Chapter III: The Z Transform

III.1 Introducing the Z transform

Preliminaries: complex numbers

- real and imaginary part
- **modulus and phase**
- graphical interpretation
- Euler formula
- modulus and phase of e^{jx}

Definition of Z transform

- The Z Transform of a signal $x[n]$, called $X(z)$, is defined as:

$$X(z) = \sum_{-\infty}^{\infty} x[n] z^{-n}$$

- It is defined only for the values of z where the sum is finite (called *region of convergence)*
- Notation:

$$\mathcal{Z}\left(x[n]\right) = X(Z)$$

$$x[n] \xleftrightarrow{\mathrm{Z}} X(Z)$$

- Similar to the Laplace transform for analog signals
- The Z transform associates **a polynomial** to a signal (think Decision and Estimation class)
- Why?
  - Convolution of two signals = multiplication of polynomials
  - Short descriptions of complicated signals (i.e. exponential signals)

Examples

$x[n] = 1, 2, 5, 7, 0,$ (with time origin in 1 or in 5)
$\delta[n]$, $\delta[n-k]$, $\delta[n+k]$
$\left(\frac{1}{2}\right)^n$
$x[n] = a^n u[n]$
$x[n] = -a^n u[-n-1]$

Region of convergence

- For finite-support signals, the CR is the whole Z plane, possibly except 0 or ∞
- For causal signals, the CR is *the outside of a circle*:

$$|z| > r_1$$

- For anti-causal signals, the CR is *the inside of a circle*:

$$|z| < r_2$$

- For bilateral signals, both the causal and the anti-causal terms of the sum must converges —> the CR is the area between two circles:

$$r_1 < |z| < r_2$$

- For finite-support signals, the two "circles" are 0 and ∞
- Two different signals can have the same expression of $X(z)$, but with different RC!
  - RC is an essential part in specifying a Z transform
  - should never be omitted

The Inverse Z Transform

- From a purely mathematical point of view, $X(z)$ is a complex function
- Proper definition of inverse transform is based on the theory of complex functions

$$X(z) = \sum_{-\infty}^{\infty} x[k] z^{-k}$$

- Multiply with z^{n-1} and integrate along a contour C inside the Convergence Region:

$$\oint_C X(z) z^{n-1} dz = \oint_C \sum_{-\infty}^{\infty} x[k] z^{n-k-1} dz = \sum_{-\infty}^{\infty} x[k] \oint_C z^{n-k-1} dz$$

The Inverse Z Transform

- The Cauchy integral theorem says that:

$$\frac{1}{2\pi j}\oint_C z^{n-k-1}dz = \begin{cases} 1, & \text{if } k = n \\ 0, & \text{if } k \neq n \end{cases}$$

- And therefore:

$$x[n] = \frac{1}{2\pi j}\oint_C X(z)z^{n-1}dz$$

- We will not use this relation in practice, but instead will rely on **partial fraction decomposition**

Properties of Z transform

1. Linearity

If $x_1[n] \overset{\mathrm{Z}}{\longleftrightarrow} X_1(z)$ with CR1, and $x_2[n] \overset{\mathrm{Z}}{\longleftrightarrow} X_2(z)$ with CR2, then:

$$ax_1[n] + bx_2[n] \overset{\mathrm{Z}}{\longleftrightarrow} aX_1(z) + bX_2(z)$$

and CR is at least the intersection of CR1 and CR2.
Proof: use definition

Properties of Z transform

2. Shifting in time

If $x[n] \overset{Z}{\longleftrightarrow} X(z)$ with CR, then:

$$x[n-k] \overset{Z}{\longleftrightarrow} z^{-k}X(z)$$

with same RC, possibly except 0 and ∞.
Proof: by definition

- valid for all k, also for $k < 0$
- delay of 1 sample $= z^{-1}$

Properties of Z transform

3. Modulation in time

If $x[n] \xleftrightarrow{\text{Z}} X(z)$ with CR, then:

$$e^{j\omega_0 n} x[n] \xleftrightarrow{\text{Z}} X\left(e^{-j\omega_0} z\right)$$

with same CR

Proof: by definition

Properties of Z transform

4. Reflected signal

If $x[n] \xleftrightarrow{\mathrm{Z}} X(z)$ with CR $r_1 < |z| < r_2$, then:

$$x[-n] \xleftrightarrow{\mathrm{Z}} X\left(z^{-1}\right)$$

with CR $\frac{1}{r_2} < |z| \frac{1}{r_1}$
Proof: by definition

Properties of Z transform

5. Derivative of Z transform

If $x[n] \overset{\mathrm{Z}}{\longleftrightarrow} X(z)$ with CR, then:

$$nx[n] \overset{\mathrm{Z}}{\longleftrightarrow} -z\frac{dX\left(z\right)}{dz}$$

with same CR
Proof: by derivating the difference

Properties of Z transform

6. Transform of difference

If $x[n] \xleftrightarrow{\mathrm{Z}} X(z)$ with CR, then:

$$x[n] - x[n-1] \xleftrightarrow{\mathrm{Z}} (1 - z^{-1})X(z)$$

with same CR except $z = 0$.
Proof: using linearity and time-shift property

Properties of Z transform

7. Accumulation in time

If $x[n] \overset{\mathrm{Z}}{\longleftrightarrow} X(z)$ with CR, then:

$$y[n] = \sum_{k=-\infty}^{n} x[k] \overset{\mathrm{Z}}{\longleftrightarrow} \frac{X(z)}{(1 - z^{-1})}$$

with same CR except $z = 1$.
Proof: $x[n] = y[n] - y[n-1]$, apply previous property

Properties of Z transform

8. Complex conjugation

If $x[n] \xleftrightarrow{\mathrm{Z}} X(z)$ with CR, and $x[n]$ is a complex signal, then:

$$x^*[n] \xleftrightarrow{\mathrm{Z}} X^*(z^*)$$

with same CR except $z = 0$.
Proof: apply definition

Consequence

If $x[n]$ is a real signal, the poles / zeroes are either real or in complex pairs

Properties of Z transform

9. Convolution in time

If $x_1[n] \xleftrightarrow{\mathrm{Z}} X_1(z)$ with CR1, and $x_2[n] \xleftrightarrow{\mathrm{Z}} X_2(z)$ with CR2, then:

$$x[n] = x_1[n] * x_2[n] \xleftrightarrow{\mathrm{Z}} X(z) = X_1(z) \cdot X_2(z)$$

and CR the intersection of CR1 and CR2.
Proof: use definition

- **Very important property!**
- Can compute the convolution of two signals via the Z transform

Properties of Z transform

10. Correlation in time

If $x_1[n] \overset{Z}{\longleftrightarrow} X_1(z)$ with CR1, and $x_2[n] \overset{Z}{\longleftrightarrow} X_2(z)$ with CR2, then:

$$r_{x_1 x_2}[l] = \sum_{n=-\infty}^{\infty} x_1[n] x_2[n-l] \overset{Z}{\longleftrightarrow} R_{x_1 x_2}(z) = X_1(z) \cdot X_2(z^{-1})$$

and CR the intersection of CR1 and with the CR of $X_2(z^{-1})$ (see reflection property)
Proof: correlation = convolution with second signal reflected, use convolution and reflection properties

Properties of Z transform

11. Initial value theorem

If $x[n]$ is a causal signal, then

$$x[0] = \lim_{z\to\infty} X(z)$$

Proof:

$$X(z) = \sum_{n=0}^{\infty} x[n]z^{-n} = x[0] + x[1]z^{-1} + x[2]z^{-2} + \ldots$$

When $z \to \infty$, all terms z^{-k} vanish.

Common Z transform pairs

- Easily found all over the Internet

Sequence	Transform	ROC
$\delta[n]$	1	All z
$u[n]$	$\frac{1}{1-z^{-1}}$	$\|z\| > 1$
$-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$\|z\| < 1$
$\delta[n-m]$	z^{-m}	All z except 0 or ∞
$a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\|z\| > \|a\|$
$-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\|z\| < \|a\|$
$na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| > \|a\|$
$-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| < \|a\|$
$\begin{cases} a^n & 0 \le n \le N-1, \\ 0 & \text{otherwise} \end{cases}$	$\frac{1-a^N z^{-N}}{1-az^{-1}}$	$\|z\| > 0$
$\cos(\omega_0 n)u[n]$	$\frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$\|z\| > 1$
$r^n \cos(\omega_0 n)u[n]$	$\frac{1-r\cos(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2z^{-2}}$	$\|z\| > r$

III.2. Z transforms which are Rational Functions

Rational functions

- Many Z transforms are in the form of a rational function, i.e. a fraction where
  - numerator = polynomial in z^{-1} or z
  - denominator = polynomial in z^{-1} or z

$$X(z) = \frac{B(z)}{A(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} = \frac{b_0 + b_1 z^{-1} + b_2 z{-2} + ... + b_M z^{-M}}{a_0 + a_1 z^{-1} + a_2 z{-2} + ... + a_N z^{-N}}$$

- Example:

$$X(z) = \frac{3 + 2z^{-1} + 4z^{-2}}{1 - 5z^{-2} + 7z^{-4}}$$

Poles and zeros

- A polynomial is completely determined by its roots and a scaling factor
- **Definition**: the **zeros** of $X(z)$ are the roots of the numerator $B(z)$
- **Definition**: the **poles** of $X(z)$ are the roots of the denominator $A(z)$
- The zeros are usually named $z_1, z_2, ...z_M$, and the poles $p_1, p_2, ...p_N$.

The transform $X(z)$ can be rewritten as:

$$X(z) = \frac{b_0}{a_0} \cdot z^{N-M} \cdot \frac{(z - z_1)...(z - z_M)}{(z - p_1)...(z - z_N)} = \frac{b_0}{a_0} \cdot \frac{(1 - z_1 z^{-1})...(1 - z_M z^{-1})}{(1 - p_1 z^{-1})...(1 - z_N z^{-1})}$$

It has:

- M zeros with finite values
- N poles with finite values
- and either N-M zeros in 0, if N $>$ M, or N-M poles in 0, if N $<$ M (trivial poles/zeros)

Graphical representation

- The graphical representation of poles and zeros in the complex place is called **the pole-zero plot**
- Graphical: poles = “x”, zeros = “0”
- CR cannot contain poles
- Example: at whiteboard

III.3 Inverse Z transform for rational functions

Methods for computing the Inverse Z Transform

1. Direct evaluation using the Cauchy integral

$$x[n] = \frac{1}{2\pi j}\oint_C X(z)z^{n-1}dz$$

2. Decomposition as continuous power series
3. **Partial fraction decomposition**

Partial fraction decomposition

Any rational function

$$\frac{b_0 + b_1 z^{-1} + b_2 z{-}2 + ... + b_M z^{-M}}{a_0 + a_1 z^{-1} + a_2 z{-}2 + ... + a_N z^{-N}}$$

can be decomposed in **partial fractions**:

$$c_0 + c_1 z^{-1} + ...c_{N-M} z^{-(M-N)} + \frac{A_1}{z - p_1} + ... \frac{A_N}{z - p_N}$$

- Each pole has a corresponding partial fraction
- First terms appear if $M \leq N$
- Based on linearity, we invert each term separately (simple)

Procedure for Inverse Z Transform

$$X(z) = \frac{B(z)}{A(z)}$$

1. If $M \geq N$, divide numerator to denominator to obtain the first terms. The remaining fraction is $X_1(z) = \frac{B_1(z)}{A(z)}$, with numerator degree strictly smaller then denominator
2. In the remaining fraction, eliminate the negative powers of z by multiplying with z^N
3. Divide by z,
$$\frac{X_1(z)}{z} = \frac{B_1(z)}{zA(z)}$$
4. Compute the poles of $\frac{X_1(z)}{z}$ and decompose in partial fractions:
$$\frac{X_1(z)}{z} = \frac{A_1}{z - p_1} + ...$$

Procedure for Inverse Z Transform

5. Multiply back with z:

$$X_1(z) = A_1 \frac{z}{z - p_1} + ...$$

6. Convert each term back to the time domain

Computation of partial fractions coefficients

- If all poles are distinct:

$$A_k = (z - p_k)\frac{X(z)}{z}|_{z=p_k}$$

- If poles are in complex conjugate pairs
  - group the two fractions into a single fraction of degree 2
- If there exist m **multiple poles of same value** (pole order $m > 1$):

$$\frac{A_{1k}}{z - p_k} + \frac{A_{2k}}{(z - p_k)^2} + ... + \frac{A_{mk}}{(z - p_k)^m}$$

$$A_{ik} = \frac{1}{(m-i)!}\frac{d^{m-i}}{dz^{m-i}}\left[(z - p_k)^m \cdot \frac{X(z)}{z}\right]|_{z=p_k}$$

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* example for m = 2
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Real signals and complex poles/zeros

- Consequence of the complex-conjugate property of Z transform:
- A real signal $x[n]$ can have only
  - real poles or zeroes
  - complex poles and zeroes in conjugate pairs, which can be grouped into a single term of degree 2, with real coefficients
- If a Z transform has a complex pole / zero without its conjugate pair, then the corresponding signal $x[n]$ is complex

Position of poles and time behaviour - 1 pole

- Consider a Z transform with **1 pole**, analyze the look of the corresponding signal
- Consider the pole value is a
  - Consider only real signals $x[n]$ —> a is real
  - Consider causal signal $x[n$ —> CR is $|z| > |a|$
- Therefore the Z transform is of the type:

$$X(z) = \frac{1}{1 - az^{-1}} = \frac{z}{z-a}, CR : |z| > |a|$$

- Therefore the signal $x[n]$ is of the type:

$$x[n] = a^n u[n]$$

Position of poles and time behavior - 1 pole

Scenarios for a single real pole in a:

- Pole inside the unit circle ($|a| < 1$) –> exponential decreasing signal
- Pole outside the unit circle ($|a| > 1$) –> exponential increasing signal
- Pole exactly on unit circle ($|a| = 1$) –> not increasing, not decreasing
- Negative pole ($a < 0$)–> alternating signal
- Positive value ($a > 0$) –> non-alternating signal

Position of poles and time behavior - 1 pole

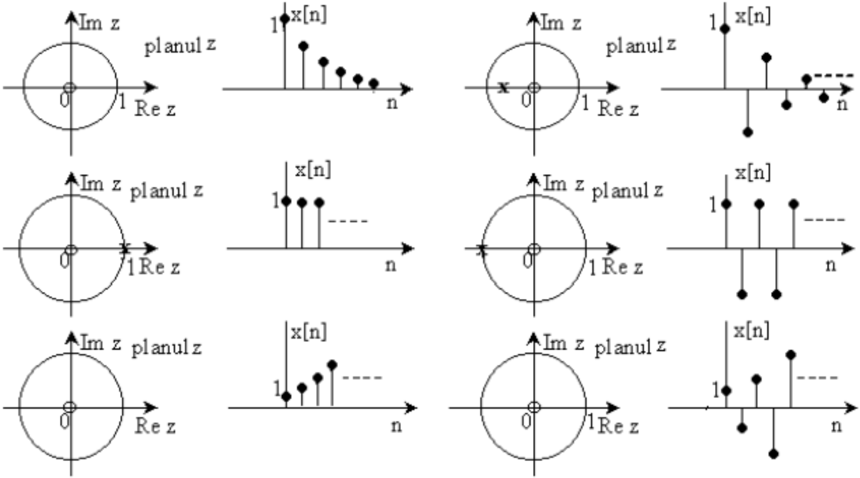

Figure 2: Signal behavior for 1 pole

Position of poles and time behavior - 1 double pole

$$X(z) = \frac{az^{-1}}{(1-az^{-1})^2}, CR : |z| > |a|$$

$$x[n] = na^n u[n]$$

A double pole in a:

- Pole inside the unit circle ($|a| < 1$) –> decreasing signal
- Pole outside the unit circle ($|a| > 1$) –> increasing signal
- Pole exactly on unit circle ($|a| = 1$) –> **increasing signal**
- Negative pole ($a < 0$)–> alternating signal
- Positive value ($a > 0$) –> non-alternating signal

Position of poles and time behavior - 1 double pole

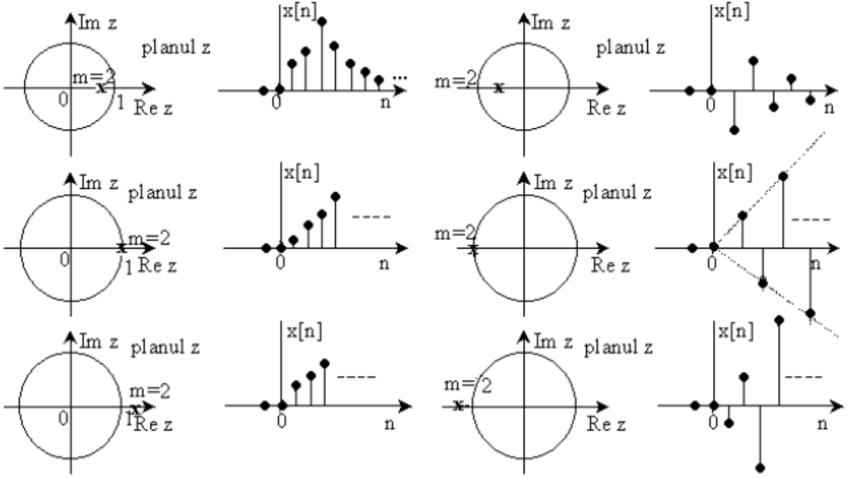

Figure 3: Signal behavior for 1 double pole

Position of poles and time behavior - conjugate poles

$$X(z) = \frac{1 - az^{-1}\cos\omega_0}{1 - 2z^{-1}\cos\omega_0 + z^{-2}}, CR : |z| > |a|$$

$$x[n] = na^n u[n]$$

A pair of complex conjugate poles:

- a sinusoidal with exponential envelope
  - phase of poles –> frequency of sinusoidal signal
  - modulus of poles –> exponential envelope
  - poles inside unit circle –> decreasing signal
  - poles outside unit circle –> increasing signal
  - poles on unit circle –> oscillating signal, constant amplitude

What if poles are double?

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* poles on unit circle --> increasing signal
* otherwise similar
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Position of poles and time behavior - conjugate poles

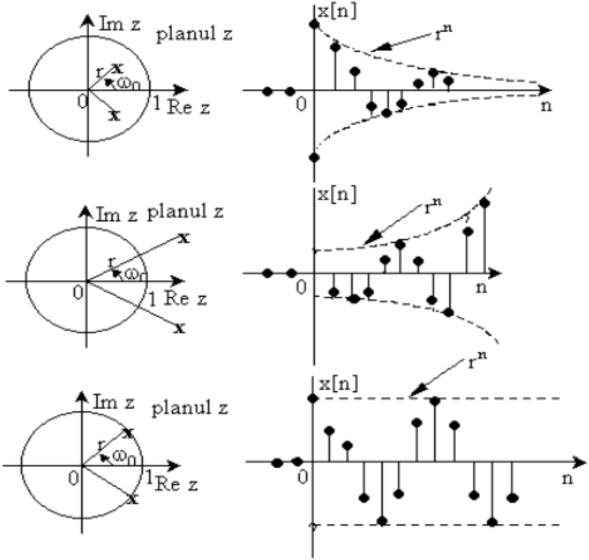

Figure 4: Signal behavior for 1 double pole

Position of poles and time behavior

- A Z transform can be decomposed into partial fractions, i.e. separate poles
- Analyzing the individual behavior of poles –> tells something about whole signal
- Conclusions (for real signals, causal):
  - **all poles inside unit circle –> bounded signal**
  - *simple* poles on unit circle –> bounded signal
  - otherwise –> unbounded signal
  - poles inside unit circle, closer to origin –> fast decrease of signal
  - poles inside unit circle, closer to unit circle –> slow decrease of signal

System function of a LTI system

- Considering a LTI system with $h[n]$, input signal $x[n]$ −> output is convolution

$$y[n] = x[n] * h[n]$$

- In Z transform, convolution = product of transforms

$$Y(z) = X(z) \cdot H(z)$$

- **The system function of a LTI system is the Z transform of the impulse response** $h[n]$
- The system function of a LTI system is:

$$H(z) = \frac{Y(z)}{X(z)}$$

System function and the difference equation

- Any LTI system is characterized by a **difference equation**:

$$
\begin{aligned}
y[n] &= -\sum_{k=1}^{N} a_k y[n-k] + \sum_{k=0}^{M} b_k x[n-k] \\
&= -a_1 y[n-1] - a_2 y[n-2] - \ldots - a_N y[n-N] + b_0 x[n] + b_1 x[n-1] +
\end{aligned}
$$

or

$$
y[n] + \sum_{k=1}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]
$$

$$
y[n]+a_1 y[n-1]+a_2 y[n-2]+\ldots+a_N y[n-N] = b_0 x[n]+b_1 x[n-1]+\ldots+b_M x[n-
$$

System function and the difference equation

- The system function $H(z)$ can be derived directly from the difference equation:

$$Y(z)\left(1+\sum_{k=1}^{N} a_k z^{-k}\right) = X(z)\left(\sum_{k=0}^{M} b_k z^{-k}\right)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1+\sum_{k=1}^{N} a_k z^{-k}}$$

Particular cases of system functions

- FIR systems: $a_k = 0$
  - has only zeroes, no poles (all-zero system)

$$H(z) = \frac{Y(z)}{X(z)} = \sum_{k=0}^{M} b_k z^{-k}$$

- All-pole system: $b_k = 0, k \geq 1$ (must have at least $b_0 \neq 0$)
  - has only poles

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

- Othwerise, in general, we have a *pole-zero system*, with both poles and zeroes