Digital Signal Processing

Chapter V. Frequency Analysis of Discrete Systems

Response of LTI systems to harmonic signals

- We consider an LTI system with $h[n]$
- Input signal = complex harmonic (exponential) signal $x[n] = Ae^{j\omega_0 n}$
- Output signal = convolution

$$
\begin{aligned}
y[n] &= \sum_{k=-\infty}^{\infty} h[k]x[n-k] \\
&= \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega_0 k} Ae^{j\omega_0 n} \\
&= H(\omega_0) \cdot x[n]
\end{aligned}
$$

- $H(\omega_0)$ = Fourier transform of $h[[n]$ evaluated for $\omega = \omega_0$

Eigen-function

- Complex exponential signals are **eigen-functions** (functii proprii) of LTI systems:
  - output signal $=$ input signal $\times$ a (complex) constant
- $H(\omega_0)$ is a constant that multiplies the input signal
  - Amplitude of input gets multiplies by $|H(\omega_0)|$
  - Phase of input signal is added with $\angle H(\omega_0)$
- Why are sin/cos/exp functions important?
  - If input signal $=$ sum of complex exponential ($=$ coses $+$ sinuses),
  - since the system is linear,
  - then output $=$ same sum of complex exponentials, each scaled with some coefficients

Response to cosine and sine

- Cosine / sine = sum of two exponentials, via Euler
- System is linear and real =>
  - amplitude is multiplied by $|H(\omega_0)|$
  - phase increases by $\angle H(\omega_0)$
- See proof at blackboard

Frequency response

- Names
  - $H(\omega)$ = frequency response of the system
  - $|H(\omega)|$ = amplitude response
  - $\angle H(\omega)$ = phase response
- Phase response might have jumps of 2π
- Stitching the pieces in a continuous function = phase *unwrapping*
  - Example: at blackboard
- Wrapped phase: $\in [-\pi, \pi]$, may have jumps of 2π
- Unwrapped phase: continuous function, may go outside interval

Permanent and transient response

- The above harmonic signals start at $n = -\infty$, not at 0.
- What if the signal starts at some time $n = 0$?
- Total response = transient response + permanent response
  - transient response goes towards 0 as $n \to \infty$
  - permanent response = the above
- So the above relations are valid only in **permanent regime**
  - i.e. after the transient regime has passed
  - i.e. after the transient response has practically vanished
  - i.e. when the signal started very long ago (from $n = -\infty$)
- Example at blackboard

Permanent response of LTI systems to periodic inputs

- Assume the input $x[n]$ is periodic with period N
- Then it can be represented as a Fourier series:

$$x[n] = \sum_{k=0}^{N-1} c_k e^{j2\pi kn/N}$$

- Since the system is linear, the output to each component k is

$$c_k H\left(\frac{2\pi}{N}k\right) e^{j2\pi kn/N}$$

- So the total output is:

$$x[n] = \sum_{k=0}^{N-1} c_k H\left(\frac{2\pi}{N}k\right) e^{j2\pi kn/N}$$

- The output is still periodic, same period, same frequencies

Response of LTI systems to non-periodic signals

- The output = input convolution with impulse response

$$y[n] = x[n] * h[n]$$

$$Y(\omega) = X(\omega) \cdot H(\omega)$$

- In polar form:
  - modulus get multiplied
  - phases get added
- The system **attenuates/amplifies** the input frequencies and **changes their phases**
- $H(\omega)$ = the **transfer function**
- $H(z)$ = the **system function**
- $H(\omega) = H(z = e^{j\omega})$ if unit circle is in CR

Power spectral density

- $S_{zz}(\omega) = |Y(\omega)|^2 = |H(\omega)|^2 \cdot S_{xx}(\Omega)$
- The poles and zeros of $S(\omega)$ come in pairs ($z, 1/z$ and $p, 1/p$)

Digital filters

- LTI systems are also known as **filters** because their transfer function shapes (*filters*) the frequencies of the input signals
- The transfer function can be found from $H(z)$ and $z = e^{j\omega}$
- Alternatively, the transfer function can be found by the **geometrical method** based on the locations of poles and zeros

Ideal filters

- Ideal low-pass filter: example at whiteboard
- Ideal band-pass filter: example at whiteboard
- Ideal high- pass filter: example at whiteboard
- Ideal band-stop filter: example at whiteboard
- Ideal all-pass filter (*changes the phase*): idem

Linear-phase filters

- Consider a constant filter with **linear phase** function:

$$H(\omega) = C \cdot e^{-j\omega n_0}$$

- The output signal is

$$Y(\omega) = X(\omega) \cdot C \cdot e^{-j\omega n_0}$$

$$y[n] = C \cdot x[n - n_0]$$

- Linear phase means **just a delaying** of the input signal
  - Fourier property: $x[n - n_0] < -- > X(\omega) e^{-j\omega n_0}$

Group delay

- = The time delay experienced by a component of frequency ω when passing through the filter
- **Group delay** of the filter:

$$\tau_g(\omega) = \frac{d\Theta(\omega)}{d\omega}$$

- Linear phase = constant group delay = all frequencies delayed the same = whole signal delayed

Filter distortions

- When a filter is non-ideal:
  - non-constant amplitude –> amplitude distortions
  - non-linear phase –> phase distortions
- Phase distortions may be tolerated by certain applications
  - e.g. human ears are insensitive to phase distortions of sounds

Filter order

- The **order** of a filter = maximum degree in numerator or denominator of $H(z)$
  - i.e. largest power of z
- Any filter can be implemented, in general, with this number of unit delay blocks (z^{-1})
- Higher order -> better filter transfer function
  - closer to ideal filter
  - more complex to implement
  - more delays (bad)
- Lower order
  - worse transfer function (not close to ideal)
  - simpler, cheaper
  - faster response

Filter design by pole and zero placements

- Based on geometric method
- The gain coefficient must be found by separate condition
  - i.e. specify the desired magnitude response at one frequency
- Examples at blackboard

Zero-phase transfer function

- Normally, $|H(\omega|)$ is strictly positive

$$|H(\omega)| \geq 0$$

- When $H(\omega)$ the function passes through 0:
  - $|H(\omega)|$ remains positive
  - $\angle H(\omega)$ has a jump of π
- **Zero-phase transfer function**
  - $H_R(\omega) = \pm|H(\omega)|$, including the sign (can be positive or negative)
  - $\Theta_R(\omega)$ doesn't have anymore the jumps of π

$$H(\omega) = H_R(\omega) e^{j\Theta_R(\omega)}$$

- Everything else still applies
  - $H_R(\omega)$ is even
  - $\Theta_R(\omega)$ is odd

Linear-phase FIR filters

- Only FIR filters can have linear phase!
- IIR filters cannot have linear phase (no proof)

Symmetry conditions for linear-phase FIR

- Let filter order be M
- The filter coefficients are $h[0], \ldots \; h[M-1]$
- Linear-phase is guaranteed in two cases
- **Positive symmetry**

$$h[n] = h[M-1-n]$$

- **Negative symmetry (anti-symmetry)**

$$h[n] = -h[M-1-n]$$

Cases of linear-phase FIR

- Proofs at blackboard

1. Positive symmetry, M = odd
2. Positive symmetry, M = even
3. Negative symmetry, M = odd
4. Negative symmetry, M = even

- Check constraints for $H(0)$ and $H(\pi)$
- For what types of filters is each case appropriate?

Particular classes of filters

- Digital resonators
  - = very selective band pass filters
  - poles very close to unit circle
  - may have zeros in 0 or at 1/-1
- Notch filters
  - have zeros exactly on unit circle
  - will completely reject certain frequencies
  - has additional poles to make the rejection band very narrow
- Comb filters
  - = periodic notch filters

Digital oscillators

- Oscillator = a system which produces an output signal even in absence of input
- Has a pair of complex conjugate poles **exactly on unit circle**
- Example at blackboard

Inverse filters

- Sometimes is necessary to **undo** a filtering
  - undo attenuation of a signal passed through a channel
- Inverse filter: has inverse system function:

$$H_I(z) = \frac{1}{H(z)}$$

- Problem: if $H(z)$ has zeros outside unit circle, $H_I(z)$ has poles outside unit circle –> unstable
- Examples at blackboard

Minimum and maximum phase filters

- Start with example
- If replace a zero z with $1/z$, the modulus does not change!
  - only phase is affected
- Minimum phase: all zeros are inside the unit circle
  - minimum phase = minimum derivative = minimum delay
  - inverse filter is stable
- Maximum phase: all zeros are outside unit circle
- Mixed phase: some zeros inside, some outside