference

Properties of Z transform

6. Transform of difference

If $x[n] \xleftrightarrow{\mathrm{Z}} X(z)$ with CR, then:

$$x[n] - x[n-1] \xleftrightarrow{\mathrm{Z}} (1 - z^{-1})X(z)$$

with same CR except $z = 0$.
Proof: using linearity and time-shift property

Properties of Z transform

7. Accumulation in time

If $x[n] \overset{\mathrm{Z}}{\longleftrightarrow} X(z)$ with CR, then:

$$y[n] = \sum_{k=-\infty}^{n} x[k] \overset{\mathrm{Z}}{\longleftrightarrow} \frac{X(z)}{(1 - z^{-1})}$$

with same CR except $z = 1$.

Proof: $x[n] = y[n] - y[n-1]$, apply previous property

Properties of Z transform

8. Complex conjugation

If $x[n] \xleftrightarrow{\mathrm{Z}} X(z)$ with CR, and $x[n]$ is a complex signal, then:

$$x^*[n] \xleftrightarrow{\mathrm{Z}} X^*\left(z^*\right)$$

with same CR except $z = 0$.
Proof: apply definition

Properties of Z transform

9. Convolution in time

If $x_1[n] \xleftrightarrow{\mathrm{Z}} X_1(z)$ with CR1, and $x_2[n] \xleftrightarrow{\mathrm{Z}} X_2(z)$ with CR2, then:

$$x[n] = x_1[n] * x_2[n] \xleftrightarrow{\mathrm{Z}} X(z) = X_1(z) \cdot X_2(z)$$

and CR the intersection of CR1 and CR2.
Proof: use definition

- **Very important property!**
- Can compute the convolution of two signals via the Z transform

10. Correlation in time

If $x_1[n] \overset{\mathrm{Z}}{\longleftrightarrow} X_1(z)$ with CR1, and $x_2[n] \overset{\mathrm{Z}}{\longleftrightarrow} X_2(z)$ with CR2, then:

$$r_{x_1 x_2}[l] = \sum_{n=-\infty}^{\infty} x_1[n] x_2[n-l] \overset{\mathrm{Z}}{\longleftrightarrow} R_{x_1 x_2}(z) = X_1(z) \cdot X_2(z^{-1})$$

and CR the intersection of CR1 and with the CR of $X_2(z^{-1})$ (see reflection property)
Proof: correlation = convolution with second signal reflected, use convolution and reflection properties

Properties of Z transform

11. Initial value theorem

If $x[n]$ is a causal signal, then

$$x[0] = \lim_{z \to \infty} X(z)$$

Proof:

$$X(z) = \sum_{n=0}^{\infty} x[n]z^{-n} = x[0] + x[1]z^{-1} + x[2]z^{-2} + ...$$

When $z \to \infty$, all terms z^{-k} vanish.

Common Z transform pairs

- Easily found all over the Internet

Sequence	Transform	ROC
$\delta[n]$	1	All z
$u[n]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
$-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
$\delta[n-m]$	z^{-m}	All z except 0 or ∞
$a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
$-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
$na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
$-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
$\begin{cases} a^n & 0 \le n \le N-1, \\ 0 & \text{otherwise} \end{cases}$	$\frac{1-a^N z^{-N}}{1-az^{-1}}$	$\lvert z\rvert > 0$
$\cos(\omega_0 n)u[n]$	$\frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
$r^n \cos(\omega_0 n)u[n]$	$\frac{1-r\cos(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2 z^{-2}}$	$\lvert z\rvert > r$

III.2. Z transforms which are Rational Functions

Rational functions

- Many Z transforms are in the form of a rational function, i.e. a fraction where
  - numerator = polynomial in z^{-1} or z
  - denominator = polynomial in z^{-1} or z

$$X(z) = \frac{B(z)}{A(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} = \frac{b_0 + b_1 z^{-1} + b_2 z{-2} + ... + b_M z^{-M}}{a_0 + a_1 z^{-1} + a_2 z{-2} + ... + a_N z^{-N}}$$

- Example:

$$X(z) = \frac{3 + 2z^{-1} + 4z^{-2}}{1 - 5z^{-2} + 7z^{-4}}$$

Poles and zeros

- A polynomial is completely determined by its roots and a scaling factor
- **Definition**: the **zeros** of $X(z)$ are the roots of the numerator $B(z)$
- **Definition**: the **poles** of $X(z)$ are the roots of the denominator $A(z)$
- The zeros are usually named $z_1, z_2, ...z_M$, and the poles $p_1, p_2, ...p_N$.

The transform $X(z)$ can be rewritten as:

$$X(z) = \frac{b_0}{a_0} \cdot z^{N-M} \cdot \frac{(z - z_1)...(z - z_M)}{(z - p_1)...(z - z_N)} = \frac{b_0}{a_0} \cdot \frac{(1 - z_1 z^{-1})...(1 - z_M z^{-1})}{(1 - p_1 z^{-1})...(1 - z_N z^{-1})}$$

It has:

- M zeros with finite values
- N poles with finite values
- and either N-M zeros in 0, if N $>$ M, or N-M poles in 0, if N $<$ M (trivial poles/zeros)

Graphical representation

- The graphical representation of poles and zeros in the complex place is called **the pole-zero plot**
- Graphical: poles = "x", zeros = "0"
- CR cannot contain poles
- Example: at whiteboard

III.3 Inverse Z transform for rational functions

Methods for computing the Inverse Z Transform

1. Direct evaluation using the Cauchy integral

$$x[n] = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz$$

2. Decomposition as continuous power series
3. **Partial fraction decomposition**

Partial fraction decomposition

Any rational function

$$\frac{b_0 + b_1 z^{-1} + b_2 z{-2} + \dots + b_M z^{-M}}{a_0 + a_1 z^{-1} + a_2 z{-2} + \dots + a_N z^{-N}}$$

can be decomposed in **partial fractions**:

$$c_0 + c_1 z^{-1} + \dots c_{N-M} z^{-(M-N)} + \frac{A_1}{z - p_1} + \dots \frac{A_N}{z - p_N}$$

- Each pole has a corresponding partial fraction
- First terms appear if $M \leq N$
- Based on linearity, we invert each term separately (simple)

Procedure for Inverse Z Transform

$$X(z) = \frac{B(z)}{A(z)}$$

1. If $M \geq N$, divide numerator to denominator to obtain the first terms. The remaining fraction is $X_1(z) = \frac{B_1(z)}{A(z)}$, with numerator degree strictly smaller then denominator
2. In the remaining fraction, eliminate the negative powers of z by multiplying with z^N
3. Divide by z,
$$\frac{X_1(z)}{z} = \frac{B_1(z)}{zA(z)}$$
4. Compute the poles of $\frac{X_1(z)}{z}$ and decompose in partial fractions:
$$\frac{X_1(z)}{z} = \frac{A_1}{z - p_1} + ...$$

Procedure for Inverse Z Transform

5. Multiply back with z:

$$X_1(z) = A_1 \frac{z}{z - p_1} + \dots$$

6. Convert each term back to the time domain

Computation of partial fractions coefficients

- If all poles are distinct:

$$A_k = (z - p_k)\frac{X(z)}{z}|_{z=p_k}$$

- If poles are in complex conjugate pairs
  - group the two fractions into a single fraction of degree 2
- If there exist m **multiple poles of same value** (pole order $m > 1$):

$$\frac{A_{1k}}{z-p_k} + \frac{A_k}{(z-p_k)^2} + ... + \frac{A_{mk}}{(z-p_k)^m}$$

$$A_{ik} = \frac{1}{(m-i)!}\frac{d^{m-i}}{dz^{m-i}}\left[(z-p_k)^m \cdot \frac{X(z)}{z}\right]|_{z=p_k}$$

`* example for m = 2`

Position of poles and time behaviour