Chapter IV: Frequency Analysis of Discrete Systems

IV.1 Reminder: Frequency analysis of analog signals

Introduction

- Very useful to analyze signals in **frequency domain**
- The **spectrum** of a signal indicates the frequency contents
- Mathematical tools:
  - periodicac signals: **Fourier series**
  - non-periodical signals: **Fourier transform**

Analog periodical signals

- Periodical signal:
$$x(t) = x(t + T)$$
- The fundamental frequency is
$$F_0 = \frac{1}{T}$$
- **The signal can be decomposed as a sum of complex exponential signals, with multiples of the fundamental frequency,** kF_0
$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi kF_0 t}$$
- The coefficients c_k are the **spectrum** of the signal
$$c_k = \frac{1}{T} \int_{T/2}^{T/2} x(t) e^{-j2\pi kF_0 t}$$
- The coefficients c_k are complex values
  - their modulus = "amplitude spectrum"
  - their phase = "phase spectrum""

Conditions for convergence

- When is the Fourier series convergent to the signal?
  - i.e. when is the relation correct,
  - i.e. when is the sum actually equal to $x(t)$?
- Dirichlet conditions: the sum is convergent in all continuity points if:
  1. $x(t)$ is continuous or has a finite number of discontinuities in any finite interval
  2. $x(t)$ has a finite number of maxima and minima in any period
  3. $x(t)$ is absolutely integrable in any period, i.e.:

$$\int_T |x(t)| dt < \infty$$

- Weaker condition:
  - if $x(t)$ is square summable

$$\int_T x(t)^2 dt < \infty$$

  - then the he difference $d(t) = x(t) - \sum_{k=-\infty}^{\infty} c_k e^{j2\pi kF_0 t}$ has zero energy
  - Does not guarantee *pointwise* convergence

Signal spectrum

- The coefficients c_k are complex numbers
- If the signal is **real** $x(t) \in \mathbb{R}$, then the c_k are **even**
  - $|c_k| = |c_{-k}|$
  - $\angle c_k = -\angle c_{-k}$
  - group the terms with c_k with c_{-k} –> **cosine with amplitude |c_k| and phase** $\angle c_k$
- Average power of signal = energy of coefficients

$$P_T = \frac{1}{T} \int_T |x(t)|^2 = \sum_{-infty}^{\infty} |c_k|^2$$

- Interpretation of Fourier series for real signal
  - **the signal is the sum of cosine signals with frequency** $0, F_0, 2F_0 ...$**, with amplitudes** $|c_k|$ **and phase** $\angle c_k$
- No other frequencies appear in spectrum –> spectrum is made of “lines”

Time-frequency duality

- Time-frequency **duality**:
  - Real signal –> Even spectrum
  - Periodic signal –> Discrete spectrum

Analog non-periodical signals

- The signal is composed of all frequencies (inverse Fourier transform)

$$x(t) = \int_{infty}^{\infty} X(F) e^{j2\pi ft} dF$$

- The frequency content is found by the Fourier transform

$$X(F) = \int_{infty}^{\infty} x(t) e^{-j2\pi ft} dt$$

- (Can use instead $\Omega = 2\pi F$)
- $X(F)$ is a complex function
  - $|X(F)|$ is the amplitude spectrum
  - $\angle X(F)$ is the phase spectrum

Conditions for convergence

- When is the Fourier series convergent to the signal?
  - i.e. when is the relation correct,
  - i.e. when is the sum actually equal to $x(t)$?
- Dirichlet conditions: the sum is convergent in all continuity points if:
  1. $x(t)$ is continuous or has a finite number of discontinuities
  2. $x(t)$ has a finite number of maxima and minima
  3. $x(t)$ is absolutely integrable:

$$\int_{\infty}^{\infty} |x(t)| dt < \infty$$

- Weaker condition:
  - if $x(t)$ is square summable

$$\int_{\infty}^{\infty} x(t)^2 dt < \infty$$

  - then the he difference $d(t) = x(t) - \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k F_0 t}$ has zero energy
  - Does not guarantee *pointwise* convergence

Signal spectrum

- $X(F)$ is a complex function
- If the signal is **real** $x(t) \in \mathbb{R}$, then the $X(F)$ is **even**
  - $|X(F)| = |X(-F)|$
  - $\angle X(F) = -\angle X(-F)$
  - group the terms with c_k with c_{-k} **–> cosine with amplitude $|\mathbf{X(F)}|$ and phase** $\angle X(F)$
- Signal energy is the same in time and frequency domains

$$E = \int_{\infty}^{\infty} |x(t)|^2 dt = \int_{\infty}^{\infty} |X(F)|^2 dF$$

- The power spectral density of $x(t)$ is

$$S_{xx}(F) = |X(F)|^2$$

IV.2 Frequency analysis of discrete signals

Fourier series of discrete periodical signals

- A discrete signal of period N: $x[n] = x[n+N]$
- Decomposed as a **sum of complex exponentials**:

$$x[n] = \sum_{k=0}^{N-1} c_k e^{j2\pi kn/N}, n = 0, 1, ...N-1$$

- Finding the coefficients:

$$c_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N}$$

Comparison with analog Fourier series

- Compared to analog signals:
  - consider fundamental frequency $f_0 = 1/N$
  - only N terms, with frequencies $k \cdot f_0$:
    - $0, f_0, 2f_0, ...(N-1)f_0$
  - only N distinct coefficients c_k
  - the N coefficients c_k can be chosen like $-\frac{N}{2} < k \leq \frac{N}{2}$ => the frequencies span the range $-1/2...1/2$

$$-\frac{1}{2} < f_k \leq \frac{1}{2}$$

$$-\pi < \omega_k \leq \pi$$

Basic properties of Fourier coefficients

1. Signal is **discrete** –> coefficients are **periodic** with period N

$$c_{k+N} = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-j2\pi(k+N)n} = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-j2\pi kn}$$

2. If signal is real $x[n] \in \mathbb{R}$, the coefficients are **even**:
    - $c_k^* = c_{-k}$
    - $|c_k| = |c_{-k}|$
    - $\angle c_k = \angle c_{-k}$

- Together with periodicity:
    - $|c_k| = |c_{-k}| = |c_{N-k}|$
    - $\angle c_k = -\angle c_{-k} = -\angle c_{N-k}$

Expressing as sum of sinusoids

- Grouping terms with c_k and c_{-k} we get

$$x[n] = c_0 + 2\sum_{k=1}^{L} |c_k| cos(2\pi \frac{k}{N} + \angle c_k)$$

  where $L = N/2$ or $L = (N-1)/2$ depending if N is even or odd
- Signal = DC value + a finite sum of sinusoids with frequencies kf_0
  - $|c_k|$ give the amplitudes (x 2)
  - $\angle c_k$ give the phases

Power spectral density

- The average power of a discrete periodic signal

$$P = \frac{1}{N} \sum_{n=0}^{N-1} |x[n]|^2$$

- Is the same in the frequency domain (with proof):

$$P = \sum_{k=0}^{N-1} |c_k|^2$$

- Power spectral density of the signal is

$$S_x x[k] = |c_k|^2$$

- Energy over one period is

$$E = \sum_{n-0}^{N-1} |x[n]|^2 = N \sum_{k-0}^{N-1} |c_k|^2$$

Examples

- Examples:

$$x_1[n] = cos(\sqrt{5}\pi n)$$
$$x_2[n] = 2sin(\frac{\pi}{3} n)$$
$$x_3[n] = \{1, 1, 0, 0\}$$

Example in Python

```
>>> import numpy as np
>>> from scipy.fftpack import fft, ifft
>>> x = np.array([1.0, 1.0, 0.0, 0.0])
>>> y = 1.0/4.0 * fft(x)
>>> y
array([ 0.50+0.j  ,  0.25-0.25j,  0.00+0.j  ,  0.25+0.25j])
```

Properties of Fourier series

1. Linearity

If the signal $x_1[n]$ has the Fourier series coefficients $\{c_k^{(1)}\}$, and $x_2[n]$ has $\{c_k^{(2)}\}$, then their sum has
$a \cdot x_1[n] + b \cdot x_2[n] \leftrightarrow \{a \cdot c_k^{(1)} + b \cdot c_k^{(2)}\}$
Proof: via definition

Properties of Fourier series

2. Shifting in time

If $x[n] \leftrightarrow \{c_k\}$, then

$$x[n - n_0] \leftrightarrow \{e^{(-j2\pi k n_0/N) c_k}\}$$

Proof: via definition

- The amplitudes $|c_k|$ is not affected, shifting in time affects only the phase

Properties of Fourier series

3. Modulation in time

$$e^{(j2pik_0 n/N} \leftrightarrow \{c_{k-k_0}\}$$

4. Complex conjugation

$$x^*[n] \leftrightarrow \{c^*_{-k}\}$$

Properties of Fourier series

5. Circular convolution

Circular convolution of two signals $\leftrightarrow$ product of coefficients

$$x_1[n] \otimes x_2[n] \leftrightarrow \{N \cdot c_k^{(1)} \cdot c_k^{(2)}\}$$

Circular convolution:

$$x_1[n] \otimes x_2[n] = \sum_{k=0}^{N-1} x_1[k] x_2[(n-k)_N]$$

- takes two periodic signals of period N, result is the same
- Example at the whiteboard: how it is computed

Properties of Fourier series

6. Product in time

Product in time $\leftrightarrow$ circular convolution of Fourier series coefficients

$$x_1[n] \cdot x_2[n] \leftrightarrow \sum_{m=0}^{N-1} c_m^{(1)} c_{(k-m)_N}^{(2)} = c_k^{(1)} \otimes c_k^{(2)}$$

Fourier transform of discrete non-periodical signals