

## Chapter IV: Frequency Analysis of Discrete Systems

## IV.1 Reminder: Frequency analysis of analog signals

# Introduction

- ▶ Very useful to analyze signals in **frequency domain**
- ▶ The **spectrum** of a signal indicates the frequency contents
- ▶ Mathematical tools:
  - ▶ periodic signals: **Fourier series**
  - ▶ non-periodical signals: **Fourier transform**

# Analog periodical signals

- ▶ Periodical signal:

$$x(t) = x(t + T)$$

- ▶ The fundamental frequency is

$$F_0 = \frac{1}{T}$$

- ▶ **The signal can be decomposed as a sum of complex exponential signals, with multiples of the fundamental frequency,  $kF_0$**

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k F_0 t}$$

- ▶ The coefficients  $c_k$  are the **spectrum** of the signal

$$c_k = \frac{1}{T} \int_{T/2}^{T/2} x(t) e^{-j2\pi k F_0 t} dt$$

- ▶ The coefficients  $c_k$  are complex values
  - ▶ their modulus = “amplitude spectrum”
  - ▶ their phase = “phase spectrum”

# Conditions for convergence

- ▶ When is the Fourier series convergent to the signal?
  - ▶ i.e. when is the relation correct,
  - ▶ i.e. when is the sum actually equal to  $x(t)$ ?
- ▶ Dirichlet conditions: the sum is convergent in all continuity points if:
  1.  $x(t)$  is continuous or has a finite number of discontinuities in any finite interval
  2.  $x(t)$  has a finite number of maxima and minima in any period
  3.  $x(t)$  is absolutely integrable in any period, i.e.:

$$\int_T |x(t)| dt < \infty$$

- ▶ Weaker condition:
  - ▶ if  $x(t)$  is square summable

$$\int_T x(t)^2 dt < \infty$$

- ▶ then the difference  $d(t) = x(t) - \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k F_0 t}$  has zero energy
- ▶ Does not guarantee *pointwise* convergence

# Signal spectrum

- ▶ The coefficients  $c_k$  are complex numbers
- ▶ If the signal is **real**  $x(t) \in \mathbb{R}$ , then the  $c_k$  are **even**
  - ▶  $|c_k| = |c_{-k}|$
  - ▶  $\angle c_k = -\angle c_{-k}$
  - ▶ group the terms with  $c_k$  with  $c_{-k} \rightarrow$  **cosine with amplitude  $|c_k|$  and phase  $\angle c_k$**
- ▶ Average power of signal = energy of coefficients

$$P_T = \frac{1}{T} \int_T |x(t)|^2 = \sum_{-\infty}^{\infty} |c_k|^2$$

- ▶ Interpretation of Fourier series for real signal
  - ▶ **the signal is the sum of cosine signals with frequency  $0, F_0, 2F_0, \dots$ , with amplitudes  $|c_k|$  and phase  $\angle c_k$**
- ▶ No other frequencies appear in spectrum  $\rightarrow$  spectrum is made of "lines"

# Time-frequency duality

- ▶ Time-frequency **duality**:
  - ▶ Real signal  $\rightarrow$  Even spectrum
  - ▶ Periodic signal  $\rightarrow$  Discrete spectrum

# Analog non-periodical signals

- ▶ The signal is composed of all frequencies (inverse Fourier transform)

$$x(t) = \int_{-\infty}^{\infty} X(F) e^{j2\pi ft} dF$$

- ▶ The frequency content is found by the Fourier transform

$$X(F) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt$$

- ▶ ( Can use instead  $\Omega = 2\pi F$ )
- ▶  $X(F)$  is a complex function
  - ▶  $|X(F)|$  is the amplitude spectrum
  - ▶  $\angle X(F)$  is the phase spectrum



# Conditions for convergence

- ▶ When is the Fourier series convergent to the signal?
  - ▶ i.e. when is the relation correct,
  - ▶ i.e. when is the sum actually equal to  $x(t)$ ?
- ▶ Dirichlet conditions: the sum is convergent in all continuity points if:
  1.  $x(t)$  is continuous or has a finite number of discontinuities
  2.  $x(t)$  has a finite number of maxima and minima
  3.  $x(t)$  is absolutely integrable:

$$\int_{-\infty}^{\infty} |x(t)| dt < \infty$$

- ▶ Weaker condition:
  - ▶ if  $x(t)$  is square summable

$$\int_{-\infty}^{\infty} x(t)^2 dt < \infty$$

- ▶ then the difference  $d(t) = x(t) - \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k F_0 t}$  has zero energy
- ▶ Does not guarantee *pointwise* convergence

# Signal spectrum

- ▶  $X(F)$  is a complex function
- ▶ If the signal is **real**  $x(t) \in \mathbb{R}$ , then the  $X(F)$  is **even**
  - ▶  $|X(F)| = |X(-F)|$
  - ▶  $\angle X(F) = -\angle X(-F)$
  - ▶ group the terms with  $c_k$  with  $c_{-k} \rightarrow$  **cosine with amplitude  $|X(F)|$  and phase  $\angle X(F)$**
- ▶ Signal energy is the same in time and frequency domains

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(F)|^2 dF$$

- ▶ The power spectral density of  $x(t)$  is

$$S_{xx}(F) = |X(F)|^2$$

## IV.2 Frequency analysis of discrete signals

# Fourier series of discrete periodical signals

- ▶ A discrete signal of period  $N$ :  $x[n] = x[n + N]$
- ▶ Decomposed as a **sum of complex exponentials**:

$$x[n] = \sum_{k=0}^{N-1} c_k e^{j2\pi kn/N}, n = 0, 1, \dots, N-1$$

- ▶ Finding the coefficients:

$$c_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N}$$

# Comparison with analog Fourier series

- ▶ Compared to analog signals:
  - ▶ consider fundamental frequency  $f_0 = 1/N$
  - ▶ only  $N$  terms, with frequencies  $k \cdot f_0$ :
    - ▶  $0, f_0, 2f_0, \dots, (N-1)f_0$
  - ▶ only  $N$  distinct coefficients  $c_k$
  - ▶ the  $N$  coefficients  $c_k$  can be chosen like  $-\frac{N}{2} < k \leq \frac{N}{2} \Rightarrow$  the frequencies span the range  $-1/2 \dots 1/2$

$$-\frac{1}{2} < f_k \leq \frac{1}{2}$$

$$-\pi < \omega_k \leq \pi$$

# Basic properties of Fourier coefficients

1. Signal is **discrete**  $\rightarrow$  coefficients are **periodic** with period  $N$

$$c_{k+N} = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j2\pi(k+N)n/N} = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N}$$

2. If signal is real  $x[n] \in \mathbb{R}$ , the coefficients are **even**:

- ▶  $c_k^* = c_{-k}$
- ▶  $|c_k| = |c_{-k}|$
- ▶  $\angle c_k = \angle c_{-k}$

- ▶ Together with periodicity:

- ▶  $|c_k| = |c_{-k}| = |c_{N-k}|$
- ▶  $\angle c_k = -\angle c_{-k} = -\angle c_{N-k}$

# Expressing as sum of sinusoids

- ▶ Grouping terms with  $c_k$  and  $c_{-k}$  we get

$$x[n] = c_0 + 2 \sum_{k=1}^L |c_k| \cos(2\pi \frac{k}{N} + \angle c_k)$$

where  $L = N/2$  or  $L = (N - 1)/2$  depending if  $N$  is even or odd

- ▶ Signal = DC value + a finite sum of sinusoids with frequencies  $kf_0$ 
  - ▶  $|c_k|$  give the amplitudes ( $\times 2$ )
  - ▶  $\angle c_k$  give the phases

# Power spectral density

- ▶ The average power of a discrete periodic signal

$$P = \frac{1}{N} \sum_{n=0}^{N-1} |x[n]|^2$$

- ▶ Is the same in the frequency domain (with proof):

$$P = \sum_{k=0}^{N-1} |c_k|^2$$

- ▶ Power spectral density of the signal is

$$S_{xx}[k] = |c_k|^2$$

- ▶ Energy over one period is

$$E = \sum_{n=0}^{N-1} |x[n]|^2 = N \sum_{k=0}^{N-1} |c_k|^2$$



# Examples

► Examples:

$$x_1[n] = \cos(\sqrt{5}\pi n)$$

$$x_2[n] = 2\sin\left(\frac{\pi}{3}n\right)$$

$$x_3[n] = \{1, 1, 0, 0\}$$

## Example in Python

```
>>> import numpy as np
>>> from scipy.fftpack import fft, ifft
>>> x = np.array([1.0, 1.0, 0.0, 0.0])
>>> y = 1.0/4.0 * fft(x)
>>> y
array([ 0.50+0.j   ,  0.25-0.25j,  0.00+0.j   ,  0.25+0.25j])
```

# Properties of Fourier series

## 1. Linearity

If the signal  $x_1[n]$  has the Fourier series coefficients  $\{c_k^{(1)}\}$ , and  $x_2[n]$  has  $\{c_k^{(2)}\}$ , then their sum has

$$a \cdot x_1[n] + b \cdot x_2[n] \leftrightarrow \{a \cdot c_k^{(1)} + b \cdot c_k^{(2)}\}$$

Proof: via definition

# Properties of Fourier series

## 2. Shifting in time

If  $x[n] \leftrightarrow \{c_k\}$ , then

$$x[n - n_0] \leftrightarrow \{e^{(-j2\pi kn_0/N)} c_k\}$$

Proof: via definition

- ▶ The amplitudes  $|c_k|$  are not affected, shifting in time affects only the phase

# Properties of Fourier series

## 3. Modulation in time

$$e^{j2\pi k_0 n/N} \leftrightarrow \{c_{k-k_0}\}$$

## 4. Complex conjugation

$$x^*[n] \leftrightarrow \{c_{-k}^*\}$$

# Properties of Fourier series

## 5. Circular convolution

Circular convolution of two signals  $\leftrightarrow$  product of coefficients

$$x_1[n] \otimes x_2[n] \leftrightarrow \{N \cdot c_k^{(1)} \cdot c_k^{(2)}\}$$

Circular convolution:

$$x_1[n] \otimes x_2[n] = \sum_{k=0}^{N-1} x_1[k] x_2[(n-k)_N]$$

- ▶ takes two periodic signals of period  $N$ , result is the same
- ▶ Example at the whiteboard: how it is computed

# Properties of Fourier series

## 6. Product in time

Product in time  $\leftrightarrow$  circular convolution of Fourier series coefficients

$$x_1[n] \cdot x_2[n] \leftrightarrow \sum_{m=0}^{N-1} c_m^{(1)} c_{(k-m)_N}^{(2)} = c_k^{(1)} \otimes c_k^{(2)}$$

# Fourier transform of discrete non-periodical signals

- ▶ Non-periodical signals contain all frequencies, not only the multiples of  $f_0$
- ▶ The Fourier transform of a discrete signal:

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

- ▶ The inverse Fourier transform:

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega)e^{j\omega n}d\omega$$



# Comparison

- ▶ Compared with the Fourier transform of analog signals
  - ▶ sum instead of integral in Fourier transform
  - ▶ spectrum is only in range:

$$\omega \in [-\pi, \pi]$$

$$f \in [-\frac{1}{2}, \frac{1}{2}]$$

- ▶ Compared with the Fourier series of discrete periodical signals
  - ▶ general  $\omega$  instead of  $2\pi kf_0$
  - ▶ spectrum is continuous, not discrete
  - ▶ integral, not sum in inverse Fourier transform

# Parseval theorem

- ▶ **Parseval theorem:** energy of the signal is the same in time and frequency domains

$$E = \sum_{-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(\omega)|^2$$

- ▶ Proof: on whiteboard
- ▶ The **energy spectral density** gives the energy contained for each frequency

$$S_{xx}(\omega) = |X(\omega)|^2$$

# Basic properties of Fourier transform

- ▶ It is **periodical** with period  $2\pi$

$$X(\omega + 2\pi) = X(\omega)$$

- ▶ If the signal  $x[n]$  is real, the Fourier transform is **even**

$$x[n] \in \mathbb{R} \rightarrow X^*(\omega) = X(-\omega)$$

- ▶ This means

- ▶ modulus is even:  $|X(\omega)| = |X(-\omega)|$
- ▶ phase is odd:  $X(\omega) = -X(-\omega)$

# Convergence of the Fourier transform

- ▶ When are the relations valid?
- ▶ Assume we compute the Fourier transform with only  $2M + 1$  samples:

$$X_M(\omega) = \sum_{-M}^M x[n]e^{-j\omega n}$$

- ▶ If a signal  $x[n]$  is **absolutely summable**:

$$\sum_{-\infty}^{\infty} |x[n]| < \infty$$

- ▶ then the Fourier series is **uniform convergent** for every  $\omega$  (OK):

$$\lim_{M \rightarrow \infty} X(\omega) - X_M(\omega) = 0$$

# Convergence for square-summable signals

- ▶ Signals that are only **square summable**

$$\sum_{-\infty}^{\infty} |x[n]|^2 < \infty$$

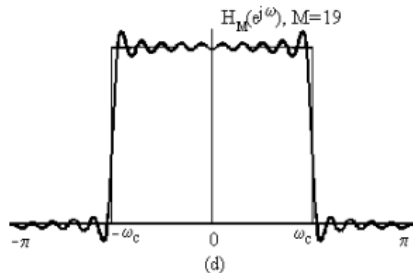
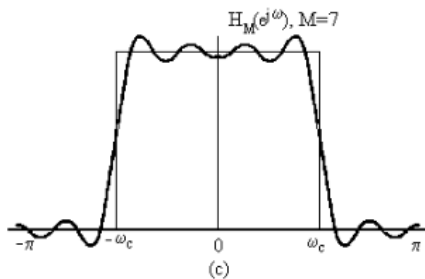
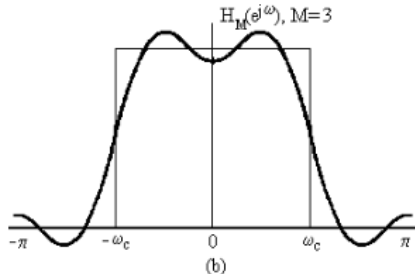
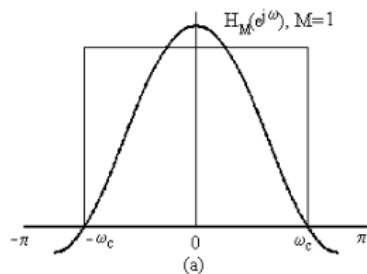
- ▶ have a weaker convergence:

$$\lim_{M \rightarrow \infty} \int_{-\pi}^{\pi} |X(\omega) - X_M(\omega)|^2 d\omega = 0$$

# The Gibbs phenomenon

- ▶ When  $H(\omega)$  **has discontinuities**, then  $h[n]$  is **not** absolutely summable, only square summable
- ▶ Problem: if we only use  $M$  samples, even if  $M$  is very large, we will obtain **small oscillations around the discontinuity**
- ▶ As  $M \rightarrow \infty$ , the oscillations do not become smaller, but thinner  $\rightarrow$  they don't go away!
- ▶ The Fourier transform will always *overshoot* with about 9% below and above
- ▶ Known as the **Gibbs phenomenon**

# Gibbs phenomenon



# Relation between Fourier series and Fourier transform

- ▶ If apply Fourier transform to periodical discrete signals,  $X(\omega)$  contains Diracs
- ▶ The Diracs are at frequencies  $kf_0$ , just like the Fourier series
- ▶ The value of an impulse = the coefficient  $c_k$  of the Fourier series
- ▶ **The Fourier series  $\approx$  the Fourier transform of periodic signals**
  - ▶ we directly compute the coefficients  $c_k$  of the impulses in the spectrum



# Fourier transform and Z transform

- ▶ Definition of Fourier transform = Z transform with:

$$z = e^{j\omega}$$

- ▶  $e^{j\omega}$  = points on the unit circle
- ▶ Fourier transform = Z transform evaluated **on the unit circle**
  - ▶ if the unit circle is in the convergence region of Z transform
  - ▶ otherwise, equivalence does not hold
- ▶ This is true for most usual signals we work with
  - ▶ there are exceptions, but they are outside the scope of this class

# Properties of Fourier transform

## 1. Linearity

$$a \cdot x_1[n] + b \cdot x_2[n] \leftrightarrow a \cdot X_1(\omega) + b \cdot X_2(\omega)$$

Proof: via definition

# Properties of Fourier transform

## 2. Shifting in time

$$x[n - n_0] \leftrightarrow e^{-j\omega n_0} X(\omega)$$

Proof: via definition

- ▶ The amplitudes  $|X(\omega)|$  is not affected, shifting in time affects only the phase

# Properties of Fourier transform

## 3. Modulation in time

$$e^{j\omega_0 n} x[n] \leftrightarrow X(\omega - \omega_0)$$

## 4. Complex conjugation

$$x^*[n] \leftrightarrow X^*(-\omega)$$

## 5. Convolution

$$x_1[n] * x_2[n] \leftrightarrow X_1(\omega) \cdot X_2(\omega)$$

- Not circular convolution, this is the normal convolution

# Properties of Fourier transform

## 6. Product in time

Product in time  $\leftrightarrow$  convolution of Fourier transforms

$$x_1[n] \cdot x_2[n] \leftrightarrow \frac{1}{2\pi} \int_{-\pi}^{\pi} X_1(\lambda) X_2(\omega - \lambda) d\lambda$$

# Properties of Fourier transform

## Correlation theorem

$$r_{x_1 x_2}[l] \leftrightarrow X_1(\omega)X_2(-\omega)$$

## Wiener Khinchin theorem

Autocorrelation of a signal  $\leftrightarrow$  Power spectral density

$$r_{xx}[l] \leftrightarrow S_{xx}(\omega) = |X(\omega)|^2$$

# Properties of Fourier transform

## Parseval theorem

Energy is the same when computed in the time or frequency domain

$$\sum |x[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(\omega)|^2 d\omega$$



# Relationship of Fourier transform and Fourier series

- ▶ How are they related?
  - ▶ Fourier transform: for non-periodical signals
  - ▶ Fourier series: for periodical series
- ▶ Duality: periodic in time  $\leftrightarrow$  discrete in frequency
- ▶ If we **periodize** a signal  $x[n]$  by repeating with period  $N$ :

$$x_N[n] = \sum_{k=-\infty}^{\infty} x[n - kN]$$

- ▶ then the Fourier transform  $w$  is discrete (made of Diracs):

$$X_N(\omega) = 2\pi c_k \delta(\omega - k \frac{2\pi}{N})$$

- ▶ The coefficients of the Diracs = exactly the Fourier series coefficients

# Relationship of Fourier transform and Fourier series

- ▶ So, Fourier transform can be considered for both periodic and non-periodic signals
- ▶ Fourier transform for periodic signals = discrete (sum of Diracs with some coefficients)
  - ▶ Diracs at frequencies  $f_0 = 1/N$  and its multiplies
- ▶ Fourier series for periodic signals = gives the coefficients of the Diracs directly
  - ▶ it just omits to write the Diracs explicitly in the equation

# Relation of Fourier transform and Z transform

- ▶ Fourier transform:  $X(\omega) = \sum_{-\infty}^{\infty} x[n]e^{-j\omega n}$
- ▶ Z transform:  $X(z) = \sum_{-\infty}^{\infty} x[n]z^{-n}$
- ▶ **Fourier transform = Z transform for  $z = e^{j\omega}$**
- ▶  $z = e^{j\omega}$  means **evaluated on the unit circle**:
  - ▶  $|z| = |e^{j\omega}| = 1$  (*modulus*)
  - ▶  $\angle z = \angle e^{j\omega} = \omega$  (*phase*)
- ▶ Conditions:
  - ▶ unit circle must be in the Convergence Region of Z transform
  - ▶ some signals can have Fourier transform even though unit circle not in CR
- ▶ If signal has pole on unit circle  $\rightarrow$  Dirac (infinite) in Fourier transform
  - ▶ e.g.  $u[n]$
  - ▶ some signals are non-convergent on unit circle, but have Fourier transform (e.g.  $u[n]$ )

# Geometric interpretation of Fourier transform

$$X(z) = C \cdot \frac{(z - z_1) \cdots (z - z_M)}{(z - p_1) \cdots (z - p_N)}$$

$$X(\omega) = C \cdot \frac{(e^{j\omega} - z_1) \cdots (e^{j\omega} - z_M)}{(e^{j\omega} - p_1) \cdots (e^{j\omega} - p_N)}$$

- Modulus:

$$|X(\omega)| = |C| \cdot \frac{|e^{j\omega} - z_1| \cdots |e^{j\omega} - z_M|}{|e^{j\omega} - p_1| \cdots |e^{j\omega} - p_N|}$$

- Phase:

$$\angle X = \angle C + \angle(e^{j\omega} - z_1) + \cdots + \angle(e^{j\omega} - z_M) - \angle(e^{j\omega} - p_1) - \cdots - \angle(e^{j\omega} - p_N)$$

# Geometric interpretation of Fourier transform

- ▶ For complex numbers:
  - ▶ modulus of  $|a - b|$  = the length of the segment between  $a$  and  $b$
  - ▶ phase of  $|a - b|$  = the angle of the segment from  $b$  to  $a$  (direction is important)
- ▶ So, for a point on the unit circle  $z = e^{j\omega}$ 
  - ▶ modulus  $|X(\omega)|$  is **given by the distances to the zeros and to the poles**
  - ▶ phase  $\angle X(\omega)$  is **given by the angles from the zeros and poles to  $z$**

# Geometric interpretation of Fourier transform

- ▶ Consequences:
  - ▶ when a **pole** is very close to unit circle  $\rightarrow$  Fourier transform is **large** at this point
  - ▶ when a **zero** is very close to unit circle  $\rightarrow$  Fourier transform is **small** at this point
- ▶ Examples:...
- ▶ Simple interpretation for modulus  $|X(\omega)|$ :
  - ▶ Z transform  $X(z)$  is a “landscape”
    - ▶ poles = mountains of infinite height
    - ▶ zeros = valleys of zero height
  - ▶ Fourier transform  $X(\omega) =$  “Walking over this landscape along the unit circle”  $\rightarrow$  the heights give the Fourier transform
  - ▶ When close to a mountain  $\rightarrow$  road is high  $\rightarrow$  Fourier transform has large amplitude
  - ▶ When close to a valley  $\rightarrow$  road is low  $\rightarrow$  Fourier transform has small amplitude
- ▶ Enough to sketch the Fourier transform for signals with few poles/zeros

# Geometric interpretation of Fourier transform

- ▶ Note:  $X(z)$  might also have a constant  $C$  in front!
  - ▶ It does not appear in pole-zero plot
  - ▶ The value of  $|C|$  and  $\angle C$  must be determined separately
- ▶ This “geometric method” can be applied for both modulus and phase

# Time-frequency duality

- ▶ **Duality** properties related to Fourier transform/series
- ▶ Discrete  $\leftrightarrow$  Periodic
  - ▶ **discrete** in time  $\rightarrow$  **periodic** in frequency
  - ▶ **periodic** in time  $\rightarrow$  **discrete** in frequency
- ▶ Continuous  $\leftrightarrow$  Non-periodic
  - ▶ **continuous** in time  $\rightarrow$  **non-periodic** in frequency
  - ▶ **non-periodic** in time  $\rightarrow$  **continuous** in frequency



# Frequency-based classification of signals

- ▶ Based on frequency content:
  - ▶ **low-frequency** signals
  - ▶ **mid-frequency** signals (band-pass)
  - ▶ **high-frequency** signals
- ▶ **Band-limited** signals: spectrum is 0 over some frequency  $f_{max}$
- ▶ **Time-limited** signals: signal value is 0 outside some time interval
- ▶ **Bandwidth**  $B$ : frequency interval  $[F_1, F_2]$  which contains 95% of energy
  - ▶  $B = F_2 - F_1$
- ▶ Based on bandwidth  $B$ :
  - ▶ **Narrow-band** signals:  $B \ll$  central frequency  $\frac{F_1 + F_2}{2}$
  - ▶ **Wide-band** signals: not narrow-band