# Section 1

## Digital Signal Processing

Section 2

## Chapter V. Frequency Analysis of Discrete Systems

# Response of LTI systems to harmonic signals

- We consider an LTI system with $h[n]$
- Input signal = complex harmonic (exponential) signal $x[n] = Ae^{j\omega_0 n}$
- Output signal = convolution

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

$$= \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega_0 k}Ae^{j\omega_0 n}$$

$$= H(\omega_0) \cdot x[n]$$

- $H(\omega_0)$ = Fourier transform of $h[[n]$ evaluated for $\omega = \omega_0$

## Eigen-function

- Complex exponential signals are **eigen-functions** (functii proprii) of LTI systems:
  - output signal = input signal $\times$ a (complex) constant
- $H(\omega_0)$ is a constant that multiplies the input signal
  - Amplitude of input gets multiplies by $|H(\omega_0)|$
  - Phase of input signal is added with $\angle H(\omega_0)$
- Why are sin/cos/exp functions important?
  - If input signal = sum of complex exponential (= coses + sinuses),
  - since the system is linear,
  - then output = same sum of complex exponentials, each scaled with some coefficients

# Response to cosine and sine

- Cosine / sine = sum of two exponentials, via Euler
- System is linear and real =>
    - amplitude is multiplied by $|H(\omega_0)|$
    - phase increases by $\angle H(\omega_0)$
- See proof at blackboard

# Frequency response

- ▶ Names
  - ▶ $H(\omega)$ = frequency response of the system
  - ▶ $|H(\omega)|$ = amplitude response
  - ▶ $\angle H(\omega)$ = phase response
- ▶ Phase response might have jumps of $2\pi$
- ▶ Stitching the pieces in a continuous function = phase *unwrapping*
  - ▶ Example: at blackboard
- ▶ Wrapped phase: $\in [-\pi, \pi]$, may have jumps of $2\pi$
- ▶ Unwrapped phase: continuous function, may go outside interval

## Permanent and transient response

▶ The above harmonic signals start at $n = -\infty$, not at 0.

▶ What if the signal starts at some time $n = 0$?

▶ Total response = transient response + permanent response

    ▶ transient response goes towards 0 as $n \to \infty$
    ▶ permanent response = the above

▶ So the above relations are valid only in **permanent regime**

    ▶ i.e. after the transient regime has passed
    ▶ i.e. after the transient response has practically vanished
    ▶ i.e. when the signal started very long ago (from $n = -\infty$)

▶ Example at blackboard

# Permanent response of LTI systems to periodic inputs

- ▶ Assume the input $x[n]$ is periodic with period N

- ▶ Then it can be represented as a Fourier series:

$$x[n] = \sum_{k=0}^{N-1} c_k e^{j2\pi kn/N}$$

- ▶ Since the system is linear, the output to each component $k$ is

$$c_k H\left(\frac{2\pi}{N}k\right) e^{j2\pi kn/N}$$

- ▶ So the total output is:

$$y[n] = \sum_{k=0}^{N-1} c_k H\left(\frac{2\pi}{N}k\right) e^{j2\pi kn/N}$$

- ▶ The output is still periodic, same period, same frequencies

## Response of LTI systems to non-periodic signals

- The output = input convolution with impulse response

$$y[n] = x[n] * h[n]$$

$$Y(\omega) = X(\omega) \cdot H(\omega)$$

- In polar form:
    - modulus get multiplied
    - phases get added

- The system **attenuates/amplifies** the input frequencies and **changes their phases**

- $H(\omega)$ = the **transfer function**

- $H(z)$ = the **system function**

- $H(\omega) = H(z = e^{j\omega})$ if unit circle is in CR

# Power spectral density

- $S_{zz}(\omega) = |Y(\omega)|^2 = |H(\omega)|^2 \cdot S_{xx}(\Omega)$
- The poles and zeros of $S(\omega)$ come in pairs ($z, 1/z$ and $p, 1/p$)

# Digital filters

- LTI systems are also known as **filters** because their transfer function shapes (*filters*) the frequencies of the input signals

- The transfer function can be found from $H(z)$ and $z = e^{j\omega}$

- Alternatively, the transfer function can be found by the **geometrical method** based on the locations of poles and zeros

## Ideal filters

- ▶ Ideal low-pass filter: example at whiteboard
- ▶ Ideal band-pass filter: example at whiteboard
- ▶ Ideal high- pass filter: example at whiteboard
- ▶ Ideal band-stop filter: example at whiteboard
- ▶ Ideal all-pass filter (*changes the phase*): idem

## Linear-phase filters

▶ Consider a constant filter with **linear phase** function:

$$H(\omega) = C \cdot e^{-j\omega n_0}$$

▶ The output signal is

$$Y(\omega) = X(\omega) \cdot C \cdot e^{-j\omega n_0}$$

$$y[n] = C \cdot x[n - n_0]$$

▶ Linear phase means **just a delaying** of the input signal
  ▶ Fourier property: $x[n - n_0] <-- > X(\omega)e^{-j\omega n_0}$

# Group delay

- $=$ The time delay experienced by a component of frequency $\omega$ when passing through the filter

- **Group delay** of the filter:

$$\tau_g(\omega) = \frac{d\Theta(\omega)}{d\omega}$$

- Linear phase $=$ constant group delay $=$ all frequencies delayed the same $=$ whole signal delayed

# Filter distortions

- When a filter is non-ideal:
  - non-constant amplitude –> amplitude distortions
  - non-linear phase –> phase distortions
- Phase distortions may be tolerated by certain applications
  - e.g. human ears are insensitive to phase distortions of sounds

# Filter order

- The **order** of a filter = maximum degree in numerator or denominator of $H(z)$
    - i.e. largest power of $z$
- Any filter can be implemented, in general, with this number of unit delay blocks $(z^{-1})$
- Higher order -> better filter transfer function
    - closer to ideal filter
    - more complex to implement
    - more delays (bad)
- Lower order
    - worse transfer function (not close to ideal)
    - simpler, cheaper
    - faster response

# Filter design by pole and zero placements

- Based on geometric method
- The gain coefficient must be found by separate condition
  - i.e. specify the desired magnitude response at one frequency
- Examples at blackboard

## Zero-phase transfer function

- Normally, $|H(\omega)|$ is strictly positive

$$|H(\omega)| \geq 0$$

- When $H(\omega)$ the function passes through 0:
  - $|H(\omega)|$ remains positive
  - $\angle H(\omega)$ has a jump of $\pi$

- **Zero-phase transfer function**
  - $H_R(\omega) = \pm|H(\omega)|$, including the sign (can be positive or negative)
  - $\Theta_R(\omega)$ doesn't have anymore the jumps of $\pi$

$$H(\omega) = H_R(\omega)e^{j\Theta_R(\omega)}$$

- Everything else still applies
  - $H_R(\omega)$ is even
  - $\Theta_R(\omega)$ is odd

# Linear-phase FIR filters

- Only FIR filters can have linear phase!
- IIR filters cannot have linear phase (no proof)

## Symmetry conditions for linear-phase FIR

- Let filter order be $M$
- The filter coefficients are $h[0], \ldots h[M-1], h[M]$
- Linear-phase is guaranteed in two cases
- **Positive symmetry**
$$h[n] = h[M-n]$$

- **Negative symmetry (anti-symmetry)**
$$h[n] = -h[M-n]$$

▶ Proofs at blackboard

1. Positive symmetry, M = odd
2. Positive symmetry, M = even
3. Negative symmetry, M = odd
4. Negative symmetry, M = even

▶ Check constraints for $H(0)$ and $H(\pi)$
▶ For what types of filters is each case appropriate?

## Particular classes of filters

- Digital resonators
  - $=$ very selective band pass filters
  - poles very close to unit circle
  - may have zeros in 0 or at 1/-1
- Notch filters
  - have zeros exactly on unit circle
  - will completely reject certain frequencies
  - has additional poles to make the rejection band very narrow
- Comb filters
  - $=$ periodic notch filters

## Digital oscillators

- Oscillator $=$ a system which produces an output signal even in absence of input
- Has a pair of complex conjugate poles **exactly on unit circle**
- Example at blackboard

# Inverse filters

- Sometimes is necessary to **undo** a filtering
  - undo attenuation of a signal passed through a channel
- Inverse filter: has inverse system function:

$$H_I(z) = \frac{1}{H(z)}$$

- Problem: if $H(z)$ has zeros outside unit circle, $H_I(z)$ has poles outside unit circle $->$ unstable
  - In this case the filtering cannot be undone
- Examples at blackboard

## Minimum and maximum phase filters

- ▶ Start with example
- ▶ If replace a zero $z_k$ with $1/z_k$, the modulus of the transfer function does not change! (Not true; TO CHECK what's up)
  - ▶ only the phase is affected
- ▶ Minimum phase: all zeros are inside the unit circle
  - ▶ minimum phase = minimum derivative = minimum delay
  - ▶ inverse filter is stable
- ▶ Maximum phase: all zeros are outside unit circle
- ▶ Mixed phase: some zeros inside, some outside