

Digital Signal Processing

Chapter III: The Z Transform

III.1 Introducing the Z transform

Preliminaries: complex numbers

- ▶ real and imaginary part
- ▶ **modulus and phase**
- ▶ graphical interpretation
- ▶ Euler formula
- ▶ modulus and phase of e^{jx}

Definition of Z transform

- ▶ The Z Transform of a signal $x[n]$, called $X(z)$, is defined as:

$$X(z) = \sum_{-\infty}^{\infty} x[n]z^{-n}$$

- ▶ It is defined only for the values of z where the sum is finite (called *region of convergence*)
- ▶ Notation:

$$\mathcal{Z}(x[n]) = X(Z)$$

$$x[n] \xleftrightarrow{Z} X(Z)$$

Definition of Z transform

- ▶ Similar to the Laplace transform for analog signals
- ▶ The Z transform associates a **polynomial** to a signal (think Decision and Estimation class)
- ▶ Why?
 - ▶ Convolution of two signals = multiplication of polynomials
 - ▶ Short descriptions of complicated signals (i.e. exponential signals)

Examples

$x[n] = 1, 2, 5, 7, 0$, (with time origin in 1 or in 5)

$\delta[n], \delta[n - k], \delta[n + k]$

$\left(\frac{1}{2}\right)^n$

$x[n] = a^n u[n]$

$x[n] = -a^n u[-n - 1]$

Region of convergence

- ▶ For finite-support signals, the CR is the whole Z plane, possibly except 0 or ∞
- ▶ For causal signals, the CR is *the outside of a circle*:

$$|z| > r_1$$

- ▶ For anti-causal signals, the CR is *the inside of a circle*:

$$|z| < r_2$$

- ▶ For bilateral signals, both the causal and the anti-causal terms of the sum must converge \longrightarrow the CR is the area between two circles:

$$r_1 < |z| < r_2$$

Region of convergence

- ▶ For finite-support signals, the two “circles” are 0 and ∞
- ▶ Two different signals can have the same expression of $X(z)$, but with different RC!
 - ▶ RC is an essential part in specifying a Z transform
 - ▶ should never be omitted

The Inverse Z Transform

- ▶ From a purely mathematical point of view, $X(z)$ is a complex function
- ▶ Proper definition of inverse transform is based on the theory of complex functions

$$X(z) = \sum_{-\infty}^{\infty} x[k]z^{-k}$$

- ▶ Multiply with z^{n-1} and integrate along a contour C inside the Convergence Region:

$$\oint_C X(z)z^{n-1}dz = \oint_C \sum_{-\infty}^{\infty} x[k]z^{n-k-1}dz = \sum_{-\infty}^{\infty} x[k] \oint_C z^{n-k-1}dz$$

The Inverse Z Transform

- ▶ The Cauchy integral theorem says that:

$$\frac{1}{2\pi j} \oint_C z^{n-k-1} dz = \begin{cases} 1, & \text{if } k = n \\ 0, & \text{if } k \neq n \end{cases}$$

- ▶ And therefore:

$$x[n] = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz$$

- ▶ We will not use this relation in practice, but instead will rely on **partial fraction decomposition**

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- ▶ Properties of Z Transform: only 1, 2, and 9

Properties of Z transform

1. Linearity

If $x_1[n] \xleftrightarrow{Z} X_1(z)$ with CR1, and $x_2[n] \xleftrightarrow{Z} X_2(z)$ with CR2, then:

$$ax_1[n] + bx_2[n] \xleftrightarrow{Z} aX_1(z) + bX_2(z)$$

and CR is at least the intersection of CR1 and CR2.

Proof: use definition

Properties of Z transform

2. Shifting in time

If $x[n] \xleftrightarrow{Z} X(z)$ with CR, then:

$$x[n - k] \xleftrightarrow{Z} z^{-k} X(z)$$

with same RC, possibly except 0 and ∞ .

Proof: by definition

- ▶ valid for all k , also for $k < 0$
- ▶ delay of 1 sample $= z^{-1}$

Properties of Z transform

3. Modulation in time

If $x[n] \xleftrightarrow{Z} X(z)$ with CR, then:

$$e^{j\omega_0 n} x[n] \xleftrightarrow{Z} X(e^{-j\omega_0} z)$$

with same CR

Proof: by definition

Properties of Z transform

4. Reflected signal

If $x[n] \xrightarrow{Z} X(z)$ with CR $r_1 < |z| < r_2$, then:

$$x[-n] \xrightarrow{Z} X(z^{-1})$$

with CR $\frac{1}{r_2} < |z| < \frac{1}{r_1}$

Proof: by definition

Properties of Z transform

5. Derivative of Z transform

If $x[n] \xleftrightarrow{Z} X(z)$ with CR, then:

$$nx[n] \xleftrightarrow{Z} -z \frac{dX(z)}{dz}$$

with same CR

Proof: by derivating the difference

Properties of Z transform

6. Transform of difference

If $x[n] \xleftrightarrow{Z} X(z)$ with CR, then:

$$x[n] - x[n-1] \xleftrightarrow{Z} (1 - z^{-1})X(z)$$

with same CR except $z = 0$.

Proof: using linearity and time-shift property

Properties of Z transform

7. Accumulation in time

If $x[n] \xleftrightarrow{Z} X(z)$ with CR, then:

$$y[n] = \sum_{k=-\infty}^n x[k] \xleftrightarrow{Z} \frac{X(z)}{(1 - z^{-1})}$$

with same CR except $z = 1$.

Proof: $x[n] = y[n] - y[n - 1]$, apply previous property

Properties of Z transform

8. Complex conjugation

If $x[n] \xleftrightarrow{Z} X(z)$ with CR, and $x[n]$ is a complex signal, then:

$$x^*[n] \xleftrightarrow{Z} X^*(z^*)$$

with same CR except $z = 0$.

Proof: apply definition

Consequence

If $x[n]$ is a real signal, the poles / zeroes are either real or in complex pairs

Properties of Z transform

9. Convolution in time

If $x_1[n] \xleftrightarrow{Z} X_1(z)$ with CR1, and $x_2[n] \xleftrightarrow{Z} X_2(z)$ with CR2, then:

$$x[n] = x_1[n] * x_2[n] \xleftrightarrow{Z} X(z) = X_1(z) \cdot X_2(z)$$

and CR the intersection of CR1 and CR2.

Proof: use definition

- ▶ **Very important property!**
- ▶ Can compute the convolution of two signals via the Z transform

Properties of Z transform

10. Correlation in time

If $x_1[n] \xleftrightarrow{Z} X_1(z)$ with CR1, and $x_2[n] \xleftrightarrow{Z} X_2(z)$ with CR2, then:

$$r_{x_1 x_2}[l] = \sum_{n=-\infty}^{\infty} x_1[n] x_2[n-l] \xleftrightarrow{Z} R_{x_1 x_2}(z) = X_1(z) \cdot X_2(z^{-1})$$

and CR the intersection of CR1 and with the CR of $X_2(z^{-1})$ (see reflection property)

Proof: correlation = convolution with second signal reflected, use convolution and reflection properties

Properties of Z transform

11. Initial value theorem

If $x[n]$ is a causal signal, then

$$x[0] = \lim_{z \rightarrow \infty} X(z)$$

Proof:

$$X(z) = \sum_{n=0}^{\infty} x[n]z^{-n} = x[0] + x[1]z^{-1} + x[2]z^{-2} + \dots$$

When $z \rightarrow \infty$, all terms z^{-k} vanish.

Common Z transform pairs

- Easily found all over the Internet

Sequence	Transform	ROC
$\delta[n]$	1	All z
$u[n]$	$\frac{1}{1-z^{-1}}$	$ z > 1$
$-u[-n-1]$	$\frac{1}{1-z^{-1}}$	$ z < 1$
$\delta[n-m]$	z^{-m}	All z except 0 or ∞
$a^n u[n]$	$\frac{1}{1-az^{-1}}$	$ z > a $
$-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$ z < a $
$na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$ z > a $
$-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$ z < a $
$\begin{cases} a^n & 0 \leq n \leq N-1, \\ 0 & \text{otherwise} \end{cases}$	$\frac{1-a^N z^{-N}}{1-az^{-1}}$	$ z > 0$
$\cos(\omega_0 n) u[n]$	$\frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}}$	$ z > 1$
$r^n \cos(\omega_0 n) u[n]$	$\frac{1-r\cos(\omega_0)z^{-1}}{1-2r\cos(\omega_0)z^{-1}+r^2 z^{-2}}$	$ z > r$

III.2. Z transforms which are Rational Functions

Rational functions

- ▶ Many Z transforms are in the form of a rational function, i.e. a fraction where
 - ▶ numerator = polynomial in z^{-1} or z
 - ▶ denominator = polynomial in z^{-1} or z

$$X(z) = \frac{B(z)}{A(z)} = \frac{\sum_{k=0}^M b_k z^{-k}}{\sum_{k=0}^N a_k z^{-k}} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \dots + b_M z^{-M}}{a_0 + a_1 z^{-1} + a_2 z^{-2} + \dots + a_N z^{-N}}$$

- ▶ Example:

$$X(z) = \frac{3 + 2z^{-1} + 4z^{-2}}{1 - 5z^{-2} + 7z^{-4}}$$

Poles and zeros

- ▶ A polynomial is completely determined by its roots and a scaling factor
- ▶ **Definition:** the **zeros** of $X(z)$ are the roots of the numerator $B(z)$
- ▶ **Definition:** the **poles** of $X(z)$ are the roots of the denominator $A(z)$
- ▶ The zeros are usually named z_1, z_2, \dots, z_M , and the poles p_1, p_2, \dots, p_N .

The transform $X(z)$ can be rewritten as:

$$X(z) = \frac{b_0}{a_0} \cdot z^{N-M} \cdot \frac{(z - z_1) \dots (z - z_M)}{(z - p_1) \dots (z - p_N)} = \frac{b_0}{a_0} \cdot \frac{(1 - z_1 z^{-1}) \dots (1 - z_M z^{-1})}{(1 - p_1 z^{-1}) \dots (1 - p_N z^{-1})}$$

It has:

- ▶ M zeros with finite values
- ▶ N poles with finite values
- ▶ and either N-M zeros in 0, if $N > M$, or N-M poles in 0, if $N < M$
(trivial poles/zeros)

Graphical representation

- ▶ The graphical representation of poles and zeros in the complex plane is called **the pole-zero plot**
- ▶ Graphical: poles = "x", zeros = "0"
- ▶ CR cannot contain poles
- ▶ Example: at whiteboard

III.3 Inverse Z transform for rational functions

Methods for computing the Inverse Z Transform

1. Direct evaluation using the Cauchy integral

$$x[n] = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz$$

2. Decomposition as continuous power series
3. **Partial fraction decomposition**

Partial fraction decomposition

Any rational function

$$\frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \dots + b_M z^{-M}}{a_0 + a_1 z^{-1} + a_2 z^{-2} + \dots + a_N z^{-N}}$$

can be decomposed in **partial fractions**:

$$c_0 + c_1 z^{-1} + \dots c_{N-M} z^{-(M-N)} + \frac{A_1}{z - p_1} + \dots \frac{A_N}{z - p_N}$$

- ▶ Each pole has a corresponding partial fraction
- ▶ First terms appear if $M \leq N$
- ▶ Based on linearity, we invert each term separately (simple)

Procedure for Inverse Z Transform

$$X(z) = \frac{B(z)}{A(z)}$$

1. If $M \geq N$, divide numerator to denominator to obtain the first terms. The remaining fraction is $X_1(z) = \frac{B_1(z)}{A(z)}$, with numerator degree strictly smaller than denominator
2. In the remaining fraction, eliminate the negative powers of z by multiplying with z^N

3. Divide by z ,

$$\frac{X_1(z)}{z} = \frac{B_1(z)}{zA(z)}$$

4. Compute the poles of $\frac{X_1(z)}{z}$ and decompose in partial fractions:

$$\frac{X_1(z)}{z} = \frac{A_1}{z - p_1} + \dots$$

Procedure for Inverse Z Transform

5. Multiply back with z :

$$X_1(z) = A_1 \frac{z}{z - p_1} + \dots$$

6. Convert each term back to the time domain

Computation of partial fractions coefficients

- ▶ If all poles are distinct:

$$A_k = (z - p_k) \frac{X(z)}{z} \Big|_{z=p_k}$$

- ▶ If poles are in complex conjugate pairs
 - ▶ group the two fractions into a single fraction of degree 2
- ▶ If there exist m **multiple poles of same value** (pole order $m > 1$):

$$\frac{A_{1k}}{z - p_k} + \frac{A_{2k}}{(z - p_k)^2} + \dots + \frac{A_{mk}}{(z - p_k)^m}$$

$$A_{ik} = \frac{1}{(m-i)!} \frac{d^{m-i}}{dz^{m-i}} \left[(z - p_k)^m \cdot \frac{X(z)}{z} \right] \Big|_{z=p_k}$$

* example for $m = 2$

Real signals and complex poles/zeros

- ▶ Consequence of the complex-conjugate property of Z transform:
- ▶ A real signal $x[n]$ can have only
 - ▶ real poles or zeroes
 - ▶ complex poles and zeroes in conjugate pairs, which can be grouped into a single term of degree 2, with real coefficients
- ▶ If a Z transform has a complex pole / zero without its conjugate pair, then the corresponding signal $x[n]$ is complex

Position of poles and time behavior

- ▶ When the Z transform is a ratio of polynomials, the signal $x[n]$ is a sum of exponential signals
 - ▶ A rational Z transform $X(z)$ = sum of partial fractions, as we just saw
 - ▶ Each partial fraction (pole) generates an exponential signal
 - ▶ $a^n u[n]$, or
 - ▶ $-a^n u[-n - 1]$
- ▶ For a single partial fraction (one pole only), we will analyze the relation between the position of the pole and the signal in time

Position of poles and time behaviour - 1 pole

- ▶ Consider a Z transform with **1 pole**, analyze the look of the corresponding signal
- ▶ Consider the pole value is a
 - ▶ Consider only real signals $x[n] \rightarrow a$ is real
 - ▶ Consider causal signal $x[n] \rightarrow$ CR is $|z| > |a|$
- ▶ Therefore the Z transform is of the type:

$$X(z) = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}, \text{ CR : } |z| > |a|$$

- ▶ Therefore the signal $x[n]$ is of the type:

$$x[n] = a^n u[n]$$

Position of poles and time behavior - 1 pole

Scenarios for a single real pole in a :

- ▶ Pole inside the unit circle ($|a| < 1$) \rightarrow exponential decreasing signal
- ▶ Pole outside the unit circle ($|a| > 1$) \rightarrow exponential increasing signal
- ▶ Pole exactly on unit circle ($|a| = 1$) \rightarrow not increasing, not decreasing
- ▶ Negative pole ($a < 0$) \rightarrow alternating signal
- ▶ Positive value ($a > 0$) \rightarrow non-alternating signal

Position of poles and time behavior - 1 pole

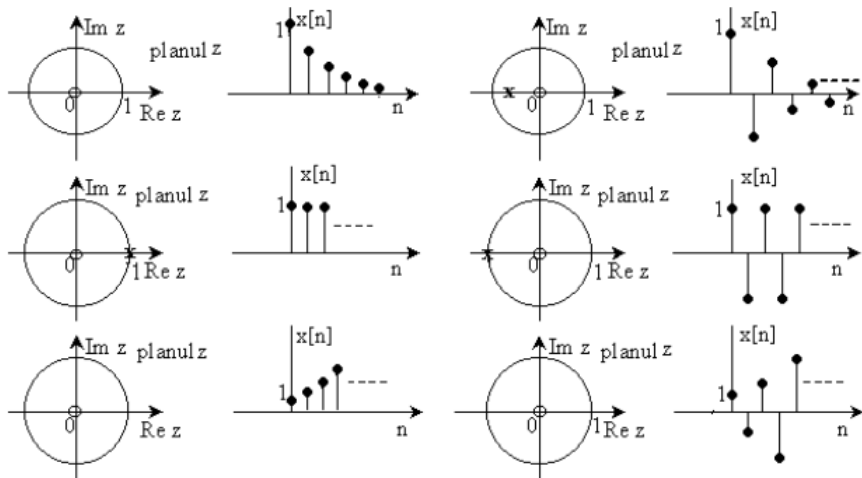


Figure 2: Signal behavior for 1 pole

Position of poles and time behavior - 1 double pole

$$X(z) = \frac{az^{-1}}{(1 - az^{-1})^2}, CR : |z| > |a|$$
$$x[n] = na^n u[n]$$

A double pole in a :

- ▶ Pole inside the unit circle ($|a| < 1$) \rightarrow decreasing signal
- ▶ Pole outside the unit circle ($|a| > 1$) \rightarrow increasing signal
- ▶ Pole exactly on unit circle ($|a| = 1$) \rightarrow **increasing signal**
- ▶ Negative pole ($a < 0$) \rightarrow alternating signal
- ▶ Positive value ($a > 0$) \rightarrow non-alternating signal

Position of poles and time behavior - 1 double pole

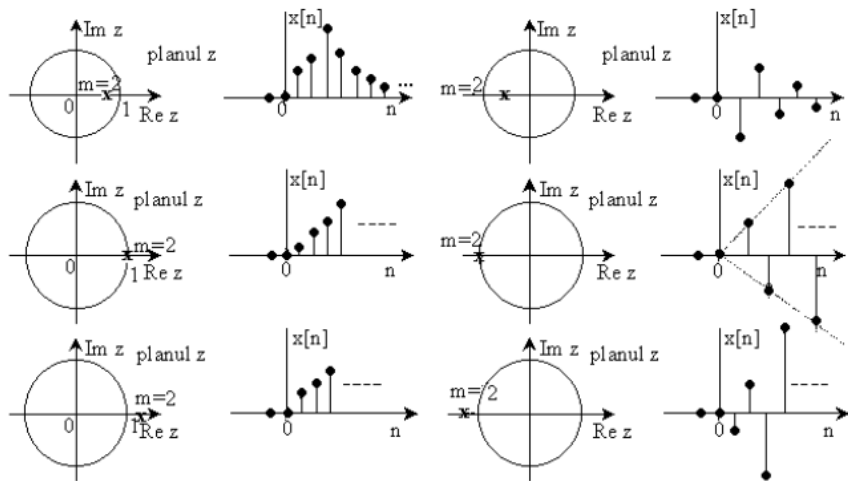


Figure 3: Signal behavior for 1 double pole

Position of poles and time behavior - conjugate poles

$$X(z) = \frac{1 - a \cos \omega_0 z^{-1}}{1 - 2a \cos \omega_0 z^{-1} + a^2 z^{-2}}, CR : |z| > |a|$$
$$x[n] = a^n \cos(\omega_0 n) u[n]$$

A pair of complex conjugate poles:

- ▶ a sinusoidal with exponential envelope
 - ▶ phase of poles \rightarrow frequency of sinusoidal signal
 - ▶ modulus of poles \rightarrow exponential envelope
 - ▶ poles inside unit circle \rightarrow decreasing signal
 - ▶ poles outside unit circle \rightarrow increasing signal
 - ▶ poles on unit circle \rightarrow oscillating signal, constant amplitude

What if poles are double?

- ▶ poles on unit circle \rightarrow increasing signal
- ▶ otherwise, similar to above

Position of poles and time behavior - conjugate poles

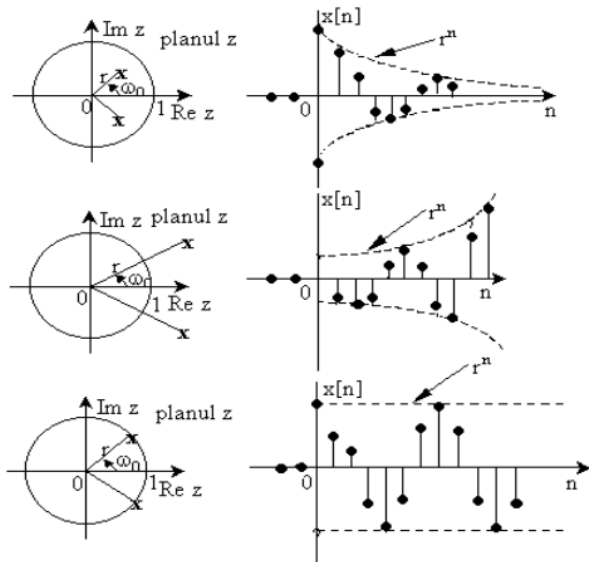


Figure 4: Signal behavior for 1 double pole

Position of poles and time behavior

- ▶ A Z transform can be decomposed into partial fractions, i.e. separate poles
- ▶ Analyzing the individual behavior of poles \rightarrow tells something about whole signal
- ▶ Conclusions (for real signals, causal):
 - ▶ **all poles inside unit circle \rightarrow bounded signal**
 - ▶ *simple* poles on unit circle \rightarrow bounded signal
 - ▶ otherwise \rightarrow unbounded signal
 - ▶ poles inside unit circle, closer to origin \rightarrow fast decrease of signal
 - ▶ poles inside unit circle, closer to unit circle \rightarrow slow decrease of signal

III.4 LTI systems and the Z Transform

System function of a LTI system

- ▶ Considering a LTI system with $h[n]$, input signal $x[n] \rightarrow$ output is convolution

$$y[n] = x[n] * h[n]$$

- ▶ In Z transform, convolution = product of transforms

$$Y(z) = X(z) \cdot H(z)$$

- ▶ **The system function** of a LTI system = the Z transform of the impulse response $h[n]$
- ▶ The system function of a LTI system is:

$$H(z) = \frac{Y(z)}{X(z)}$$

System function and the difference equation

- ▶ Any LTI system is characterized by a **difference equation**:

$$\begin{aligned}y[n] &= - \sum_{k=1}^N a_k y[n-k] + \sum_{k=0}^M b_k x[n-k] \\&= -a_1 y[n-1] - a_2 y[n-2] - \dots - a_N y[n-N] + b_0 x[n] + b_1 x[n-1] - \dots\end{aligned}$$

or

$$y[n] + \sum_{k=1}^n a_k y[n-k] = \sum_{k=0}^m b_k x[n-k]$$

$$y[n] + a_1 y[n-1] + a_2 y[n-2] + \dots + a_N y[n-N] = b_0 x[n] + b_1 x[n-1] + \dots + b_M x[n-M]$$

System function and the difference equation

- The system function $H(z)$ can be derived directly from the difference equation:

$$Y(z) \left(1 + \sum_{k=1}^N a_k z^{-k} \right) = X(z) \left(\sum_{k=0}^M b_k z^{-k} \right)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^M b_k z^{-k}}{1 + \sum_{k=1}^N a_k z^{-k}}$$

Particular cases of system functions

Particular cases of system functions:

► **FIR systems:** $a_k = 0$

► has only zeroes, no poles (all-zero system)

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^M b_k z^{-k}}{1}$$

► Otherwise we have an **IIR system**

► **All-pole system:** $b_k = 0, k \geq 1$ (numerator = a constant = b_0)

► has only poles

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0}{1 + \sum_{k=1}^N a_k z^{-k}}$$

► Or a general IIR system with both poles and zeros

Output of the system, no initial conditions

- ▶ Consider a causal LTI system with initial conditions = 0 (relaxed system)
 - ▶ Remember: I.C. are relevant for IIR implementations ($y[n - k]$ part), not FIR

- ▶ Input signal:

$$x[n] \xleftrightarrow{Z} X(z) = \frac{N(z)}{Q(z)}$$

- ▶ Impulse response / System function:

$$h[n] \xleftrightarrow{Z} H(z) = \frac{B(z)}{A(z)}$$

- ▶ Output signal:

$$y[n] = x[n] * h[n] \xleftrightarrow{Z} Y(z) = X(z)H(z) = \frac{N(z)B(z)}{Q(z)A(z)}$$

- ▶ Some poles and zeros might simplify, if exactly identical

Natural and forced response

- ▶ Assume all poles are *simple* (i.e. no multiplicity)
- ▶ Assume all poles \neq all zeros, so no simplification
- ▶ Call the poles of $X(z)$ q_i and the poles of $H(z)$, p_i
- ▶ Then

$$Y(z) = \sum_{k=1}^N \frac{A_k}{1 - p_k z^{-1}} + \sum_{k=1}^L \frac{Q_k}{1 - q_k z^{-1}}$$

and $y[n]$ is

$$y[n] = \underbrace{\sum_{k=1}^N A_k (p_k)^n u[n]}_{\text{natural response}} + \underbrace{\sum_{k=1}^L Q_k (q_k)^n u[n]}_{\text{forced response}}$$

Natural and forced response

The output $y[n]$ is the sum of two signals:

- ▶ Natural response $y_{nr}[n]$ = the part given by the poles of **the system** itself
- ▶ Forced response $y_{fr}[n]$ = given by the poles of **the input signal**
- ▶ Together they form the **zero-state response** of the system (no initial conditions)

If there are initial conditions, there is a third component:

- ▶ Zero-input response $y_{fr}[n]$ = given by the initial conditions of the system
 - ▶ It behaves similarly to the natural response

Transient and permanent response

- ▶ For a stable system, all system poles $|p_k| < 1$, so natural response (including initial conditions) is made of decreasing exponentials
- ▶ For a stable system, the natural response dies out exponentially
- ▶ Thus, the natural response is called a **transient response**
 - ▶ it fades away
- ▶ Input signals typically last longer, or infinitely \Rightarrow the forced response is a **permanent response**
- ▶ Operating regimes:
 - ▶ when the input signal is first applied, and the transient response is present, the system is in **transient regime**
 - ▶ When the transient response has died out, the system remains in **permanent regime**, where only the input signal determines the output
- ▶ Example: apply a infinitely long sinusoidal, starting from $n = 0$

Stability of a system and $H(z)$

- ▶ Stable system: a bounded input implies a bounded output
- ▶ Reminder: A system is stable if

$$\sum |h[n]| < \infty (\text{convergent})$$

- ▶ For a stable system, with $H(z)$

$$|H(z)| \leq \sum |h[n]| \cdot |z^{-n}| = \sum |h[n]| < \infty$$

considering $|z| = 1$, i.e. **on the unit circle**.

Stability of a system and $H(z)$

- ▶ A LTI system is stable if the **unit circle is inside the Region of Convergence**
 - ▶ one can also prove the reciprocal, so there is equivalence
- ▶ If the system is causal, $\text{RoC} = \text{exterior of a circle given by the largest pole}$,
 - ▶ therefore all poles must be **inside** unit circle
- ▶ A causal LTI system is stable if **all the poles are inside the unit circle**

Stability of a system and $H(z)$

- ▶ Alternative explanation: if one pole is outside unit circle, the term corresponding to its partial fraction will be increasing \rightarrow whole signal is unbounded