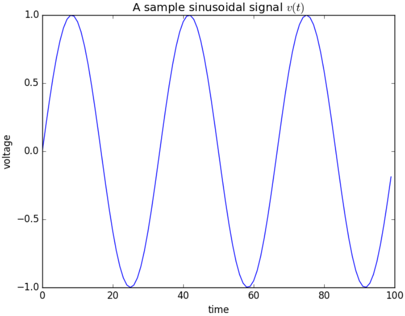
I. Analog and Digital Signals

I.1 Signals

Definition

- **Signal** $=$ a measurable quantity which varies in time, space or some other variable
- Examples:
  - a voltage which varies in time (1D voltage signal)
  - sound pressure which varies in time (sound signal)
  - intensity of light which varies across a photo (2D image)
- Usually represented as mathematical functions, e.g. $v(t)$.

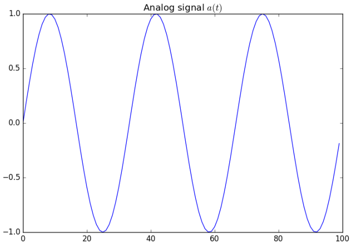


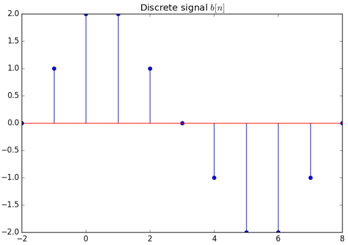
Unidimensional and multidimensional signals

- **Unidimensional** (1D) signal = a function of a single variable
  - Example: a voltage signal $v(t)$ only varies in time.
- **Multidimensional** (2D, 3D ... M-D) signal = a function of a multiple variables
  - Example: intensity of a grayscale image $I(x, y)$ across the surface of a photo
- In these lectures we consider only 1D signals
- The techniques which you will learn for 1D signals can also be used for multidimensional signals (usually 2D signals, images).

Analog and discrete signals

- **Analog** signals = functions of continuous variables
  - there exists a signal value for any value of the variable within the defined range
- **Discrete** signals = functions of discrete variables
  - have values only at certain discrete values, typically indexed with integer numbers ($x[-1]$, $x[0]$, $x[1]\dots$)





Discrete signals

- The values of a discrete signal are usually called **samples**
- The spacing between the samples is usually uniform
- **Note:** A discrete signal has **no value defined** in the space between samples
  - in these areas, it simply does not exist.
- Discrete signals are usually obtained by **sampling** analog signals.

Notation

- Analog signal $a(t)$
  - use **round brackets**
  - variable typically denoted with t (from *time*), e.g. $a(t)$.
- Discrete signal $x[n]$
  - use **square brackets**
  - variable is typically denoted with n, or sometimes k (suggest natural numbers)
- Examples:
  - $a(1)$ and $a(3.23542)$ are the values of signal $a(t)$ at time $t = 2$ and $t = 3.23542$
  - $b[-1]$ and $b[2]$ are the values of $b[n]$ at discrete time $n = -1$ and $n = 2$
  - Writing $x[1.3]$ is incorrect: $x[n]$ is only defined for integer values of n.

Signals with continuous and discrete values

- Besides the variable, the *value* of the signal can also be continuous or discrete
- Signal with continuous values: can have any value in a certain defined range
  - Example: the voltage in one point of a circuit: any value between, for example, 0V and 5V.
- Signal with discrete values: can only have a value from a discrete set of possible values
  - Example: the number of bits received in a second over a binary communication channel

I.2 Discrete and analog frequency

Periodicity

- **Periodic** signal: if its values repeat themselves after a certain time (known as **period**)
- For an analog signal:

$$x(t) = x(t+T), \forall t$$

- For a discrete signal:

$$y[n] = y[n+N]), \forall t$$

- **Fundamental period** = the minimum value of T or N
  - multiples of T or N are also periods, but non-fundamental
  - shorthand notation: from now on, by *"period"* we will mean the fundamental period.
- For analog signals, the period has unit *seconds*
  - T is time
- For digital signals the period is *adimensional*
  - N is just a number, has no unit attached

Frequency

- Fundamental frequency of a periodic signal = inverse of the period
- For **analog signals**:
$$F = \frac{1}{T}$$
- The unit is:
$$[F] = \frac{1}{s} = Hz$$
- For **discrete signals**:
$$f = \frac{1}{N}$$
- Has no unit, since N has no unit also.
- Notation:
    - frequencies of analog signals with F (capital letter)
    - frequencies of discrete signals with f (small letter)

Frequency example

- An analog cosine with a frequency of $F = 0.1Hz$

$$x(t) = cos(2 \cdot \pi \cdot 0.1 \cdot t)$$

- A discrete cosine with a frequency of $f = 0.1$.

$$x[n] = cos(2 \cdot \pi \cdot 0.1 \cdot t)$$

Pulsation

- For harmonic signals, the pulsation $= \omega = 2\pi f$ for both analog and discrete signals
- We use ω when treating harmonic signals like *cos()* or *sin()*, e.g.

$$cos(\omega t) = cos(2\pi ft)$$

.

Domain of existence

- Analog signals
  - the period can be as small as desired, $T \to 0$
  - therefore the frequency can be very large, $F_{max} = \infty$.

$$F \in (-\infty, \infty)$$

- Discrete signals
  - **the smallest period is** $N = 2$ (excluding N=1 which means a constant signal)
  - therefore, the largest possible frequency is $f_{max} = \frac{1}{2}$

$$f \in [-\frac{1}{2}, \frac{1}{2}]$$

- For mathematical reasons: we will consider negative frequencies as well (remember SCS)
  - they mirror the positive frequencies.

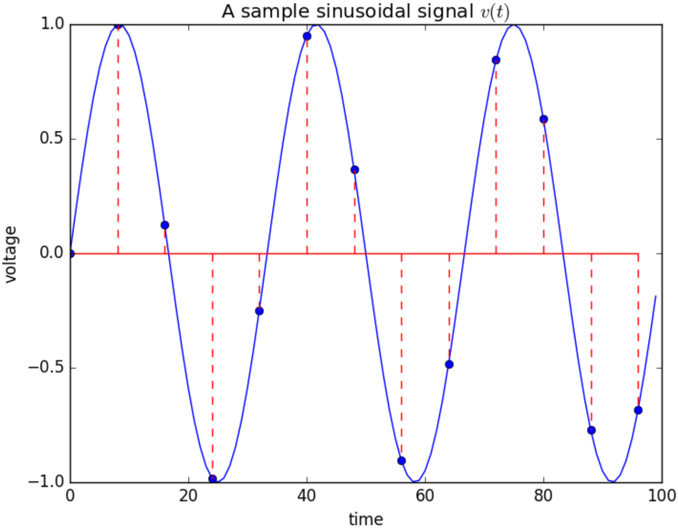
I.3 Sampling of analog signals

Definition

- Discrete signals are obtained from analog signals through **sampling**
- Sampling = taking the values from the analog signal at discrete moments of time (usually periodic)
- The time between two samples is the **sampling period** T_s
- The corresponding frequency is the **sampling frequency**

$$F_s = \frac{1}{T_s}$$

Graphical example



Sampling equation

- Sampling equation:
$$x[n] = x_a(n \cdot Ts).$$
- Interpretation:
  - The n-th value in the discrete signal x[n] = the value of the analog signal taken after n sampling periods, at time $t = nT_s$.

Sampling of harmonic signals

- Consider a cosine signal $x_a(t) = cos(2\pi Ft)$, sampling frequency is F_s
- What is the resulting discrete signal $x[n]$?
- Applying the sampling equation:

$$\begin{aligned} x[n] = x_a(n \cdot T_s) = x_a(n \cdot \frac{1}{F_s}) &= cos(2\pi Fn\frac{1}{Fs}) \\ &= cos(2\pi \frac{F}{Fs} n) = cos(2\pi fn), \end{aligned}$$

- Sampling an analog cosine/sine produces a discrete cosine/sine of similar form, but with discrete frequency

$$f = \frac{F}{F_s}$$

Sampling example

- Analog signal: $x_a(t) = cos(2\pi 100t)$
- Sampling frequency: $F_s = 300Hz$
- Result:

$$x[n] = cos(2\pi \frac{1}{3} t)$$

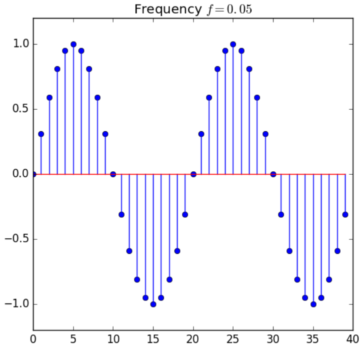
- The discrete frequency is $f = \frac{1}{3}$
- Sampling with a different $F_s = 500Hz$ produces a different signal

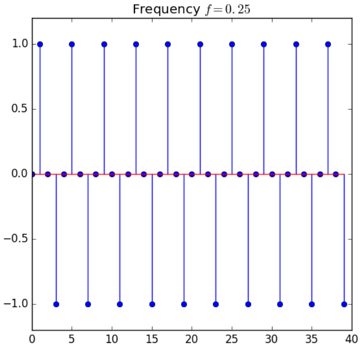
$$x[n] = cos(2\pi \frac{1}{5} t)$$

- Explain: why does sampling with a higher sampling frequency produce a signal with lower discrete frequency?
- Note: in both cases the resulting f is smaller than $f_{max} = \frac{1}{2}$

False friends

- **Note:** A discrete sinusoidal signal might not *look* sinosoidal, when its frequency is high (close to $\frac{1}{2}$).





Sampling theorem

The Nyquist-Shannon sampling theorem:
If a signal having maximum frequency F_{max} is sampled with a a sampling frequency $F_s \geq 2F_{max}$, then it can be perfectly reconstructed from its samples using the formula:

$$x_a(t) = \sum_{n=-\infty}^{+\infty} x[n] \cdot \frac{sin(\pi(F_s t - n))}{\pi(F_s t - n)}.$$

Comments

- All the information in the original signal is contained in the samples, provided that they are sampled correctly
- We can process discrete samples instead of the original analog signals
- Sampling with $F_s \geq 2F_{max}$ means the resulting discrete frequencies are smaller than $1/2$

$$f = \frac{F}{F_s} \leq \frac{F_{max}}{F_s} \leq \frac{1}{2}$$

Aliasing

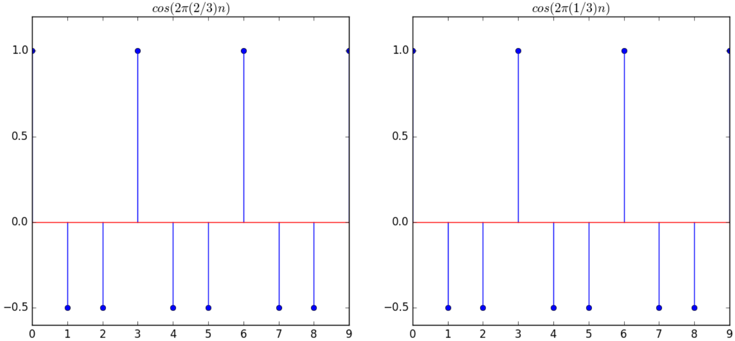
- What if the sampling frequency is not high enough?
- Obtain a discrete frequency higher than $\frac{1}{2}$
- A discrete frequency higher than $1/2$ is **identical** to a frequency smaller than $\frac{1}{2}$
- This phenomenon is known as **aliasing**

Aliasing example

Example

- $x_a(t) = cos(2\pi 10t)$ is sampled with $F_s = 15Hz$
- The result is
$$x[n] = cos(2\pi\frac{10}{15}n) = cos(2\pi\frac{2}{3}n)$$
- But, the frequency $\frac{2}{3}$ is actually identical to $\frac{1}{3}$:
$$cos(2\pi\frac{2}{3}n) = cos(2\pi\frac{1}{3}n)$$
- Proof: since *cos()* is periodical, we can subtract a multiple of 2π:
$$\begin{aligned} cos(2\pi\frac{2}{3}n) = cos(2\pi\frac{2}{3}n - 2\pi n) &= cos(2\pi(\frac{2}{3} - 1)n) \\ &= cos(-2\pi\frac{1}{3}n) = cos(2\pi\frac{1}{3}n) \end{aligned}$$

Aliasing example



Aliasing

- http://www.dictionary.com/browse/alias:

"alias": *a false name used to conceal one's identity; an assumed name*

- Every discrete frequency above $f_{max} = \frac{1}{2}$ is equivalent (an **alias**) to a frequency that smaller than $f_{max} = \frac{1}{2}$:

$$cos(2\pi(\frac{1}{2} + \epsilon)n) = cos(2\pi(\frac{1}{2} - \epsilon)n)$$

$$sin(2\pi(\frac{1}{2} + \epsilon)n) = -sin(2\pi(\frac{1}{2} - \epsilon)n)$$

- Proof: at whiteboard
- Every frequency $f \in \mathbb{R}$ is actually identical to a frequency $f \in [-\frac{1}{2}, \frac{1}{2}]$, up to a different phase
- Note: Aliasing is only valid for discrete frequencies, not analog!

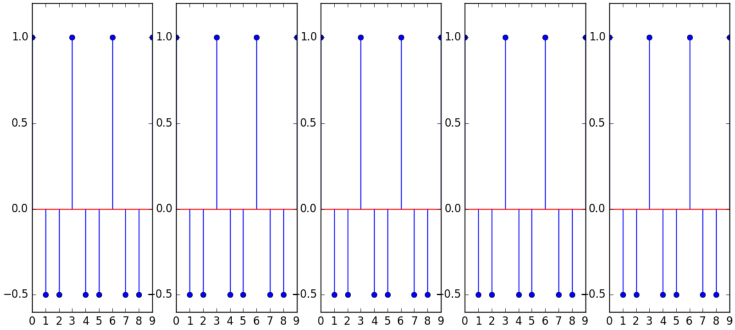
What's the problem with aliasing?

- Problem: sampling analog signals with different frequencies, will lead to exactly the same samples
- How can we know the original frequency? Impossible
- **Note:** Always, there is only a single analog frequency $F \in [-\frac{Fs}{2}, \frac{Fs}{2}]$ that corresponds to a discrete frequency in $f \in [-\frac{1}{2}, \frac{1}{2}]$
  - No confusion if every $F < F_s/2$, i.e. sampled according to the theorem
  - Problems are only for analog frequencies which are not sampled according to the theorem

Example

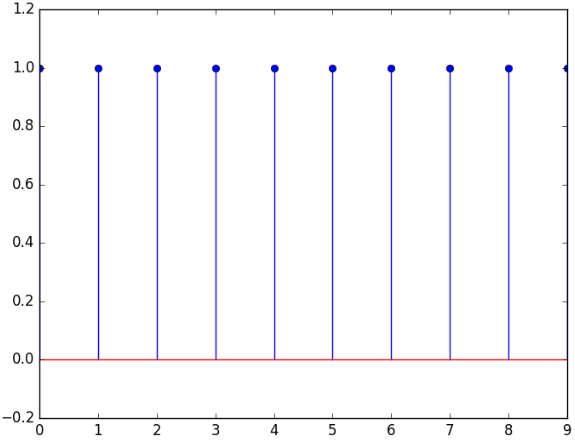
Example:

- If sampling frequency is $F_s = 15Hz$, all the following signals produce identical discrete signals:
  - $cos(2\pi 5t)$, $cos(2\pi 10t)$, $cos(2\pi 20t)$, $cos(2\pi 30t)$
- **Exercise**: which are the next signals in the above sequence, which produce the same samples?



Exercise

- What signals produce the following samples?



Ideal reconstruction of analog signals from samples

- If the discrete frequency $f \in [-\frac{1}{2}, \frac{1}{2}]$, use the following:

$$x_r(t) = x[\frac{t}{T_s}] = x[t \cdot F_s].$$

- If f outside this interval: the same equation, but **replace** f with its equivalent from the interval $[-\frac{1}{2}, \frac{1}{2}]$.
- Consequence: **reconstruction leads only to analog signals with frequencies** $F \in [-\frac{F_s}{2}, \frac{F_s}{2}]$
    - Analog signals which were sampled with at least $2F_{max}$: reconstructed identically
    - Analog signals which are not sampled correctly: **not reconstructed identically**

Anti-alias filtering

- If a signal has frequencies $F > \frac{F_s}{2}$, it is better to eliminate these frequencies from the signal before sampling
  - If left, they will just overlap with the samples of a frequency from the base interval $[-\frac{Fs}{2}, \frac{Fs}{2}]$
- **Anti-alias filter**: an analog low-pass filter, before a sampling circuit, designed to reject all frequencies $F > \frac{F_s}{2}$ from the signal before sampling.

I.4 Signal quantization and coding

Definition

- After sampling, the samples can have any real value
- **Quantization** = adjusting a value to a limited set of predefined values (**quantization levels**)
- **Quantization error** = the difference between quantized value and the original value
- Quantization methods:
  - **truncation**: choose the quantization level immediately smaller
  - **rounding**: choose the nearest quantization level

Examples

- Example 1 :
  - The grade of a student is 8.75, but it is adjusted to 8 or 9 (closest quantization levels)
- Example 2:
  - A voltage sample can be between 0V and 10V, but we need to store the value on one byte (8 bits)
  - With 8 bits: $2^8 = 256$ different possible values (quantization levels)
  - Divide the interval $[0 - 10]$ in 255 equal sub-intervals of size

$$\Delta = \frac{10}{255} \approx 0.039V$$

  - Each level corresponds to one of 256 possible numeric values: $0 = 0V$, $1 \approx 0.039V$, $2 \approx 0.078V$, $\ldots 256 = 10V$
  - The sample value can be rounded to the closest quantization level

Coding

- **Coding** = converting a quantized value in binary form
  - a binary form can be handled by a processor or microcontroller in a digital system
- Sampling + quantization + coding are usually done by an **Analog to Digital Converter (ADC)**.
- Reconstruction of a analog signal from numeric samples: by a **Digital to Analig Converter (DAC)**
  - Usually the reconstruction is not done based on the ideal reconstruction equations above
  - Simpler approximative methods are preferred