# II. Discrete signals and systems

# II.1 Discrete signals

# Representation

A discrete signal can represented:

- graphically
- in table form
- as a vector: $x[n] = [..., 0, 0, 1, 3, 4, 5, 0, ...]$
  - an **arrow** indicates the origin of time ($n = 0$). -if the arrow is missing, the origin of time is at the first element -the dots ... indicate that the value remains the same from that point onwards

Examples: at blackboard
Notation: $x[4]$ represents the value of the fourth sample in the signal $x[n]$

# Basic signals

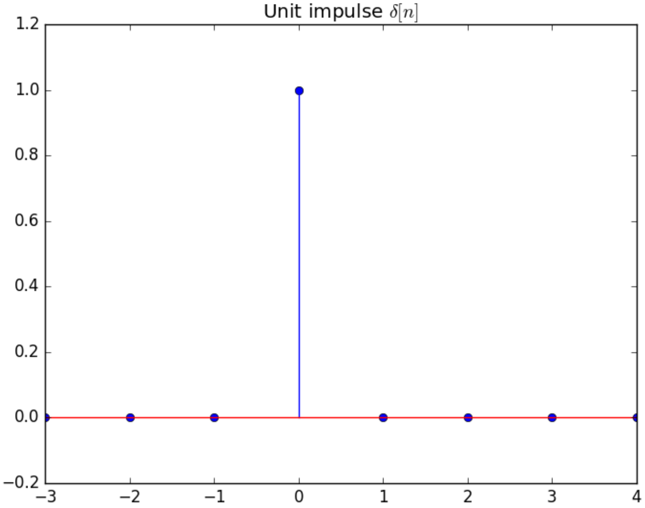
Some elementary signals are presented below.

## Unit impulse

Contains a single non-zero value of 1 located at time 0. It is denoted with $\delta[n]$.

$$\delta[n] = \begin{cases} 1 & \text{if } n = 0 \\ 0 & \text{otherwise} \end{cases}.$$

# Representation



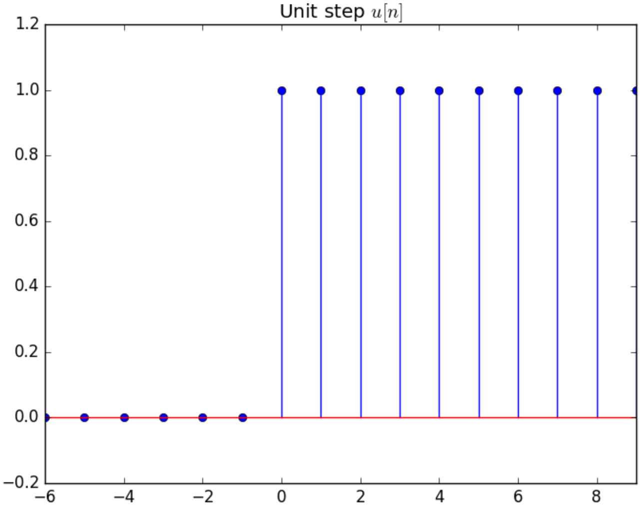
# Unit step

## Unit step

It is denoted with $u[n]$.

$$u[n] = \begin{cases} 1 & \text{if } n \geq 0 \\ 0 & \text{otherwise} \end{cases}.$$

# Representation



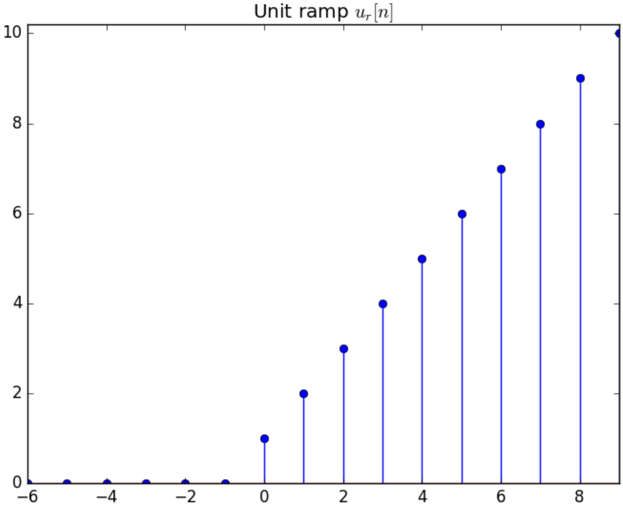
# Unit ramp

## Unit ramp

It is denoted with $u_r[n]$.

$$u_r[n] = \begin{cases} n & \text{if } n \geq 0 \\ 0 & \text{otherwise} \end{cases}.$$

# Representation



# Exponential signal

## Exponential signal

It does not have a special notation. It is defined by:

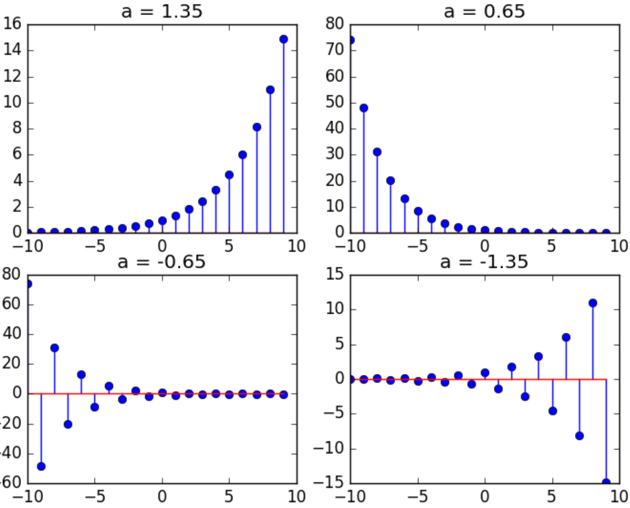
$$x[n] = a^n.$$

$a$ can be a real or a complex number. Here we consider only the case when $a$ is real.

Depending on the value of $a$, we have four possible cases:

1. $a \geq 1$
2. $0 \leq a < 1$
3. $-1 < a < 0$
4. $a \leq 1$

# Representation



# II.2 Types of discrete signals

# Signals with finite energy

- The **energy of a discrete signal** is defined as

$$E = \sum_{n=-\infty}^{\infty} (x[n])^2.$$

- If $E$ is finite, the signal is said to have finite energy.
- Examples:
  - unit impulse has finite energy
  - unit step does not

# Signals with finite power

- The **average power of a discrete signal** is defined as

$$P = \lim_{N \to \infty} \frac{\sum_{n=-N}^{N} (x[n])^2}{2N+1}.$$

- In other words, the average power is the average energy per sample.
- If $P$ is finite, the signal is said to have finite power.
- A signal with finite energy has finite power ($P = 0$ if the signal has infinite length). A signal with infinite energy can have finite or infinite power.
- Example: unit step has finite power $P = \frac{1}{2}$ (see proof at blackboard).

# Periodic and non-periodic signals

- A signal is called **periodic** if its values repeat themselves after a certain time (known as **period**).

$$x[n] = x[n+N]), \forall t$$

- The **fundamental period** of a signal is the minimum value of $N$.
- Periodic signals have infinite energy, and finite power equal to the power of a single period.

# Even and odd signals

- A real signal is **even** if it satisfies the following symmetry:

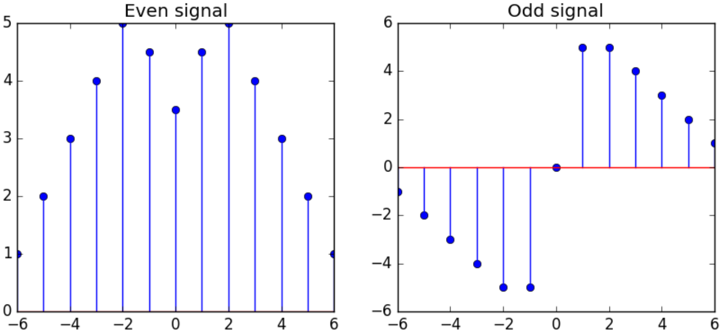
$$x[n] = x[-n], \forall n.$$

- A real signal is **odd** if it satisfies the following anti-symmetry:

$$-x[n] = x[-n], \forall n.$$

- There exist signals which are neither even nor odd.

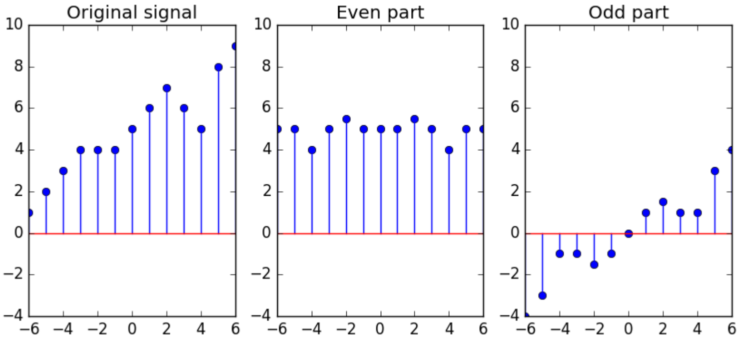
# Even and odd signals: example



# Even and odd parts of a signal

- Every signal can be written as the sum of an even signal and an odd signal:
$$x[n] = x_e[n] + x_o[n]$$
- The even and the odd parts of the signal can be found as follows:
$$x_e[n] = \frac{x[n] + x[-n]}{2}.$$
$$x_o[n] = \frac{x[n] - x[-n]}{2}.$$
- Proof: check that $x_e[n]$ is even, $x_o[n]$ is odd, and their sum is $x[n]$

# Even and odd parts: example

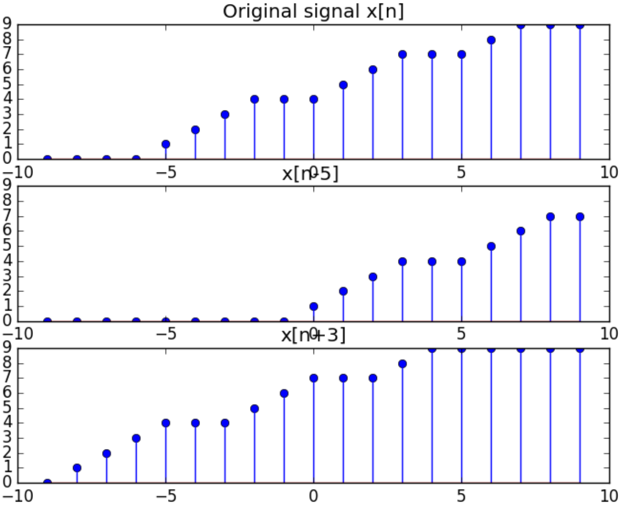


# II.3 Basic operations with discrete signals

# Time shifting

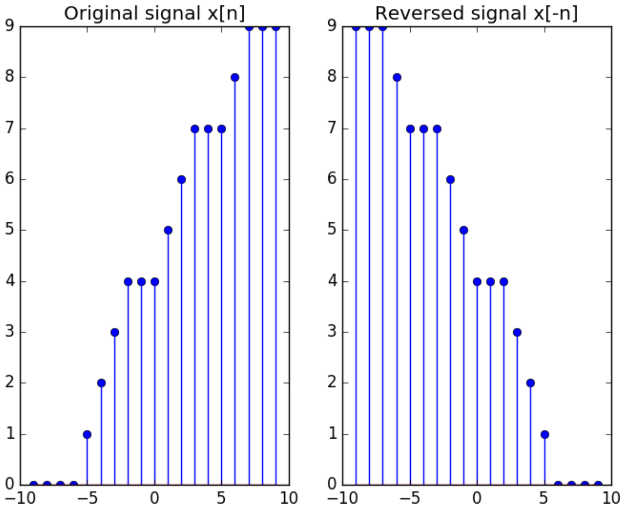
- The signal $x[n-k]$ is $x[n]$ **delayed with** $k$ **time units**
  - Graphically, $x[n-k]$ is shifted $k$ units to the **right** compared to the original
- The signal $x[n+k]$ is $x[n]$ **anticipated with** $k$ **time units**
  - Graphically, $x[n+k]$ is shifted $k$ units to the **left** compared to the original signal.

# Time shifting: representation



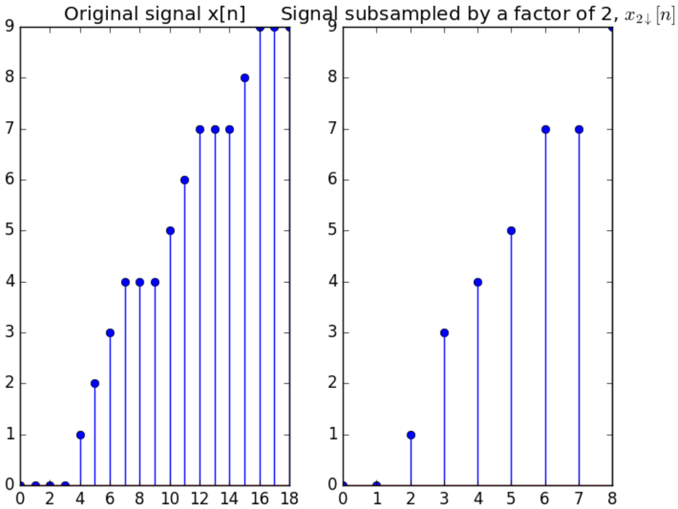
# Time reversal

- Changing the variable $n$ to $-n$ produces a signal $x[-n]$ which mirrors $x[n]$.



# Subsampling

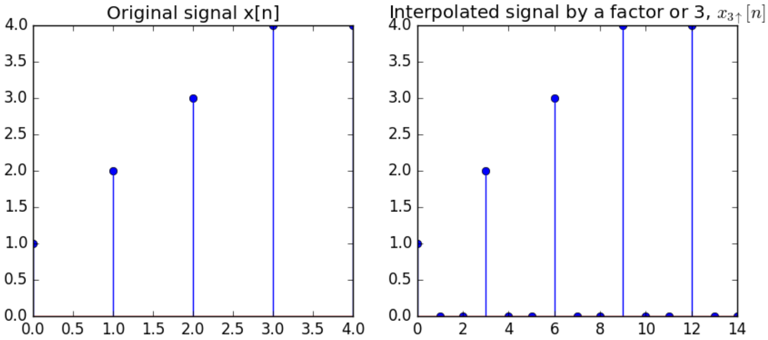
- $x_{M\downarrow}[n] = x[Mn]$ is a **subsampled** version of $x[n]$ with a factor of $M$
  - Keep only 1 sample out of $M$ samples from the original signal $x[n]$



# Interpolation

- **Interpolation** by a factor of $L$ adds $L$ of zeros between two samples in the original signal.

$$x_{L\uparrow} = \begin{cases} x[\frac{n}{L}] & \text{if } \frac{n}{L} \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}.$$



# Mathematical operations

- A signal $x[n]$ can be **scaled** by a constant A, i.e. each sample is multipled by A:

$$y[n] = Ax[n].$$

- Two signals $x_1[n]$ and $x_2[n]$ can be **summed** by summing the individual samples:

$$y[n] = x_1[n] + x_2[n]$$

- Two signals $x_1[n]$ and $x_2[n]$ can be **multiplied** by multiypling the individual samples:

$$y[n] = x_1[n] \cdot x_2[n]$$

# II.4 Discrete systems

# Definition

- **System** = a device or algorithm which produces an **output signal** based on an **input signal**
- We will only consider systems with a single input and a single output
- Figure here: blackboard.
- Common notation:
  - x[n] is the input
  - y[n] is the output
  - H is the system.

# Notations

- Notations:

$$y[n] = H\left[x[n]\right]$$

(“*the system H applied to the input x[n] produces the output y[n]*”)

$$x[n] \xrightarrow{H} y[n]$$

(“*the input* $x[n]$ *is transformed by the system H into y[n]*”)

# Equations

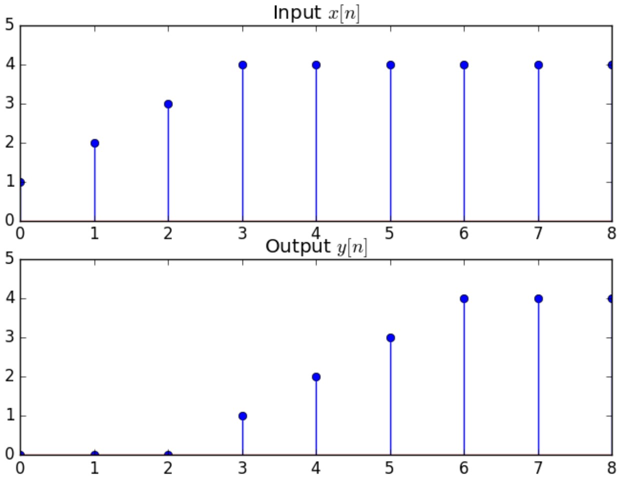
- Usually, a system is described by the **input-output equation** (or **difference equation**) which expains how $y[n]$ is defined in terms of $x[n]$.

Examples:

1. $y[n] = x[n]$ (the identity system)
2. $y[n] = x[n-3]$
3. $y[n] = x[n+1]$
4. $y[n] = \frac{1}{3}(x[n+1] + x[n] + x[n-1])$
5. $y[n] = \max(x[n+1], x[n], x[n-1])$
6. $y[n] = (x[n])^2 + \log_{10} x[n-1]$
7. $y[n] = \sum_{k=-\infty}^{n} x[k] = x[n] + x[n-1] + x[n-2] + ...$

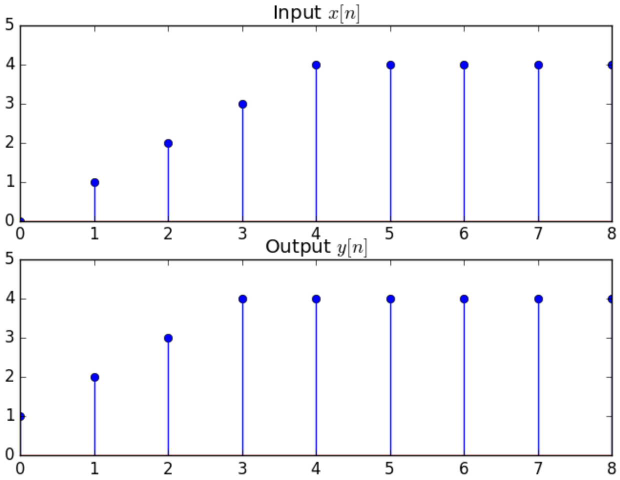
# Example

$$y[n] = x[n-3]$$



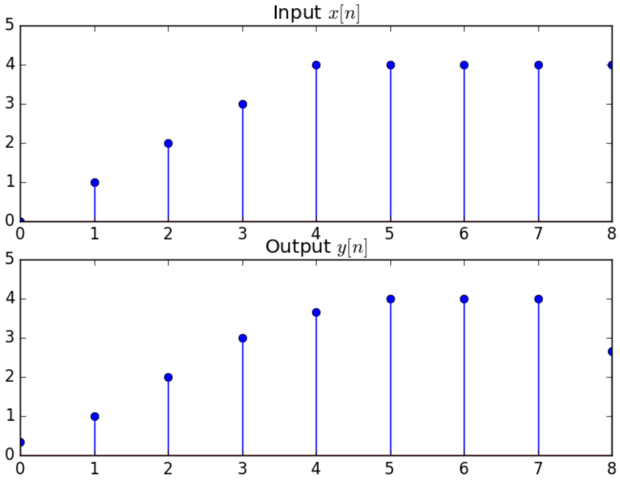
# Example

$$y[n] = x[n+1]$$



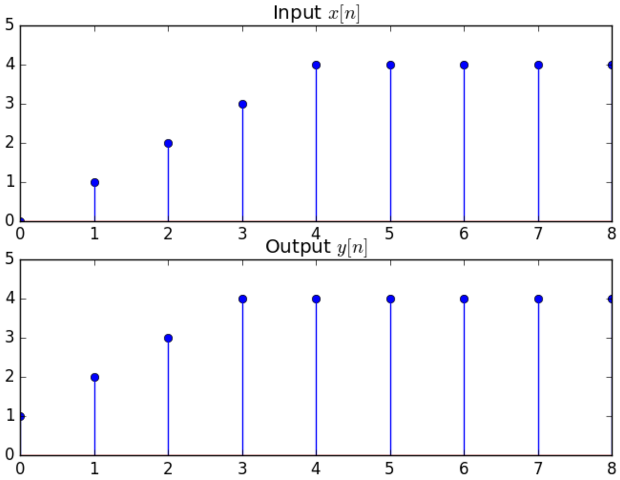
# Example

$$y[n] = (x[n+1] + x[n] + x[n-1])/3$$



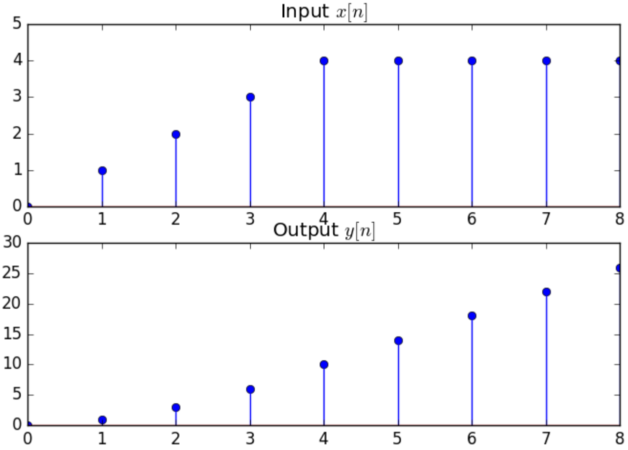
# Example

$$y[n] = \max(x[n+1], x[n], x[n-1])$$



# Example

$$y[n] = \sum_{k=-\infty}^{n} x[k] = x[n] + x[n-1] + x[n-2] + \dots$$



# Recursive systems

- The last system $y[n] = \sum_{k=-\infty}^{n} x[k] = x[n] + x[n-1] + x[n-2] + \ldots$ can be also written in **recursive form**

$$y[n] = y[n-1] + x[n],$$

- Need to start from an **initial condition**

$$y[n_0] = \sum_{k=-\infty}^{n_0} x[k]$$

- Recursive systems always have one or more initial conditions.
- For recursive systems, the output signal depends on:
  - the input signal
  - **and** on initial conditions
- The initial conditions must always be specified for a recursive system
  - If not specified : implicitly assumed they are 0 (**relaxed** system)
- A recursive system with non-zero initial conditions can produce an output signal even in the absence of an input ($x[n] = 0$)

# Representation of systems

- The operation of a system can be described graphically (see examples on blackboard):
  - summation of two signals
  - scaling of a signal with a constant
  - multiplication of two signals
  - delay element
  - anticipation element
  - other blocks for more complicated math operations

# II.4 Classification of discrete systems

# Memoryless / systems with memory

- **Memoryless (or static)**: output at time $n$ depends only on the input **from the same moment** $n$
- Otherwise, the system **has memory (dynamic)**
- Examples:
  - memoryless: $y[n] = (x[n])^3 + 5$
  - with memory: $y[n] = (x[n])^3 + x[n-1]$
- Memory of size $N$:
  - output at time $n$ $y[n]$ depends only up to the last $N$ inputs, $x[n-N], x[n-(N-1)], ...x[n]$,
  - if $N$ is finite: the system has **finite memory**
  - if $N = \infty$, the system has infinite memory
- Examples:
  - finite memory of order 4: $y[n] = x[n] + x[n-2] + x[n-4]$
  - infinite memory: $y[n] = \sum_{k=-\infty}^{n} x[k] = x[n] + x[n-1] + x[n-2] + ...$

# Time-Invariant and Time-Variant systems

- A relaxed system $H$ is **time-invariant** if and only if:

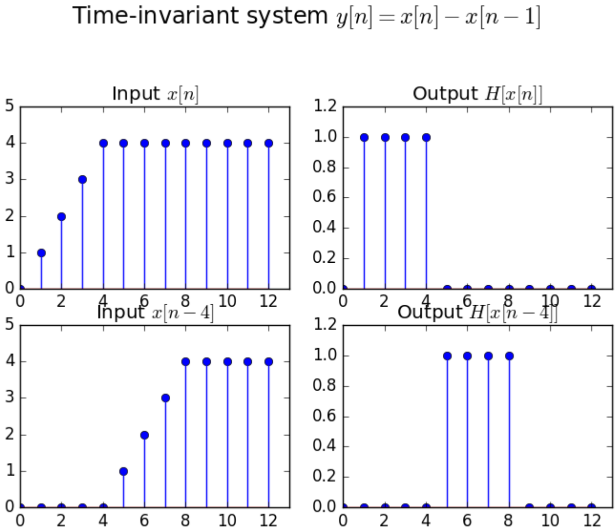
  $$x[n] \xrightarrow{H} y[n]$$

  implies

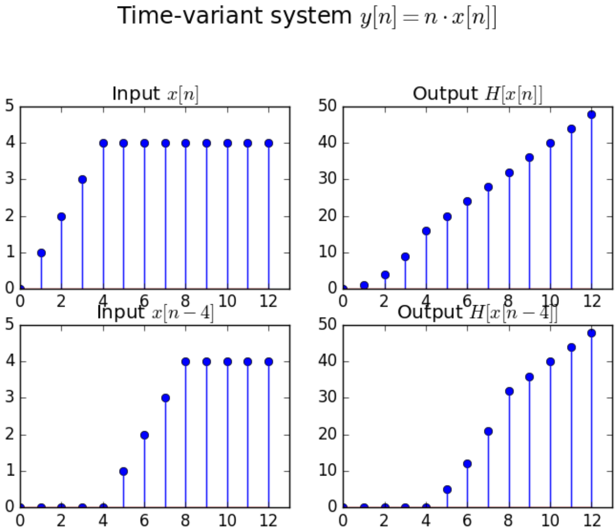
  $$x[n-k] \xrightarrow{H} y[n-k],$$

  $\forall x[n], \forall k$.
- **Delaying the input signal with $k$ will only delay the output with the same amount, otherwise the output is not affected**
  - Must be true for all input signals, for all possible delays (positive or negative)
- Otherwise, the system is said to be **time-variant**
- Examples:
  - $y[n] = x[n] - x[n-1]$ is time-invariant
  - $y[n] = n \cdot x[n]$ is not time-invariant
- A system is time-invariant if it depends on $n$ only through the input signal $x[n]$

# Example


Time-invariant system $y[n] = x[n] - x[n-1]$
Input $x[n]$
Output $H[x[n]]$
Input $x[n-4]$
Output $H[x[n-4]]$
0
1
2
3
4
5
0.0
0.2
0.4
0.6
0.8
1.0
1.2
0 2 4 6 8 10 12

# Another example



# Linear and nonlinear systems

- A system $H$ is **linear** if it satisfies:

$$H\left[ax_1[n] + bx_2[n]\right] = aH\left[x_1[n]\right] + bH\left[x_2[n]\right].$$

- **Applying the system to a sum of two signals = applying the system to each signal, and adding the results**
- **Scaling the input signal with a constant $a$ is the same as scaling the output signal with $a$**
- The same relation will be true for a sum of many signals, not just two
- Advantage of linear systems
  - Complicated input signals can be decomposed into a sum of smaller parts
  - The system can be applied to each part independently
  - Then the results are added back
- Examples:
  - linear system: $y[n] = 3x[n] + 5x[n-2]$
  - nonlinear system: $y[n] = 3(x[n])^2 + 5x[n-2]$

# Linear and nonlinear systems

- For a system to be linear, the input samples $x[n]$ must not undergo non-linear transformations.
- **The only transformations** of the input x[n] allowed to take place in a linear system are:
  - scaling (multiplication) with a constant
  - delaying
  - summing different delayed versions of the signal (not summing with a constant)
- Examples: at blackboard

# Causal and non-causal systems

- **Causal**: the output $y[n]$ depends only on the current input $x[n]$ and the past values $x[n-1]$, $x[n-2] \ldots$, but not on the future samples $x[n+1]$, $x[n+2] \ldots$
- Otherwise the system is **non-causal**.
- A causal system can operate in real-time
  - we need only the input samples from the past
  - non-causal systems need samples from the future
- Examples:
  - $y[n] = x[n] - x[n-1]$ is causal
  - $y[n] = x[n+1] - x[n-1]$ is non-causal
  - $y[n] = x[-n]$ is non-causal

# Stable and unstable systems

- **Bounded** signal: if there exists a value $M$ such that all the samples of the signal or smaller than M, in absolute values

$$x[n] \in [-M, M]$$

$$|x[n]| \leq M$$

- **Stable** system: if for any bounded input signal it produces a bounded output signal
  - not necessarily with the same $M$
  - known as BIBO (Bounded Input –> Bounded Output)
- In other words: when the input signal has bounded values, the output signal does not go towards $\infty$ or $-\infty$.
- Examples:
  - $y[n] = (x[n])^3 - x[n+4]$ is stable
  - $y[n] = \frac{1}{x[n]-x[n-1]}$ is unstable
  - $y[n] = \sum_{k=-\infty}^{n} x[k] = x[n] + x[n-1] + x[n-2] + ...$ is unstable

# Impulse response of Linear Time-Invariant (LTI) systems

# Linear Time-Invariant (LTI) systems

- Notation: An **LTI** system (**L**inear **T**ime-**I**nvariant) is a system which is simultaneously **linear** and **time-invariant**.
- LTI systems can be described via either (or both):
  1. the **impulse response** $h[n]$
  2. the **difference equation**

$$
\begin{aligned}
y[n] &= -\sum_{k=1}^{N} a_k y[n-k] + -\sum_{k=1}^{M} b_k x[n-k] \\
&= -a_1 y[n-1] - a_2 y[n-2] - \ldots - a_N y[n-N] + b_0 x[n] + b_1 x[n -
\end{aligned}
$$

# The impulse response

- **Impulse response** of a system = output (response) of when the input signal is the impulse $\delta[n]$:

$$h[n] = H(\delta[n])$$

- The impulse response of a LTI system **fully characterizes the system**:
  - based on $h[n]$ we can compute the response of the system to **any** input signal
  - all the properties of LTI systems can be described via characteristics of the impulse response

# Signals are a sum of impulses

- Any signal can be composed as **a sum of scaled and delayed impulses** $\delta[n]$.
- Example:
  $x[n] = \{3, 1, -5, 0, 2\} = 3\delta[n] + \delta[n-1] - 5\delta[n-2] + 2\delta[n-2]$
- In general

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

i.e. a sum of impulses $\delta[n]$, delayed with $k$ and scaled with the corresponding value $x[k]$

# Convolution

- The response of a LTI system to a sum of impulses, delayed with $k$ and scaled with $x[k]$, **is a sum of impulse responses, delayed with $k$ and scaled with $x[k]$.**
  - The input signal is composed of a "bunch" of impulses
  - LTI system –> each impulse will generate its own response
  - output signal is the sum of impulse responses, delayed and scaled appropriately

$$\begin{aligned} y[n] &= H\left(x[n]\right) \\ &= H\left(\sum_{k=-\infty}^{\infty} x[k]\delta[n-k]\right) \\ &= \sum_{k=-\infty}^{\infty} x[k]H\left(\delta[n-k]\right) \\ &= \sum_{k=-\infty}^{\infty} x[k]h[n-k]. \end{aligned}$$

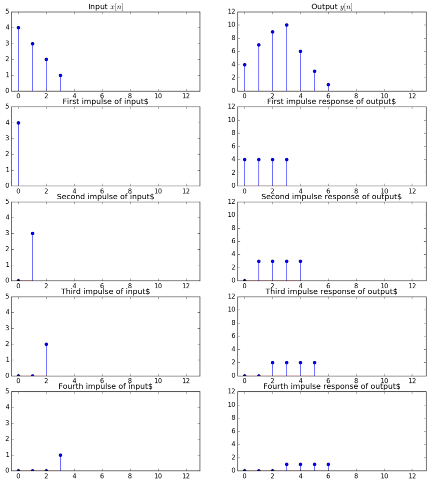
# Convolution

- This operation $=$ the **convolution** of two signals $x[n]$ and $h[n]$

$$x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

- **The response of a LTI system to an input signal x[n] is the convolution of x[n] with the system's impulse response h[n]**

# Example


Input $x[n]$
Output $y[n]$
First impulse of input$
First impulse response of output$
Second impulse of input$
Second impulse response of output$
Third impulse of input$
Third impulse response of output$
Fourth impulse of input$
Fourth impulse response of output$

# Properties of convolution

- Convolution is commutative (the order of the two signals doesn't matter):

$$x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = h[n] * x[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

.

Proof: make variable change $(n-k) \to l$, change all in equation

- Convolution is associative

$$(a[n] * b[n]) * c[n] = a[n] * (b[n] * c[n])$$

.

(No proof)

- The unit impulse is neutral element for convolution

$$a[n] * \delta[n] = \delta[n] * a[n] = a[n]$$

.

# Properties of LTI systems expressed with $h[n]$

## 1. Identity system

- A system with $h[n] = \delta[n]$ produces an response equal to the input, $y[n] = x[n], \forall x[n]$.
- Proof: $\delta[n]$ is neutral element for convolution.

# Properties of LTI systems expressed with $h[n]$

## 2. Series connection is commutative

- LTI systems connected in series can be interchanged in any order
- Proof: by commutativity of convolution.
- LTI systems connected in series are equivalent to a single system with

$$h_{equiv}[n] = h_1[n] * h_2[n] * ... * h_N[n]$$

# Properties of LTI systems expressed with $h[n]$

## 3. Parallel connection means sum

LTI systems connected in parallel are equivalent to a single system with

$$h_{equiv}[n] = h_1[n] + h_2[n] + ... + h_N[n]$$

# Properties of LTI systems expressed with $h[n]$

## 4. Response of LTI systems to unit step

- If the input signal is $u[n]$, the response of the system is

$$s[n] = u[n] * h[n] = h[n] * u[n] = \sum_{k=-\infty}^{\infty} h[k]u[n-k] = \sum_{k=-\infty}^{n} h[k]$$

.

# Properties of LTI systems expressed with $h[n]$

- Proof:
  - The signal $\sum_{k=-\infty}^{n} h[k]$ is a *discrete-time integration* of $h[n]$
  - The unit step $u[n]$ iteslf is the discrete-time integral of the unit impulse:

$$u[n] = \sum_{k=-\infty}^{n} \delta[k]$$

$$\delta[n] = u[n] - u[n-1]$$

  - Therefore the system response to the integral of the impulse = the integral of the system response to the impulse
  - The interchanging of the integration with the system is due to the linearity of the system and is valid for all signals:

$$H\left(\sum_{k=-\infty}^{n} x[k]\right) = \sum_{k=-\infty}^{n} H\left(x[k]\right)$$

# Relation between LTI system properties and $h[n]$

# 1. Causal LTI systems and their $h[n]$

If a LTI system is causal, then

$$h[n] = 0, \forall n < 0$$

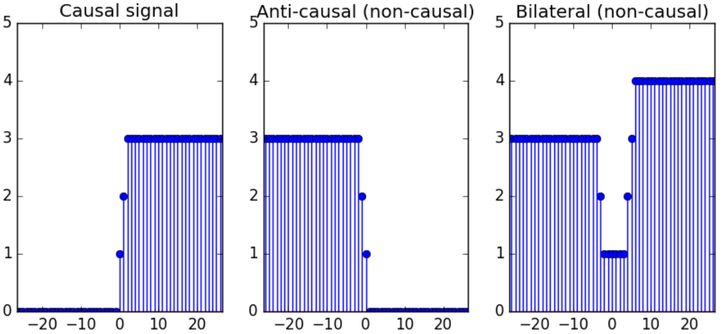
- Proof:
  - $y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$,
  - $y[n]$ does not depend on $x[n+1], x[n+2], \ldots$
  - it means that these terms are multiplied with 0
  - the value $x[n+1]$ is multiplied with $h[n-(n+1)] = h[-1]$, $x[n+2]$ is multiplied with $h[n-(n+2)] = h[-2]$, and so on
  - Therefore:
  $$h[n] = 0, \forall n < 0$$

  .

# Causal signals and causal systems

- A signal which is 0 for $n < 0$ is called a *causal* ***signal***
- Otherwise the signal is *non-causal*
- We can say that *a system is causal if and only if it has a causal impulse response*
- Further definitions:
  - a signal which 0 for $n > 0$ is called an *anti-causal* signal
  - a signal which has non-zero values both for some $n > 0$ and for some $n < 0$ (and thus is neither causal nor non-causal) is called *bilateral*.

# Example



Causal signal
Anti-causal (non-causal)
Bilateral (non-causal)
5
4
3
2
1
0
−20
−10
0
10
20

## 2. Stable systems and their $h[n]$

- Considering a bounded input signal, $|x[n]| \leq A$, the absolute value of the output is:

$$
\begin{aligned}
|y[n]| =& \left|\sum_{k=-\infty}^{\infty} x[k]h[n-k]\right| \\
\leq& \sum_{k=-\infty}^{\infty} |x[k]h[n-k]| \\
=& \sum_{k=-\infty}^{\infty} |x[k]||h[n-k]| \\
\leq& A \sum_{k=-\infty}^{\infty} |h[n-k]|
\end{aligned}
$$

- Therefore **a LTI system is stable if**

$$\sum_{k=-\infty}^{\infty} |h[n]| < \infty$$

# 3. Memoryless systems and their $h[n]$ (Exercise)

**Exercises:**

- What can we say about the impulse response $h[n]$ of a memoryless system?
- What about a system with finite memory $M$?

# FIR and IIR systems

# Support

- The **support** of a discrete signal = the smallest interval of $n$ such that the signal is 0 everywhere outside the interval.
- Examples: at whiteboard
- Depending on the support of the impulse response, discrete LTI systems can be **FIR** or **IIR** systems.

# FIR systems

- A **F**inite **I**mpulse **R**esponse (**FIR**) system has an impulse response with finite support
  - i.e. the impulse response is 0 outside a certain interval.
- For a causal system:
  - $h[n] = 0$ for $n < 0$
  - therefore $h[n] = 0$ for $n < 0$ or $n \geq M$, for some $M$
  - The convolution becomes:

$$y[n] = \sum_{k=0}^{M} h[k]x[n-k] = h[0] \cdot x[n] + h[1] \cdot x[n-1] + ...h[M] \cdot x[n-M]$$

- For a causal FIR system, the output is a linear combination of the last $M$ input samples (has finite memory $M$)

# IIR systems

- An **I**nfinite **I**mpulse **R**esponse (**FIR**) system has an impulse response with infinite support
  - i.e. the impulse response never becomes completely 0 forever.
- Causal system: the output $y[n]$ potentially depends on all the preceding input samples
  - from the convolution equation
- An IIR system has infinite memory

# Recursive / non-recursive implementations

- **Recursive** implementation: compute $y[n]$ based partly on the previous output samples $y[n-1], y[n-2], ....$
- Every LTI system can be expressed only based on the input samples $x[n], x[n-1], ...$
  - but sometimes we need an infinite amount of memory
  - recursive expression may be more efficient
- Example:

$$y[n] = \frac{1}{n+1} \sum_{0}^{n} x[n]$$

can be rewritten in recursive form:

$$y[n] = n \cdot y[n-1] + x[n]$$

# Recursive / non-recursive implementations

- In general, the output $y[n]$ of a recursive system depends on:
  - the last $N$ samples of the output, y[n-1], ... y[n-N]
  - and the current and the last $M$ samples of the input, x[0], x[1], ... x[n-M].
- ***Non-recursive*** system: the output $y[n]$ is computed based on last $M$ samples of the input, x[0], x[1], ... x[n-M].
- FIR systems can always be implemented non-recursively, but may also be implemented in a recursive way
- **IIR systems can only be implemented recursively!**
  - otherwise they would need infinite memory

# Initial conditions for recursive systems

- Recursive systems rely on previous outputs −> the previous values must be always available
- We need some starting values at the start moment (*the initial conditions* of the system)
- Notes:
  - Output signal depends on the input **and** on the initial conditions
  - A system with non-zero initial conditions produces an output even when the input signal is zero
  - This output is called *zero-input response*, $y_{zi}[n]$
  - A system with initial conditions equal to 0 is called *relaxed*
  - The output of a relaxed system to an input signal is called *zero-state response*, $y_{zs}[n]$ (also called *forced response*)
- For linear systems, the output of a system is always the sum of the forced response and the natural response:

$$y[n] = y_{zs}[n] + y_{zi}[n]$$