

Chapter V. Frequency Analysis of Discrete Systems

Response of LTI systems to harmonic signals

- ▶ We consider an LTI system with $h[n]$
- ▶ Input signal = complex harmonic (exponential) signal $x[n] = Ae^{j\omega_0 n}$
- ▶ Output signal = convolution

$$\begin{aligned}y[n] &= \sum_{k=-\infty}^{\infty} h[k]x[n-k] \\&= \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega_0 k} Ae^{j\omega_0 n} \\&= H(\omega_0) \cdot x[n]\end{aligned}$$

- ▶ $H(\omega_0)$ = Fourier transform of $h[n]$ evaluated for $\omega = \omega_0$

Eigen-function

- ▶ Complex exponential signals are **eigen-functions** (functii proprii) of LTI systems:
 - ▶ output signal = input signal \times a (complex) constant
- ▶ $H(\omega_0)$ is a constant that multiplies the input signal
 - ▶ Amplitude of input gets multiplied by $|H(\omega_0)|$
 - ▶ Phase of input signal is added with $\angle H(\omega_0)$
- ▶ Why are sin/cos/exp functions important?
 - ▶ If input signal = sum of complex exponential (= cosines + sines),
 - ▶ since the system is linear,
 - ▶ then output = same sum of complex exponentials, each scaled with some coefficients

Response to cosine and sine

- ▶ Cosine / sine = sum of two exponentials, via Euler
- ▶ System is linear and real \Rightarrow
 - ▶ amplitude is multiplied by $|H(\omega_0)|$
 - ▶ phase increases by $\angle H(\omega_0)$
- ▶ See proof at blackboard

Frequency response

- ▶ Names
 - ▶ $H(\omega)$ = frequency response of the system
 - ▶ $|H(\omega)|$ = amplitude response
 - ▶ $\angle H(\omega)$ = phase response
- ▶ Phase response might have jumps of 2π
- ▶ Stitching the pieces in a continuous function = phase *unwrapping*
 - ▶ Example: at blackboard
- ▶ Wrapped phase: $\in [-\pi, \pi]$, may have jumps of 2π
- ▶ Unwrapped phase: continuous function, may go outside interval

Permanent and transient response

- ▶ The above harmonic signals start at $n = -\infty$, not at 0.
- ▶ What if the signal starts at some time $n = 0$?
- ▶ Total response = transient response + permanent response
 - ▶ transient response goes towards 0 as $n \rightarrow \infty$
 - ▶ permanent response = the above
- ▶ So the above relations are valid only in **permanent regime**
 - ▶ i.e. after the transient regime has passed
 - ▶ i.e. after the transient response has practically vanished
 - ▶ i.e. when the signal started very long ago (from $n = -\infty$)
- ▶ Example at blackboard

Permanent response of LTI systems to periodic inputs

- ▶ Assume the input $x[n]$ is periodic with period N
- ▶ Then it can be represented as a Fourier series:

$$x[n] = \sum_{k=0}^{N-1} c_k e^{j2\pi kn/N}$$

- ▶ Since the system is linear, the output to each component k is

$$c_k H\left(\frac{2\pi}{N}k\right) e^{j2\pi kn/N}$$

- ▶ So the total output is:

$$x[n] = \sum_{k=0}^{N-1} c_k H\left(\frac{2\pi}{N}k\right) e^{j2\pi kn/N}$$

- ▶ The output is still periodic, same period, same frequencies

Response of LTI systems to non-periodic signals

- ▶ The output = input convolution with impulse response

$$y[n] = x[n] * h[n]$$

$$Y(\omega) = X(\omega) \cdot H(\omega)$$

- ▶ In polar form:
 - ▶ modulus get multiplied
 - ▶ phases get added
- ▶ The system **attenuates/amplifies** the input frequencies and **changes their phases**
- ▶ $H(\omega)$ = the **transfer function**
- ▶ $H(z)$ = the **system function**
- ▶ $H(\omega) = H(z = e^{j\omega})$ if unit circle is in CR

Power spectral density

- ▶ $S_{zz}(\omega) = |Y(\omega)|^2 = |H(\omega)|^2 \cdot S_{xx}(\Omega)$
- ▶ The poles and zeros of $S(\omega)$ come in pairs $(z, 1/z$ and $p, 1/p)$

- ▶ LTI systems are also known as **filters** because their transfer function shapes (*filters*) the frequencies of the input signals
- ▶ The transfer function can be found from $H(z)$ and $z = e^{j\omega}$
- ▶ Alternatively, the transfer function can be found by the **geometrical method** based on the locations of poles and zeros

Ideal filters

- ▶ Ideal low-pass filter: example at whiteboard
- ▶ Ideal band-pass filter: example at whiteboard
- ▶ Ideal high-pass filter: example at whiteboard
- ▶ Ideal band-stop filter: example at whiteboard
- ▶ Ideal all-pass filter (*changes the phase*): idem

Linear-phase filters

- ▶ Consider a constant filter with **linear phase** function:

$$H(\omega) = C \cdot e^{-j\omega n_0}$$

- ▶ The output signal is

$$Y(\omega) = X(\omega) \cdot C \cdot e^{-j\omega n_0}$$

$$y[n] = C \cdot x[n - n_0]$$

- ▶ Linear phase means **just a delaying** of the input signal
 - ▶ Fourier property: $x[n - n_0] \longleftrightarrow X(\omega)e^{-j\omega n_0}$

Group delay

- ▶ = The time delay experienced by a component of frequency ω when passing through the filter
- ▶ **Group delay** of the filter:

$$\tau_g(\omega) = \frac{d\Theta(\omega)}{d\omega}$$

- ▶ Linear phase = constant group delay = all frequencies delayed the same = whole signal delayed

Filter distortions

- ▶ When a filter is non-ideal:
 - ▶ non-constant amplitude \rightarrow amplitude distortions
 - ▶ non-linear phase \rightarrow phase distortions
- ▶ Phase distortions may be tolerated by certain applications
 - ▶ e.g. human ears are insensitive to phase distortions of sounds

Filter order

- ▶ The **order** of a filter = maximum degree in numerator or denominator of $H(z)$
 - ▶ i.e. largest power of z
- ▶ Any filter can be implemented, in general, with this number of unit delay blocks (z^{-1})
- ▶ Higher order \rightarrow better filter transfer function
 - ▶ closer to ideal filter
 - ▶ more complex to implement
 - ▶ more delays (bad)
- ▶ Lower order
 - ▶ worse transfer function (not close to ideal)
 - ▶ simpler, cheaper
 - ▶ faster response

Filter design by pole and zero placements

- ▶ Based on geometric method
- ▶ The gain must be found by separate condition
- ▶ Examples at blackboard

Linear-phase FIR filters

- ▶ Only FIR filters can have linear phase!
- ▶ IIR filters cannot have linear phase (no proof)

Zero-phase transfer function

- ▶ Normally, $|H(\omega)|$ is strictly positive

$$|H(\omega)| \geq 0$$

- ▶ When $H(\omega)$ the function passes through 0:
 - ▶ $|H(\omega)|$ remains positive
 - ▶ $\angle H(\omega)$ has a jump of π
- ▶ Zero phase transfer function $H(\omega) = H_R(\omega)e^{j\Theta_R(\omega)}$
 - ▶ $H_R(\omega) = \pm |H(\omega)|$, including the sign (can be positive or negative)
 - ▶ $\Theta_R(\omega)$ doesn't have anymore the jumps of π
- ▶ Everything else still applies
 - ▶ $H_R(\omega)$ is even
 - ▶ $\Theta_R(\omega)$ is odd

Symmetries for linear-phase FIR

- ▶ Let filter order be M
- ▶ The filter coefficients are $h[0], \dots, h[M-1]$
- ▶ Linear-phase is guaranteed in two cases
- ▶ **Positive symmetry**

$$h[n] = h[M-1-n]$$

- ▶ **Negative symmetry (anti-symmetry)**

$$h[n] = -h[M-1-n]$$

Cases of linear-phase FIR

- ▶ Proofs at blackboard

1. Positive symmetry, $M = \text{odd}$
2. Positive symmetry, $M = \text{even}$
3. Negative symmetry, $M = \text{odd}$
4. Negative symmetry, $M = \text{even}$

- ▶ Check constraints for $H(0)$ and $H(\pi)$
- ▶ For what types of filters is each case appropriate?

Particular classes of filters

- ▶ Digital resonators
 - ▶ = very selective band pass filters
 - ▶ poles very close to unit circle
 - ▶ may have zeros in 0 or at $1/-1$
- ▶ Notch filters
 - ▶ have zeros exactly on unit circle
 - ▶ will completely reject certain frequencies
 - ▶ has additional poles to make the rejection band very narrow
- ▶ Comb filters
 - ▶ = periodic notch filters

Digital oscillators

- ▶ Oscillator = a system which produces an output signal even in absence of input
- ▶ Has a pair of complex conjugate poles **exactly on unit circle**
- ▶ Example at blackboard

Inverse filters

- ▶ Sometimes is necessary to **undo** a filtering
 - ▶ undo attenuation of a signal passed through a channel
- ▶ Inverse filter: has inverse system function:

$$H_I(z) = \frac{1}{H(z)}$$

- ▶ Problem: if $H(z)$ has zeros outside unit circle, $H_I(z)$ has poles outside unit circle \rightarrow unstable
- ▶ Examples at blackboard

Minimum and maximum phase filters

- ▶ Start with example
- ▶ If replace a zero z with $1/z$, the modulus does not change!
 - ▶ only phase is affected
- ▶ Minimum phase: all zeros are inside the unit circle
 - ▶ minimum phase = minimum derivative = minimum delay
 - ▶ inverse filter is stable
- ▶ Maximum phase: all zeros are outside unit circle
- ▶ Mixed phase: some zeros inside, some outside

System identification

- ▶ How to find system/transfer function of an unknown system?
- ▶ Via the Z transform
 - ▶ Observe the output $y[n]$ to a known input $x[n]$
 - ▶ Then $H(z) = Y(z)/X(z)$
 - ▶ Example
- ▶ Direct in time domain (for causal systems)
 - ▶ use convolution equation to find $h[0]$, then $h[1]$ etc
- ▶ Using correlation .