II. Discrete signals and systems

II.1 Discrete signals

A discrete signal can represented:

- graphically
- in table form
- as a vector: $x[n] = [..., 0, 0, 1, 3, 4, 5, 0, ...]$, with an **arrow** indicating the origin of time ($n = 0$). If the arrow is missing, the origin of time is at the first element. The dots ... indicate that the value remains the same from that point onwards

Examples: blackboard

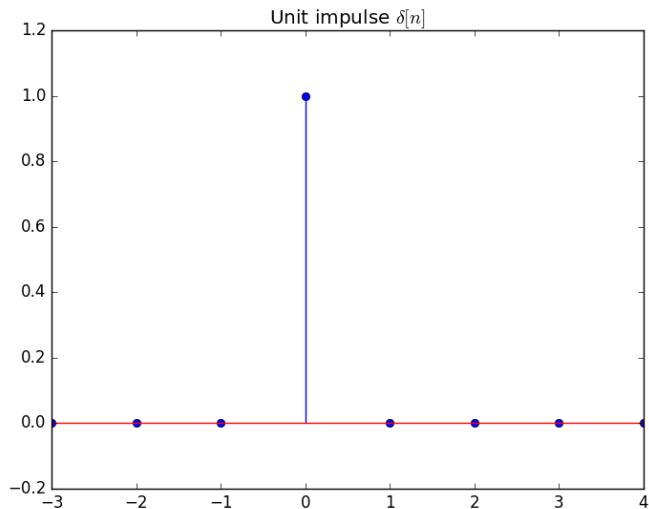Notation: $x[4]$ represents the value of the fourth sample in the signal $x[n]$

Some elementary signals are presented below.

Unit impulse

Contains a single non-zero value of 1 located at time 0. It is denoted with $\delta[n]$.

$$\delta[n] = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases}.$$

Unit impulse $\delta[n]$
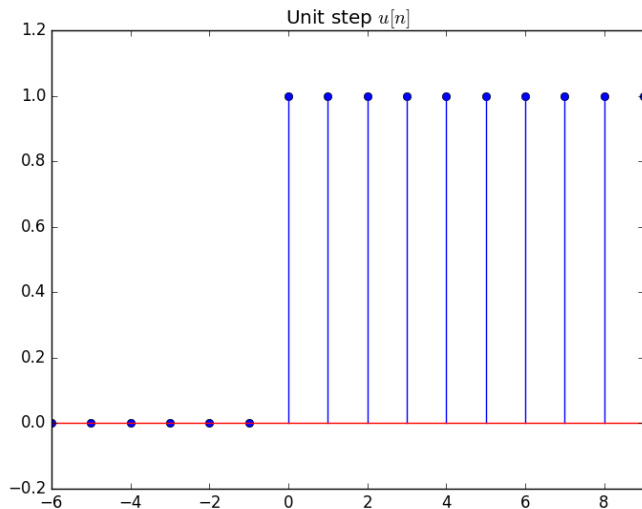
Unit step

It is denoted with $u[n]$.

$$u[n] = \begin{cases} 1 & \text{if } n \geq 0 \\ 0 & \text{otherwise} \end{cases}.$$
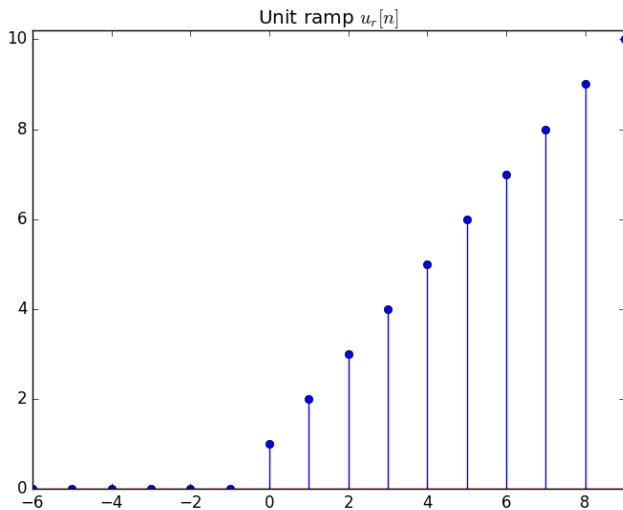
Representation

Unit ramp

It is denoted with $u_r[n]$.

$$u_r[n] = \begin{cases} n & \text{if } n \geq 0 \\ 0 & \text{otherwise} \end{cases}.$$

Unit ramp $u_r[n]$

Exponential signal

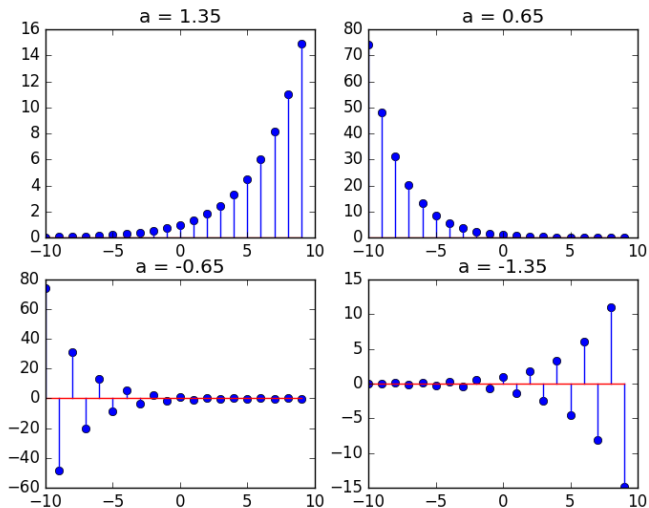It does not have a special notation. It is defined by:

$$x[n] = a^n.$$

a can be a real or a complex number. Here we consider only the case when a is real.

Depending on the value of a, we have four possible cases:

1. $a \geq 1$
2. $0 \leq a < 1$
3. $-1 < a < 0$
4. $a \leq 1$

Representation

II.2 Types of discrete signals

- The **energy of a discrete signal** is defined as

$$E = \sum_{n=-\infty}^{\infty} (x[n])^2.$$

- If E is finite, the signal is said to have finite energy.
- Examples: unit impulse has finite energy; unit step does not.

- The **average power of a discrete signal** is defined as

$$P = \lim_{N \to \infty} \frac{\sum_{n=-N}^{N} (x[n])^2}{2N + 1}.$$

- In other words, the average power is the average energy per sample.
- If P is finite, the signal is said to have finite power.
- A signal with finite energy has finite power ($P = 0$ if the signal has infinite length). A signal with infinite energy can have finite or infinite power.
- Example: unit step has finite power $P = \frac{1}{2}$ (see proof at blackboard).

Periodic and non-periodic signals

- A signal is called **periodic** if its values repeat themselves after a certain time (known as **period**).

$$x[n] = x[n + N]), \forall t$$

- The **fundamental period** of a signal is the minimum value of N.
- Periodic signals have infinite energy, and finite power equal to the power of a single period.

Even and odd signals

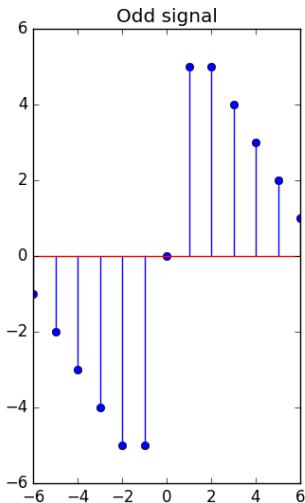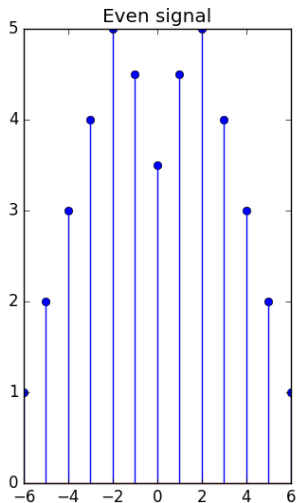- A signal is **even** if it satisfies the following symmetry:

$$x[n] = x[-n], \forall n.$$

- A signal is **odd** if it satisfies the following anti-symmetry:

$$-x[n] = x[-n], \forall n.$$

- There exist signals which are neither even nor odd.

Even and odd parts of a signal

- Every signal can be written as the sum of an even signal and an odd signal:
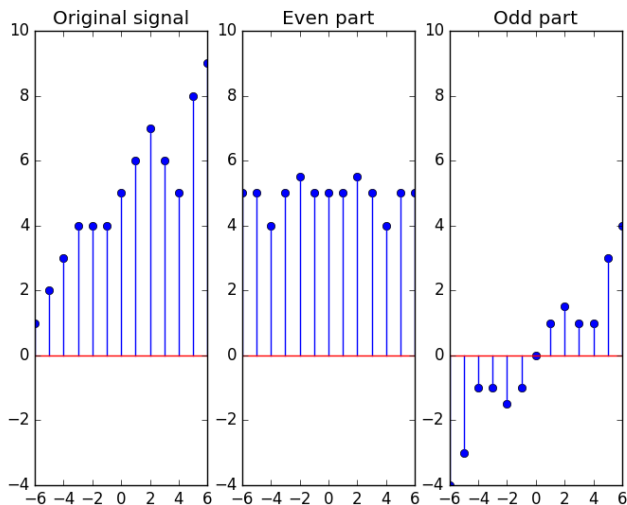
$$x[n] = x_e[n] + x_o[n]$$

- The even and the odd parts of the signal can be found as follows:

$$x_e[n] = \frac{x[n] + x[-n]}{2}.$$

$$x_o[n] = \frac{x[n] - x[-n]}{2}.$$

- Proof: check that $x_e[n]$ is even, $x_o[n]$ is odd, and their sum is $x[n]$

Even and odd parts: example

II.3 Basic operations with discrete signals

- Let $x[n]$ be a signal.
- The signal $x[n - k]$ **is** $x[n]$ **delayed with** k **time units**. Graphically, $x[n - k]$ is shifted k units to the **right** compared to the original signal.
- The signal $x[n + k]$ **is** $x[n]$ **anticipated with** k **time units**. Graphically, $x[n + k]$ is shifted k units to the **left** compared to the original signal.
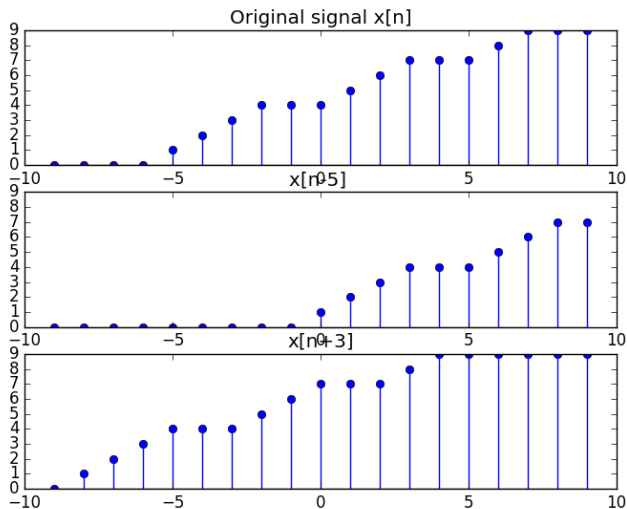
Time reversal

- Changing the variable n to $-n$ produces a signal $x[-n]$ which mirrors $x[n]$.
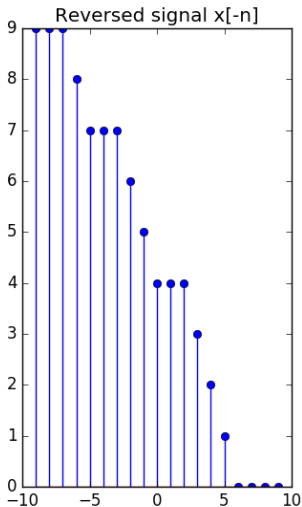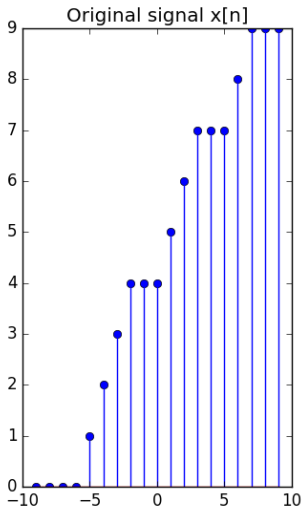
Subsampling

- $x_{M\downarrow}[n] = x[Mn]$ is a **subsampled** version of $x[n]$ with a factor of M.
- Only 1 sample out of M are kept from the original signal $x[n]$, the rest are discarded.
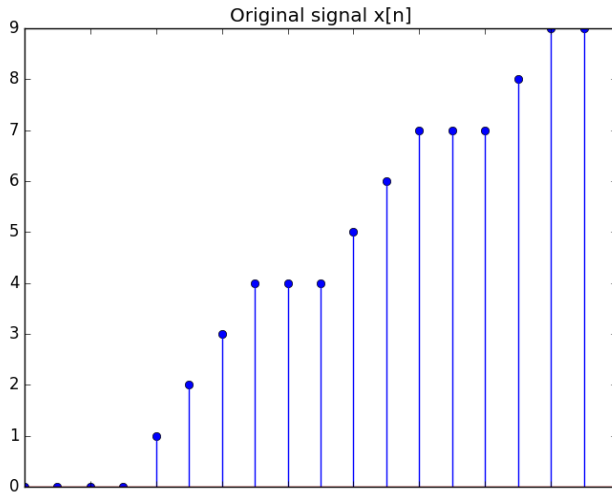
Original signal x[n]

▶ **Interpolation** by a factor of L adds L of zeros between two samples in the original signal.

$$x_{L\uparrow} = \begin{cases} x[\frac{n}{L}] & \text{if } \frac{n}{L} \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}.$$

Original signal x[n]
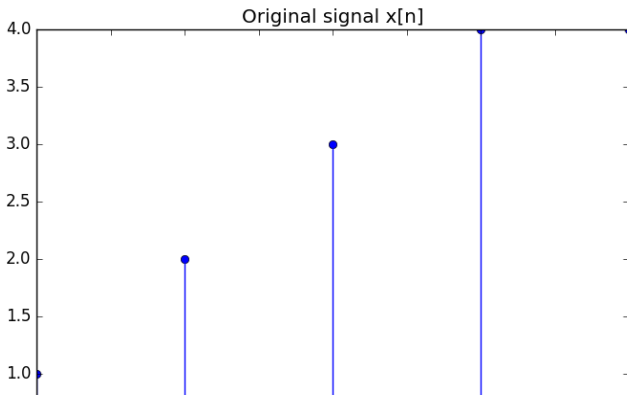
Mathematical operations

- A signal $x[n]$ can be **scaled** by a constant A, i.e. each sample is multipled by A.

$$y[n] = Ax[n].$$

- Two signals $x_1[n]$ and $x_2[n]$ can be **summed** by summing the individual samples:

$$y[n] = x_1[n] + x_2[n]$$

- Two signals $x_1[n]$ and $x_2[n]$ can be **multiplied** by multiypling the individual samples:

$$y[n] = x_1[n] \cdot x_2[n]$$

II.4 Discrete systems

Definition

- **System** = a device or algorithm which produces an **output signal** based on an **input signal**.
- We will only consider systems with a single input and a single output
- Figure here: blackboard.
- Common notation:
  - x[n] is the input
  - y[n] is the output
  - H is the system.

Notations

- The relation between the signals can be written as

$$y[n] = H[x[n]],$$

meaning "*the system H applied to the input x[n] produces the output y[n]*".

- It can also be represented as

$$x[n] \xrightarrow{H} y[n],$$

suggesting "*the input x[n] is transformed by the system H into y[n]*".

- Usually, a system is described by the **input-output equation** which expains how $y[n]$ is defined in terms of $x[n]$.

Examples:

1. $y[n] = x[n]$ (the identity system)
2. $y[n] = x[n-3]$
3. $y[n] = x[n+1]$
4. $y[n] = \frac{1}{3}(x[n+1] + x[n] + x[n-1])$
5. $y[n] = \max(x[n+1], x[n], x[n-1])$
6. $y[n] = (x[n])^2 + \ldots = x[n-1]$

Example

$y[n] = x[n - 3]$
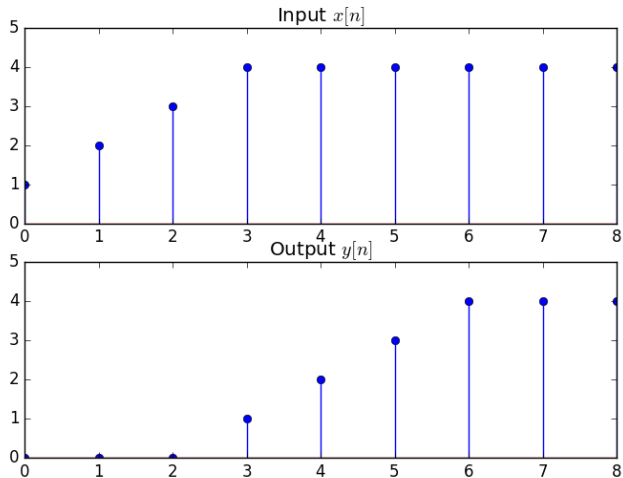
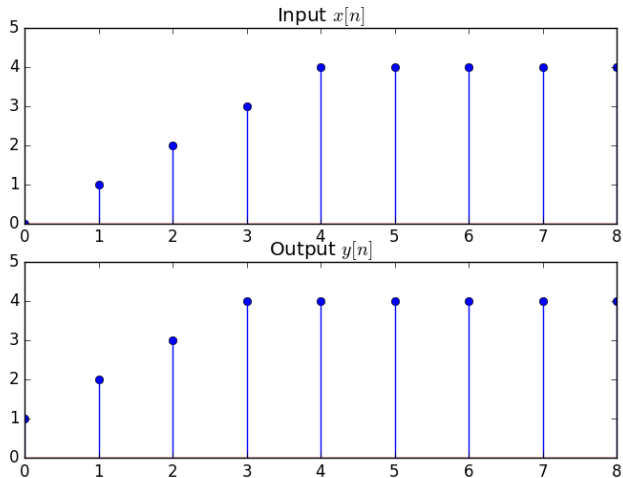# Example

$y[n] = x[n+1]$

$$y[n] = x[n+1]$$

Example

$$y[n] = \frac{1}{3}(x[n+1] + x[n] + x[n-1])$$

$$y[n] = (x[n+1] + x[n] + x[n-1])/3$$
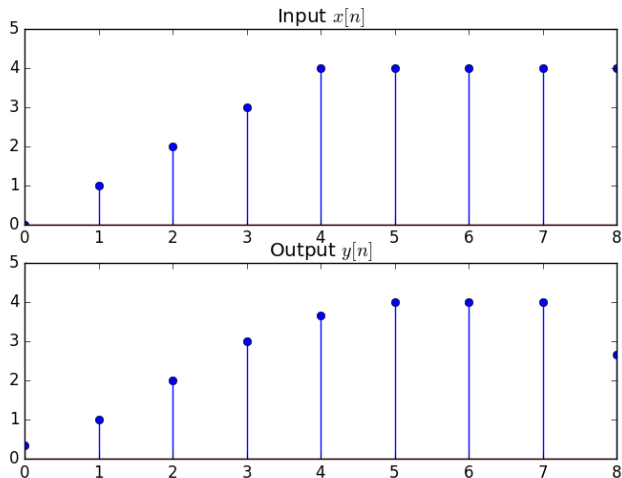
Example

$$y[n] = \max(x[n+1], x[n], x[n-1])$$

$$y[n] = \max(x[n+1], x[n], x[n-1])$$
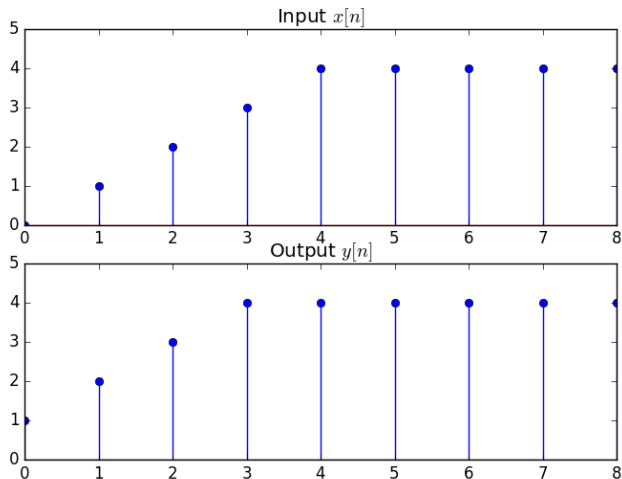
Example

$y[n] = \sum_{k=-\infty}^{n} x[k] = x[n] + x[n-1] + x[n-2] + ...$

$$y[n] = \sum_{k=-\infty}^{n} x[k] = x[n] + x[n-1] + x[n-2] + \ldots$$
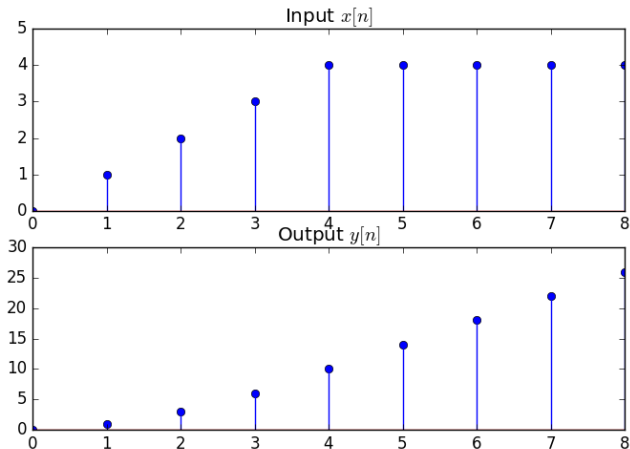
Recursive systems

- The last system $y[n] = \sum_{k=-\infty}^{n} x[k] = x[n] + x[n-1] + x[n-2] + ...$ can be also written in **recursive form**

$$y[n] = y[n-1] + x[n],$$

- Need to start from an **initial condition**

$$y[n_0] = \sum_{k=-\infty}^{n_0} x[k]$$

- Recursive systems always have one or more initial conditions.
- For recursive systems, the output signal depends on:
- the input signal
- **and** on initial conditions
- The initial conditions must always be specified for a recursive system
    - If not specified : implicitly assumed they are 0.
- **Relaxed** system: when the initial conditions are all 0
- A recursive system with non-zero initial conditions can produce an output signal even in the absence of an input ($x[n] = 0$)

- The operation of a system can be described graphically (see examples on blackboard):
  - summation of two signals
  - scaling of a signal with a constant
  - multiplication of two signals
  - delay element
  - anticipation element
  - other blocks for more complicated math operations

II.4 Classification of discrete systems

- **Memoryless (or static)**: output at time n depends only on the input **from the same moment** n
- Otherwise, the system **has memory (dynamic)**
- Examples:
    - memoryless: $y[n] = (x[n])^3 + 5$
    - with memory: $y[n] = (x[n])^3 + x[n-1]$
- Memory of size N:
    - output at time n $y[n]$ depends only up to the last N inputs, $x[n-N], x[n-(N-1)], ... x[n]$,
    - if N is finite: the system has **finite memory**
    - if $N = \infty$, the system has infinite memory
- Examples:
    - finite memory of order 4: $y[n] = x[n] + x[n-2] + x[n-4]$
    - infinite memory: $y[n] = \sum_{k=-\infty}^{n} x[k] = x[n] + x[n-1] + x[n-2] + ...$

Time-Invariant and Time-Variant systems

▶ A relaxed system H is **time-invariant** if and only if:

$$x[n] \overset{H}{\to} y[n]$$

implies

$$x[n-k] \overset{H}{\to} y[n-k],$$

$\forall x[n], \forall k$.

▶ **Delaying the input signal with k will only delay the output with the same amount, otherwise the output is not affected**
  ▶ Must be true for all input signals, for all possible delays (positive or negative).
▶ Otherwise, the system is said to be **time-variant**.
▶ Examples:
  ▶ $y[n] = x[n] - x[n-1]$ is time-invariant
  ▶ $y[n] = n \cdot x[n]$ is not time-invariant
▶ A system is time-invariant if it depends on n only through the input signal $x[n]$.

Example

Time-invariant system $y[n] = x[n] - x[n-1]$
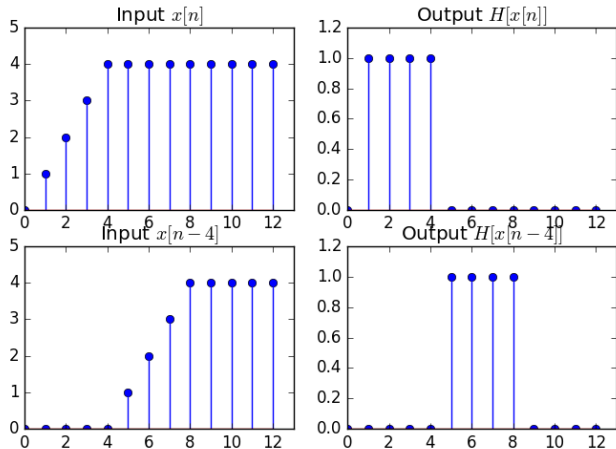
Another example

Time-variant system $y[n] = n \cdot x[n]$
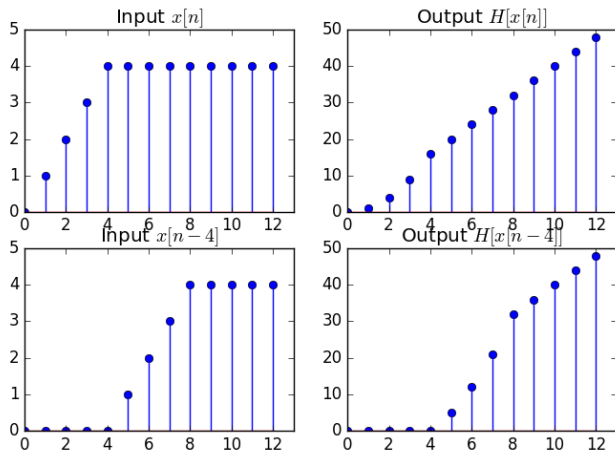
H[x[n]] = [0, 1, 4, 9, 16, 20, 24, 28, 32, 36, 40, 44, 48]

Linear and nonlinear systems

- A system H is **linear** if it satisfies:

$$H\left[ax_1[n] + bx_2[n]\right] = aH\left[x_1[n]\right] + bH\left[x_2[n]\right].$$

- **Applying the system to a sum of two signals = applying the system to each signal, and adding the results.**
- **Scaling the input signal with a constant a is the same as scaling the output signal with a.**
- The same relation will be true for a sum of many signals, not just two
- Advantage of linear systems
    - Complicated input signals can be decomposed into a sum of smaller parts
    - The system can be applied to each part independently
    - Then the results are added back
- Examples:
    - linear system: $y[n] = 3x[n] + 5x[n-2]$
    - nonlinear system: $y[n] = 3(x[n])^2 + 5x[n-2]$

- For a system to be linear, the input samples $x[n]$ must not undergo non-linear transformations.
- **The only transformations** of the input x[n] allowed to take place in a linear system are:
    - scaling (multiplication) with a constant
    - delaying
    - summing different delayed versions of the signal (not summing with a constant)
- Examples: at blackboard

Causal and non-causal systems

- **Causal**: the output $y[n]$ depends only on the current input $x[n]$ and the past values $x[n-1]$, $x[n-2]$..., but not on the future samples $x[n+1]$, $x[n+2]$...
- Otherwise the system is **non-causal**.
- A causal system can operate in real-time
    - we need only the input samples from the past
    - non-causal systems need samples from the future
- Examples:
    - $y[n] = x[n] - x[n-1]$ is causal
    - $y[n] = x[n+1] - x[n-1]$ is non-causal
    - $y[n] = x[-n]$ is non-causal

Stable and unstable systems

- **Bounded** signal: if there exists a value M such that all the samples of the signal or smaller than M, in absolute values

$$x[n] \in [-M, M]$$

$$|x[n]| \leq M$$

- **Stable** system: if for any bounded input signal it produces a bounded output signal
    - not necessarily with the same M
    - known as BIBO (Bounded Input $->$ Bounded Output)
- In other words: when the input signal has bounded values, the output signal does not go towards ∞ or $-\infty$.
- Examples:
    - $y[n] = (x[n])^3 - x[n+4]$ is stable
    - $y[n] = \frac{1}{x[n] - x[n-1]}$ is unstable
    - $y[n] = \sum_{k=-\infty}^{n} x[k] = x[n] + x[n-1] + x[n-2] + ...$ is unstable

Impulse response of Linear Time-Invariant (LTI) systems

Linear Time-Invariant (LTI) systems

- Notation: An **LTI** system (**L**inear **T**ime-**I**nvariant) is a system which is simultaneously **linear** and **time-invariant**.
- LTI systems can be described via either (or both):
  1. the **impulse response** $h[n]$
  2. the **difference equation**

$$y[n] = -\sum_{k=1}^{N} a_k y[n-k] + -\sum_{k=1}^{M} b_k x[n-k]$$
$$= -a_1 y[n-1] - a_2 y[n-2] - ... - a_N y[n-N] + b_0 x[n] + b_1 x[n-$$

The impulse response

- **Impulse response** of a system = output (response) of when the input signal is the impulse $\delta[n]$:

$$h[n] = H(\delta[n]).$$

- The impulse response of a LTI system **fully characterizes the system**:
  - based on $h[n]$ we can compute the response of the system to **any** input signal
  - all the properties of LTI systems can be described via characteristics of the impulse response

Signals are a sum of impulses

- Any signal can be composed as **a sum of scaled and delayed impulses** $\delta[n]$.
- Example:
  $x[n] = 3, 1, -5, 0, 2 = 3\delta[n] + \delta[n-1] - 5\delta[n-2] + 2\delta[n-2]$
- In general

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k],$$

i.e. a sum of impulses $\delta[n]$, delayed with k and scaled with the corresponding value $x[k]$.

Convolution

▶ The system is linear and time-invariant , the response of the system to a sum of impulses, delayed with k and scaled with $x[k]$, **is a sum of impulse responses, delayed with k and scaled with $x[k]$.**

  ▶ The input signal is composed of a "bunch" of impulses
  ▶ LTI system −> each impulse will generate its own response
  ▶ output signal is the sum of impulse responses, delayed and scaled appropriately

$$\begin{aligned}
y[n] &= H\left(x[n]\right) \\
&= H\left(\sum_{k=-\infty}^{\infty} x[k]\delta[n-k]\right) \\
&= \sum_{k=-\infty}^{\infty} x[k]H\left(\delta[n-k]\right) \\
&= \sum_{k=-\infty}^{\infty} x[k]h[n-k].
\end{aligned}$$

Example
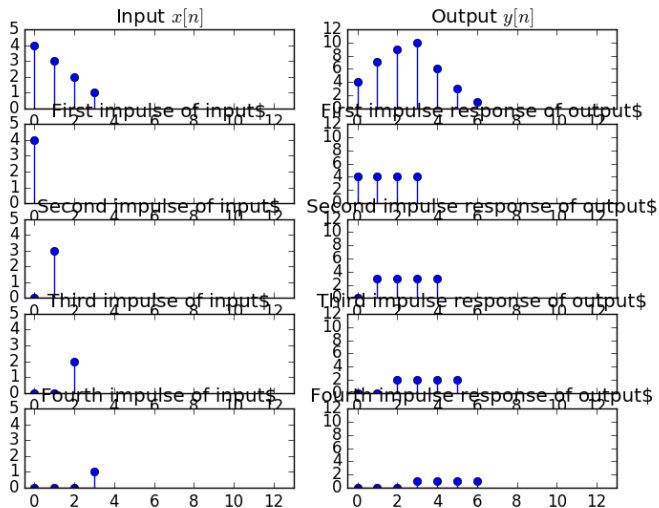
[-0.5, 13, 0, 12]

Properties of convolution

▶ Convolution is commutative (the order of the two signals doesn't matter):

$$x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = h[n] * x[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

.

Proof: make variable change $(n-k) \to l$, change all in equation

▶ Convolution is associative

$$(a[n] * b[n]) * c[n] = a[n] * (b[n] * c[n])$$

.

(No proof)

▶ The unit impulse is neutral element for convolution

$$a[n] * \delta[n] = \delta[n] * a[n] = a[n]$$

.

1. Identity system

A system with $h[n] = \delta[n]$ produces an response equal to the input, $y[n] = x[n], \forall x[n]$.

Proof: $\delta[n]$ is neutral element for convolution.

2. Series connection is commutative

LTI systems connected in series can be interchanged in any order. Proof: by commutativity of convolution.

LTI systems connected in series are equivalent to a single system with

$$h_{equiv}[n] = h_1[n] * h_2[n] * \ldots * h_N[n]$$

3. Parallel connection means sum

LTI systems connected in parallel are equivalent to a single system with

$$h_{equiv}[n] = h_1[n] + h_2[n] + ... * h_N[n]$$

Properties of LTI systems expressed with $h[n]$

4. Response of LTI systems to unit step

▶ If the input signal is $u[n]$, the response of the system is

$$s[n] = u[n] * h[n] = h[n] * u[n] = \sum_{k=-\infty}^{\infty} h[k]u[n-k] = \sum_{k=-\infty}^{n} h[k]$$

.

▶ The signal $\sum_{k=-\infty}^{n} h[k]$ is a *discrete-time integration* of $h[n]$

▶ (equivalent to the integral for continuous signals)

▶ It follows that

$$h[n] = s[n] - s[n-1].$$

▶ Note that the unit step $u[n]$ iteslf is the discrete-time integral of the unit impulse:

$$u[n] = \sum_{k=-\infty}^{n} \delta[k]$$

$$\delta[n] = u[n] - u[n-1]$$

Relation between LTI system properties and $h[n]$

1. Causal LTI systems and their $h[n]$

- If a LTI system is causal, then $h[n] = 0, \forall n < 0$.
- Proof: If $y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$, but $y[n]$ does not depend on $x[n+1], x[n+2], ...$, it means that these terms are multiplied with 0. The value $x[n+1]$ is multiplied with $h[n-(n+1)] = h[-1]$, $x[n+2]$ is multiplied with $h[n-(n+2)] = h[-2]$, and so on. Therefore:

$$h[n] = 0, \forall n < 0$$

  .
- A signal which is 0 for $n < 0$ is called a *causal **signal***
- Otherwise the signal is *non-causal*.
- We can say that *a system is causal if and only if it has a causal impulse response*
- Further definitions:
    - a signal which 0 for $n > 0$ is called an *anti-causal* signal
    - a signal which has non-zero values both for some $n > 0$ and for some $n < 0$ (and thus is neither causal nor non-causal) is called *bilateral*.

Causal signal   Anti-causal signal (non-causal)   Bilateral signal (non-causal)
5   5   5

2. Stable systems and their $h[n]$

- Considering a bounded input signal, $|x[n]| \leq A$, the absolute value of the output is:

$$
\begin{aligned}
|y[n]| =& |\sum_{k=-\infty}^{\infty} x[k]h[n-k]| \\
\leq& \sum_{k=-\infty}^{\infty} |x[k]h[n-k]| \\
=& \sum_{k=-\infty}^{\infty} |x[k]||h[n-k]| \\
\leq& A \sum_{k=-\infty}^{\infty} |h[n-k]|
\end{aligned}
$$

- The output signal is bounded (and hence the system is stable) if $\sum_{k=-\infty}^{\infty} |h[n]|$ is finite.
- Therefore **a LTI system is stable if**

Exercise: What ca we say about the impulse response $h[n]$ of a memoryless system? What about a system with finite memory M?

FIR and IIR systems

- The *support* of a discrete signal = the smallest interval of n such that the signal is 0 everywhere outside the interval.
- Examples: at whiteboard
- Depending on the support of the impulse response, discrete LTI systems can be **FIR** or **IIR** systems.

FIR systems

- A **F**inite **I**mpulse **R**esponse (**FIR**) system has an impulse response with finite support
    - i.e. the impulse response is 0 outside a certain interval.
- For a causal system:
    - $h[n] = 0$ for $n < 0$
    - therefore $h[n] = 0$ for $n < 0$ or $n \geq M$, for some M.
    - The convolution becomes:

$$y[n] = \sum_{k=0}^{M} h[k]x[n-k] = h[0] \cdot x[n] + h[1] \cdot x[n-1] + ... h[M] \cdot x[n-M].$$

- For a causal FIR system, the output is a linear combination of the last M input samples (has finite memory M).

- An **I**nfinite **I**mpulse **R**esponse (**FIR**) system has an impulse response with infinite support
    - i.e. the impulse response never becomes completely 0 forever.
- Causal system: the output $y[n]$ potentially depends on all the preceding input samples
    - from the convolution equation
- An IIR system has infinite memory.

Recursive / non-recursive implementations

- **Recursive** implementation: compute $y[n]$ based partly on the previous output samples $y[n-1], y[n-2], ...$
- Every LTI system can be expressed only based on the input samples $x[n], x[n-1], ...$
  - but sometimes we need an infinite amount of memory
  - recursive expression may be more efficient
- Example:

$$y[n] = \frac{1}{n+1} \sum_{0}^{n} x[n]$$

can be rewritten in recursive form:

$$y[n] = n \cdot y[n-1] + x[n]$$

- **In general, the output $y[n]$ of a recursive system depends on:**
  - the last N samples of the output, y[n-1], ... y[n-N]
  - and the current and the last M samples of the input, x[0], x[1], ... x[n-M].**
- **Non-recursive** system: the output $y[n]$ is computed based on last M samples of the input, x[0], x[1], ... x[n-M].
- FIR systems can always be implemented non-recursively, but may also be implemented in a recursive way
- **IIR systems can only be implemented recursively!**
  - otherwise they would need infinite memory

- ▶ Recursive systems rely on previous outputs $->$ the previous values must be always available.
- ▶ The previous output values needed at the start moment are *the initial conditions* of the system
- ▶ Notes
  - ▶ The output of a system always depends on the initial conditions, besides the input signal
  - ▶ A system with initial conditions equal to 0 is called *relaxed*
  - ▶ The output of a relaxed system to an input signal is called *zero-state response*, $y_{zs}[n]$, because the initial conditions (initial state) is 0, and the output depends only in the input signal. This is also called *forced response*.
  - ▶ A system with non-zero initial conditions produces an output even when the input signal is zero. This output is called *zero-input response*, $y_{zi}[n]$, because the input signal is 0, and the output depends only on the initial conditions. This is also called *natural response*.
  - ▶ For linear systems, the output of a system is always the sum of the forced response and the natural response:

$$y[n] = y_{zs}[n] + y_{zi}[n]$$