Chapter IV: Frequency Analysis of Discrete Systems

IV.1 Reminder: Frequency analysis of analog signals

Introduction

- Very useful to analyze signals in **frequency domain**
- The **spectrum** of a signal indicates the frequency contents
- Mathematical tools:
  - periodicac signals: **Fourier series**
  - non-periodical signals: **Fourier transform**

Analog periodical signals

- Periodical signal:
$$x(t) = x(t + T)$$
- The fundamental frequency is
$$F_0 = \frac{1}{T}$$
- **The signal can be decomposed as a sum of complex exponential signals, with multiples of the fundamental frequency,** kF_0
$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k F_0 t}$$
- The coefficients c_k are the **spectrum** of the signal
$$c_k = \frac{1}{T} \int_{T/2}^{T/2} x(t) e^{-j2\pi k F_0 t}$$
- The coefficients c_k are complex values
  - their modulus = "amplitude spectrum"
  - their phase = "phase spectrum""

Conditions for convergence

- When is the Fourier series convergent to the signal?
  - i.e. when is the relation correct,
  - i.e. when is the sum actually equal to $x(t)$?
- Dirichlet conditions: the sum is convergent in all continuity points if:
  1. $x(t)$ is continuous or has a finite number of discontinuities in any finite interval
  2. $x(t)$ has a finite number of maxima and minima in any period
  3. $x(t)$ is absolutely integrable in any period, i.e.:

$$\int_T |x(t)| dt < \infty$$

- Weaker condition:
  - if $x(t)$ is square summable

$$\int_T x(t)^2 dt < \infty$$

  - then the he difference $d(t) = x(t) - \sum_{k=-\infty}^{\infty} c_k e^{j 2\pi k F_0 t}$ has zero energy
  - Does not guarantee *pointwise* convergence

Signal spectrum

- The coefficients c_k are complex numbers
- If the signal is **real** $x(t) \in \mathbb{R}$, then the c_k are **even**
  - $|c_k| = |c_{-k}|$
  - $\angle c_k = -\angle c_{-k}$
  - group the terms with c_k with c_{-k} –> **cosine with amplitude |c_k| and phase** $\angle c_k$
- Average power of signal = energy of coefficients

$$P_T = \frac{1}{T} \int_T |x(t)|^2 = \sum_{-infty}^{\infty} |c_k|^2$$

- Interpretation of Fourier series for real signal
  - **the signal is the sum of cosine signals with frequency** $0, F_0, 2F_0 ...$**, with amplitudes** $|c_k|$ **and phase** $\angle c_k$
- No other frequencies appear in spectrum –> spectrum is made of "lines"
- Time-frequency **duality**:
  - Real signal –> Even spectrum
  - Periodic signal –> Discrete spectrum

Analog non-periodical signals

- The signal is composed of all frequencies (inverse Fourier transform)

$$x(t) = \int_{infty}^{\infty} X(F) e^{j2\pi ft} dF$$

- The frequency content is found by the Fourier transform

$$X(F) = \int_{infty}^{\infty} x(t) e^{-j2\pi ft} dt$$

- (Can use instead $\Omega = 2\pi F$)
- $X(F)$ is a complex function
  - $|X(F)|$ is the amplitude spectrum
  - $\angle X(F)$ is the phase spectrum ### Conditions for convergence
- When is the Fourier series convergent to the signal?
  - i.e. when is the relation correct,
  - i.e. when is the sum actually equal to $x(t)$?
- Dirichlet conditions: the sum is convergent in all continuity points if:
  1. $x(t)$ is continuous or has a finite number of discontinuities
  2. $x(t)$ has a finite number of maxima and minima
  3. $x(t)$ is absolutely integrable:

Signal spectrum

- $X(F)$ is a complex function
- If the signal is **real** $x(t) \in \mathbb{R}$, then the $X(F)$ is **even**
  - $|X(F)| = |X(-F)|$
  - $\angle X(F) = -\angle X(-F)$
  - group the terms with c_k with c_{-k} **–> cosine with amplitude |X(F)| and phase** $\angle X(F)$
- Signal energy is the same in time and frequency domains

$$E = \int_{\infty}^{\infty} |x(t)|^2 dt = \int_{\infty}^{\infty} |X(F)|^2 dF$$

- The power spectral density of $x(t)$ is

$$S_{xx}(F) = |X(F)|^2$$

IV.2 Frequency analysis of discrete signals

Fourier series of discrete periodical signals

- A discrete signal of period N: $x[n] = x[n+N]$
- Decomposed as a **sum of complex exponentials**:

$$x[n] = \sum_{k=0}^{N-1} c_k e^{j2\pi kn/N}, n = 0, 1, ... N-1$$

- Finding the coefficients:

$$c_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn}$$

- Compared to analog signals:
  - consider fundamental frequency $f_0 = 1/N$
  - only N terms: $0, f_0 = 1/N, 2f_0, ...(N-1)f_0 = (N-1)/N$
  - only N distinct coefficients c_k
  - the N coefficients c_k can be chosen like $-\frac{N}{2} < k \leq \frac{N}{2}$ −> the frequencies span the range $-1/2 ... 1/2$

$$-\frac{1}{2} < f_k \leq \frac{1}{2}$$

$$-\pi < \omega_k \leq \pi$$

Signal spectrum

- Signal is **discrete** –> coefficients are **periodic** with period N

$$c_{k+N} = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j2\pi(k+N)n} = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn}$$

- If signal is real $x[n] \in \mathbb{R}$, the coefficients are even:
  - $c_k^* = c_{-k}$
  - $|c_k| = |c_{-k}|$
  - $\angle c_k = \angle c_{-k}$
- Together with periodicity:
  - $|c_k| = |c_{-k}| = |c_{N-k}|$
  - $\angle c_k = -anglec_{-k} = -\angle c_{N-k}$
- Grouping terms with c_k and c_{-k} we get

$$x[n] = c_0 + 2 \sum_{k=1}^{L} |c_k| cos(\frac{2\pi k}{N} + \angle c_k)$$

where $L = N/2$ or $L = (N-1)/2$ depending if N is even or odd

Power spectral density

- The average power of a discrete periodic signal

$$P = \frac{1}{N} \sum_{n=0}^{N-1} |x[n]|^2$$

- Is the same in the frequency domain (with proof):

$$P = \sum_{k=0}^{N-1} |c_k|^2$$

- Power spectral density of the signal is

$$S_x x[k] = |c_k|^2$$

- Energy over one period is

$$E = \sum_{n-0}^{N-1} |x[n]|^2 = N \sum_{k-0}^{N-1} |c_k|^2$$

Examples

- Examples

Properties of Fourier series

Linearity

If the signal $x_1[n]$ has the Fourier series coefficients $\{c_k^{(1)}\}$, and $x_2[n]$ has $\{c_k^{(2)}\}$, then