Digital Signal Processing

Chapter IV: The Fourier Transform and its applications

IV.1 Vector spaces of signals (crash course)

Vector spaces

- **Vector space** $=$ a set $V\{v_i\}$ with the following two properties:
  - one element $+$ another element $=$ still an element of the same space
  - a scalar constant $\times$ an element $=$ still an element of the same space
- You **can't escape** a vector space by summing or scaling
- The elements of a vector space are called **vectors**

Examples of vector spaces

- Geometric spaces are great intuitive examples:
  - a line, or the set $\mathbb{R}$ (one-dimensional)
  - a plane, or the set $\mathbb{C}$ (two-dimensional)
  - 3D space (three-dimensional)
  - 4D space (four-dimensional, like the spatio-temporal universe)
  - arrays with N numbers (N-dimensional)
  - space of continuous signals (∞-dimensional)
- The **dimension** of the space = "how many numbers you need in order to specify one element" (informal)
- A "vector" like in maths = a sequence of N numbers = a "vector" like in programming
  - e.g. a point in a plane has two coordinates = a vector of size $N = 2$
  - e.g. a point in a 3D-space has three coordinates = a vector of size $N = 3$

Inner product

- Many vector spaces have a fundamental operation: **the (Euclidean) inner product**
  - for **discrete** signals:
$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_i x_i y_i^*$$
  - for **continuous** signals:
$$\langle \mathbf{x}, \mathbf{y} \rangle = \int x(t) y^*(t)$$
- * represents **complex conjugate** (has no effect for real signals)
- The result is one number (real or complex)
- Also known as **dot product** or **scalar product** (“produs scalar”)

Inner product

- Each entry in $\mathbf{x}$ times the complex conjugate of the one in $\mathbf{y}$, all summed
- For discrete signals, it can be understood as a row $\times$ column multiplication
- Discrete vs continuous: just change sum/integral depending on signal type

Inner product properties

- Inner product is **linear** in both terms:

$$\langle \mathbf{x_1} + \mathbf{x_2}, \mathbf{y} \rangle = \langle \mathbf{x_1}, \mathbf{y} \rangle + \langle \mathbf{x_2}, \mathbf{y} \rangle$$

$$\langle c \cdot \mathbf{x}, \mathbf{y} \rangle = c \cdot \langle \mathbf{x_1}, \mathbf{y} \rangle$$

$$\langle \mathbf{x}, \mathbf{y_1} + \mathbf{y_2} \rangle = \langle \mathbf{x}, \mathbf{y_1} \rangle + \langle \mathbf{x}, \mathbf{y_2} \rangle$$

$$\langle \mathbf{x}, c \cdot \mathbf{y} \rangle = c^* \cdot \langle \mathbf{x_1}, \mathbf{y} \rangle$$

The distance between two vectors

- An inner product induces a **norm** and a **distance** function
- **The (Euclidean) distance** between two vectors =

$$d(\mathbf{x}, \mathbf{y}) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + ... + (x_N - y_N)^2}$$

- This distance is the **usual geometric distance** you know from geometry
- It has the exact same intuition like in **normal geometry**:
  - if two vectors have small distance, they are close, they are similar
  - two vectors with large distance are far away, not similar
  - two identical vectors have zero distance

The norm of a vector

- An inner product induces a **norm** and a **distance** function
- The **norm** (length) of a vector = sqrt(inner product with itself)

$$\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle} = \sqrt{x_1^2 + x_2^2 + ... + x_N^2}$$

- The **norm** of a vector is the distance from $\mathbf{x}$ to point $\mathbf{0}$.
- It has the exact same intuition like in **normal geometry**:
  - vector has large norm = has big values, is far from $\mathbf{0}$
  - vector has small norm = has small values, is close to $\mathbf{0}$
  - vector has zero norm = it is the vector $\mathbf{0}$
- Norm of a vector = sqrt(the signal **energy**)

Norm and distance

- The norm and distance are related
- The distance between **a** and **b** = norm (length) of their difference

$$d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\| = \sqrt{x_1^2 + x_2^2 + \ldots + x_N^2}$$

- Just like in geometry: distance = length of the difference vector

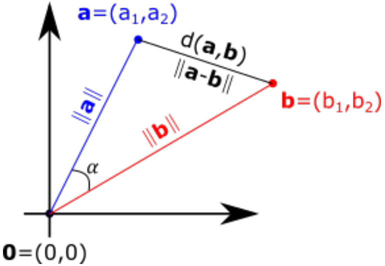

Figure 1: Norm and distance in vector spaces

Angle between vectors

- The **angle** between two vectors is:

$$\cos(\alpha) = \frac{\langle x, y \rangle}{||x|| \cdot ||y||}$$

  - is a value between -1 and 1
- **Otrhogonal vectors** $=$ two vectors with $\langle x, y \rangle = 0$
  - their angle $= 90\,\deg$
  - in geometric language, the two vectors are **perpendicular**

Why vector space

- Why are all these useful?
- They are a very general **framework** for different kinds of signals
- We can have **generic** algorithms expressed in terms of distances, norms, angles, and they will work the same in all vector spaces
  - Example in DEDP class: ML decision with 1, 2, N samples

Vector spaces in DSP class

We deal mainly with the following vector spaces:

- The vector space of all infinitely-long real signals $x[n]$
- The vector space of all infinitely-long periodic signals $x[n]$ with period N
  - for each N we have a different vector space
- The vector space of all finite-length signals $x[n]$ with only N samples
  - for each N we have a different vector space

Bases

- A **basis** = a set of N linear independent elements from a vector space

$$B = \{\mathbf{b}^1, \mathbf{b}^2 ... \mathbf{b}^N\}$$

- Any vector in a vector space is expressed as a **linear combination** of the basis elements:

$$\mathbf{x} = \alpha_1 \mathbf{b}^1 + \alpha_2 \mathbf{b}^2 + ... + \alpha_N \mathbf{b}^N$$

- The vector is defined by these coefficients:

$$\mathbf{x} = (\alpha_1, \alpha_2, ... \alpha_N)$$

Bases and coordinate systems

- Bases are just like **coordinate systems** in a geometric space
  - any point is expressed w.r.t. a coordinate system

$$\mathbf{x} = x_1\mathbf{i} + x_2\mathbf{j}$$

  - any vector is expressed w.r.t. a basis

$$\mathbf{x} = \alpha_1\mathbf{b}^1 + \alpha_2\mathbf{b}^2 + \cdots + \alpha_N\mathbf{b}^N$$

- N = The number of basis elements = The dimension of the space
- Example: any color = RGB values (monitor) or Cyan-Yellow-Magenta values (printer)

Bases and coordinate systems

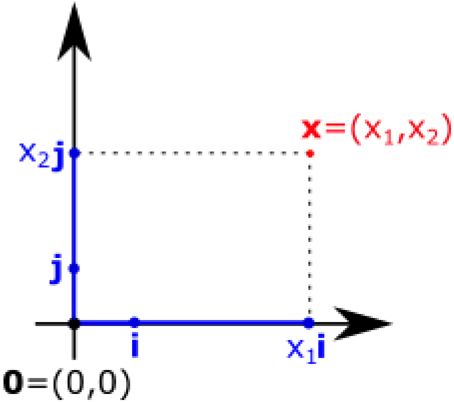

Figure 2: Basis expansion of a vector x

Choice of bases

- There is typically an infinite choice of bases
- The **canonical basis** = all basis vectors are full of zeros, just with one 1
- You used it already in an exercise:
  - any signal $x[n]$ can be expressed of a sum of $\delta[n-k]$

    $$\{\ldots, 3, 6, 2, \ldots\} = \cdots + 3\delta[n] + 6\delta[n-1] + 2\delta[n-2] + \ldots$$

  - the canonical basis is $B = \{..., \delta[n], \delta[n-1], \delta[n-2], \ldots\}$

Orthonormal bases

- An **orthonormal basis** a basis where all elements $\mathbf{b}^i$ are:
  - orthogonal to each other:

$$\langle \mathbf{b}^i, \mathbf{b}^j \rangle = 0, \forall i \neq j$$

  - **normalized** (their norm $= 1$):

$$||\mathbf{b}^i|| = \sqrt{\langle \mathbf{b}^i, \mathbf{b}^i \rangle} = 1, \forall i$$

- Example: the canonical basis $\{\delta[n-k]\}$ is orthonormal:
  - $\langle \delta[n-k], \delta[n-l] \rangle = 0, \forall k \neq l$
  - $\langle \delta[n-k], \delta[n-k] \rangle = 1, \forall k$

Orthonormal bases

- Orthonormal basis = like a coordinate system with orthogonal vectors, of length 1

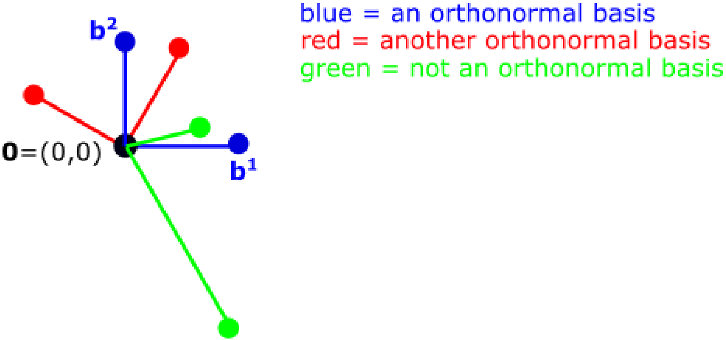

Figure 3: Sample bases in a 2D space

Basis expansion of a vector

- Suppose we have an **orthonormal basis** $B = \{\mathbf{b}^i\}$
- Suppose we have a vector $\mathbf{x}$
- We can write (expand) $\mathbf{x}$ as:

$$\mathbf{x} = \alpha_1 \mathbf{b}^1 + \alpha_2 \mathbf{b}^2 + \cdots + \alpha_N \mathbf{b}^N$$

- Question: how to **find** the coefficients α_i?

Basis expansion of a vector

- If the basis is **orthonormal**, we have:

$$\begin{aligned}
\langle \mathbf{x}, \mathbf{b}^i \rangle =& \langle \alpha_1 \mathbf{b}^1 + \alpha_2 \mathbf{b}^2 + \cdots + \alpha_N \mathbf{b}^N, \mathbf{b}^i \rangle \\
&= \langle \alpha_1 \mathbf{b}^1, \mathbf{b}^i \rangle + \langle \alpha_2 \mathbf{b}^2, \mathbf{b}^i \rangle + \cdots + \langle \alpha_N \mathbf{b}^N, \mathbf{b}^i \rangle \\
&= \alpha_1 \langle \mathbf{b}^1, \mathbf{b}^i \rangle + \alpha_2 \langle \mathbf{b}^2, \mathbf{b}^i \rangle + \cdots + \alpha_N \langle \mathbf{b}^N, \mathbf{b}^i \rangle \\
&= \alpha_i
\end{aligned}$$

Basis expansion of a vector

- Any vector $\mathbf{x}$ can be written as:

$$\mathbf{x} = \alpha_1 \mathbf{b}^1 + \alpha_2 \mathbf{b}^2 + \cdots + \alpha_N \mathbf{b}^N$$

- For orthonormal basis: the coefficients α_i are found by inner product with the corresponding basis vector:

$$\alpha_i = \langle, \mathbf{x}, \mathbf{b}^i \rangle$$

Why bases

- How does all this talk about bases help us?
- To better understand the Fourier transform
- The signals $\{e^{j\omega n}\}$ form an **orthonormal basis**
- The Fourier Transform of a signal x = finding the coefficients of **x** in this basis
- The Inverse Fourier Transform = expanding **x** with the elements of this basis
- Same **generic** thing every time, only the type of signals differ

IV.2 The Fourier Transform

Reminder

- Reminder:

$$e^{jx} = \cos(x) + j\sin(x)$$

$$\cos(x) = \frac{e^{jx} + e^{-jx}}{2}$$

$$\sin(x) = \frac{e^{jx} - e^{-jx}}{2j}$$

$$\sin(x) = \cos(x - \frac{pi}{2})$$

$$\cos(x) = \sin(x + \frac{pi}{2})$$

Why sinusoidal signals

- Why are sinusoidal signals sin() and cos() so prevalent in signal processing?
- Answer: because they are combinations of an e^{jx} and an e^{-jx}
- Why are these e^{jx} so special?
- Answer: because they are **eigen-functions** of linear and time-invariant (LTI) systems

Response of LTI systems to harmonic signals

- Consider an LTI system with $h[n]$
- Input signal = complex harmonic (exponential) signal $x[n] = Ae^{j\omega_0 n}$
- Output signal = convolution

$$
\begin{aligned}
y[n] &= \sum_{k=-\infty}^{\infty} h[k]x[n-k] \\
&= \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega_0 k} Ae^{j\omega_0 n} \\
&= H(\omega_0) \cdot x[n]
\end{aligned}
$$

- Output signal = input signal $\times$ a (complex) constant $(H(\omega_0))$

Eigen-function

- Complex exponential signals are **eigen-functions** ("funcții proprii") of LTI systems:
  - output signal $=$ input signal $\times$ a (complex) constant
- We can understand the effect of a LTI system **very easily** if we decompose all signals as a combination of $\{e^{j\omega n}\}$
- Example: RGB color filter (listen)
- All vector space theory becomes useful now:
  - $\{e^{j\omega n}\}$ is an **orthonormal basis**
  - decomposing signals $=$ finding coefficients α_i
  - we know how to do this with the inner product

Fourier Transform of Discrete Signals

- Consider signal $\mathbf{x} = x[n]$ in the vector space of infinitely-long signals
- The signals $\{e^{j2\pi fn}\}, \forall f \in [-\frac{1}{2}, \frac{1}{2}]$ form an **orthonormal basis** in this space
- We can expand (almost) any $\mathbf{x}$ in this basis
- There are an infinite number of terms, use an integral:

$$x[n] = \int_{f=-1/2}^{1/2} \underbrace{X(f)}_{\alpha_i} e^{j2\pi fn} df$$

- The coefficient of every $e^{j2\pi fn}$ is found by inner product:

$$\alpha_i = X(f) = \langle x[n], e^{j2\pi fn} \rangle = \sum_n x[n] e^{-j2\pi fn}$$

Fourier Transform of Discrete Signals

Inverse Fourier Transform of Discrete Signals

$$x[n] = \int_{f=-1/2}^{1/2} X(f) e^{j2\pi fn} df$$

- Any $x[n]$ can be written as a linear combination of $\{e^{j2\pi fn}\}, \forall f \in [-\frac{1}{2}, \frac{1}{2}]$, with some coefficients $X(f)$

Fourier Transform of Discrete Signals

$$X(f) = \langle x[n], e^{j2\pi fn} \rangle = \sum_{n} x[n] e^{-j2\pi fn}$$

- The coefficient $X(f)$ of every $\{e^{j2\pi fn}\}$ is found using the inner product $\langle \mathbf{x}, e^{j2\pi fn} \rangle$

Fourier Transform of Discrete Signals

- We can replace $2\pi f = \omega$, and $df = \frac{1}{2\pi} d\omega$

$$x[n] = \frac{1}{2\pi} \int_{\omega=-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega$$

$$X(f) = \langle x[n], e^{j\omega n} \rangle = \sum_{n} x[n] e^{-j\omega n}$$

Basic properties of Fourier transform

- $X(\omega)$ is a complex function
- It is **periodical** with period 2π

$$X(\omega + 2\pi) = X(\omega)$$

- If the signal $x[n]$ is real, the Fourier transform is **even**

$$x[n] \in \mathbb{R} \rightarrow X(-\omega) = X^*(\omega)$$

- This means:
  - modulus is even: $|X(\omega)| = |X(-\omega)|$
  - phase is odd: $X(\omega) = -X(-\omega)$

Parseval theorem

- **Parseval theorem**: energy of the signal is the same in time and frequency domains

$$E = \sum_{-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(\omega)|^2$$

- Is true for all orthonormal bases

Expressing as sum of sinusoids

- Grouping terms with $e^{j\omega n}$ and $e^{j(-\omega)n}$ we get:

$$
\begin{aligned}
x[n] &= \frac{1}{2\pi}\int_{\pi}^{0} X(\omega)e^{j\omega n} + \frac{1}{2\pi}\int_{0}^{\pi} X(\omega)e^{j\omega n}d\omega \\
&= \frac{1}{2\pi}\int_{0}^{\pi} (X(\omega)e^{j\omega n} + X(\omega)e^{j(-\omega)n})d\omega \\
&= \frac{1}{2\pi}\int_{0}^{\pi} 2|X(\omega)|(e^{j\omega n+\angle X(\omega)} + e^{-j\omega n-\angle X(\omega)})d\omega \\
&= \frac{1}{2\pi}\int_{0}^{\pi} 2|X(\omega)|\cos(\omega n + \angle X(\omega))d\omega
\end{aligned}
$$

- Any signal $x[n]$ is **a sum of sinusoids with all frequencies** $f \in [0, \frac{1}{2}]$, or $\omega \in [0, \pi]$

Expressing as sum of sinusoids

- Any signal $x[n]$ is **a sum of sinusoids with all frequencies** $f \in [0, \frac{1}{2}]$, or $\omega \in [0, \pi]$
  - this is the fundamental practical interpretation of the Fourier transform
- The **modulus** $|X(\omega)|$ is the **amplitude** of the sinusoids ($\times$ 2)
  - for $\omega = 0$, $|X(\omega = 0)|$ = the DC component
- The **phase** $\angle X(\omega)$ gives the initial phase

Power spectral density

- The average power of a discrete periodic signal

$$P = \frac{1}{N} \sum_{n=0}^{N-1} |x[n]|^2$$

- Is the same in the frequency domain (with proof):

$$P = \sum_{k=0}^{N-1} |c_k|^2$$

- Power spectral density of the signal is

$$S_{xx}[k] = |c_k|^2$$

- Energy over one period is

$$E = \sum_{n-0}^{N-1} |x[n]|^2 = N \sum_{k-0}^{N-1} |c_k|^2$$

IV.3 The Fourier Series

IV.4 The Discrete Fourier Transform

IV.2 Frequency analysis of discrete signals

Fourier series of discrete periodical signals

- A discrete signal of period N:

$$x[n] = x[n+N]$$

- Can always be decomposed as a **sum of complex exponentials**:

$$x[n] = \sum_{k=0}^{N-1} c_k e^{j2\pi kn/N}, n = 0, 1, ...N-1$$

- Finding the coefficients c_k:

$$c_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N}$$

Comparison with analog Fourier series

- Compared to analog signals:
  - consider fundamental frequency $f_0 = 1/N$
  - only N terms, with frequencies $k \cdot f_0$:
    - $0, f_0, 2f_0, ...(N-1)f_0$
  - only N distinct coefficients c_k
  - the N coefficients c_k can be chosen like $-\frac{N}{2} < k \leq \frac{N}{2} =>$ the frequencies span the range $-1/2...1/2$

$$-\frac{1}{2} < f_k \leq \frac{1}{2}$$

$$-\pi < \omega_k \leq \pi$$

Basic properties of Fourier coefficients

1. Signal is **discrete** –> coefficients are **periodic** with period N

$$c_{k+N} = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-j2\pi(k+N)n/N} = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N}$$

2. If signal is real $x[n] \in \mathbb{R}$, the coefficients are **even**:
   - $c_k^* = c_{-k}$
   - $|c_k| = |c_{-k}|$
   - $\angle c_k = \angle c_{-k}$

- Together with periodicity:
  - $|c_k| = |c_{-k}| = |c_{N-k}|$
  - $\angle c_k = -\angle c_{-k} = -\angle c_{N-k}$

Expressing as sum of sinusoids

- Grouping terms with c_k and c_{-k} we get

$$x[n] = c_0 + 2\sum_{k=1}^{L} |c_k| cos(2\pi \frac{k}{N} + \angle c_k)$$

  where $L = N/2$ or $L = (N-1)/2$ depending if N is even or odd
- Signal = DC value + a finite sum of sinusoids with frequencies kf_0
  - $|c_k|$ give the amplitudes (x 2)
  - $\angle c_k$ give the phases

Power spectral density

- The average power of a discrete periodic signal

$$P = \frac{1}{N} \sum_{n=0}^{N-1} |x[n]|^2$$

- Is the same in the frequency domain (with proof):

$$P = \sum_{k=0}^{N-1} |c_k|^2$$

- Power spectral density of the signal is

$$S_{xx}[k] = |c_k|^2$$

- Energy over one period is

$$E = \sum_{n-0}^{N-1} |x[n]|^2 = N \sum_{k-0}^{N-1} |c_k|^2$$

2018-2019 Exam

2018-2019 Exam

- Properties of Fourier series: only 1, 2, and 5

Properties of Fourier series

1. Linearity

If the signal $x_1[n]$ has the Fourier series coefficients $\{c_k^{(1)}\}$, and $x_2[n]$ has $\{c_k^{(2)}\}$, then their sum has
$a \cdot x_1[n] + b \cdot x_2[n] \leftrightarrow \{a \cdot c_k^{(1)} + b \cdot c_k^{(2)}\}$
Proof: via definition

Properties of Fourier series

2. Shifting in time

If $x[n] \leftrightarrow \{c_k\}$, then

$$x[n - n_0] \leftrightarrow \{e^{(-j2\pi k n_0/N)} c_k\}$$

Proof: via definition

- The amplitudes $|c_k|$ are not affected, shifting in time affects only the phase

Properties of Fourier series

3. Modulation in time

$$e^{j2\pi k_0 n/N} \leftrightarrow \{c_{k-k_0}\}$$

4. Complex conjugation

$$x^*[n] \leftrightarrow \{c^*_{-k}\}$$

Properties of Fourier series

5. Circular convolution

Circular convolution of two signals $\leftrightarrow$ product of coefficients

$$x_1[n] \otimes x_2[n] \leftrightarrow \{N \cdot c_k^{(1)} \cdot c_k^{(2)}\}$$

Circular convolution:

$$x_1[n] \otimes x_2[n] = \sum_{k=0}^{N-1} x_1[k] x_2[(n-k)_N]$$

- takes two periodic signals of period N, result is also periodic with period N
- Example at the whiteboard: how it is computed

Properties of Fourier series

6. Product in time

Product in time $\leftrightarrow$ circular convolution of Fourier series coefficients

$$x_1[n] \cdot x_2[n] \leftrightarrow \sum_{m=0}^{N-1} c_m^{(1)} c_{(k-m)_N}^{(2)} = c_k^{(1)} \otimes c_k^{(2)}$$

Fourier transform of discrete non-periodical signals

- Non-periodical signals contain all frequencies, not only the multiples of f_0
- The Fourier transform of a discrete signal:

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

- The inverse Fourier transform:

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega$$

Comparison

- Compared with the Fourier transform of analog signals
  - sum instead of integral in Fourier transform
  - spectrum is only in range:

$$\omega \in [-\pi, \pi]$$

$$f \in [-\frac{1}{2}, \frac{1}{2}]$$

- Compared with the Fourier series of discrete periodical signals
  - general ω instead of $2\pi k f_0$
  - spectrum is continuous, not discrete
  - integral, not sum in inverse Fourier transform

Parseval theorem

- **Parseval theorem**: energy of the signal is the same in time and frequency domains

$$E = \sum_{-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(\omega)|^2$$

- Proof: on whiteboard
- The **energy spectral density** gives the energy contained for each frequency

$$S_{xx}(\omega) = |X(\omega)|^2$$

Basic properties of Fourier transform

- It is **periodical** with period 2π

$$X(\omega + 2\pi) = X(\omega)$$

- If the signal $x[n]$ is real, the Fourier transform is **even**

$$x[n] \in \mathbb{R} \rightarrow X^*(\omega) = X(-\omega)$$

- This means
  - modulus is even: $|X(\omega)| = |X(-\omega)|$
  - phase is odd: $X(\omega) = -X(-\omega)$

Relation between Fourier series and Fourier transform

- If apply Fourier transform to periodical discrete signals, $X(\omega)$ contains Diracs
- The Diracs are at frequencies kf_0, just like the Fourier series
- The value of an impulse $=$ the coefficient c_k of the Fourier series
- **The Fourier series $\approx$ the Fourier transform of periodic signals**
  - we directly compute the coefficients c_k of the impulses in the spectrum

Fourier transform and Z transform

- Definition of Fourier transform = Z transform with:

$$z = e^{j\omega}$$

- $e^{j\omega}$ = points on the unit circle
- Fourier transform = Z transform evaluated **on the unit circle**
  - if the unit circle is in the convergence region of Z transform
  - otherwise, equivalence does not hold
- This is true for most usual signals we work with
  - there are exceptions, but they are outside the scope of this class

2018-2019 Exam

2018-2019 Exam

- Properties of Fourier transform: only 1, 2, 5, and Parseval theorem

Properties of Fourier transform

1. Linearity

$$a \cdot x_1[n] + b \cdot x_2[n] \leftrightarrow a \cdot X_1(\omega) + b \cdot X_2(\omega)$$

Proof: via definition

Properties of Fourier transform

2. Shifting in time

$$x[n - n_0] \leftrightarrow e^{-j\omega n_0} X(\omega)$$

Proof: via definition

- The amplitudes $|X(\omega)|$ is not affected, shifting in time affects only the phase

Properties of Fourier transform

3. Modulation in time

$$e^{j\omega_0 n}x[n] \leftrightarrow X(\omega - \omega_0)$$

4. Complex conjugation

$$x^*[n] \leftrightarrow X^*(-\omega)$$

Properties of Fourier transform

5. Convolution

$$x_1[n] * x_2[n] \leftrightarrow X_1(\omega) \cdot X_2(\omega)$$

- Not circular convolution, this is the normal convolution

Properties of Fourier transform

6. Product in time

Product in time $\leftrightarrow$ convolution of Fourier transforms

$$x_1[n] \cdot x_2[n] \leftrightarrow \frac{1}{2\pi} \int_{-\pi}^{\pi} X_1(\lambda) X_2(\omega - \lambda) d\lambda$$

Properties of Fourier transform

Correlation theorem

$$r_{x_1x_2}[l] \leftrightarrow X_1(\omega)X_2(-\omega)$$

Wiener Khinchin theorem

Autocorrelation of a signal $\leftrightarrow$ Power spectral density

$$r_{xx}[l] \leftrightarrow S_{xx}(\omega) = |X(\omega)|^2$$

Properties of Fourier transform

Parseval theorem

Energy is the same when computed in the time or frequency domain

$$\sum |x[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(\omega)|^2 d\omega$$

Relationship of Fourier transform and Fourier series

- How are they related?
  - Fourier transform: for non-periodical signals
  - Fourier series: for periodical series
- Duality: periodic in time $\leftrightarrow$ discrete in frequency
- If we **periodize** a signal $x[n]$ by repeating with period N:

$$x_N[n] = \sum_{k=-\infty}^{\infty} x[n - kN]$$

- then the Fourier transform w is discrete (made of Diracs):

$$X_N(\omega) = 2\pi c_k \delta(\omega - k\frac{2\pi}{N})$$

- The coefficients of the Diracs = exactly the Fourier series coefficients

Relationship of Fourier transform and Fourier series

- So, Fourier transform can be considered for both periodic and non-periodic signals
- Fourier transform for periodic signals = discrete (sum of Diracs with some coefficients)
  - Dirac at frequencies $f_0 = 1/N$ and its multiplies
- Fourier series for periodic signals = gives the coefficients of the Diracs directly
  - it just omits to write the Diracs explicitly in the equation

Relation of Fourier transform and Z transform

- Fourier transform: $X(\omega) = \sum_{-\infty}^{\infty} x[n]e^{-j\omega n}$
- Z transform: $X(z) = \sum_{-\infty}^{\infty} x[n]z^{-n}$
- **Fourier tranform = Z transform for** $z = e^{j\omega}$
- $z = e^{j\omega}$ means **evaluated on the unit circle**:
  - $|z| = |e^{j\omega}| = 1(modulus)$
  - $\angle z = \angle e^{j\omega} = \omega(phase)$
- Conditions:
  - unit circle must be in the Convergence Region of Z transform
  - some signals can have Fourier transform even though unit circle not in CR
- If signal has pole on unit circle $->$ Dirac (infinite) in Fourier transform
  - e.g. $u[n]$
  - some signals are non-convergent on unit circle, but have Fourier transform (e.g. $u[n]$)

Geometric interpretation of Fourier transform

$$X(z) = C \cdot \frac{(z - z_1) \cdots (z - z_M)}{(z - p_1) \cdots (z - p_N)}$$

$$X(\omega) = C \cdot \frac{(e^{j\omega} - z_1) \cdots (e^{j\omega} - z_M)}{(e^{j\omega} - p_1) \cdots (e^{j\omega} - p_N)}$$

- Modulus:

$$|X(\omega)| = |C| \cdot \frac{|e^{j\omega} - z_1| \cdots |e^{j\omega} - z_M|}{|e^{j\omega} - p_1| \cdots |e^{j\omega} - p_N|}$$

- Phase:

$$\angle X = \angle C + \angle(e^{j\omega} - z_1) + \cdots + \angle(e^{j\omega} - z_M) - \angle(e^{j\omega} - p_1) - \cdots - \angle(e^{j\omega} - p_N)$$

Geometric interpretation of Fourier transform

- For complex numbers:
  - modulus of $|a - b|$ = the length of the segment between a and b
  - phase of $|a - b|$ = the angle of the segment from b to a (direction is important)
- So, for a point on the unit circle $z = e^{j\omega}$
  - modulus $|X(\omega)|$ is **given by the distances to the zeros and to the poles**
  - phase $\angle X(\omega)$ is **given by the angles from the zeros and poles to z**

Geometric interpretation of Fourier transform

- Consecutive:
  - when a **pole** is very close to unit circle –> Fourier transform is **large** at this point
  - when a **zero** is very close to unit circle –> Fourier transform is **small** at this point
- Examples:...

Geometric interpretation of Fourier transform

- Simple interpretation for modulus $|X(\omega)|$:
  - Z transform $X(z)$ is a *"landscape"*
    - poles = mountains of infinite height
    - zeros = valleys of zero height
  - Fourier transform $X(\omega)$ = "*Walking over this landscape along the unit circle*" −> the heights give the Fourier transform
  - When close to a mountain −> road is high −> Fourier transform has large amplitude
  - When close to a valley −> road is low −> Fourier transform has small amplitude
- Enough to sketch the Fourier transform for signals with few poles/zeros

Geometric interpretation of Fourier transform

- Note: $X(z)$ might also have a constant C in front!
  - It does not appear in pole-zero plot
  - The value of $|C|$ and $\angle C$ must be determined separately
- This "geometric method" can be applied for both modulus and phase

Time-frequency duality

- **Duality** properties related to Fourier transform/series
- Discrete $\leftrightarrow$ Periodic
  - **discrete** in time –> **periodic** in frequency
  - **periodic** in time –> **discrete** in frequency
- Continuous $\leftrightarrow$ Non-periodic
  - **continous** in time –> **non-periodic** in frequency
  - **non-periodic** in time –> **continuous** in frequency

Frequency-based classification of signals

- Based on frequency content:
  - **low-frequency** signals
  - **mid-frequency** signals (band-pass)
  - **high-frequency** signals
- **Band-limited** signals: spectrum is 0 over some frequency f_{max}
- **Time-limited** signals: signal value is 0 outside some time interval
- **Bandwitdh** B: frequency interval $[F_1, F_2]$ which contains 95% of energy
  - $B = F_2 - F_1$
- Based on bandwidth B:
  - **Narrow-band** signals: $B « $ central frequency $\frac{F_1+F_2}{2}$
  - **Wide-band** signals: not narrow-band