

# Digital Signal Processing

## Chapter V. Frequency Analysis of Discrete Systems

# Response of LTI systems to harmonic signals

- ▶ We consider an LTI system with  $h[n]$
- ▶ Input signal = complex harmonic (exponential) signal  $x[n] = Ae^{j\omega_0 n}$
- ▶ Output signal = convolution

$$\begin{aligned}y[n] &= \sum_{k=-\infty}^{\infty} h[k]x[n-k] \\&= \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega_0 k} Ae^{j\omega_0 n} \\&= H(\omega_0) \cdot x[n]\end{aligned}$$

- ▶  $H(\omega_0)$  = Fourier transform of  $h[n]$  evaluated for  $\omega = \omega_0$

# Eigen-function

- ▶ Complex exponential signals are **eigen-functions** (functii proprii) of LTI systems:
  - ▶ output signal = input signal  $\times$  a (complex) constant
- ▶  $H(\omega_0)$  is a constant that multiplies the input signal
  - ▶ Amplitude of input gets multiplied by  $|H(\omega_0)|$
  - ▶ Phase of input signal is added with  $\angle H(\omega_0)$
- ▶ Why are sin/cos/exp functions important?
  - ▶ If input signal = sum of complex exponential (= cosines + sines),
  - ▶ since the system is linear,
  - ▶ then output = same sum of complex exponentials, each scaled with some coefficients

# Response to cosine and sine

- ▶ Cosine / sine = sum of two exponentials, via Euler
- ▶ System is linear and real  $\Rightarrow$ 
  - ▶ amplitude is multiplied by  $|H(\omega_0)|$
  - ▶ phase increases by  $\angle H(\omega_0)$
- ▶ See proof at blackboard

# Frequency response

- ▶ Names
  - ▶  $H(\omega)$  = frequency response of the system
  - ▶  $|H(\omega)|$  = amplitude response
  - ▶  $\angle H(\omega)$  = phase response
- ▶ Phase response might have jumps of  $2\pi$
- ▶ Stitching the pieces in a continuous function = phase *unwrapping*
  - ▶ Example: at blackboard
- ▶ Wrapped phase:  $\in [-\pi, \pi]$ , may have jumps of  $2\pi$
- ▶ Unwrapped phase: continuous function, may go outside interval

# Permanent and transient response

- ▶ The above harmonic signals start at  $n = -\infty$ , not at 0.
- ▶ What if the signal starts at some time  $n = 0$ ?
- ▶ Total response = transient response + permanent response
  - ▶ transient response goes towards 0 as  $n \rightarrow \infty$
  - ▶ permanent response = the above
- ▶ So the above relations are valid only in **permanent regime**
  - ▶ i.e. after the transient regime has passed
  - ▶ i.e. after the transient response has practically vanished
  - ▶ i.e. when the signal started very long ago (from  $n = -\infty$ )
- ▶ Example at blackboard

# Permanent response of LTI systems to periodic inputs

- ▶ Assume the input  $x[n]$  is periodic with period  $N$
- ▶ Then it can be represented as a Fourier series:

$$x[n] = \sum_{k=0}^{N-1} c_k e^{j2\pi kn/N}$$

- ▶ Since the system is linear, the output to each component  $k$  is

$$c_k H\left(\frac{2\pi}{N}k\right) e^{j2\pi kn/N}$$

- ▶ So the total output is:

$$y[n] = \sum_{k=0}^{N-1} c_k H\left(\frac{2\pi}{N}k\right) e^{j2\pi kn/N}$$

- ▶ The output is still periodic, same period, same frequencies



# Response of LTI systems to non-periodic signals

- ▶ The output = input convolution with impulse response

$$y[n] = x[n] * h[n]$$

$$Y(\omega) = X(\omega) \cdot H(\omega)$$

- ▶ In polar form:
  - ▶ modulus get multiplied
  - ▶ phases get added
- ▶ The system **attenuates/amplifies** the input frequencies and **changes their phases**
- ▶  $H(\omega)$  = the **transfer function**
- ▶  $H(z)$  = the **system function**
- ▶  $H(\omega) = H(z = e^{j\omega})$  if unit circle is in CR

# Power spectral density

- ▶  $S_{zz}(\omega) = |Y(\omega)|^2 = |H(\omega)|^2 \cdot S_{xx}(\Omega)$
- ▶ The poles and zeros of  $S(\omega)$  come in pairs  $(z, 1/z$  and  $p, 1/p)$

- ▶ LTI systems are also known as **filters** because their transfer function shapes (*filters*) the frequencies of the input signals
- ▶ The transfer function can be found from  $H(z)$  and  $z = e^{j\omega}$
- ▶ Alternatively, the transfer function can be found by the **geometrical method** based on the locations of poles and zeros

# Ideal filters

- ▶ Ideal low-pass filter: example at whiteboard
- ▶ Ideal band-pass filter: example at whiteboard
- ▶ Ideal high-pass filter: example at whiteboard
- ▶ Ideal band-stop filter: example at whiteboard
- ▶ Ideal all-pass filter (*changes the phase*): idem

# Linear-phase filters

- ▶ Consider a constant filter with **linear phase** function:

$$H(\omega) = C \cdot e^{-j\omega n_0}$$

- ▶ The output signal is

$$Y(\omega) = X(\omega) \cdot C \cdot e^{-j\omega n_0}$$

$$y[n] = C \cdot x[n - n_0]$$

- ▶ Linear phase means **just a delaying** of the input signal
  - ▶ Fourier property:  $x[n - n_0] \longleftrightarrow X(\omega)e^{-j\omega n_0}$

# Group delay

- ▶ = The time delay experienced by a component of frequency  $\omega$  when passing through the filter
- ▶ **Group delay** of the filter:

$$\tau_g(\omega) = \frac{d\Theta(\omega)}{d\omega}$$

- ▶ Linear phase = constant group delay = all frequencies delayed the same = whole signal delayed

# Filter distortions

- ▶ When a filter is non-ideal:
  - ▶ non-constant amplitude  $\rightarrow$  amplitude distortions
  - ▶ non-linear phase  $\rightarrow$  phase distortions
- ▶ Phase distortions may be tolerated by certain applications
  - ▶ e.g. human ears are insensitive to phase distortions of sounds

# Filter order

- ▶ The **order** of a filter = maximum degree in numerator or denominator of  $H(z)$ 
  - ▶ i.e. largest power of  $z$
- ▶ Any filter can be implemented, in general, with this number of unit delay blocks ( $z^{-1}$ )
- ▶ Higher order  $\rightarrow$  better filter transfer function
  - ▶ closer to ideal filter
  - ▶ more complex to implement
  - ▶ more delays (bad)
- ▶ Lower order
  - ▶ worse transfer function (not close to ideal)
  - ▶ simpler, cheaper
  - ▶ faster response



# Filter design by pole and zero placements

- ▶ Based on geometric method
- ▶ The gain coefficient must be found by separate condition
  - ▶ i.e. specify the desired magnitude response at one frequency
- ▶ Examples at blackboard

# Zero-phase transfer function

- ▶ Normally,  $|H(\omega)|$  is strictly positive

$$|H(\omega)| \geq 0$$

- ▶ When  $H(\omega)$  the function passes through 0:

- ▶  $|H(\omega)|$  remains positive
- ▶  $\angle H(\omega)$  has a jump of  $\pi$

- ▶ **Zero-phase transfer function**

- ▶  $H_R(\omega) = \pm |H(\omega)|$ , including the sign (can be positive or negative)
- ▶  $\Theta_R(\omega)$  doesn't have anymore the jumps of  $\pi$

$$H(\omega) = H_R(\omega)e^{j\Theta_R(\omega)}$$

- ▶ Everything else still applies

- ▶  $H_R(\omega)$  is even
- ▶  $\Theta_R(\omega)$  is odd

2018-2019 Exam

- ▶ Skip next 3 slides (up to “Particular classes of filters”)

# Linear-phase FIR filters

- ▶ Only FIR filters can have linear phase!
- ▶ IIR filters cannot have linear phase (no proof)

# Symmetry conditions for linear-phase FIR

- ▶ Let filter order be  $M$
- ▶ The filter coefficients are  $h[0], \dots, h[M-1]$
- ▶ Linear-phase is guaranteed in two cases
- ▶ **Positive symmetry**

$$h[n] = h[M-1-n]$$

- ▶ **Negative symmetry (anti-symmetry)**

$$h[n] = -h[M-1-n]$$

# Cases of linear-phase FIR

- ▶ Proofs at blackboard
- 1. Positive symmetry,  $M = \text{odd}$
- 2. Positive symmetry,  $M = \text{even}$
- 3. Negative symmetry,  $M = \text{odd}$
- 4. Negative symmetry,  $M = \text{even}$
- ▶ Check constraints for  $H(0)$  and  $H(\pi)$
- ▶ For what types of filters is each case appropriate?

# Particular classes of filters

- ▶ Digital resonators
  - ▶ = very selective band pass filters
  - ▶ poles very close to unit circle
  - ▶ may have zeros in 0 or at  $1/-1$
- ▶ Notch filters
  - ▶ have zeros exactly on unit circle
  - ▶ will completely reject certain frequencies
  - ▶ has additional poles to make the rejection band very narrow
- ▶ Comb filters
  - ▶ = periodic notch filters

# Digital oscillators

- ▶ Oscillator = a system which produces an output signal even in absence of input
- ▶ Has a pair of complex conjugate poles **exactly on unit circle**
- ▶ Example at blackboard



# Inverse filters

- ▶ Sometimes is necessary to **undo** a filtering
  - ▶ e.g. undo attenuation of a signal passed through a channel
- ▶ Inverse filter: has inverse system function:

$$H_I(z) = \frac{1}{H(z)}$$

- ▶ Problem: if  $H(z)$  has zeros outside unit circle,  $H_I(z)$  has poles outside unit circle  $\rightarrow$  unstable
- ▶ Examples at blackboard