Information Theory

Chapter IV: Discrete transmission channels

What are they?

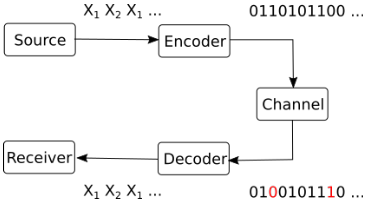

Figure 1: Communication system

- A device that transmits data from one place to another
- The data undergoes **distortions / errors**
- We consider that transmission is instantaneous

How do they work?

- A random variable $X \in \{x_1, x_2, ...\}$ is put at the input of the channel
- A random variable $Y \in \{y_1, y_2, ...\}$ appears immediately at the output of the channel
  - Y is related to X
- The receiver wants to find X, but can see only Y

Naming:

- The inputs $\{x_1, x_2, ...\}$ and outputs $\{y_1, y_2, ...\}$ are called **symbols**
- Symbols $\neq$ messages s_i from the source S
  - The encoder might convert s_i to a different representation
  - Example: source messages = characters, but channel symbols = 0/1 (encoder converts characters to binary)

What do we want?

- A successful communication = deduce the X which was sent from the Y that was received
- We are interested in **deducing X when knowing just Y**
- Main topic: How much does knowing Y tell us about X?
  - Depends on the relation between them
  - Is the same as how much X tells us about Y (symmetrical)

Probabilistic description

From a probabilistic point of view:

- A system of two related random variables
  - Input random variable $X \in \{x_1, x_2, ...\}$
  - Output random variable $Y \in \{y_1, y_2, ...\}$
  - It doesn't matter that one is *input* and other is *output*, we just care about the relation between the two random variables
- X and Y are *related* probabilistically, but still *random* (because of noise / errors / distortions)
  - All the probabilities are known
- We need to analyze the relation of X with Y

Intuitive examples

- Binary channel with errors
  - Send 0's and 1's, receive 0's and 1's, but with errors
- Pipe
  - Send colored balls over the pipe, but someone may be re-painting them
- Grandma calling!
  - She says *"cat"* / *"hat"* / *"pet"*, but sometimes you hear her wrong
- Living near stadium
  - You don't actually see the game, but try to deduce the score from the shouts you hear

Nomenclature

We only deal with discrete memoryless stationary channels

- Discrete: number of input and output symbols is finite
- Memoryless: the output symbol depends only on the current input symbol
- Stationary: the probabilities involved do not change in time

Systems of two random variables

- Two random variables: $X = \{x_1, x_2, ...\}$, $Y = \{y_1, y_2, ...\}$.
- Example: throw a dice (X) and a coin (Y) simultaneously
- How to describe this system?

A single joint information source:

$$X \cap Y : \begin{pmatrix} x_1 \cap y_1 & x_1 \cap y_2 & ... & x_i \cap y_j \\ p(x_1 \cap y_1) & p(x_1 \cap y_2) & ... & p(x_i \cap y_j) \end{pmatrix}$$

Arrange in a nicer form (table):

	y_1	y_2	y_3
x_1	...	...	...
x_2	...	...	...
x_3	...	...	...

- Elements of the table: $p(x_i \cap y_j)$

Joint probability matrix

The table constitutes the **joint probability matrix**:

$$P(X,Y) = \begin{bmatrix} p(x_1 \cap y_1) & p(x_1 \cap y_2) & \cdots & p(x_1 \cap y_M) \\ p(x_2 \cap y_1) & p(x_2 \cap y_2) & \cdots & p(x_2 \cap y_M) \\ \vdots & \vdots & \cdots & \vdots \\ p(x_N \cap y_1) & p(x_N \cap y_2) & \cdots & p(x_N \cap y_M) \end{bmatrix}$$

$$\sum_i \sum_j p(x_i \cap y_j) = 1$$

- This matrix completely defines the two-variable system
- This matrix completely defines the communication process

Joint entropy

- The distribution $X \cap Y$ determines the **joint entropy**:

$$H(X,Y) = -\sum_i \sum_j p(x_i \cap y_j) \cdot \log(p(x_i \cap y_j))$$

- This is the global entropy of the system (knowing the input and the output)

Marginal distributions

- $p(x_i) = \sum_j p(x_i \cap y_j) =$ sum of row i from P(X,Y)
- $p(y_j) = \sum_i p(x_i \cap y_j) =$ sum of column j from P(X,Y)
- The distributions $p(x)$ and $p(y)$ are called **marginal distributions** ("summed along the margins")

Examples [marginal distributions not enough]

Marginal distributions don't tell everything about the system:

- Example 1:

$$P(X,Y) = \begin{bmatrix} 0.3 & 0 \\ 0 & 0.7 \end{bmatrix}$$

- Example 2:

$$P(X,Y) = \begin{bmatrix} 0.15 & 15 \\ 0.15 & 0.55 \end{bmatrix}$$

- Both have identical $p(x)$ and $p(y)$, but are completely different
- Which one is better for a transmission?
- Marginal distribution are useful, but not enough. Essential is the *relation* between X and Y.

Bayes formula

$$p(A \cap B) = p(A) \cdot p(B|A)$$

$$p(B|A) = \frac{p(A \cap B)}{p(A)}$$

- "The conditional probability of B **given A**" (i.e. given that event A happened)
- Examples: listen to the lecture

When A and B are independent events:

$$p(A \cap B) = p(A)p(B)$$

$$p(B|A) = p(B)$$

- The fact that event A happened doesn't influence B at all

Three examples

Examples to help you remember conditional probabilities

- Gambler's paradox
- CNN: Crippled cruise ship returns; passengers happy to be back

Channel matrix

Noise (or channel) matrix:

$$P(Y|X) = \begin{bmatrix} p(y_1|x_1) & p(y_2|x_1) & \cdots & p(y_M|x_1) \\ p(y_1|x_2) & p(y_2|x_2) & \cdots & p(y_M|x_2) \\ \vdots & \vdots & \cdots & \vdots \\ p(y_1|x_N) & p(y_2|x_N) & \cdots & p(y_M|x_N) \end{bmatrix}$$

- Defines the probability of an output **given an input**
- Each row = a separate distribution that indicates the probability of the outputs **if the input is** x_i
- The sum of each row is 1 (there must be some output if the input is x_i

Relation of channel matrix and joint probability matrix

- $P(Y|X)$ is obtained from $P(X,Y)$ by dividing every row to its sum ($p(x_i)$)
- This is known as *normalization* of rows
- $P(X,Y)$ can be obtained back from $P(Y|X)$ by multiplying each row with $p(x_i)$
- $P(Y|X)$ contains less information than $P(X,Y)$
  - it doesn't tell us the probabilities $p(x_i)$ anymore

Definition of a discrete transmission channel

Definition: A discrete transmission channel is defined by three items:

1. The input alphabet $X = \{x_1, x_2, \ldots\}$
2. The output alphabet $Y = \{y_1, y_2, \ldots\}$
3. The noise (channel) matrix $P(Y|X)$ which defines the conditional probabilities of the outputs y_j for every possible input x_i

Graphical representation of a channel

- Nice picture with arrows :)

Intuitive examples

- Postal service
- Play and win the lottery
  - $+$ funny joke

Conditional entropy H(Y|X) (mean error)

- Since each row in $P(Y|X)$ is a distribution, each row has an entropy
- Entropy of row x_i:

$$H(Y|x_i) = -\sum_j p(y_j|x_i)\log(p(y_j|x_i))$$

- $H(Y|x_i) =$ *"The uncertainty of the output symbol when the input symbol is x_i"*
- Example: lottery

Conditional entropy H(Y|X) (mean error)

- There may be a different value $H(Y|x_i)$ for every x_i
- Compute the average over all x_i:

$$\begin{aligned} H(Y|X) &= \sum_i p(x_i) H(Y|x_i) \\ &= -\sum_i \sum_j p(x_i) p(y_j|x_i) \log(p(y_j|x_i)) \\ &= -\sum_i \sum_j p(x_i \cap y_j) \log(p(y_j|x_i)) \end{aligned}$$

- $H(Y|X) =$ **"The uncertainty of the output symbol when we know the input symbol"** (any input, in general)
- Also known as **average error**

Equivocation matrix

Equivocation matrix:

$$P(X|Y) = \begin{bmatrix} p(x_1|y_1) & p(x_1|y_2) & \cdots & p(x_1|y_M) \\ p(x_2|y_1) & p(x_2|y_2) & \cdots & p(x_2|y_M) \\ \vdots & \vdots & \cdots & \vdots \\ p(x_N|y_1) & p(x_N|y_2) & \cdots & p(x_N|y_M) \end{bmatrix}$$

- Defines the probability of an input **given an output**
- Each column = a separate distribution that indicates the probability of the inputs **if the output is** y_j
- The sum of each column is 1 (there must be some input if the output is y_j

Relation of equivocation matrix and joint probability matrix

- $P(X|Y)$ is obtained from $P(X,Y)$ by dividing every column to its sum ($p(y_j)$)
- This is known as *normalization* of columns
- $P(X,Y)$ can be obtained back from $P(X|Y)$ by multiplying each column with $p(y_j)$
- $P(X|Y)$ contains less information than $P(X,Y)$
  - it doesn't tell us the probabilities $p(y_i)$ anymore

Conditional entropy H(X|Y) (equivocation)

- Since each column is a distribution, each column has an entropy
- Entropy of column y_j:

$$H(X|y_j) = -\sum_i p(x_i|y_j) \log(p(x_i|y_j))$$

- $H(X|y_j) =$ *"The uncertainty of the input symbol when the output symbol is y_j"*

Conditional entropy H(X|Y) (equivocation)

- A different $H(X|y_j)$ for every y_j
- Compute the average over all y_j:

$$\begin{aligned} H(X|Y) &= \sum_j p(y_j) H(X|y_j) \\ &= -\sum_i \sum_j p(y_j) p(x_i|y_j) \log(p(x_i|y_j)) \\ &= -\sum_i \sum_j p(x_i \cap y_j) \log(p(x_i|y_j)) \end{aligned}$$

- **“The uncertainty of the input symbol when we know the output symbol”** (any output, in general)
- Also known as **equivocation**
- Should be small for a good communication

Properties of conditional entropies

For a general system with two random variables X and Y:

- Conditioning always reduces entropy:

$$H(X|Y) \leq H(X)$$

$$H(Y|X) \leq H(Y)$$

  (knowing something cannot harm)

- If the variables are independent:

$$H(X|Y) = H(X)$$

$$H(Y|X) = H(Y)$$

  (knowing the second variable does not help at all)

Mutual information I(X,Y)

- Mutual information I(X,Y) = the average information that one variable has about the other
- Mutual information I(X,Y) = the average information that is transmitted on the channel
- Consider a communication channel with X as input and Y as output:
  - We are the receiver and we want to find out the X
  - When we don't know the output: H(X)
  - When we know the output: H(X|Y)
- How much information was transmitted?
  - Reduction of uncertainty:

$$I(X,Y) = H(X) - H(X|Y)$$

Mutual information I(X,Y)

$$\begin{aligned}
I(X,Y) &= H(X) - H(X|Y) \\
&= -\sum_i p(x_i)\log(p(x_i)) + \sum_i \sum_j p(x_i \cap y_j)\log(p(x_i|y_j)) \\
&= -\sum_i \sum_j p(x_i \cap y_j)\log(p(x_i)) + \sum_i \sum_j p(x_i \cap y_j)\log(p(x_i|y_j)) \\
&= \sum_i \sum_j p(x_i \cap y_j)\log(\frac{p(x_i|y_j)}{p(x_i)}) \\
&= \sum_i \sum_j p(x_i \cap y_j)\log(\frac{p(x_i \cap y_j)}{p(x_i)p(y_j)})
\end{aligned}$$

Properties of mutual information

Mutual information $I(X, Y)$ is:

- commutative: $I(X, Y) = I(Y, X)$
- non-negative: $I(X, Y) \geq 0$
- a special case of the Kullback–Leibler distance (relative entropy distance)

Relation to Kullback-Leibler distance

- $I(X, Y)$ is a special case of the Kullback-Leibler distance

$$D_{KL}(P||Q) = \sum_i p(s_i) \log(\frac{p(s_i)}{q(s_i)})$$

- In our case, the distributions are:
  - $p(s_i) = p(x_i \cap y_j)$ = joint distribution of X and Y our system
  - $q(s_i) = p(x_i) \cdot p(y_j)$ = joint distribution when X and Y are independent

$$I(X, Y) = D_{KL}(p(x_i \cap y_j)||p(x_i) \cdot p(y_j))$$

- Interpretation
  - When X and Y are independent, mutual information $I(X, Y) = 0$
  - Our mutual information = how far away are from being independent
  - Example: height of a point = how far is it from the point of 0 height

Relations between the informational measures

- Nice picture with two circles :)
- All six: $H(X)$, $H(Y)$, $H(X,Y)$, $H(X|Y)$, $H(Y|X)$, $I(X,Y)$
- All relations on the picture are valid relations:

$$H(X,Y) = H(X) + H(Y) - I(X,Y)$$

$$H(X,Y) = H(X) + H(Y|X) = H(Y) + H(X|Y)$$

$$I(X,Y) = H(X) - H(X|Y) = H(Y) - H(Y|X)$$

...

- If know three, can find the other three
- Simplest to find first H(X), H(Y), H(X,Y) —> then find others

Types of communication channels

1. Channels with zero equivocation

$$H(X|Y) = 0$$

   - Each column of the noise (channel) matrix contains only one non-zero value
   - No doubts on the input symbols when the output symbols are known
   - All input information is transmitted

$$I(X, Y) = H(X)$$

- Example: codewords...

Types of communication channels

2. Channels with zero mean error

$$H(Y|X) = 0$$

- Each row of the noise (channel) matrix contains only one non-zero value
- No doubts on the output symbols when the input symbols are known
- *The converse is not necessary true!*

- Example: AND gate

Types of communication channels

3. Channels uniform with respect to the input

$$H(Y|x_i) = same$$

- Each row of noise matrix contains the same values, possibly in different order
- $H(Y|x_i) = same = H(Y|X)$
- $H(Y|X)$ does not depend on the actual probabilities $p(x_i)$

Types of communication channels

4. Channels uniform with respect to the output

- Each column of noise matrix contains the same values, possibly in different order
- If the input symbols are equiprobable, the output symbols are also equiprobable
- Attention:

$$H(X|y_j) \neq same!$$

Types of communication channels

5. Symmetric channels
    - Uniform with respect to the input and to the output
    - Example: binary symmetric channel

Input probabilities are important

- Suppose we have a channel defined by P(Y|X)
- I(X,Y) **depends on the input probabilities** $p(x_i)$
  - For some distribution $p(x_i)$, we get a value of I(X,Y)
  - For a different distribution $p(x_i)$, we get a different I(X,Y)
- We want I(X,Y) to be as large as possible
- Questions:
  - what is the largest possible value of I(X,Y) (depending on $p(x_i)$)?
  - For what distribution $p(x_i)$?

Channel capacity

- What is the maximum information I(X,Y) we can transmit on a certain channel?
- **Definition:** the **information capacity of a channel** is the maximum value of the mutual information, where the maximization is done over the input probabilities $p(x_i)$

$$C = \max_{p(x_i)} I(X, Y)$$

- i.e. the maximum mutual information we can obtain if we are allowed to choose $p(x_i)$ as we want
- Use together with definition of $I(X, Y)$:

$$C = \max_{p(x_i)} (H(Y) - H(Y|X))$$

$$C = \max_{p(x_i)} (H(X) - H(X|Y))$$

What channel capacity means

- Channel capacity is the maximum information we can transmit on a channel, on average, with one symbol
- One of the most important notions in information theory
- Its importance comes from Shannon's second theorem (noisy channel theorem)
- It allows us to compare channels

Preview of the channel coding theorem

- For transmission with no errors, we use **error coding** of data before transmission
- How error coding usually works:
  - For each k symbols of data, coder appends additional m symbols, computed via some coding algorithm
  - All of them are sent on the channel
  - The decoder detects/corrects errors based on the additional m bits
- Coding rate:

$$R = \frac{k}{k+m}$$

  - stronger protection = bigger m = less efficient
  - weaker protection = smaller m = more efficient

Preview of the channel coding theorem

- A rate is called **achievable** for a channel if, for that rate, there exists a coding and decoding algorithm guaranteed to correct all possible errors on the channel

Shannon's noisy channel coding theorem (second theorem)

For a given channel, all rates below capacity $R < C$ are achievable. All rates above capacity, $R > C$, are not achievable.

Channel coding theorem explained

In layman terms:

- For all coding rates $R < C$, there is a way to recover the transmitted data perfectly (decoding algorithm will detect and correct all errors)
- For all coding rates $R > C$, there is no way to recover the transmitted data perfectly

Example:

- Send binary digits (0,1) on a channel with capacity 0.7 bits/message
- There exists coding schemes with R $<$ 0.7 that allow perfect recovery
  - i.e. for every 7 bits of data coding adds 3 or more bits, on average $=>$ $R = \frac{7}{7+3}$
- With less than 3 bits for every 7 bits of data $=>$ impossible to recover all the data

Efficiency and redundancy

- Efficiency of a channel:

$$\eta_C = \frac{I(X,Y)}{C}$$

- Absolute redundancy of a channel:

$$R_C = C - I(X,Y)$$

- Relative redundancy of a channel:

$$\rho_C = \frac{R_C}{C} = 1 - \frac{I(X,Y)}{C} = 1 - \eta_C$$

Computing the capacity

- Tricks for easier computation of the capacity
- Channel is uniform with respect to the input:
  - $H(Y|X)$ does not depend on the actual probabilities $p(x_i)$
  - $C = \max_{p(x_i)} I(X,Y) = \max_{p(x_i)} (H(Y) - H(Y|X)) = \max_{p(x_i)} (H(Y)) - H(Y|X)$
  - Should maximize $H(Y)$
- If channel is also uniform with respect to the output:
  - same values on columns of P(Y|X)
  - $p(y_j) = \sum_i p(y_j|x_i)p(x_i)$
  - if $p(x_i) = \text{uniform} = \frac{1}{n}$, then $p(y_j) = \frac{1}{n}\sum_i p(y_j|x_i) = \text{uniform}$
  - therefore $p(y_j)$ are constant = uniform = H(Y) is maximized
  - H(Y) is maximized when H(X) is maximized (equiprobable symbols)

Computing the capacity

- If channel is symmetric: use both tricks
  - $C = \max_{p(x_i)} (H(Y)) - H(Y|X)$
  - H(Y) is maximized when H(X) is maximized (equiprobable symbols)

Examples of channels and their capacity

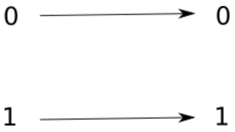

Figure 2: Noiseless binary channel

- Capacity $= 1$ bit/message, when $p(x_1) = p(x_2) = \frac{1}{2}$

Noisy binary non-overlapping channel

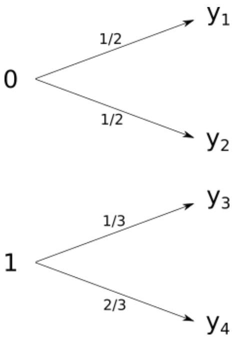

Figure 3: Noisy binary non-overlapping

- There is noise ($H(Y|X) > 0$), but can deduce the input ($H(X|Y) = 0$)
- Capacity = 1 bit/message, when $p(x_1) = p(x_2) = \frac{1}{2}$

Noisy typewriter

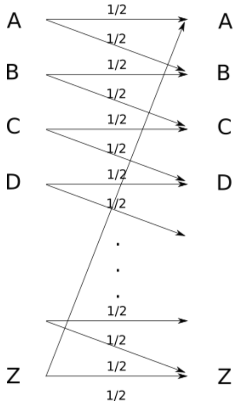

Figure 4: Noisy typewriter

$$\max I(X,Y) = \max\left(H(Y) - H(Y|X)\right) = \max H(Y) - 1$$
$$= \log(26) - 1 = \log(13)$$

Noisy typewriter

- Capacity $= \log(13)$ bit/message, when input probabilities are uniform
- Can transmit 13 letters with no errors (A, C, E, G, ...)

Binary symmetric channel

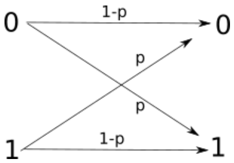

Figure 5: Binary symmetric channel (BSC)

- Capacity $= 1 - H_p = 1 + p\log(p) + (1-p)\log(1-p)$
- Capacity is reached when input distribution is uniform

Binary erasure channel

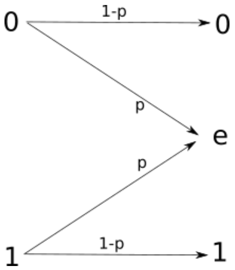

Figure 6: Binary erasure channel

- Different from BSC: here we know when errors happened
- Capacity $= 1 - p$
- Intuitive meaning: lose p bits, remaining bits $=$ capacity $= 1 - p$

Symmetric channel of n-th order

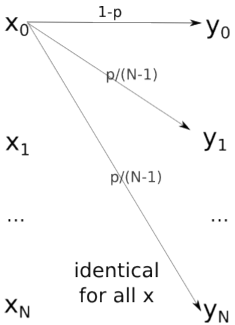

Figure 7: N-th order symmetric channel

- Extension of binary symmetric channel for n symbols
- $1 - p$ chances that symbol has no error
- p chances that symbol is changed, uniformly to any other (N-1) symbols ($\frac{p}{N-1}$ each)

Symmetric channel of n-th order

- Channel is symmetric $=>$

$$C = \max_{p(x_i)} I(X,Y) = \max_{p(x_i)} (H(Y) - H(Y|X)) = \max_{p(x_i)} (H(Y)) - H(Y|X)$$

- $\max_{p(x_i)} (H(Y)) = \log(N)$
- $H(Y|X) = H(Y|x_i) =$ entropy of any row (same values)

$==>$

$$C = \log(N) + (1-p)\log(1-p) + p\log(\frac{p}{N-1})$$

- Capacity is reached when input probabilities are uniform

Chapter summary

- Channel = Probabilistic system with two random variables X and Y
- Characterization of transmission:
  - P(X,Y) => H(X,Y) *joint entropy*
  - $p(x_i)$, $p(y_j)$ *marginal distributions* => H(X), H(Y)
  - P(Y|X) *channel matrix* => H(Y|X) *average noise*
  - P(X|Y) => H(X|Y) *equivocation*
  - I(X,Y) *mutual information*
- Channel capacity: $C = \max_{p(x_i)} I(X, Y)$
- Examples:
  - Binary symmetric channel: $C = 1 - H_p$
  - Binary erasure channel: $C = 1 - p$
  - N-th symmetric channel: $C = \log(N) - H(\textit{of a row of channel matrix})$

History

Figure 8: Claude Shannon (1916 - 2001)

- *A mathematical theory of communications*, 1948

Exercises and problems

- At blackboard only