Information Theory
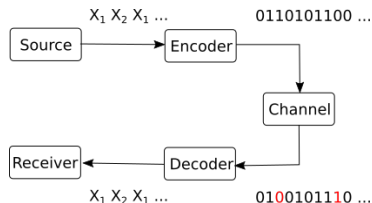
Chapter II: Discrete transmission channels

Figure 1: Communication system

▶ A device that transmits data from one place to another

▶ The data undergoes **distortions / errors**

▶ We consider that transmission is instantaneous

How do they work?

- A random variable $X \in \{x_1, x_2, ...\}$ is put at the input of the channel

- A random variable $Y \in \{y_1, y_2, ...\}$ appears immediately at the output of the channel

  - Y is related to X

- The receiver wants to find X, but can see only Y

Naming:

- The inputs $\{x_1, x_2, ...\}$ and outputs $\{y_1, y_2, ...\}$ are called **symbols**

- Symbols $\neq$ messages s_i from the source S

  - The encoder might convert s_i to a different representation
  - Example: source messages = characters, but channel symbols = 0/1 (encoder converts characters to binary)

What do we want?

- A successful communication $=$ deduce the X which was sent from the Y that was received
- We are interested in **deducing X when knowing just Y**
- Main topic: How much does knowing Y tell us about X?
    - Depends on the relation between them
    - Is the same as how much X tells us about Y (symmetrical)

Probabilistic description

From a probabilistic point of view:

- A system of two related random variables
  - Input random variable $X \in \{x_1, x_2, ...\}$
  - Output random variable $Y \in \{y_1, y_2, ...\}$
  - It doesn't matter that one is *input* and other is *output*, we just care about the relation between the two random variables
- X and Y are *related* probabilistically, but still *random* (because of noise / errors / distortions)
  - All the probabilities are known
- We need to analyze the relation of X with Y

Intuitive examples

- ▶ Binary channel with errors
  - ▶ Send 0's and 1's, receive 0's and 1's, but with errors
- ▶ Pipe
  - ▶ Send colored balls over the pipe, but someone may be re-painting them
- ▶ Grandma calling!
  - ▶ She says *"cat"* / *"hat"* / *"pet"*, but sometimes you hear her wrong
- ▶ Living near stadium
  - ▶ You don't actually see the game, but try to deduce the score from the shouts you hear

Nomenclature

- ▶ Discrete: the input alphabet and the output alphabet are finite
- ▶ Memoryless: the output symbol depends only on the current input symbol
- ▶ Stationary: the noise arising on the channel is time invariant (i.e. its statistics do not vary in time)

Systems of two random variables

- Two random variables: $X = \{x_1, x_2, ...\}$, $Y = \{y_1, y_2, ...\}$.
- Example: throw a dice (X) and a coin (Y) simultaneously
- How to describe this system?

A single joint information source:

$$X \cap Y : \begin{pmatrix} x_1 \cap y_1 & x_1 \cap y_2 & ... & x_i \cap y_j \\ p(x_1 \cap y_1) & p(x_1 \cap y_2) & ... & p(x_i \cap y_j) \end{pmatrix}$$

Arrange in a nicer form (table):

	y_1	y_2	y_3
x_1	...	...	...
x_2	...	...	...
x_3	...	...	...

- Elements of the table: $p(x_i \cap y_j)$

Joint probability matrix

The table constitutes the **joint probability matrix**:

$$P(X, Y) = \begin{bmatrix} p(x_1 \cap y_1) & p(x_1 \cap y_2) & \cdots & p(x_1 \cap y_M) \\ p(x_2 \cap y_1) & p(x_2 \cap y_2) & \cdots & p(x_2 \cap y_M) \\ \vdots & \vdots & \cdots & \vdots \\ p(x_N \cap y_1) & p(x_N \cap y_2) & \cdots & p(x_N \cap y_M) \end{bmatrix}$$

$$\sum_i \sum_j p(x_i \cap y_j) = 1$$

▶ This matrix completely defines the two-variable system
▶ This matrix completely defines the communication process

Joint entropy

▶ The distribution $X \cap Y$ determines the **joint entropy**:

$$H(X, Y) = -\sum_i \sum_j p(x_i \cap y_j) \cdot \log(p(x_i \cap y_j))$$

▶ This is the global entropy of the system (knowing the input and the output)

- $p(x_i) = \sum_j p(x_i \cap y_j) =$ sum of row i from P(X,Y)
- $p(y_j) = \sum_i p(x_i \cap y_j) =$ sum of column j from P(X,Y)
- The distributions $p(x)$ and $p(y)$ are called **marginal distributions** ("summed along the margins")

Examples [marginal distributions not enough]

Marginal distributions are not enough:

▶ Example 1:

$$P(X, Y) = \begin{bmatrix} 0.3 & 0 \\ 0 & 0.7 \end{bmatrix}$$

▶ Example 2:

$$P(X, Y) = \begin{bmatrix} 0.15 & 15 \\ 0.15 & 0.55 \end{bmatrix}$$

▶ Both have identical $p(x)$ and $p(y)$, but are completely different
▶ Which one is better for a transmission?
▶ Marginal distribution are useful, but not enough. Essential is the *relation* between X and Y.

Bayes formula

$$p(A \cap B) = p(A) \cdot p(B|A)$$

$$p(B|A) = \frac{p(A \cap B)}{p(A)}$$

► "The conditional probability of B **given A**" (i.e. given that event A happened)

► Examples: listen to the lecture

When A and B are independent events:

$$p(A \cap B) = p(A)p(B)$$

$$p(B|A) = p(B)$$

► The fact that event A happened doesn't influence B at all

Examples to help you remember conditional probabilities

▶ Gambler's paradox

▶ CNN: Crippled cruise ship returns; passengers happy to be back

Channel matrix

Noise (or channel) matrix:

$$P(Y|X) = \begin{bmatrix} p(y_1|x_1) & p(y_2|x_1) & \cdots & p(y_M|x_1) \\ p(y_1|x_2) & p(y_2|x_2) & \cdots & p(y_M|x_2) \\ \vdots & \vdots & \cdots & \vdots \\ p(y_1|x_N) & p(y_2|x_N) & \cdots & p(y_M|x_N) \end{bmatrix}$$

▶ Defines the probability of an output **given an input**

▶ Each row = a separate distribution that indicates the probability of the outputs **if the input is** x_i

▶ The sum of each row is 1 (there must be some output if the input is x_i

Relation of channel matrix and joint probability matrix

- $P(Y|X)$ is obtained from $P(X, Y)$ by dividing every row to its sum ($p(x_i)$)
- This is known as *normalization* of rows
- $P(X, Y)$ can be obtained back from $P(Y|X)$ by multiplying each row with $p(x_i)$
- $P(Y|X)$ contains less information than $P(X, Y)$
  - it doesn't tell us the probabilities $p(x_i)$ anymore

Definition: A discrete transmission channel is defined by three items:

1. The input alphabet $X = \{x_1, x_2, \ldots\}$
2. The output alphabet $Y = \{y_1, y_2, \ldots\}$
3. The noise (channel) matrix $P(Y|X)$ which defines the conditional probabilities of the outputs y_j for every possible input x_i

Graphical representation of a channel

- Nice picture with arrows :)

- Postal service
- Play and win the lottery
    - + funny joke

Conditional entropy H(Y|X) (mean error)

▶ Since each row is a distribution, each row has an entropy
▶ Entropy of row x_i:

$$H(Y|x_i) = -\sum_j p(y_j|x_i) \log(p(y_j|x_i))$$

▶ *"The uncertainty of the output message when the input message is x_i"*
▶ Example: lottery

- A different $H(Y|x_i)$ for every x_i
- Compute the average over all x_i:

$$\begin{aligned}
H(Y|X) &= \sum_i p(x_i) H(Y|x_i) \\
&= -\sum_i \sum_j p(x_i) p(y_j|x_i) \log(p(y_j|x_i)) \\
&= -\sum_i \sum_j p(x_i \cap y_j) \log(p(y_j|x_i))
\end{aligned}$$

- **"The uncertainty of the output message when we know the input message"** (any input, in general)
- Also known as **average error**

Equivocation matrix

Equivocation matrix:

$$P(X|Y) = \begin{bmatrix} p(x_1|y_1) & p(x_1|y_2) & \cdots & p(x_1|y_M) \\ p(x_2|y_1) & p(x_2|y_2) & \cdots & p(x_2|y_M) \\ \vdots & \vdots & \cdots & \vdots \\ p(x_N|y_1) & p(x_N|y_2) & \cdots & p(x_N|y_M) \end{bmatrix}$$

▶ Defines the probability of an input **given an output**

▶ Each column = a separate distribution that indicates the probability of the inputs **if the output is** y_j

▶ The sum of each column is 1 (there must be some input if the output is y_j

- $P(X|Y)$ is obtained from $P(X,Y)$ by dividing every column to its sum ($p(y_j)$)
- This is known as *normalization* of columns
- $P(X,Y)$ can be obtained back from $P(X|Y)$ by multiplying each column with $p(y_j)$
- $P(X|Y)$ contains less information than $P(X,Y)$
    - it doesn't tell us the probabilities $p(y_i)$ anymore

▶ Since each column is a distribution, each column has an entropy
▶ Entropy of column y_j:

$$H(X|y_j) = -\sum_i p(x_i|y_j) \log(p(x_i|y_j))$$

▶ *"The uncertainty of the input message when the output message is y_j"*
▶ Example: . . .

Conditional entropy H(X|Y) (equivocation)

- A different $H(X|y_j)$ for every y_j
- Compute the average over all y_j:

$$H(X|Y) = \sum_j p(y_j) H(X|y_j)$$

$$= - \sum_i \sum_j p(y_j) p(x_i|y_j) \log(p(x_i|y_j))$$

$$= - \sum_i \sum_j p(x_i \cap y_j) \log(p(x_i|y_j))$$

- **"The uncertainty of the input message when we know the output message"** (any output, in general)
- Also known as **equivocation**
- Should be small for a good communication

Properties of conditional entropies

For a general system with two random variables X and Y:

- Conditioning always reduces entropy:

$$H(X|Y) \leq H(X)$$

$$H(Y|X) \leq H(Y)$$

  (knowing something cannot harm)

- If the variables are independent:

$$H(X|Y) = H(X)$$

$$H(Y|X) = H(Y)$$

  (knowing the second variable does not help at all)

Mutual information I(X,Y)

- Mutual information I(X,Y) = the average information that one variable has about the other
- Mutual information I(X,Y) = the average information that is transmitted on the channel
- Consider a communication channel with X as input and Y as output:
  - We are the receiver and we want to find out the X
  - When we don't know the output: H(X)
  - When we know the output: H(X|Y)
- How much information was transmitted?
  - Reduction of uncertainty:

$$I(X, Y) = H(X) - H(X|Y)$$

$$I(X, Y) = H(X) - H(X|Y)$$

$$= -\sum_i p(x_i) \log(p(x_i)) + \sum_i \sum_j p(x_i \cap y_j) \log(p(x_i|y_j))$$

$$= -\sum_i \sum_j p(x_i \cap y_j) \log(p(x_i)) + \sum_i \sum_j p(x_i \cap y_j) \log(p(x_i|y_j))$$

$$= \sum_i \sum_j p(x_i \cap y_j) \log\left(\frac{p(x_i|y_j)}{p(x_i)}\right)$$

$$= \sum_i \sum_j p(x_i \cap y_j) \log\left(\frac{p(x_i \cap y_j)}{p(x_i)p(y_j)}\right)$$

Mutual information $I(X, Y)$ is:

- commutative: $I(X, Y) = I(Y, X)$
- non-negative: $I(X, Y) \geq 0$
- a special case of the Kullback–Leibler distance (relative entropy distance)

Relation to Kullback-Leibler distance

▶ $I(X, Y)$ is a special case of the Kullback-Leibler distance

$$D_{KL}(P||Q) = \sum_i p(s_i) \log(\frac{p(s_i)}{q(s_i)})$$

▶ In our case, the distributions are:

  ▶ $p(s_i) = p(x_i \cap y_j) =$ joint distribution of X and Y our system
  ▶ $q(s_i) = p(x_i) \cdot p(y_j) =$ joint distribution when X and Y are independent

$$I(X, Y) = D_{KL}(p(x_i \cap y_j)||p(x_i) \cdot p(y_j))$$

▶ Interpretation

  ▶ When X and Y are independent, mutual information $I(X, Y) = 0$
  ▶ Our mutual information $=$ how far away are from being independent
  ▶ Example: height of a point $=$ how far is it from the point of 0 height

Relations between the informational measures

▶ Nice picture with two circles :)
▶ All six: $H(X)$, $H(Y)$, $H(X,Y)$, $H(X|Y)$, $H(Y|X)$, $I(X,Y)$
▶ All relations on the picture are valid relations:

$$H(X,Y) = H(X) + H(Y) - I(X,Y)$$

$$H(X,Y) = H(X) + H(Y|X) = H(Y) + H(X|Y)$$

$$I(X,Y) = H(X) - H(X|Y) = H(Y) - H(Y|X)$$

...

▶ If know three, can find the other three
▶ Simplest to find first H(X), H(Y), H(X,Y) —> then find others

Types of communication channels

1. Channels with zero equivocation

$$H(X|Y) = 0$$

- ▶ Each column of the noise (channel) matrix contains only one non-zero value
- ▶ No doubts on the input messages when the output messages are known
- ▶ All input information is transmitted

$$I(X, Y) = H(X)$$

▶ Example: codewords. . .

Types of communication channels

2. Channels with zero mean error

$$H(Y|X) = 0$$

- ▶ Each row of the noise (channel) matrix contains only one non-zero value
- ▶ No doubts on the output messages when the input messages are known
- ▶ *The converse is not necessary true!*

▶ Example: AND gate

3. Channels uniform with respect to the input

$$H(Y|x_i) = same$$

▶ Each row of noise matrix contains the same values, possibly in different order
▶ $H(Y|x_i) = same = H(Y|X)$
▶ $H(Y|X)$ does not depend on the actual probabilities $p(x_i)$

4. Channels uniform with respect to the output

▶ Each column of noise matrix contains the same values, possibly in different order

▶ If the input messages are equiprobable, the output messages are also equiprobable

▶ Attention:

$$H(X|y_j) \neq same!$$

5. Symmetric channels
   - ▶ Uniform with respect to the input and to the output
   - ▶ Example: binary symmetric channel

Input probabilities are important

▶ Suppose we have a channel defined by $P(Y|X)$

▶ $I(X,Y)$ **depends on the input probabilities** $p(x_i)$

　▶ For some distribution $p(x_i)$, we get a value of $I(X,Y)$
　▶ For a different distribution $p(x_i)$, we get a different $I(X,Y)$

▶ We want $I(X,Y)$ to be as large as possible

▶ Questions:

　▶ what is the largest possible value of $I(X,Y)$ (depending on $p(x_i)$)?
　▶ For what distribution $p(x_i)$?

Channel capacity

- What is the maximum information I(X,Y) we can transmit on a certain channel?

- **Definition:** the **information capacity of a channel** is the maximum value of the mutual information, where the maximization is done over the input probabilities $p(x_i)$

$$C = \max_{p(x_i)} I(X, Y)$$

- i.e. the maximum mutual information we can obtain if we are allowed to choose $p(x_i)$ as we want

- Use together with definition of $I(X, Y)$:

$$C = \max_{p(x_i)} (H(Y) - H(Y|X))$$

$$C = \max_{p(x_i)} (H(X) - H(X|Y))$$

- Channel capacity is the maximum information we can transmit on a channel, on average, with one message
- One of the most important notions in information theory
- Its importance comes from Shannon's second theorem (noisy channel theorem)
- It allows us to compare channels

▶ For transmission with no errors, we use **error coding** of data before transmission
▶ How error coding usually works:
  ▶ For each k symbols of data, coder appends additional m symbols, computed via some coding algorithm
  ▶ All of them are sent on the channel
  ▶ The decoder detects/corrects errors based on the additional m bits
▶ Coding rate:
$$R = \frac{k}{k + m}$$
  ▶ stronger protection = bigger m = less efficient
  ▶ weaker protection = smaller m = more efficient

▶ A rate is called **achievable** for a channel if, for that rate, there exists a coding and decoding algorithm guaranteed to correct all possible errors on the channel

Shannon's noisy channel coding theorem (second theorem)

For a given channel, all rates below capacity $R < C$ are achievable. All rates above capacity, $R > C$, are not achievable.

Channel coding theorem explained

In layman terms:

- ▶ For all coding rates $R < C$, there is a way to recover the transmitted data perfectly (decoding algorithm will detect and correct all errors)
- ▶ For all coding rates $R > C$, there is no way to recover the transmitted data perfectly

Example:

- ▶ Send binary digits (0,1) on a channel with capacity 0.7 bits/message
- ▶ There exists coding schemes with R < 0.7 that allow perfect recovery
  - ▶ i.e. for every 7 bits of data coding adds 3 or more bits, on average $=>$ $R = \frac{7}{7+3}$
- ▶ With less than 3 bits for every 7 bits of data $=>$ impossible to recover all the data

Efficiency and redundancy

▶ Efficiency of a channel:

$$\eta_C = \frac{I(X, Y)}{C}$$

▶ Absolute redundancy of a channel:

$$R_C = C - I(X, Y)$$

▶ Relative redundancy of a channel:

$$\rho_C = \frac{R_C}{C} = 1 - \frac{I(X, Y)}{C} = 1 - \eta_C$$

Computing the capacity

- Tricks for easier computation of the capacity
- Channel is uniform with respect to the input:
    - $H(Y|X)$ does not depend on the actual probabilities $p(x_i)$
    - $C = \max_{p(x_i)} I(X, Y) = \max_{p(x_i)} (H(Y) - H(Y|X)) = \max_{p(x_i)} (H(Y)) - H(Y|X)$
    - Should maximize $H(Y)$
- If channel is also uniform with respect to the output:
    - same values on columns of $P(Y|X)$
    - $p(y_j) = \sum_i p(y_j|x_i)p(x_i)$
    - if $p(x_i) = $ uniform $ = \frac{1}{n}$, then $p(y_j) = \frac{1}{n} \sum_i p(y_j|x_i) = $ uniform
    - therefore $p(y_j)$ are constant $= $ uniform $= H(Y)$ is maximized
    - $H(Y)$ is maximized when $H(X)$ is maximized (equiprobable messages)

- If channel is symmetric: use both tricks
  - $C = \max_{p(x_i)} (H(Y)) - H(Y|X)$
  - H(Y) is maximized when H(X) is maximized (equiprobable messages)

Examples of channels and their capacity

Figure 2: Noiseless binary channel

▶ Capacity = 1 bit/message, when $p(x_1) = p(x_2) = \frac{1}{2}$
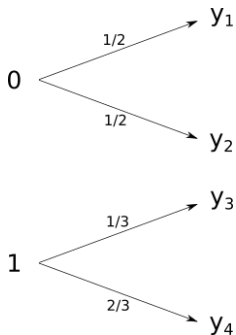
Noisy binary non-overlapping channel

Figure 3: Noisy binary non-overlapping

- There is noise ($H(Y|X) > 0$), but can deduce the input ($H(X|Y) = 0$)
- Capacity = 1 bit/message, when $p(x_1) = p(x_2) = \frac{1}{2}$

Noisy typewriter
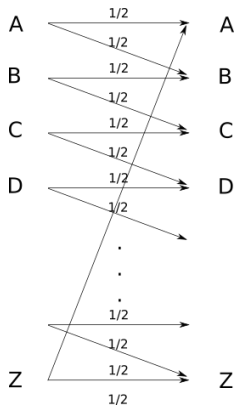
Figure 4: Noisy typewriter

$$\max I(X, Y) = \max \left(H(Y) - H(Y|X) \right) = \max H(Y) - 1$$
$$= \log(26) - 1 = \log(13)$$

▶ Capacity = log(13) bit/message, when input probabilities are uniform
▶ Can transmit 13 letters with no errors (A, C, E, G, . . .)
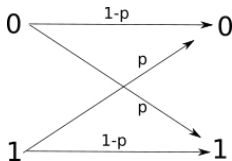
Binary symmetric channel

Figure 5: Binary symmetric channel (BSC)

▶ Capacity $= 1 - H_p = 1 + p \log(p) + (1 - p) \log(1 - p)$
▶ Capacity is reached when input distribution is uniform
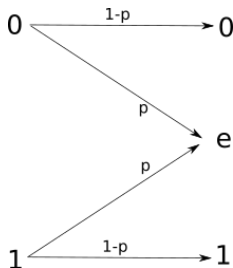
Binary erasure channel

Figure 6: Binary erasure channel

- ▶ Different from BSC: here we know when errors happened
- ▶ Capacity $= 1 - p$
- ▶ Intuitive meaning: lose p bits, remaining bits $=$ capacity $= 1 - p$
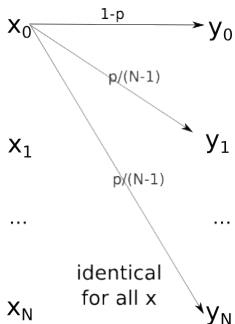
Symmetric channel of *n*-th order

Figure 7: *N*-th order symmetric channel

- Extension of binary symmetric channel for *n* messages
- $1 - p$ chances that symbol has no error
- p chances that symbol is changed, uniformly to any other (N-1) symbols ($\frac{p}{N-1}$ each)

Symmetric channel of *n*-th order

▶ Channel is symmetric $=>$

$$C = \max_{p(x_i)} I(X, Y) = \max_{p(x_i)} (H(Y) - H(Y|X)) = \max_{p(x_i)} (H(Y)) - H(Y|X)$$

▶ $\max_{p(x_i)} (H(Y)) = \log(N)$
▶ $H(Y|X) = H(Y|x_i) =$ entropy of any row (same values)

$==>$

$$C = \log(N) + (1 - p) \log(1 - p) + p \log(\frac{p}{N - 1})$$

▶ Capacity is reached when input probabilities are uniform

Chapter summary

- Channel = Probabilistic system with two random variables X and Y

- Characterization of transmission:
    - $P(X,Y) => H(X,Y)$ *joint entropy*
    - $p(x_i)$, $p(y_j)$ *marginal distributions* $=> H(X)$, $H(Y)$
    - $P(Y|X)$ *channel matrix* $=> H(Y|X)$ *average noise*
    - $P(X|Y) => H(X|Y)$ *equivocation*
    - $I(X,Y)$ *mutual information*

- Channel capacity: $C = \max_{p(x_i)} I(X, Y)$

- Examples:
    - Binary symmetric channel: $C = 1 - H_p$
    - Binary erasure channel: $C = 1 - p$
    - N-th symmetric channel: $C = \log(N) - H(\text{of a row of channel matrix})$

Figure 8: Claude Shannon (1916 - 2001)

- *A mathematical theory of communications*, 1948

- At blackboard only