

## Chapter III: The Z Transform

## III.1 Introducing the $Z$ transform

# Preliminaries: complex numbers

- ▶ real and imaginary part
- ▶ **modulus and phase**
- ▶ graphical interpretation
- ▶ Euler formula
- ▶ modulus and phase of  $e^{jx}$

# Definition of Z transform

- ▶ The Z Transform of a signal  $x[n]$ , called  $X(z)$ , is defined as:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

- ▶ It is defined only for the values of  $z$  where the sum is finite (called *region of convergence*)
- ▶ Notation:

$$\mathcal{Z}(x[n]) = X(Z)$$

$$x[n] \xleftrightarrow{\mathcal{Z}} X(Z)$$

- ▶ Similar to the Laplace transform for analog signals
- ▶ The Z transform associates a **polynomial** to a signal (think Decision and Estimation class)
- ▶ Why?
  - ▶ Convolution of two signals = multiplication of polynomials
  - ▶ Short descriptions of complicated signals (i.e. exponential signals)

# Examples

$x[n] = 1, 2, 5, 7, 0$ , (with time origin in 1 or in 5)

$\delta[n]$ ,  $\delta[n - k]$ ,  $\delta[n + k]$

$\left(\frac{1}{2}\right)^n$

$x[n] = a^n u[n]$

$x[n] = -a^n u[-n - 1]$

# Region of convergence

- ▶ For finite-support signals, the CR is the whole  $Z$  plane, possibly except 0 or  $\infty$
- ▶ For causal signals, the CR is *the outside of a circle*:

$$|z| > r_1$$

- ▶ For anti-causal signals, the CR is *the inside of a circle*:

$$|z| < r_2$$

- ▶ For bilateral signals, both the causal and the anti-causal terms of the sum must converge  $\rightarrow$  the CR is the area between two circles:

$$r_1 < |z| < r_2$$

- ▶ For finite-support signals, the two “circles” are 0 and  $\infty$
- ▶ Two different signals can have the same expression of  $X(z)$ , but with different RC!
  - ▶ RC is an essential part in specifying a  $Z$  transform
  - ▶ should never be omitted

# The Inverse Z Transform

- ▶ From a purely mathematical point of view,  $X(z)$  is a complex function
- ▶ Proper definition of inverse transform is based on the theory of complex functions

$$X(z) = \sum_{-\infty}^{\infty} x[k]z^{-k}$$

- ▶ Multiply with  $z^{n-1}$  and integrate along a contour  $C$  inside the Convergence Region:

$$\oint_C X(z)z^{n-1}dz = \oint_C \sum_{-\infty}^{\infty} x[k]z^{n-k-1}dz = \sum_{-\infty}^{\infty} x[k] \oint_C z^{n-k-1}dz$$

# The Inverse Z Transform

- ▶ The Cauchy integral theorem says that:

$$\frac{1}{2\pi j} \oint_C z^{n-k-1} dz = \begin{cases} 1, & \text{if } k = n \\ 0, & \text{if } k \neq n \end{cases}$$

- ▶ And therefore:

$$x[n] = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz$$

- ▶ We will not use this relation in practice, but instead will rely on **partial fraction decomposition**



# Properties of Z transform

## 1. Linearity

If  $x_1[n] \xleftrightarrow{Z} X_1(z)$  with CR1, and  $x_2[n] \xleftrightarrow{Z} X_2(z)$  with CR2, then:

$$ax_1[n] + bx_2[n] \xleftrightarrow{Z} aX_1(z) + bX_2(z)$$

and CR is at least the intersection of CR1 and CR2.

Proof: use definition

# Properties of Z transform

## 2. Shifting in time

If  $x[n] \xrightarrow{\mathcal{Z}} X(z)$  with CR, then:

$$x[n - k] \xrightarrow{\mathcal{Z}} z^{-k} X(z)$$

with same RC, possibly except 0 and  $\infty$ .

Proof: by definition

- ▶ valid for all  $k$ , also for  $k < 0$
- ▶ delay of 1 sample  $= z^{-1}$

# Properties of Z transform

## 3. Modulation in time

If  $x[n] \xleftrightarrow{\mathcal{Z}} X(z)$  with CR, then:

$$e^{j\omega_0 n} x[n] \xleftrightarrow{\mathcal{Z}} X(e^{-j\omega_0} z)$$

with same CR

Proof: by definition

# Properties of Z transform

## 4. Reflected signal

If  $x[n] \xrightarrow{Z} X(z)$  with CR  $r_1 < |z| < r_2$ , then:

$$x[-n] \xrightarrow{Z} X(z^{-1})$$

with CR  $\frac{1}{r_2} < |z| < \frac{1}{r_1}$

Proof: by definition

# Properties of Z transform

## 5. Derivative of Z transform

If  $x[n] \xleftrightarrow{Z} X(z)$  with CR, then:

$$nx[n] \xleftrightarrow{Z} -z \frac{dX(z)}{dz}$$

with same CR

Proof: by derivating the difference

# Properties of Z transform

## 6. Transform of difference

If  $x[n] \xleftrightarrow{Z} X(z)$  with CR, then:

$$x[n] - x[n-1] \xleftrightarrow{Z} (1 - z^{-1})X(z)$$

with same CR except  $z = 0$ .

Proof: using linearity and time-shift property

# Properties of Z transform

## 7. Accumulation in time

If  $x[n] \xleftrightarrow{Z} X(z)$  with CR, then:

$$y[n] = \sum_{k=-\infty}^n x[k] \xleftrightarrow{Z} \frac{X(z)}{(1 - z^{-1})}$$

with same CR except  $z = 1$ .

Proof:  $x[n] = y[n] - y[n-1]$ , apply previous property

# Properties of Z transform

## 8. Complex conjugation

If  $x[n] \xrightarrow{Z} X(z)$  with CR, and  $x[n]$  is a complex signal, then:

$$x^*[n] \xrightarrow{Z} X^*(z^*)$$

with same CR except  $z = 0$ .

Proof: apply definition

## Consequence

If  $x[n]$  is a real signal, the poles / zeroes are either real or in complex pairs



# Properties of Z transform

## 9. Convolution in time

If  $x_1[n] \xleftrightarrow{Z} X_1(z)$  with CR1, and  $x_2[n] \xleftrightarrow{Z} X_2(z)$  with CR2, then:

$$x[n] = x_1[n] * x_2[n] \xleftrightarrow{Z} X(z) = X_1(z) \cdot X_2(z)$$

and CR the intersection of CR1 and CR2.

Proof: use definition

- ▶ **Very important property!**
- ▶ Can compute the convolution of two signals via the Z transform

# Properties of Z transform

## 10. Correlation in time

If  $x_1[n] \xleftrightarrow{Z} X_1(z)$  with CR1, and  $x_2[n] \xleftrightarrow{Z} X_2(z)$  with CR2, then:

$$r_{x_1 x_2}[l] = \sum_{n=-\infty}^{\infty} x_1[n] x_2[n-l] \xleftrightarrow{Z} R_{x_1 x_2}(z) = X_1(z) \cdot X_2(z^{-1})$$

and CR the intersection of CR1 and with the CR of  $X_2(z^{-1})$  (see reflection property)

Proof: correlation = convolution with second signal reflected, use convolution and reflection properties

## 11. Initial value theorem

If  $x[n]$  is a causal signal, then

$$x[0] = \lim_{z \rightarrow \infty} X(z)$$

Proof:

$$X(z) = \sum_{n=0}^{\infty} x[n]z^{-n} = x[0] + x[1]z^{-1} + x[2]z^{-2} + \dots$$

When  $z \rightarrow \infty$ , all terms  $z^{-k}$  vanish.

# Common Z transform pairs

- Easily found all over the Internet

Sequence	Transform	ROC
$\delta[n]$	1	All $z$
$u[n]$	$\frac{1}{1-z^{-1}}$	$ z  > 1$
$-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$ z  < 1$
$\delta[n-m]$	$z^{-m}$	All $z$ except 0 or $\infty$
$a^n u[n]$	$\frac{1}{1-az^{-1}}$	$ z  >  a $
$-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$ z  <  a $
$na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$ z  >  a $
$-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$ z  <  a $
$\begin{cases} a^n & 0 \leq n \leq N-1, \\ 0 & \text{otherwise} \end{cases}$	$\frac{1-a^N z^{-N}}{1-az^{-1}}$	$ z  > 0$
$\cos(\omega_0 n) u[n]$	$\frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$ z  > 1$
$r^n \cos(\omega_0 n) u[n]$	$\frac{1-r\cos(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$ z  > r$

## III.2. Z transforms which are Rational Functions

# Rational functions

- ▶ Many Z transforms are in the form of a rational function, i.e. a fraction where
  - ▶ numerator = polynomial in  $z^{-1}$  or  $z$
  - ▶ denominator = polynomial in  $z^{-1}$  or  $z$

$$X(z) = \frac{B(z)}{A(z)} = \frac{\sum_{k=0}^M b_k z^{-k}}{\sum_{k=0}^N a_k z^{-k}} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \dots + b_M z^{-M}}{a_0 + a_1 z^{-1} + a_2 z^{-2} + \dots + a_N z^{-N}}$$

- ▶ Example:

$$X(z) = \frac{3 + 2z^{-1} + 4z^{-2}}{1 - 5z^{-2} + 7z^{-4}}$$

# Poles and zeros

- ▶ A polynomial is completely determined by its roots and a scaling factor
- ▶ **Definition:** the **zeros** of  $X(z)$  are the roots of the numerator  $B(z)$
- ▶ **Definition:** the **poles** of  $X(z)$  are the roots of the denominator  $A(z)$
- ▶ The zeros are usually named  $z_1, z_2, \dots, z_M$ , and the poles  $p_1, p_2, \dots, p_N$ .

The transform  $X(z)$  can be rewritten as:

$$X(z) = \frac{b_0}{a_0} \cdot z^{N-M} \cdot \frac{(z - z_1) \dots (z - z_M)}{(z - p_1) \dots (z - p_N)} = \frac{b_0}{a_0} \cdot \frac{(1 - z_1 z^{-1}) \dots (1 - z_M z^{-1})}{(1 - p_1 z^{-1}) \dots (1 - p_N z^{-1})}$$

It has:

- ▶ M zeros with finite values
- ▶ N poles with finite values
- ▶ and either N-M zeros in 0, if  $N > M$ , or N-M poles in 0, if  $N < M$  (trivial poles/zeros)

# Graphical representation

- ▶ The graphical representation of poles and zeros in the complex plane is called **the pole-zero plot**
- ▶ Graphical: poles = “x”, zeros = “0”
- ▶ CR cannot contain poles
- ▶ Example: at whiteboard



### III.3 Inverse Z transform for rational functions

# Methods for computing the Inverse Z Transform

1. Direct evaluation using the Cauchy integral

$$x[n] = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz$$

2. Decomposition as continuous power series
3. **Partial fraction decomposition**

# Partial fraction decomposition

Any rational function

$$\frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \dots + b_M z^{-M}}{a_0 + a_1 z^{-1} + a_2 z^{-2} + \dots + a_N z^{-N}}$$

can be decomposed in **partial fractions**:

$$c_0 + c_1 z^{-1} + \dots c_{N-M} z^{-(M-N)} + \frac{A_1}{z - p_1} + \dots \frac{A_N}{z - p_N}$$

- ▶ Each pole has a corresponding partial fraction
- ▶ First terms appear if  $M \leq N$
- ▶ Based on linearity, we invert each term separately (simple)

# Procedure for Inverse Z Transform

$$X(z) = \frac{B(z)}{A(z)}$$

1. If  $M \geq N$ , divide numerator to denominator to obtain the first terms. The remaining fraction is  $X_1(z) = \frac{B_1(z)}{A(z)}$ , with numerator degree strictly smaller than denominator
2. In the remaining fraction, eliminate the negative powers of  $z$  by multiplying with  $z^N$

3. Divide by  $z$ ,

$$\frac{X_1(z)}{z} = \frac{B_1(z)}{zA(z)}$$

4. Compute the poles of  $\frac{X_1(z)}{z}$  and decompose in partial fractions:

$$\frac{X_1(z)}{z} = \frac{A_1}{z - p_1} + \dots$$

# Procedure for Inverse Z Transform

5. Multiply back with  $z$ :

$$X_1(z) = A_1 \frac{z}{z - p_1} + \dots$$

6. Convert each term back to the time domain

# Computation of partial fractions coefficients

- ▶ If all poles are distinct:

$$A_k = (z - p_k) \frac{X(z)}{z} \Big|_{z=p_k}$$

- ▶ If poles are in complex conjugate pairs
  - ▶ group the two fractions into a single fraction of degree 2
- ▶ If there exist  $m$  **multiple poles of same value** (pole order  $m > 1$ ):

$$\frac{A_{1k}}{z - p_k} + \frac{A_{2k}}{(z - p_k)^2} + \dots + \frac{A_{mk}}{(z - p_k)^m}$$

$$A_{ik} = \frac{1}{(m-i)!} \frac{d^{m-i}}{dz^{m-i}} \left[ (z - p_k)^m \cdot \frac{X(z)}{z} \right] \Big|_{z=p_k}$$

\* example for  $m = 2$

# Real signals and complex poles/zeros

- ▶ Consequence of the complex-conjugate property of Z transform:
- ▶ A real signal  $x[n]$  can have only
  - ▶ real poles or zeroes
  - ▶ complex poles and zeroes in conjugate pairs, which can be grouped into a single term of degree 2, with real coefficients
- ▶ If a Z transform has a complex pole / zero without its conjugate pair, then the corresponding signal  $x[n]$  is complex

# Position of poles and time behavior

- ▶ A rational Z transform  $X(z)$  = sum of partial fractions

←

The signal  $x[n]$  is a sum of exponential signals (for each partial fraction / pole)

- ▶ In the following, we will analyze the relation between the position of the pole and the signal in time



## Position of poles and time behaviour - 1 pole

- ▶ Consider a Z transform with **1 pole**, analyze the look of the corresponding signal
- ▶ Consider the pole value is  $a$ 
  - ▶ Consider only real signals  $x[n] \rightarrow a$  is real
  - ▶ Consider causal signal  $x[n] \rightarrow$  CR is  $|z| > |a|$
- ▶ Therefore the Z transform is of the type:

$$X(z) = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}, CR : |z| > |a|$$

- ▶ Therefore the signal  $x[n]$  is of the type:

$$x[n] = a^n u[n]$$

# Position of poles and time behavior - 1 pole

Scenarios for a single real pole in  $a$ :

- ▶ Pole inside the unit circle ( $|a| < 1$ )  $\rightarrow$  exponential decreasing signal
- ▶ Pole outside the unit circle ( $|a| > 1$ )  $\rightarrow$  exponential increasing signal
- ▶ Pole exactly on unit circle ( $|a| = 1$ )  $\rightarrow$  not increasing, not decreasing
- ▶ Negative pole ( $a < 0$ )  $\rightarrow$  alternating signal
- ▶ Positive value ( $a > 0$ )  $\rightarrow$  non-alternating signal

# Position of poles and time behavior - 1 pole

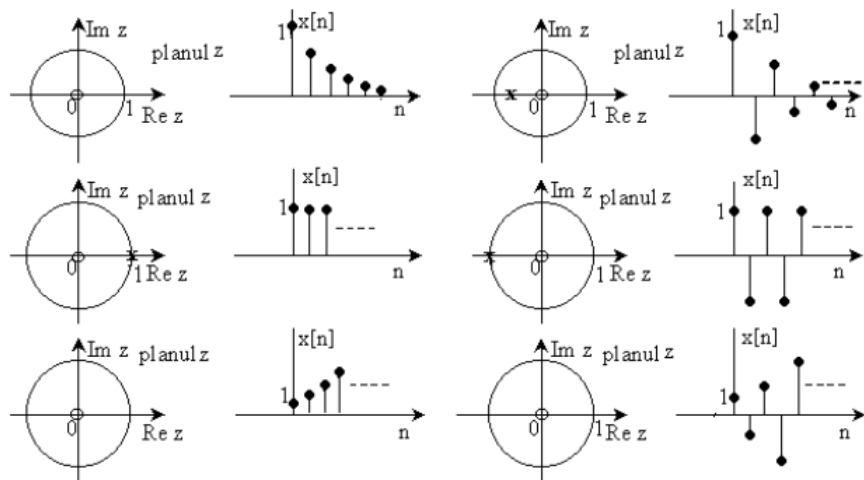


Figure 2: Signal behavior for 1 pole

## Position of poles and time behavior - 1 double pole

$$X(z) = \frac{az^{-1}}{(1 - az^{-1})^2}, CR : |z| > |a|$$

$$x[n] = na^n u[n]$$

A double pole in  $a$ :

- ▶ Pole inside the unit circle ( $|a| < 1$ )  $\rightarrow$  decreasing signal
- ▶ Pole outside the unit circle ( $|a| > 1$ )  $\rightarrow$  increasing signal
- ▶ Pole exactly on unit circle ( $|a| = 1$ )  $\rightarrow$  **increasing signal**
- ▶ Negative pole ( $a < 0$ )  $\rightarrow$  alternating signal
- ▶ Positive value ( $a > 0$ )  $\rightarrow$  non-alternating signal

# Position of poles and time behavior - 1 double pole

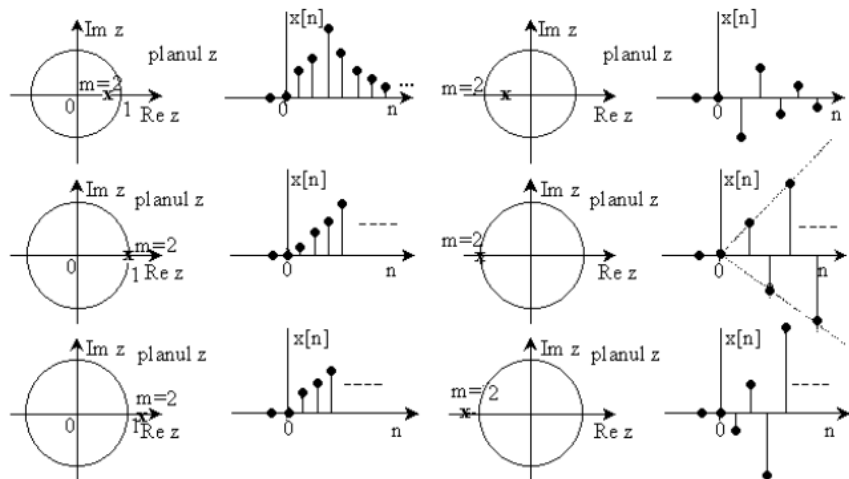


Figure 3: Signal behavior for 1 double pole

## Position of poles and time behavior - conjugate poles

$$X(z) = \frac{1 - az^{-1} \cos \omega_0}{1 - 2z^{-1} \cos \omega_0 + z^{-2}}, \text{ CR : } |z| > |a|$$

$$x[n] = na^n u[n]$$

A pair of complex conjugate poles:

- ▶ a sinusoidal with exponential envelope
  - ▶ phase of poles  $\rightarrow$  frequency of sinusoidal signal
  - ▶ modulus of poles  $\rightarrow$  exponential envelope
  - ▶ poles inside unit circle  $\rightarrow$  decreasing signal
  - ▶ poles outside unit circle  $\rightarrow$  increasing signal
  - ▶ poles on unit circle  $\rightarrow$  oscillating signal, constant amplitude

What if poles are double?

- \* poles on unit circle  $\rightarrow$  increasing signal
- \* otherwise similar

# Position of poles and time behavior - conjugate poles

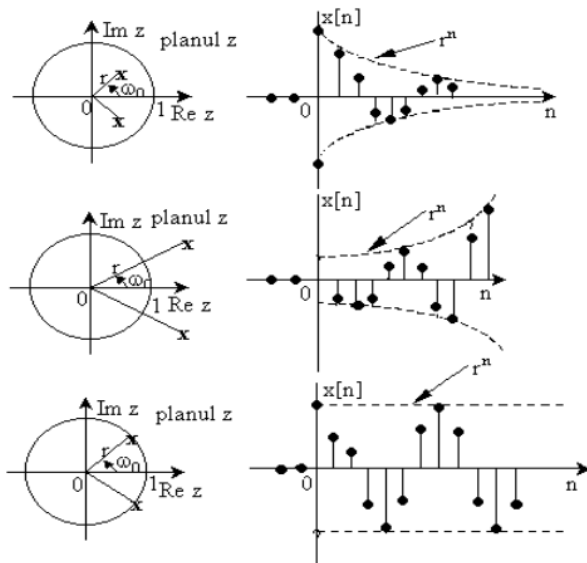


Figure 4: Signal behavior for 1 double pole

# Position of poles and time behavior

- ▶ A Z transform can be decomposed into partial fractions, i.e. separate poles
- ▶ Analyzing the individual behavior of poles  $\rightarrow$  tells something about whole signal
- ▶ Conclusions (for real signals, causal):
  - ▶ **all poles inside unit circle  $\rightarrow$  bounded signal**
  - ▶ *simple* poles on unit circle  $\rightarrow$  bounded signal
  - ▶ otherwise  $\rightarrow$  unbounded signal
  - ▶ poles inside unit circle, closer to origin  $\rightarrow$  fast decrease of signal
  - ▶ poles inside unit circle, closer to unit circle  $\rightarrow$  slow decrease of signal



## III.4 LTI systems and the Z Transform

# System function of a LTI system

- ▶ Considering a LTI system with  $h[n]$ , input signal  $x[n] \rightarrow$  output is convolution

$$y[n] = x[n] * h[n]$$

- ▶ In Z transform, convolution = product of transforms

$$Y(z) = X(z) \cdot H(z)$$

- ▶ **The system function of a LTI system is the Z transform of the impulse response  $h[n]$**
- ▶ The system function of a LTI system is:

$$H(z) = \frac{Y(z)}{X(z)}$$

# System function and the difference equation

- ▶ Any LTI system is characterized by a **difference equation**:

$$\begin{aligned}y[n] &= - \sum_{k=1}^N a_k y[n-k] + \sum_{k=0}^M b_k x[n-k] \\&= -a_1 y[n-1] - a_2 y[n-2] - \dots - a_N y[n-N] + b_0 x[n] + b_1 x[n-1] - \dots\end{aligned}$$

or

$$y[n] + \sum_{k=1}^N a_k y[n-k] = \sum_{k=0}^M b_k x[n-k]$$

$$y[n] + a_1 y[n-1] + a_2 y[n-2] + \dots + a_N y[n-N] = b_0 x[n] + b_1 x[n-1] + \dots + b_M x[n-M]$$

# System function and the difference equation

- ▶ The system function  $H(z)$  can be derived directly from the difference equation:

$$Y(z) \left( 1 + \sum_{k=1}^N a_k z^{-k} \right) = X(z) \left( \sum_{k=0}^M b_k z^{-k} \right)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^M b_k z^{-k}}{1 + \sum_{k=1}^N a_k z^{-k}}$$

# Particular cases of system functions

- ▶ FIR systems:  $a_k = 0$ 
  - ▶ has only zeroes, no poles (all-zero system)

$$H(z) = \frac{Y(z)}{X(z)} = \sum_{k=0}^M b_k z^{-k}$$

- ▶ All-pole system:  $b_k = 0, k \geq 1$  (must have at least  $b_0 \neq 0$ )
  - ▶ has only poles

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0}{1 + \sum_{k=1}^N a_k z^{-k}}$$

- ▶ Otherwise, in general, we have a *pole-zero system*, with both poles and zeroes

# Computing the system output, no initial conditions

- ▶ Consider a LTI system with initial conditions = 0 (relaxed system)
  - ▶ Remember: I.C. are relevant for IIR implementations ( $y[n-k]$ ), not FIR
- ▶ Input signal:

$$x[n] \xleftrightarrow{Z} X(z) = \frac{N(z)}{Q(z)}$$

- ▶ Impulse response / System function:

$$h[n] \xleftrightarrow{Z} H(z) = \frac{B(z)}{A(z)}$$

- ▶ Output signal:

$$x[n] * h[n] \xleftrightarrow{Z} X(z)H(z) = \frac{N(z)B(z)}{Q(z)A(z)}$$