Digital Signal Processing

Chapter V. Digital filtering

Response of LTI systems to harmonic signals

$e^{j\omega_0 n}$

- We consider an LTI system with $h[n]$
- Input signal = complex harmonic (exponential) signal $x[n] = Ae^{j\omega_0 n}$
- Output signal = convolution

$x[n-k] = A \cdot e^{j\omega_0 (n-k)} = e^{-j\omega_0 k} \cdot A \cdot e^{j\omega_0 n}$

$$
\begin{aligned}
y[n] &= \sum_{k=-\infty}^{\infty} h[k]x[n-k] \\
&= \sum_{k=-\infty}^{\infty} \underbrace{h[k]e^{-j\omega_0 k}}_{H(\omega_0)} \underbrace{Ae^{j\omega_0 n}}_{x[n]} \\
&= H(\omega_0) \cdot x[n]
\end{aligned}
$$

- $H(\omega_0)$ = Fourier transform of $h[n]$ evaluated for $\omega = \omega_0$

Response of LTI systems to harmonic signals

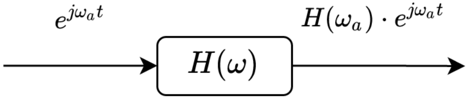


Figure 1: Output = a constant × Input

- $H(\omega)$ = Fourier transform of $h[n]$ evaluated for ω = **transfer function**

DTFT
discrete time
Fourier transform

Eigen-function

- Complex exponential signals are **eigen-functions** (funcții proprii) of LTI systems:
  - output signal $=$ input signal $\times$ a (complex) constant
- $H(\omega_0)$ is a constant that multiplies the input signal
  - Amplitude of input gets multiplies by $|H(\omega_0)|$
  - Phase of input signal is added with $\angle H(\omega_0)$
- Why are sin/cos/exp functions important?
  - If input signal $=$ sum of complex exponential (like coses $+$ sinuses),
  - then output $=$ same sum of complex exponentials, each scaled with some coefficients

$$\underbrace{H(\omega_0)}_{|H(\omega_0)| \cdot e^{j\angle H(\omega_0)}} \cdot e^{j\omega_0 n} = |H(\omega_0)| \cdot e^{j(\omega_0 n + \angle H(\omega_0))}$$

Response to cosine and sine

$$x[n] = A\cdot\cos(\omega_0 n+\varphi) \longrightarrow \boxed{H(\omega)} \longrightarrow y[n] = A\cdot|H(\omega_0)|\cdot\cos(\omega_0 n+\varphi+\angle H(\omega_0))$$

- Cosine / sine = sum of two exponentials, via Euler

$$\cos(\omega n) = \frac{e^{j\omega n} + e^{-j\omega n}}{2}$$

$$\frac{e^{j(\omega n-\pi/2)} + e^{-j(\omega n-\pi/2)}}{2} = \sin(\omega n) = \cos(\omega n - \frac{\pi}{2})$$

- System is linear and real =>
  - amplitude is multiplied by $|H(\omega_0)|$
  - phase increases by $\angle H(\omega_0)$
- See proof at blackboard

$$x[n] = A\cdot\frac{e^{j(\omega_0 n+\varphi)} + e^{-j(\omega_0 n+\varphi)}}{2}$$

$$= \frac{A}{2}\left(e^{j\cdots} + e^{j\cdots}\right)$$

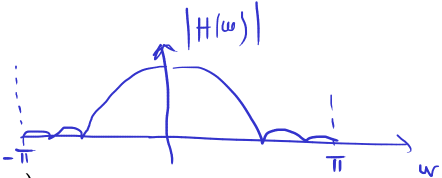
$$y[n] = \frac{A}{2}\left(\underbrace{H(\omega_0)}_{|H(\omega_0)|\cdot e^{j\angle H(\omega_0)}}\cdot e^{j(\omega_0 n+\varphi)} + \underbrace{H(-\omega_0)}_{|H(\omega_0)|\cdot e^{-j\angle H(\omega_0)}}\cdot e^{-j(\omega_0 n+\varphi)}\right)$$

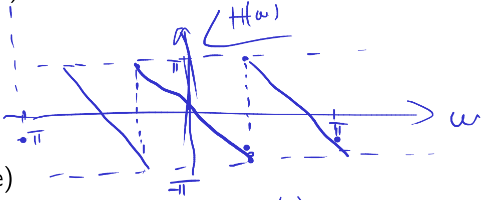
$$= |H(\omega_0)|\cdot A\cdot\frac{e^{j(\omega_0 n+\varphi+\angle H(\omega_0))} + e^{-j(\omega_0 n+\varphi+\angle H(\omega_0))}}{2}$$

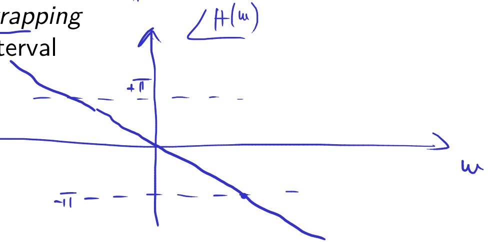
$$=$$

Frequency response

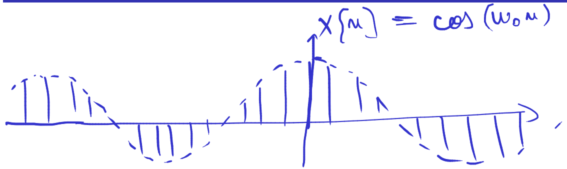
- Naming:
  - $H(\omega)$ = **frequency response** of the system
  - $|H(\omega)|$ = **amplitude response** (or magnitude response)
  - $\angle H(\omega)$ = **phase response**
- Magnitude response is non-negative: $|H(\omega)| \geq 0$
- Phase response is an angle: $\angle H(\omega) \in (-\pi, pi]$
  - Phase response may have jumps of 2π (wrapped phase)
  - Stitching the pieces in a continuous function = phase *unwrapping*
  - Unwrapped phase: continuous function, may go outside interval $(-\pi, pi]$
  - Example: at blackboard

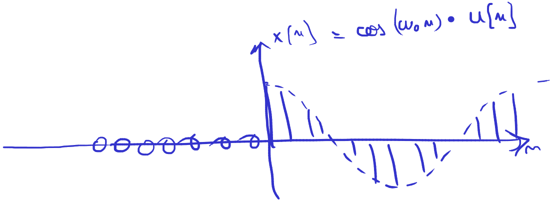




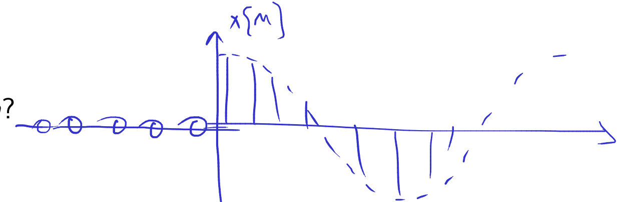


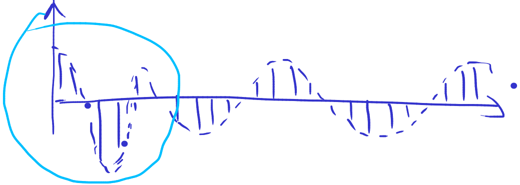
Permanent and transient response





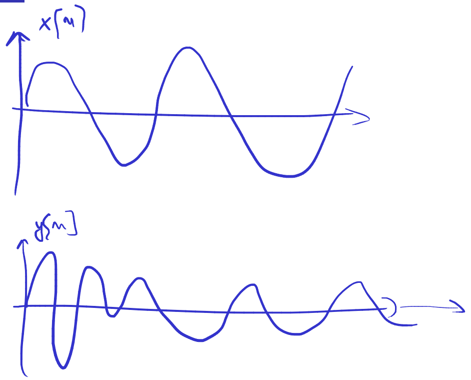
- Warning: $\cos(\omega n)$ does not start at $n = 0$
- The above harmonic signals start at $n = -\infty$.
- What's wrong if the signal starts at some time n?





Permanent and transient response

- What if the signal starts at some time n?
- Total response = transient response + permanent response
  - transient response goes towards 0 as n increases
  - permanent response = what remains
- So the above relations are valid only in **permanent regime**
  - i.e. after the transient regime has passed
  - i.e. after the transient response has practically vanished
  - i.e. when the signal started very long ago (from $n = -\infty$)
  - i.e. when only the permanent response remains in the output signal
- Example at blackboard



Permanent response of LTI systems to periodic inputs

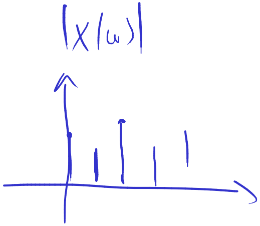
- Consider an input $x[n]$ which is periodic with period N
- Then it can be represented as a ~~Fourier series~~ DFT with coefficients c_k:

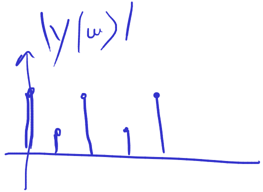
$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} c_k e^{j\overbrace{2\pi k/N}^{\omega} n}$$

- Since the system is linear, each component k gets multiplied with $H\left(\frac{2\pi}{N}k\right)$
- So the total output is:

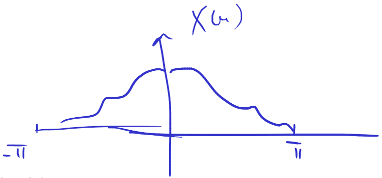
$$y[n] = \frac{1}{N}\sum_{k=0}^{N-1} c_k H\left(\overbrace{\frac{2\pi}{N}k}^{\omega}\right) e^{j2\pi kn/N}$$

- The output is still periodic, same period, same frequencies





Response of LTI systems to non-periodic signals



- Consider a general input $x[n]$ (not periodic)
- The output = input convolution with impulse response

$$y[n] = x[n] * h[n]$$

$$\Updownarrow$$

$$Y(\omega) = X(\omega) \cdot H(\omega)$$

- Output spectrum = Input spectrum × Transfer function

Response of LTI systems to non-periodic signals

- The transfer function $H(\omega)$ “shapes” the spectrum

$$Y(\omega) = X(\omega) \cdot H(\omega)$$

- In polar form:
  - modulus is multiplied

$$|Y(\omega)| = |X(\omega)| \cdot |H(\omega)|$$

  - phase~~s~~ is added:

$$\angle Y(\omega) = \angle X(\omega) + \angle H(\omega)$$

Response of LTI systems to non-periodic signals

- The system **attenuates/amplifies** the input frequencies and **changes their phases**
- $H(\omega)$ = the **transfer function**
- $H(z)$ = the **system function**
- $H(\omega) = H(z = e^{j\omega})$ if unit circle is in CR

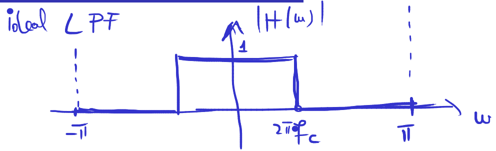
Power spectral density

- $S_{zz}(\omega) = |Y(\omega)|^2 = |H(\omega)|^2 \cdot S_{xx}(\Omega)$
- The poles and zeros of $S(\omega)$ come in pairs ($z, 1/z$ and $p, 1/p$)

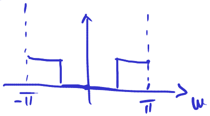
Digital filters

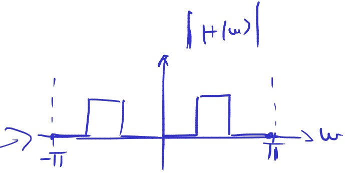
- LTI systems are also known as **filters** because their transfer function shapes ("filters") the frequencies of the input signals
- The transfer function can be found from $H(z)$ and $z = e^{j\omega}$
- Alternatively, the transfer function can be found by the **geometrical method** based on the locations of poles and zeros

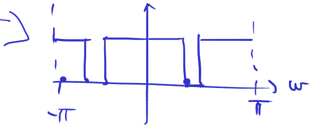
Ideal filters



- Draw at whiteboard the ideal transfer function of a:
  - low-pass filter
  - high-pass filter
  - band-pass filter
  - band-stop filter
  - all-pass filter (*changes the phase*)

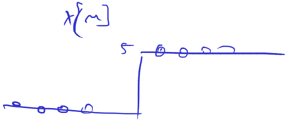


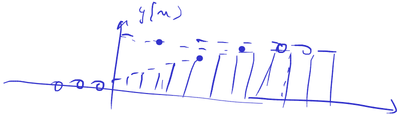


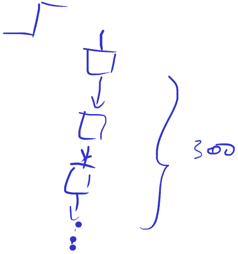


Filter order

- The **order** of a filter = maximum degree in numerator or denominator of $H(z)$
  - i.e. largest power of z or z^{-1}
- Any filter can be implemented, in general, with this number of unit delay blocks (z^{-1})
- Higher order -> better filter transfer function
  - closer to ideal filter
  - more complex to implement
  - more delays (bad)
- Lower order
  - worse transfer function (not close to ideal)
  - simpler, cheaper
  - faster response



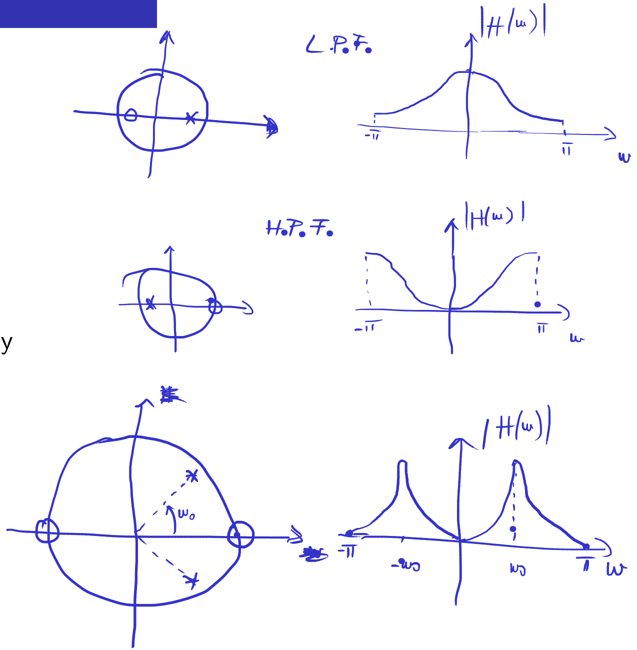




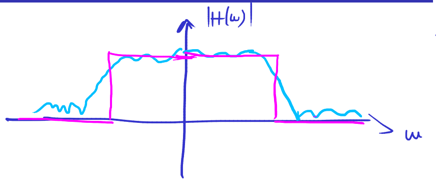
Filter design by pole and zero placements

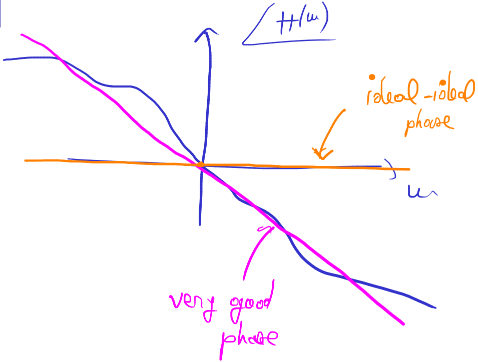
$$H(z) = \frac{b_0 + b_1 z^{-1} + \dots}{\dots}$$

- Based on geometric method
- The gain coefficient must be found by separate condition
  - i.e. specify the desired magnitude response at one frequency
- Examples at blackboard



Filter distortions



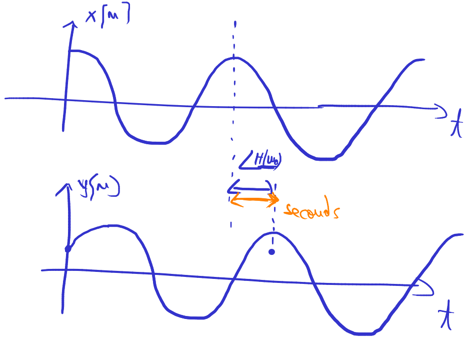


- When a filter is non-ideal:
  - non-constant amplitude –> amplitude distortions
  - non-linear phase –> phase distortions
- Phase distortions may be tolerated by certain applications
  - e.g. human auditory system is largely insensitive to phase distortions of sounds

Effect of system's phase $\angle H(\omega)$

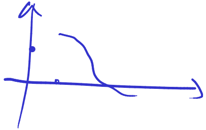
$$x[n] = \cos(\omega_0 n) \longrightarrow \boxed{H(\omega)} \longrightarrow y[n] = A\cdot\cos(\omega_0 n + \underbrace{\angle H(\omega_0)}_{\text{radians}})$$

- What is the effect of system's phase response $\angle H(\omega)$?
- $\boxed{\text{Extra phase = delay}}$
  - different frequencies are delayed differently
  - phase
- **Linear-phase** filter: delays all frequencies with the same amount of time
  - i.e. the whole signal is delayed, but otherwise not distorted
  - otherwise, we get distortions

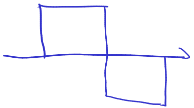


Example: $\cos(\omega_0 n)$

$\cos(\omega_0 n + \pi/2)$



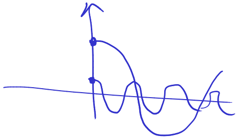
Linear-phase filters



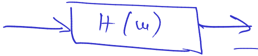
Delay 2π = delay of 1 period

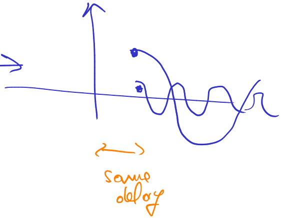
π = 1/2 period

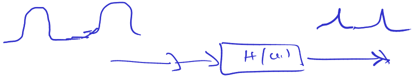
- For a sinusoidal signal, extra phase of 2π = delay of a period $N = \frac{1}{f}$
- To ensure same delay (in seconds) for all frequencies (in time), the phase $\angle H(\omega)$ must be proportional to the frequency
  - draw at blackboard
  - hence the name **linear**

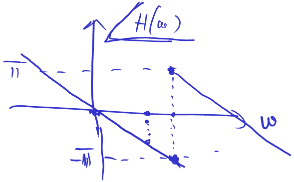


Linear-phase filter









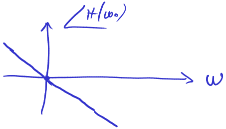
Linear-phase filters

$H(\omega) = |H(\omega)| \cdot e^{j \angle H(\omega_0)}$

- Example: consider the following filter with **linear phase** function:

$$H(\omega) = C \cdot e^{-j\omega n_0}$$

$\angle H(\omega_0) = -\omega \cdot n_0$



$x[n] \xleftrightarrow{z} X(z)$

$x[n-k] \xleftrightarrow{z} z^{-k} \cdot X(z)$

$z = e^{j\omega}$

$x[n-k] \xleftrightarrow{\mathcal{F}} e^{-j\omega k} \cdot X(\omega)$

- The output signal is

$$Y(\omega) = X(\omega) \cdot \underbrace{C \cdot e^{-j\omega n_0}}_{H(\omega)}$$

$X(\omega) \cdot e^{-j\omega n_0} \xleftrightarrow{\mathcal{F}} x[n - n_0]$

$$y[n] = C \cdot x[n - n_0]$$

- Linear phase means **just a delaying** of the input signal
  - Fourier property: $x[n - n_0] <-- > X(\omega)e^{-j\omega n_0}$

~~Group delay~~

- Group delay = The time delay experienced by a component of frequency ω when passing through the filter
  - as opposed to "phase delay" = the phase added by the filter
- **Group delay** of the filter:

$$\tau_g(\omega) = \frac{d\Theta(\omega)}{d\omega}$$

- Linear phase = constant group delay = all frequencies delayed the same = whole signal delayed

Linear-phase FIR filters

$$\text{IIR}: \quad H(z) = \frac{B(z)}{A(z)}$$

$$\text{FIR}: \quad H(z) = \frac{B(z)}{1}$$

What type of filters can have linear phase?

- IIR filters cannot have linear phase (no proof provided)
- Only FIR filters can have linear phase, and only if they satisfy some symmetry conditions

Symmetry conditions for linear-phase FIR

$$H(z) = \underbrace{b_0 + b_1 z^{-1} + b_2 z^{-2} + \dots + b_{M-1} z^{-(M-1)}}_{M}$$

$$h[n] = \{\underset{h[0]}{b_0}, \underset{h[1]}{b_1}, \dots \underset{h[M-1]}{b_{M-1}}\}$$

- Let filter have an impulse response of length M (order is $M-1$)
- The filter coefficients are $h[0], \dots \; h[M-1]$
- Linear-phase is guaranteed in two cases :
  - **Positive symmetry**

$$h[n] = h[M-1-n]$$

  - **Negative symmetry (anti-symmetry)**

$$h[n] = -h[M-1-n]$$

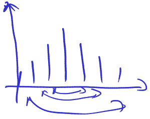
!

- The delay = the delay of the middle point of the symmetry

$$h[n] = \{1, 2, 3, (4), 3, 2, 1\}$$

$h[0] = h[M-1]$ first = last

$h[1] = h[M-1-1]$ second = second last

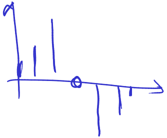


$$H(z) = 1 + 2z^{-1} + 3z^{-2} - 3z^{-3} - 2z^{-4} - 1 \cdot z^{-5}$$

$h[0] = -h[5]$

$h[1] = -h[4]$

$h[2] = -h[3]$



$$H(z) = 1 + 2z^{-1} + 3z^{-2} + 0 \cdot z^{-3} - 3z^{-4} - 2 \cdot z^{-5} - 1 \cdot z^{-6}$$

!

Cases of linear-phase FIR

DTFT: $H(\omega) = \sum_{n=-\infty}^{\infty} h[n]\, e^{-j\omega n}$

① Example: $h[n] = \ldots 0, 1, 2, 3, 4, 3, 2, 1, 0 \ldots \Rightarrow$

$H(z) = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3} + 3z^{-4} + 2z^{-5} + z^{-6}$ $\quad \boxed{z \to e^{j\omega}}$

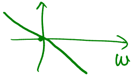
$H(\omega) = 1 + 2e^{-j\omega} + 3e^{-2j\omega} + 4\cdot e^{-3j\omega} + 3\cdot e^{-4j\omega} + 2e^{-5j\omega} + 1\cdot e^{-6j\omega} = e^{-3j\omega}\cdot\left(e^{3j\omega} + 2e^{2j\omega} + 3e^{j\omega} + 4 + 3\cdot e^{-j\omega} + 2e^{-2j\omega} + 1\cdot e^{-3j\omega}\right)$

$3\cdot 2\cdot\cos\omega \in \mathbb{R}$

$2\cdot 2\cdot\cos(2\omega) \in \mathbb{R}$

$2\cos(3\omega) \in \mathbb{R}$

$= e^{-3j\omega}\cdot \underbrace{\text{Something}}_{\in \mathbb{R}}$



$H(\omega) = |H(\omega)| \cdot e^{j\angle H(\omega)} \Rightarrow \angle H(\omega) = -3\omega$

- Proofs at blackboard

1. Positive symmetry, M = odd
2. Positive symmetry, M = even
3. Negative symmetry, M = odd
4. Negative symmetry, M = even

- Check constraints for $H(0)$ and $H(\pi)$
- For what types of filters is each case appropriate?

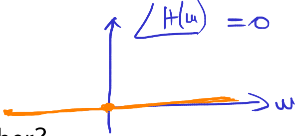
$e^{jx} + e^{-jx} = 2\cdot\cos x$

③:

Example: $h[n] = \ldots 0, 1, 2, 3, 0, -3, -2, -1, 0 \ldots \Rightarrow$ $\quad \boxed{z \to e^{j\omega}}$

$H(\omega) = 1 + 2e^{-j\omega} + 3e^{-2j\omega} - 3\cdot e^{-4j\omega} - 2e^{-5j\omega} - 1\cdot e^{-6j\omega} = e^{-3j\omega}\left(e^{3j\omega} + 2e^{2j\omega} + 3e^{j\omega} - 3e^{-j\omega} - 2e^{-2j\omega} - 1e^{-3j\omega}\right)$

Zero-phase FIR filters



- Can we avoid delay altogether?
- **Zero-phase** filter = a particular type of linear-phase filter with zero delay
- For a zero-phase filter, the phase response $\angle H(\omega) = 0$
  - (Group) delay = derivative of $\angle H(\omega)$
  - delay 0 $\Leftrightarrow$ flat $\angle H(\omega) \Leftrightarrow \angle H(\omega) = 0$
- Delay is 0 $\Leftrightarrow$ symmetry with respect to $h[0]$
  - the system cannot be causal

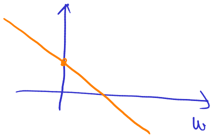
$H(\omega) = e^{-3j\omega}\left(2j\sin(3\omega) + 2\cdot 2j\sin(2\omega) + 3\cdot 2j\sin\omega\right)$

$= \underbrace{e^{-3j\omega}\cdot j}\left(\underbrace{2\sin() + 4\sin() + 6\sin()}_{\in \mathbb{R}}\right)$

$= \underbrace{e^{-3j\omega + j\pi/2}} \quad \underbrace{\qquad\qquad}_{|H(\omega)|}$

$e^{j(\underbrace{-3\omega + \pi/2}_{\angle H(\omega)})}$

$\Rightarrow \angle H(\omega) = -3\omega + \pi/2$



! PREV. SLIDE !

$e^{jx} - e^{-jx} = 2j\sin(x)$

$j = e^{j\pi/2}$

Zero-phase FIR filters

- Zero-phase filters must be non-causal
  - left side of $h[n]$ symmetrical to right side of $h[n]$
- For causal, we need to delay $h[n]$ to be wholly on the right side $=>$ delay

Example

- Linear-phase filter (low-pass):

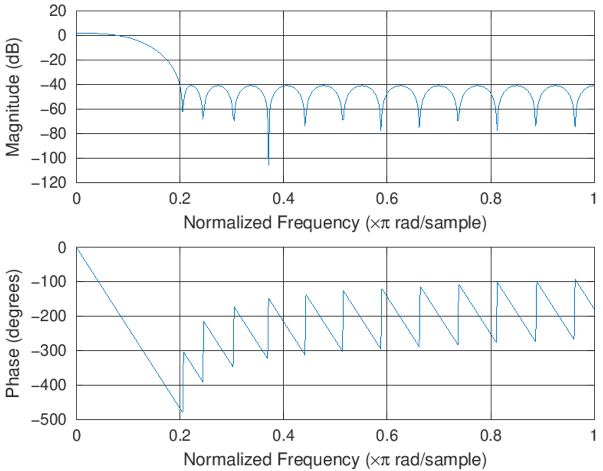


Figure 2: Transfer function of linear-phase filter

Example

- The impulse response (positive symmetry):

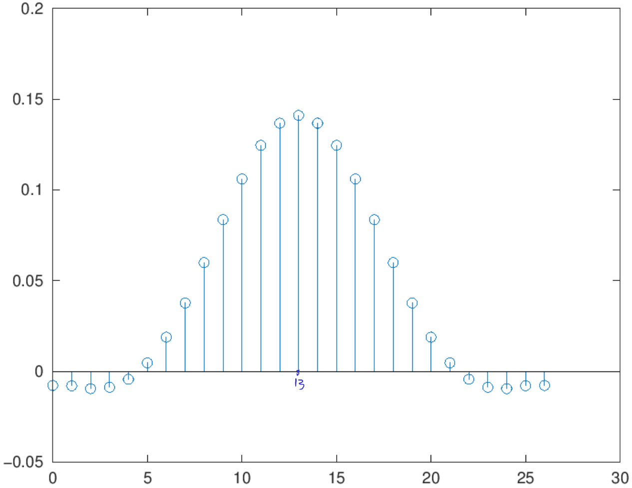


Figure 3: Impulse response of linear-phase filter

Example

- ECG signal: original and filtered. Filtering introduces **delay**

⇒ delay = 13

190 : ok

440 :

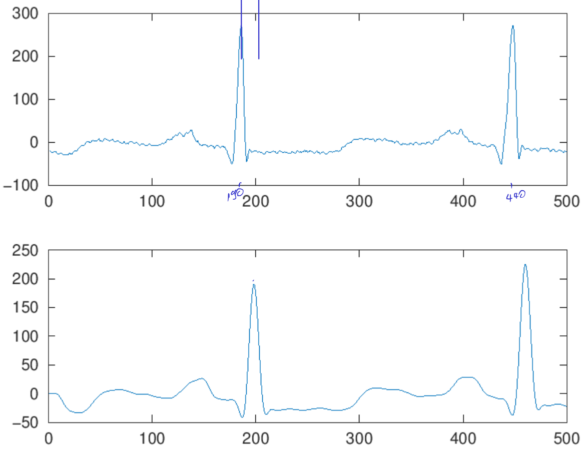
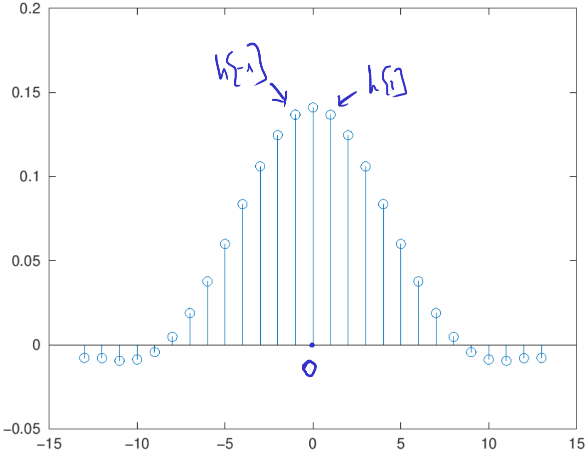


Figure 4: Delay introduced by filtering

Example

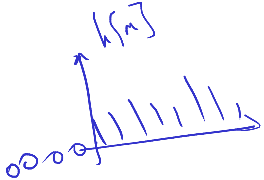
- Solution: zero-phase filter (positive symmetry, and centered in 0):
- But filter is **not causal** anymore



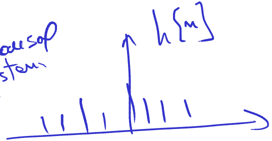
!

Figure 5: Impulse response of zero-phase filter

Causal: system



Non-causal system



Ex:

$y[n] = x[n+1] + x[n] + x[n-1]$

Example

- Filtering with zero-phase filter introduces **no delay**

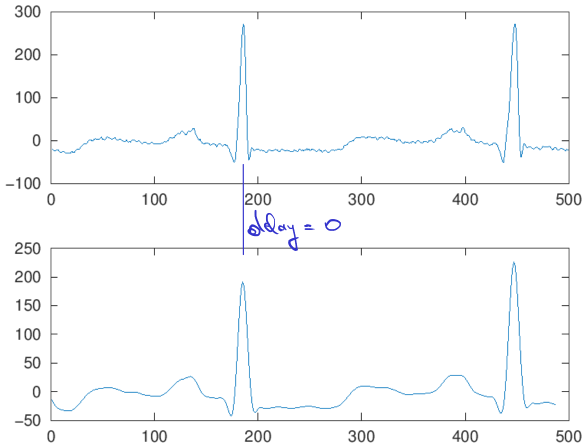
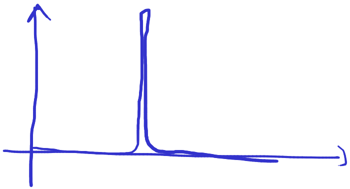


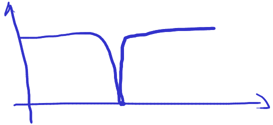
Figure 6: Zero-phase filter introduces no delay

Particular classes of filters

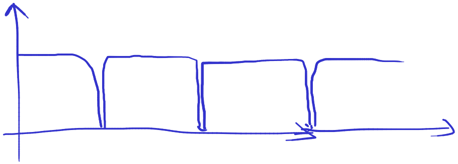
- **Digital resonators** = very selective band pass filters
  - poles very close to unit circle
  - may have zeros in 0 or at 1/-1



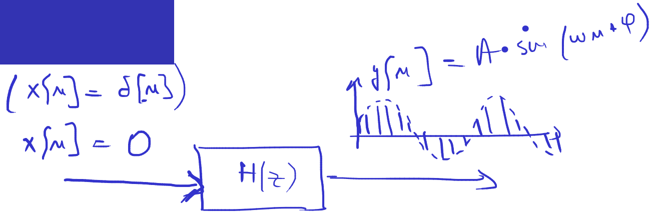
- **Notch filters**
  - have zeros exactly on unit circle
  - will completely reject certain frequencies
  - has additional poles to make the rejection band very narrow



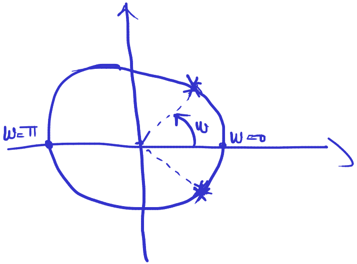
- **Comb filters**
  - = periodic notch filters



Digital oscillators



- **Oscillator** = a system which produces an output signal even in absence of input
- Has a pair of complex conjugate poles **exactly on unit circle**
- Example at blackboard



Inverse filters

Inverse filter

?

$$x[n] \longrightarrow \boxed{H(z)} \xrightarrow{y[n]} \boxed{H_I(z) = \frac{1}{H(z)}} \longrightarrow x[n]$$

inverse filter

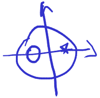
- Sometimes is necessary to **undo** a filtering
  - e.g. undo attenuation of a signal passed through a channel
- Inverse filter: has inverse system function:

$$H_I(z) = \frac{1}{H(z)}$$

$$H(z) \cdot \frac{1}{H(z)} = 1$$

- Problem: if $H(z)$ has zeros outside unit circle, $H_I(z)$ has poles outside unit circle –> unstable
- Examples at blackboard

$$H(z) = \frac{z+0.5}{z-0.7}$$



$$H_I(z) = \frac{z-0.7}{z+0.5}$$