Digital Signal Processing

I. Sampling of analog signals

I.1. Analog and Digital Signals

Signals

- Signal = a measurable quantity which varies in time, space or some other variable
- Examples:
  - a voltage which varies in time (1D voltage signal)
  - sound pressure which varies in time (sound signal)
  - intensity of light which varies across a photo (2D image)
- Represented as a mathematical function, e.g. $v(t)$.

Off Topic

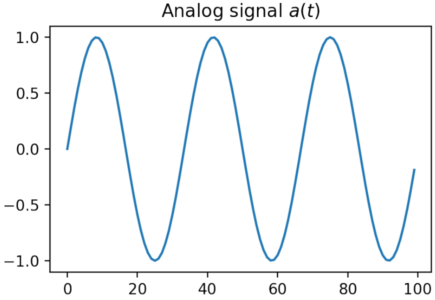
- Glossary:
  - “e.g.” = “*exampli gratia*” (lat.) = “for example” (eng.) = “de exemplu” (rom.)
  - “i.e.” = “*id est*” (lat) = “that is” (eng.) = “adică” (rom.)

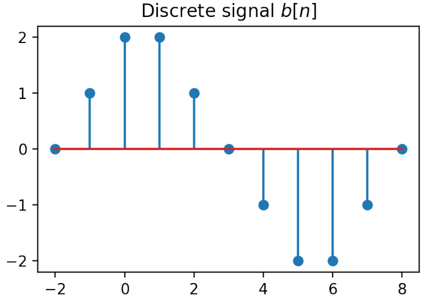
Signal dimension

- **Unidimensional** (1D) signal = a function of a single variable
  - Example: a voltage signal $v(t)$ only varies in time.
- **Multidimensional** (2D, 3D ... M-D) signal = a function of a multiple variables
  - Example: intensity of a grayscale image $I(x, y)$ across the surface of a photo
- In these lectures we consider only 1D signals, but the theory is similar

Continuous and discrete signals

- Continuous (analog) signal = function of a continuous variable
  - Signal has a value for possible value of the variable in the defined range
  - The variable may be defined only in a certain range (e.g. $t \in [0, 100]$), but it is a compact range
- Discrete signal = function of a discrete variable
  - Signal has values only at certain discrete values (*samples*)
  - Indexed with natural numbers: $x[-1]$, $x[0]$, $x[1]$ etc.
  - Outside the samples, the signal is **not defined**





Notation

- We use the following notation throughout these lectures
- Continuous signal
  - Has **round parantheses**, e.g. $x_a(t)$
  - Sometimes has the a subscript
  - The variable is usually t (time)
  - $x(2.3) =$ the value of the signal $a(t)$ at $t = 2.3$
- Discrete signal
  - Has **square brackets**, e.g. $x[n]$
  - The variables are denoted as n or k (suggest natural numbers)
  - $x[3] =$ the value of the signal $x[n]$ for $n = 3$
  - $x[1.5] =$ does not exist

Signals with continuous and discrete values

- The signal values can be continuous or discrete
  - Example: signal values stored as 8-bit or 16-bit values
- On digital systems, signals always have discrete values due to finite number precision

Discrete frequency

- A signal is **periodic** if the values repeat themselves after a certain time (**period**)
- Frequency = inverse of period
- Pulsation $\omega = 2 * \pi *$ frequency
- Continuous signals:
  - Periodic: $x_a(t) = x(t + T)$
  - T is usually measured in seconds (or some other unit)
  - $F = \frac{1}{T}$ is measured in Hz $= \frac{1}{s}$ (Hertz)
- Discrete signals:
  - Periodic: $x[n] = x[n + N]$
  - N **has no unit**, because it is just a number
  - $f = \frac{1}{N}$ **has no unit** also

Domain of existence of frequency

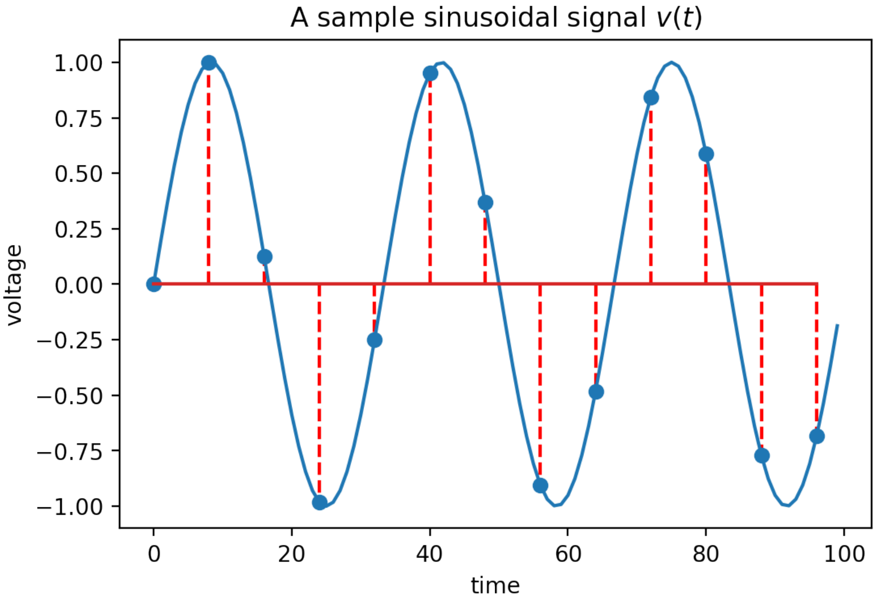
- Continuous signals
  - Period T can be as small as possible $T \to 0$
  - Therefore F could go up to ∞
- Discrete signals
  - Smallest period is $N = 2$ (excluding $N = 1$, constant signals)
  - Largest possible frequency is $f_{max} = \frac{1}{2}$
  - Consequence of using natural numbers to index the samples (x[0], x[1], x[2]...), without any physical unit attached
- For mathematical reasons: we will consider negative frequencies as well (remember SCS)
  - they mirror the positive frequencies.

I.2. Sampling

Sampling

- Taking the values from an analog signal at certain discrete moments of time, usually periodic
- Distance between two samples = **sampling period** T_s
- **Sampling frequency** $F_s = \frac{1}{T_s}$
- Why sampling?
  - Converts continuous signals to discrete
  - Processing of continuous signals is expensive
  - Processing of discrete signals is cheap (digital devices)
  - Sometimes nothing is lost due to sampling

Graphical example



Sampling equation

- Sampling of the continuous signal x_a:

$$x[n] = x_a(n \cdot T_s)$$

- The n-th value of the discrete signal $x[n]$ is the value of the analog signal $x_a(t)$ taken after n sampling periods, at $t = n \cdot T_s$

Sampling of harmonic signals

- Let's sample a cosine: $x_a(t) = cos(2\pi F t)$

$$
\begin{aligned}
x[n] =& x_a(nT_s) \\
=& cos(2\pi F n T_s) \\
=& cos(2\pi F n \frac{1}{F_s}) \\
=& cos(2\pi \underbrace{\frac{F}{F_s}}_{f} n)
\end{aligned}
$$

- Sampling a continuous cosine (or sine) produces a discrete cosine (or sine)
- The discrete frequency is $f = \frac{F}{F_s}$

Discrete frequency is relative

$$f = \frac{F}{F_s}$$

- Discrete frequency should be understood as **relative to the sampling frequency**
  - $f = \frac{1}{4}$ means "coming from an analog frequency F which was $\frac{1}{4}$ of the sampling frequency"
  - $f = 0.1$ means "one tenth of the sampling frequency", and so on

False friends

- **Note:** A discrete sinusoidal signal might not *look* sinosoidal, when its frequency is high (close to $\frac{1}{2}$).

```
  File "<ipython-input-1-f8278cf1aeb4>", line 11
    plt.stem(t,x2), use_line_collection=True; plt.title ('Fre
= 0.25$')
^
SyntaxError: can't assign to function call
```

Sampling theorem (Nyquist-Shannon)

- If a signal that has maximum frequency F_{max} is sampled with a a sampling frequency

$$F_s \geq 2F_{max},$$

then it can be perfectly reconstructed from its samples using the formula:

$$x_a(t) = \sum_{n=-\infty}^{+\infty} x[n] \cdot \frac{sin(\pi(F_s t - n))}{\pi(F_s t - n)}.$$

Comments on the sampling theorem

- All the information in the original signal is contained in the samples, provided the sampling frequency is high enough
- We can process discrete samples instead of the original analog signals
- Sampling with $F_s \geq 2F_{max}$ makes the discrete frequency smaller than $1/2$
$$f = \frac{F}{F_s} \leq \frac{F_{max}}{F_s} \leq \frac{1}{2}$$

Aliasing

- http://www.dictionary.com/browse/alias:
  - "alias": a false name used to conceal one's identity; an assumed name
- What happens when the sampling frequency is not high enough?
- Every discrete frequency f outside the interval $f \notin [-\frac{1}{2}, \frac{1}{2}]$ is **identical** (an "alias") with a frequency from this interval $f_{alias} \in [-\frac{1}{2}, \frac{1}{2}]$
- Proof: at blackboard
  - Consider $x[n] = cos(2\pi fn)$, $f \notin [-\frac{1}{2}, \frac{1}{2}]$
  - We can always add/subtract $2\pi n$ since $cos()$ is periodical, with no change
  - This means increasing/reducing f by 1
  - Thus we can always end up a frequency $f_{alias} \in [-1/2, 1/2]$ (up to a sign change)

Aliasing

- Frequency **folding**: if f exceeds the limit $\frac{1}{2}$ with ϵ, it aliases a frequency below $\frac{1}{2}$ with ϵ, symmetrically

$$cos(2\pi(\frac{1}{2}+\epsilon)n) = cos(2\pi(\frac{1}{2}-\epsilon)n)$$

$$sin(2\pi(\frac{1}{2}+\epsilon)n) = -sin(2\pi(\frac{1}{2}-\epsilon)n)$$

Aliasing

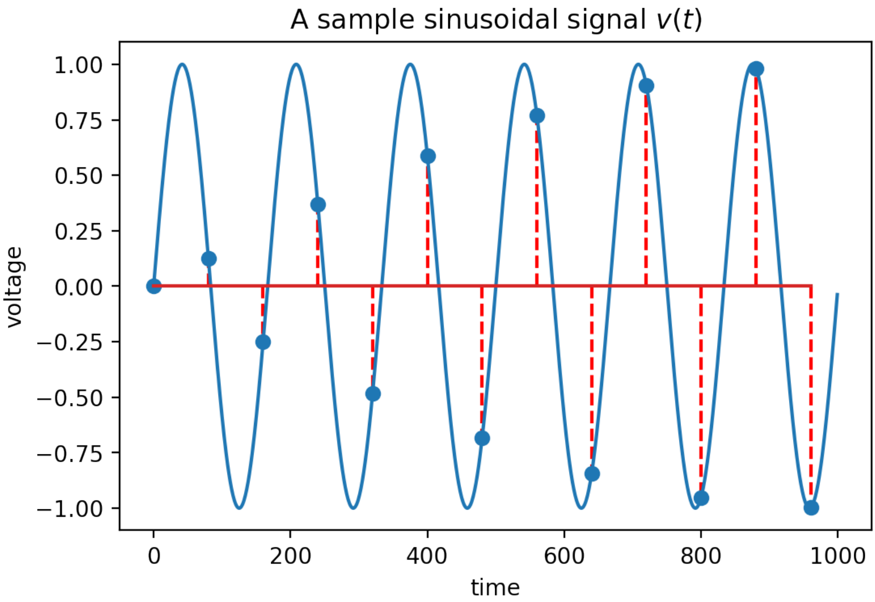
Equivalently:

- If an analog frequency F exceeds the limit $\frac{F_s}{2}$ with some ϵ, it produces the same samples (alias) as another analog frequency smaller than $\frac{F_s}{2}$ with ϵ, symmetrically
- Example: $F = 1000Hz$ sampled with $F_s = 1600Hz$ aliases $F_{alias} = 600Hz$.

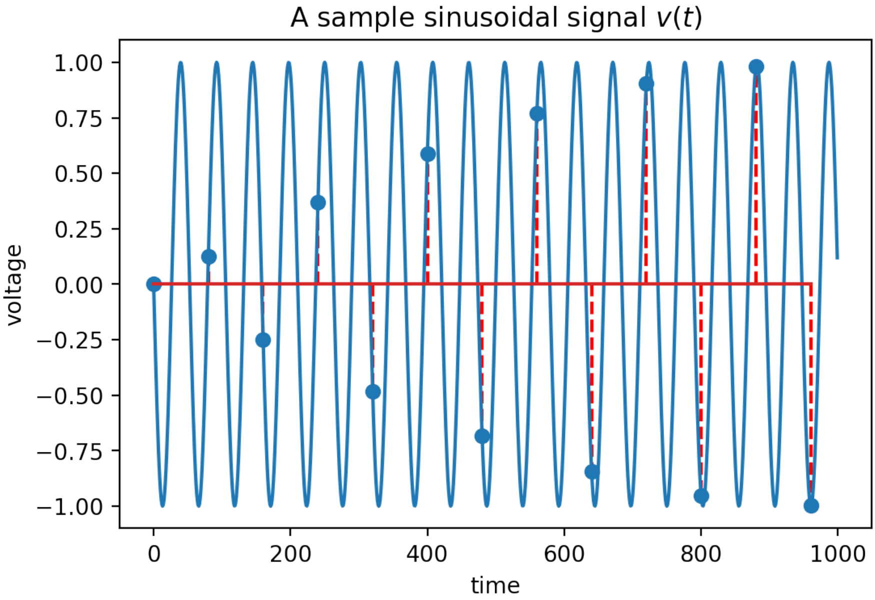
Aliasing

- Aliasing only affects digital signals, caused by sampling
- Sampling with $F_s \geq 2F_{max}$ ensures $f \leq \frac{1}{2}$, so no aliasing

Aliasing example - low frequency signal



Aliasing example - high frequency signal, same samples



The problem of aliasing

- Sampling different signals leads to exactly same samples
- How to know from what signal did the samples come from? Impossible.
- Better remove from the signal the frequencies larger than $\frac{F_s}{2}$, otherwise they will create a false frequency and bring confusion
- Anti-alias filter: a low-pass filter situated before a sampling circuit, rejecting all frequencies $F > \frac{F_s}{2}$ from the signal before sampling
  - Standard practice in the design of processing systems

Signal reconstruction from samples

- A discrete frequency $f \in [-\frac{1}{2}, \frac{1}{2}]$ will be reconstructed as follows:

$$x_r(t) = x[\frac{t}{T_s}] = x[t \cdot F_s]$$

- For a discrete frequency outside the $[-\frac{1}{2}, \frac{1}{2}]$ interval
  - Reconstruction of the original frequency is impossible
  - The frequency is replaced with the aliased frequency f' from the interval $[-\frac{1}{2}, \frac{1}{2}]$
- Reconstruction always produces signals with frequencies in $[-\frac{Fs}{2}, \frac{Fs}{2}]$
- Only signals sampled according to the sampling theorem will be reconstructed identically

Signal quantization and coding

- In practice, the values of the samples are rounded to fixed levels, e.g. 8-bit, 16-bit values.
- This "rounding" is known as **quantization**
- The "rounding error" is known as **quantization error**
- Converting the value in binary form is known as **coding**

A/D and D/A conversion

- Sampling + quantization + coding is usually done by an **Analog to Digital Converter (ADC)**
  - It takes an analog signal and produces a sequence of binary-coded values
- Reconstructing an analog signal from numeric samples is done by a **Digital to Analog Converter (DAC)**
  - Usually reconstruction is not based on sampling theorem equation, which is too complex, but with simpler empiric approaches.