Digital Signal Processing

Chapter IV: Frequency Analysis of Discrete Signals

IV.1 Reminder: Frequency analysis of analog signals

Introduction

- Very useful to analyze signals in **frequency domain**
- The **spectrum** of a signal indicates the frequency contents
- Mathematical tools:
  - periodical signals: **Fourier series**
  - non-periodical signals: **Fourier transform**

Analog periodical signals

- Periodical signal:

$$x(t) = x(t + T)$$

- The fundamental frequency is

$$F_0 = \frac{1}{T}$$

- **The signal can be decomposed as a sum of complex exponential signals, with multiples of the fundamental frequency,** kF_0

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j 2\pi k F_0 t}$$

- The coefficients c_k are the **spectrum** of the signal

$$c_k = \frac{1}{T} \int_{T/2}^{T/2} x(t) e^{-j 2\pi k F_0 t}$$

- The coefficients c_k are complex values
  - their modulus = “amplitude spectrum”
  - their phase = “phase spectrum”"

Conditions for convergence

- When is the Fourier series convergent to the signal?
  - i.e. when is the relation correct,
  - i.e. when is the sum actually equal to $x(t)$?
- Dirichlet conditions: the sum is convergent in all continuity points if:
  1. $x(t)$ is continuous or has a finite number of discontinuities in any finite interval
  2. $x(t)$ has a finite number of maxima and minima in any period
  3. $x(t)$ is absolutely integrable in any period, i.e.:

$$\int_T |x(t)| dt < \infty$$

- Weaker condition:
  - if $x(t)$ is square summable

$$\int_T x(t)^2 dt < \infty$$

  - then the he difference $d(t) = x(t) - \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k F_0 t}$ has zero energy
  - Does not guarantee *pointwise* convergence

Signal spectrum

- The coefficients c_k are complex numbers
- If the signal is **real** $x(t) \in \mathbb{R}$, then the c_k are **even**
  - $|c_k| = |c_{-k}|$
  - $\angle c_k = -\angle c_{-k}$
  - group the terms with c_k with c_{-k} **–> cosine with amplitude |c_k| and phase** $\angle c_k$
- Average power of signal = energy of coefficients

$$P_T = \frac{1}{T} \int_T |x(t)|^2 = \sum_{-\infty}^{\infty} |c_k|^2$$

- Interpretation of Fourier series for real signal
  - **the signal is the sum of cosine signals with frequency** $0, F_0, 2F_0 ...$**, with amplitudes** $|c_k|$ **and phase** $\angle c_k$
- No other frequencies appear in spectrum –> spectrum is made of "lines"

Time-frequency duality

- Time-frequency **duality**:
  - Real signal –> Even spectrum
  - Periodic signal –> Discrete spectrum

Analog non-periodical signals

- The signal is composed of all frequencies (inverse Fourier transform)

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

- The frequency content is found by the Fourier transform

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

- (Remember: $\omega = 2\pi F$)
- $X(\omega)$ is a complex function
  - $|X(\omega)|$ is the amplitude spectrum
  - $\angle X(\omega)$ is the phase spectrum

Conditions for convergence

- When is the Fourier series convergent to the signal?
  - i.e. when is the relation correct,
  - i.e. when is the sum actually equal to $x(t)$?
- Dirichlet conditions: the sum is convergent in all continuity points if:
  1. $x(t)$ is continuous or has a finite number of discontinuities
  2. $x(t)$ has a finite number of maxima and minima
  3. $x(t)$ is absolutely integrable:

$$\int_{\infty}^{\infty} |x(t)| dt < \infty$$

- Weaker condition:
  - if $x(t)$ is square summable

$$\int_{\infty}^{\infty} x(t)^2 dt < \infty$$

  - then the he difference $d(t) = x(t) - \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k F_0 t}$ has zero energy
  - Does not guarantee *pointwise* convergence

Signal spectrum

- $X(\omega)$ is a complex function
- If the signal is **real** $x(t) \in \mathbb{R}$, then the $X(\omega)$ is **even**
  - $|X(\omega)| = |X(-\omega)|$
  - $\angle X(\omega) = -\angle X(-\omega)$
  - group the terms with c_k with c_{-k} **–>** **cosine with amplitude** $|X(\omega)|$ **and phase** $\angle X(\omega)$
- Signal energy is the same in time and frequency domains

$$E = \int_{\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

- The power spectral density of $x(t)$ is

$$S_{xx}(\omega) = |X(\omega)|^2$$

IV.2 Frequency analysis of discrete signals

Fourier series of discrete periodical signals

- A discrete signal of period N: $x[n] = x[n+N]$
- Decomposed as a **sum of complex exponentials**:

$$x[n] = \sum_{k=0}^{N-1} c_k e^{j2\pi kn/N}, n = 0, 1, ...N-1$$

- Finding the coefficients:

$$c_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N}$$

Comparison with analog Fourier series

- Compared to analog signals:
  - consider fundamental frequency $f_0 = 1/N$
  - only N terms, with frequencies $k \cdot f_0$:
    - $0, f_0, 2f_0, ...(N-1)f_0$
  - only N distinct coefficients c_k
  - the N coefficients c_k can be chosen like $-\frac{N}{2} < k \leq \frac{N}{2}$ => the frequencies span the range $-1/2...1/2$

$$-\frac{1}{2} < f_k \leq \frac{1}{2}$$

$$-\pi < \omega_k \leq \pi$$

Basic properties of Fourier coefficients

1. Signal is **discrete** –> coefficients are **periodic** with period N

$$c_{k+N} = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-j2\pi(k+N)n/N} = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N}$$

2. If signal is real $x[n] \in \mathbb{R}$, the coefficients are **even**:
    - $c_k^* = c_{-k}$
    - $|c_k| = |c_{-k}|$
    - $\angle c_k = \angle c_{-k}$

- Together with periodicity:
    - $|c_k| = |c_{-k}| = |c_{N-k}|$
    - $\angle c_k = -\angle c_{-k} = -\angle c_{N-k}$

Expressing as sum of sinusoids

- Grouping terms with c_k and c_{-k} we get

$$x[n] = c_0 + 2\sum_{k=1}^{L} |c_k| cos(2\pi\frac{k}{N} + \angle c_k)$$

  where $L = N/2$ or $L = (N-1)/2$ depending if N is even or odd
- Signal = DC value + a finite sum of sinusoids with frequencies kf_0
  - $|c_k|$ give the amplitudes (x 2)
  - $\angle c_k$ give the phases

Power spectral density

- The average power of a discrete periodic signal

$$P = \frac{1}{N} \sum_{n=0}^{N-1} |x[n]|^2$$

- Is the same in the frequency domain (with proof):

$$P = \sum_{k=0}^{N-1} |c_k|^2$$

- Power spectral density of the signal is

$$S_{xx}[k] = |c_k|^2$$

- Energy over one period is

$$E = \sum_{n-0}^{N-1} |x[n]|^2 = N \sum_{k-0}^{N-1} |c_k|^2$$

Examples

- Examples:

$$x_1[n] = cos(\sqrt{5}\pi n)$$
$$x_2[n] = 2sin(\frac{\pi}{3}n)$$
$$x_3[n] = \{1, 1, 0, 0\}$$

Example in Python

```
>>> import numpy as np
>>> from scipy.fftpack import fft, ifft
>>> x = np.array([1.0, 1.0, 0.0, 0.0])
>>> y = 1.0/4.0 * fft(x)
>>> y
array([ 0.50+0.j  ,  0.25-0.25j,  0.00+0.j  ,  0.25+0.25j])
```

Properties of Fourier series

1. Linearity

If the signal $x_1[n]$ has the Fourier series coefficients $\{c_k^{(1)}\}$, and $x_2[n]$ has $\{c_k^{(2)}\}$, then their sum has
$a \cdot x_1[n] + b \cdot x_2[n] \leftrightarrow \{a \cdot c_k^{(1)} + b \cdot c_k^{(2)}\}$
Proof: via definition

Properties of Fourier series

2. Shifting in time

If $x[n] \leftrightarrow \{c_k\}$, then

$$x[n - n_0] \leftrightarrow \{e^{(-j2\pi k n_0/N)} c_k\}$$

Proof: via definition

- The amplitudes $|c_k|$ are not affected, shifting in time affects only the phase

Properties of Fourier series

3. Modulation in time

$$e^{j2\pi k_0 n/N} \leftrightarrow \{c_{k-k_0}\}$$

4. Complex conjugation

$$x^*[n] \leftrightarrow \{c^*_{-k}\}$$

Properties of Fourier series

5. Circular convolution

Circular convolution of two signals $\leftrightarrow$ product of coefficients

$$x_1[n] \otimes x_2[n] \leftrightarrow \{N \cdot c_k^{(1)} \cdot c_k^{(2)}\}$$

Circular convolution:

$$x_1[n] \otimes x_2[n] = \sum_{k=0}^{N-1} x_1[k] x_2[(n-k)_N]$$

- takes two periodic signals of period N, result is the same
- Example at the whiteboard: how it is computed

Properties of Fourier series

6. Product in time

Product in time $\leftrightarrow$ circular convolution of Fourier series coefficients

$$x_1[n] \cdot x_2[n] \leftrightarrow \sum_{m=0}^{N-1} c_m^{(1)} c_{(k-m)_N}^{(2)} = c_k^{(1)} \otimes c_k^{(2)}$$

Fourier transform of discrete non-periodical signals

- Non-periodical signals contain all frequencies, not only the multiples of f_0
- The Fourier transform of a discrete signal:

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

- The inverse Fourier transform:

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega$$

Comparison

- Compared with the Fourier transform of analog signals
  - sum instead of integral in Fourier transform
  - spectrum is only in range:

$$\omega \in [-\pi, \pi]$$

$$f \in [-\frac{1}{2}, \frac{1}{2}]$$

- Compared with the Fourier series of discrete periodical signals
  - general ω instead of $2\pi k f_0$
  - spectrum is continuous, not discrete
  - integral, not sum in inverse Fourier transform

Parseval theorem

- **Parseval theorem**: energy of the signal is the same in time and frequency domains

$$E = \sum_{-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(\omega)|^2$$

- Proof: on whiteboard
- The **energy spectral density** gives the energy contained for each frequency

$$S_{xx}(\omega) = |X(\omega)|^2$$

Basic properties of Fourier transform

- It is **periodical** with period 2π

$$X(\omega + 2\pi) = X(\omega)$$

- If the signal $x[n]$ is real, the Fourier transform is **even**

$$x[n] \in \mathbb{R} \rightarrow X^*(\omega) = X(-\omega)$$

- This means
  - modulus is even: $|X(\omega)| = |X(-\omega)|$
  - phase is odd: $X(\omega) = -X(-\omega)$

Convergence of the Fourier transform

- When are the relations valid?
- Assume we compute the Fourier transform with only $2M + 1$ samples:

$$X_M(\omega) = \sum_{-M}^{M} x[n]e^{-j\omega n}$$

- If a signal $x[n]$ is **absolutely summable**:

$$\sum_{\infty}^{\infty} |x[n]| < \infty$$

- then the Fourier series is **uniform convergent** for every ω (OK):

$$\lim_{M\to\infty} X(\omega) - X_M(\omega) = 0$$

Convergence for square-summable signals

- Signals that are only **square summable**

$$\sum_{\infty}^{\infty} |x[n]|^2 < \infty$$

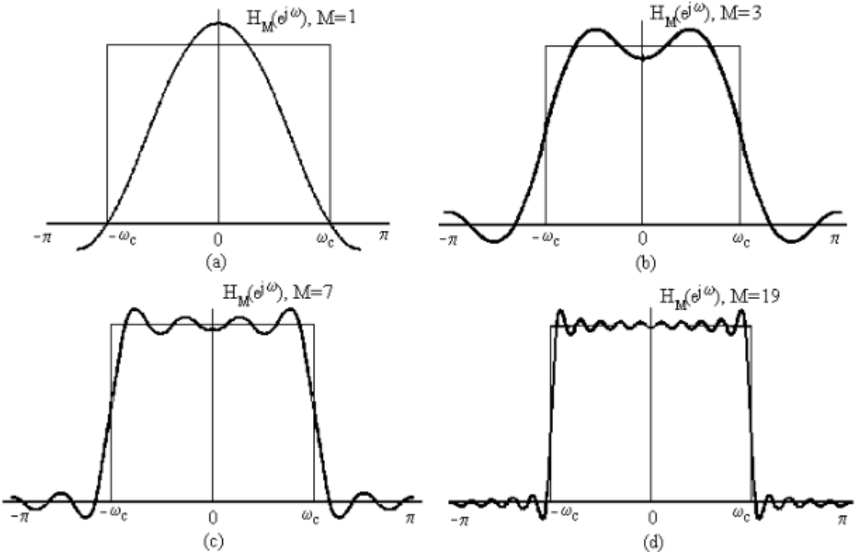
- have a weaker convergence:

$$\lim_{M\to\infty} \int_{-\pi}^{\pi} |X(\omega) - X_M(\omega)|^2 d\omega = 0$$

The Gibbs phenomenon

- When $H(\omega)$ **has discontinuities**, then $h[n]$ is **not** absolutely summable, only square summable
- Problem: if we only use M samples, even if M is very large, we will obtain **small oscillations around the discontinuity**
- As $M \to \infty$, the oscillations do not become smaller, but thinner –> they don't go away!
- The Fourier transform will always *overshoot* with about 9% below and above
- Known as the **Gibbs phenomenon**

Gibbs phenomenon



Relation between Fourier series and Fourier transform

- If apply Fourier transform to periodical discrete signals, $X(\omega)$ contains Diracs
- The Diracs are at frequencies kf_0, just like the Fourier series
- The value of an impulse $=$ the coefficient c_k of the Fourier series
- **The Fourier series $\approx$ the Fourier transform of periodic signals**
  - we directly compute the coefficients c_k of the impulses in the spectrum

Fourier transform and Z transform

- Definition of Fourier transform = Z transform with:

$$z = e^{j\omega}$$

- $e^{j\omega}$ = points on the unit circle
- Fourier transform = Z transform evaluated **on the unit circle**
    - if the unit circle is in the convergence region of Z transform
    - otherwise, equivalence does not hold
- This is true for most usual signals we work with
    - there are exceptions, but they are outside the scope of this class

Properties of Fourier transform

1. Linearity

$$a \cdot x_1[n] + b \cdot x_2[n] \leftrightarrow a \cdot X_1(\omega) + b \cdot X_2(\omega)$$

Proof: via definition

Properties of Fourier transform

2. Shifting in time

$$x[n-n_0] \leftrightarrow e^{-j\omega n_0} X(\omega)$$

Proof: via definition

- The amplitudes $|X(\omega)|$ is not affected, shifting in time affects only the phase

Properties of Fourier transform

3. Modulation in time

$$e^{j\omega_0 n}x[n] \leftrightarrow X(\omega - \omega_0)$$

4. Complex conjugation

$$x^*[n] \leftrightarrow X^*(-\omega)$$

Properties of Fourier transform

5. Convolution

$$x_1[n] * x_2[n] \leftrightarrow X_1(\omega) \cdot X_2(\omega)$$

- Not circular convolution, this is the normal convolution

Properties of Fourier transform

6. Product in time

Product in time $\leftrightarrow$ convolution of Fourier transforms

$$x_1[n] \cdot x_2[n] \leftrightarrow \frac{1}{2\pi} \int_{-\pi}^{\pi} X_1(\lambda) X_2(\omega - \lambda) d\lambda$$

Properties of Fourier transform

Correlation theorem

$$r_{x_1 x_2}[l] \leftrightarrow X_1(\omega) X_2(-\omega)$$

Wiener Khinchin theorem

Autocorrelation of a signal $\leftrightarrow$ Power spectral density

$$r_{xx}[l] \leftrightarrow S_{xx}(\omega) = |X(\omega)|^2$$

Properties of Fourier transform

Parseval theorem

Energy is the same when computed in the time or frequency domain

$$\sum |x[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(\omega)|^2 d\omega$$

Relationship of Fourier transform and Fourier series

- How are they related?
  - Fourier transform: for non-periodical signals
  - Fourier series: for periodical series
- Duality: periodic in time $\leftrightarrow$ discrete in frequency
- If we **periodize** a signal $x[n]$ by repeating with period N:

$$x_N[n] = \sum_{k=-\infty}^{\infty} x[n - kN]$$

- then the Fourier transform w is discrete (made of Diracs):

$$X_N(\omega) = 2\pi c_k \delta(\omega - k\frac{2\pi}{N})$$

- The coefficients of the Diracs = exactly the Fourier series coefficients

Relationship of Fourier transform and Fourier series

- So, Fourier transform can be considered for both periodic and non-periodic signals
- Fourier transform for periodic signals = discrete (sum of Diracs with some coefficients)
  - Diracs at frequencies $f_0 = 1/N$ and its multiples
- Fourier series for periodic signals = gives the coefficients of the Diracs directly
  - it just omits to write the Diracs explicitly in the equation

Relation of Fourier transform and Z transform

- Fourier transform: $X(\omega) = \sum_{-\infty}^{\infty} x[n]e^{-j\omega n}$
- Z transform: $X(z) = \sum_{-\infty}^{\infty} x[n]z^{-n}$
- **Fourier tranform = Z transform for** $z = e^{j\omega}$
- $z = e^{j\omega}$ means **evaluated on the unit circle**:
  - $|z| = |e^{j\omega}| = 1(modulus)$
  - $\angle z = \angle e^{j\omega} = \omega(phase)$
- Conditions:
  - unit circle must be in the Convergence Region of Z transform
  - some signals can have Fourier transform even though unit circle not in CR
- If signal has pole on unit circle –> Dirac (infinite) in Fourier transform
  - e.g. $u[n]$
  - some signals are non-convergent on unit circle, but have Fourier transform (e.g. $u[n]$)

Geometric interpretation of Fourier transform

$$X(z) = C \cdot \frac{(z - z_1)\cdots(z - z_M)}{(z - p_1)\cdots(z - p_N)}$$

$$X(\omega) = C \cdot \frac{(e^{j\omega} - z_1)\cdots(e^{j\omega} - z_M)}{(e^{j\omega} - p_1)\cdots(e^{j\omega} - p_N)}$$

- Modulus:

$$|X(\omega)| = |C| \cdot \frac{|e^{j\omega} - z_1|\cdots|e^{j\omega} - z_M|}{|e^{j\omega} - p_1|\cdots|e^{j\omega} - p_N|}$$

- Phase:

$$\angle X = \angle C + \angle(e^{j\omega} - z_1) + \cdots + \angle(e^{j\omega} - z_M) - \angle(e^{j\omega} - p_1) - \cdots - \angle(e^{j\omega} - p_N)$$

Geometric interpretation of Fourier transform

- For complex numbers:
  - modulus of $|a - b|$ = the length of the segment between a and b
  - phase of $|a - b|$ = the angle of the segment from b to a (direction is important)
- So, for a point on the unit circle $z = e^{j\omega}$
  - modulus $|X(\omega)|$ is **given by the distances to the zeros and to the poles**
  - phase $\angle X(\omega)$ is **given by the angles from the zeros and poles to z**

Geometric interpretation of Fourier transform

- Consequences:
  - when a **pole** is very close to unit circle –> Fourier transform is **large** at this point
  - when a **zero** is very close to unit circle –> Fourier transform is **small** at this point
- Examples:...
- Simple interpretation for modulus $|X(\omega)|$:
  - Z transform $X(z)$ is a *"landscape"*
    - poles = mountains of infinite height
    - zeros = valleys of zero height
  - Fourier transform $X(\omega)$ = "*Walking over this landscape along the unit circle*" –> the heights give the Fourier transform
  - When close to a mountain –> road is high –> Fourier transform has large amplitude
  - When close to a valley –> road is low –> Fourier transform has small amplitude
- Enough to sketch the Fourier transform for signals with few poles/zeros

Geometric interpretation of Fourier transform

- Note: $X(z)$ might also have a constant C in front!
  - It does not appear in pole-zero plot
  - The value of $|C|$ and $\angle C$ must be determined separately
- This "geometric method" can be applied for both modulus and phase

Time-frequency duality

- **Duality** properties related to Fourier transform/series
- Discrete $\leftrightarrow$ Periodic
  - **discrete** in time –> **periodic** in frequency
  - **periodic** in time –> **discrete** in frequency
- Continuous $\leftrightarrow$ Non-periodic
  - **continous** in time –> **non-periodic** in frequency
  - **non-periodic** in time –> **continuous** in frequency

Frequency-based classification of signals

- Based on frequency content:
  - **low-frequency** signals
  - **mid-frequency** signals (band-pass)
  - **high-frequency** signals
- **Band-limited** signals: spectrum is 0 over some frequency f_{max}
- **Time-limited** signals: signal value is 0 outside some time interval
- **Bandwitdh** B: frequency interval $[F_1, F_2]$ which contains 95% of energy
  - $B = F_2 - F_1$
- Based on bandwidth B:
  - **Narrow-band** signals: \$B $<<$ \$ central frequency $\frac{F_1+F_2}{2}$
  - **Wide-band** signals: not narrow-band