Information Theory

Chapter I: Discrete information sources

Block diagram of a communication system

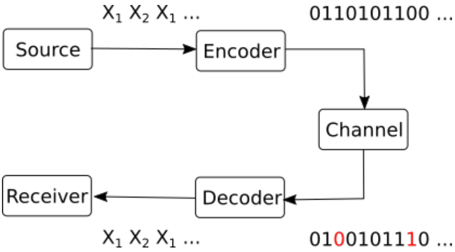


Figure 1: Block diagram of a communication system

- Source: creates information messages
- Encoder: converts messages into symbols for transmission (i.e bits)
- Channel: delivers the symbols, introduces errors
- Decoder: detects/corrects the errors, rebuilds the information messages

What is information?

Example:

- Consider the sentence: "your favorite football team lost the last match"
- Does this message carry information? How, why, how much?
- Consider the following facts:
    - the message carries information only when you don't already know the result
    - if you already known the result, the message is useless (brings no information)
    - if the result was to be expected, there is little information. If the result is highly unusual, there is more information in this message (think betting)

Information source

- Information is related to probability theory:
  - there is a *probabilistic source* that can produce a number of different *events*
  - each event has a certain probability. All probabilities are known beforehand
  - at one time, an event is randomly selected according to its probability
- The source is called an **information source** and the selected event is a **message**
- A message carries the information that **it** happened, and not the other possible message events that could have also happened
- The quantity of information is dependent on its probability

Discrete memoryless source

- A discrete memoryless source (DMS) is an information source which produces a sequence of **independent** messages
  - i.e. the choice of a message at one time does not depend on the previous messages
- Each message has a fixed probability. The set of probabilities is the **distribution** of the source

$$S : \begin{pmatrix} s_1 & s_2 & s_3 \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}$$

- Terminology:
  - Discrete: it can take a value from a discrete set (alphabet)
  - Complete: $\sum p(s_i) = 1$
  - Memoryless: succesive values are independent of previous values (e.g. successive throws of a coin)
- A message from a DMS is also called a **random variable** in probabilistics.

Examples

- A coin is a discrete memoryless source (DMS) with two messages:

$$S : \begin{pmatrix} heads & tails \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

- A dice is a discrete memoryless source (DMS) with six messages:

$$S : \begin{pmatrix} s_1 & s_2 & s_3 & s_4 & s_5 & s_6 \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \end{pmatrix}$$

- Playing the lottery can be modeled as DMS:

$$S : \begin{pmatrix} s_1 & s_2 \\ 0.9999 & 0.0001 \end{pmatrix}$$

Examples

- An extreme type of DMS containing the certain event:

$$S : \begin{pmatrix} s_1 & s_2 \\ 1 & 0 \end{pmatrix}$$

- Receiving an unknown *bit* (0 or 1) with equal probabilities:

$$S : \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

Information

- When a DMS provides a new message, it **creates information**: the information that a particular message took place
- The information attached to a particular event (message) is rigorously defined as:

$$i(s_i) = -\log_2(p(s_i))$$

- Properties:
  - $i(s_i) \geq 0$
  - lower probability (rare events) means higher information
  - higher probability (frequent events) means lower information
  - a certain event brings no information: $-\log(1) = 0$
  - an event with probability 0 brings infinite information (but it never happens...)

Entropy of a DMS

- We usually don't care about a single message. We are interested in a large number of them (think millions of bits of data)
- We are interested in the *average* information of a message from a DMS
- Definition: the **entropy** of a DMS source S is **the average information of a message**:

$$H(S) = \sum_k p(s_k) i(s_k) = -\sum_k p(s_k) \log_2(p_k)$$

where $p(s_k)$ is the probability of message k

The choice of logarithm

- Any base of logarithm can be used in the definition.
- Usual convention: use binary logarithm $\log_2()$
- In this case, H(S) is measured in **bits** (**bits / message**)
- If using natural logarithm *ln*(), H(S) is measured in *nats*.
- Logarithm bases can be converted to/from one another:

$$\log_b(x) = \frac{\log_a(x)}{\log_a(b)}$$

- Entropies using different logarithms differ only in scaling:

$$H_b(S) = \frac{H_a(S)}{\log_a(b)}$$

Examples

- Coin: $H(S) = 1 bit/message$
- Dice: $H(S) = \log(6) bits/message$
- Lottery: $H(S) = -0.9999 \log(0.9999) - 0.0001 \log(0.0001)$
- Receiving 1 bit: $H(S) = 1 bit/message$ (hence the name!)

Interpretation of the entropy

All the following interpretations of entropy are true:

- H(S) is the *average uncertainty* of the source S
- H(S) is the *average information* of the messages from source S
- A long sequence of N messages from S has total information $\approx N \cdot H(S)$
- H(S) is the minimum number of bits (0,1) required to uniquely represent an average message from source S

Properties of entropy

We prove the following **properties of entropy**:

1. $H(S) \geq 0$ (non-negative)

Proof: via definition

2. $H(S)$ is maximum when all n messages have equal probability $\frac{1}{n}$. The maximum value is $\max H(S) = \log(n)$

Proof: only for the case of 2 messages, use derivative in definition

3. *Diversification* of the source always increases the entropy

Proof: compare entropies in both cases

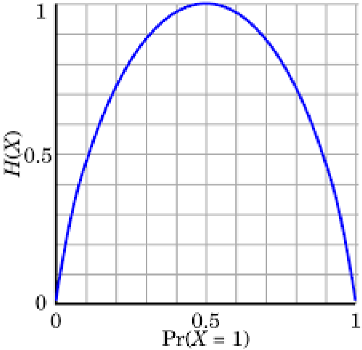
The entropy of a binary source

- Consider a general DMS with two messages:

$$S : \begin{pmatrix} s_1 & s_2 \\ p & 1-p \end{pmatrix}$$

- It's entropy is:

$$H(S) = -p \cdot \log(p) - (1-p) \cdot \log(1-p)$$



Example - Game

Game: I think of a number between 1 and 8. You have to guess it by asking yes/no questions.

- How much uncertainty does the problem have?
- How is the best way to ask questions? Why?
- What if the questions are not asked in the best way?
- On average, what is the number of questions required to find the number?

Example - Game v2

- Suppose I choose a number according to the following distribution:

$$S : \begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{8} \end{pmatrix}$$

  - On average, what is the number of questions required to find the number?
  - What questions would you ask?

- What if the distribution is:

$$S : \begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ 0.14 & 0.29 & 0.4 & 0.17 \end{pmatrix}$$

- In general:
  - What distribution makes guessing the number the most difficult?
  - What distribution makes guessing the number the easiest?

Efficiency and redundancy

- Efficiency of a DMS:

$$\eta = \frac{H(S)}{H_{max}} = \frac{H(S)}{\log(n)}$$

- Absolute redundancy of a DMS:

$$R = H_{max} - H(S)$$

- Relative redundancy of a DMS:

$$\rho = \frac{H_{max} - H(S)}{H_{max}} = 1 - \eta$$

Extended DMS

- Definition: the **n-th order extension** of a DMS S, S^n is a source which has as messages all the combinations of n messages of S:

$$\sigma_i = \underbrace{s_j s_k ... s_l}_{n}$$

- If S has k messages, S^n has k^n messages
- Since S is DMS, probabilities multiply:

$$p(\sigma_i) = p(s_j) \cdot p(s_k) \cdot ... \cdot p(s_l)$$

Extended DMS - Example

- Examples:

$$S : \begin{pmatrix} s_1 & s_2 \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix}$$

$$S^2 : \begin{pmatrix} \sigma_1 = s_1 s_1 & \sigma_2 = s_1 s_2 & \sigma_3 = s_2 s_1 & \sigma_4 = s_2 s_2 \\ \frac{1}{16} & \frac{3}{16} & \frac{3}{16} & \frac{9}{16} \end{pmatrix}$$

$$S^3 : \begin{pmatrix} s_1 s_1 s_1 & s_1 s_1 s_2 & s_1 s_2 s_1 & s_1 s_2 s_2 & s_2 s_1 s_1 & s_2 s_1 s_2 & s_2 s_2 s_1 & s_2 s_2 s_2 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

Extended DMS - Another example

- Long sequence of binary messages:

  010011001110010100...

  - Can be grouped in bits, half-bytes, bytes, 16-bit words, 32-bit long words, and so on
  - Can be considered:
    - N messages from a binary source (with 1 bit), or
    - N/2 messages from a source with 4 messages (with 2 bits)...
    - etc

Property of DMS

- Theorem: The entropy of a n-th order extension is n times larger than the entropy of the original DMS

$$H(S^n) = nH(S)$$

- Interpretation: grouping messages from a long sequence in blocks of n does not change total information (e.g. groups of 8 bits = 1 byte)

An example [memoryless is not enough]

- The distribution (frequencies) of letters in English:

letter	probability	letter	probability
A	.082	N	.067
B	.015	O	.075
C	.028	P	.019
D	.043	Q	.001
E	.127	R	.060
F	.022	S	.063
G	.020	T	.091
H	.061	U	.028
I	.070	V	.010
J	.002	W	.023
K	.008	X	.001
L	.040	Y	.020
M	.024	Z	.001

- Text from a memoryless source with these probabilities:

OCRO HLI RGWR NMIELWIS EU LL NBNESEBYA TH EEI
ALHENHTTPA OOBTTVA NAH BRL

(taken from Elements of Information Theory, Cover, Thomas)

- What's wrong? **Memoryless**

Sources with memory

- **Definition**: A source has **memory of order** m if the probability of a message depends on the last m messages.
- The last m messages $=$ the **state** of the source (notation S_i).
- A source with n messages and memory m $=>$ has n^m states in all.
- For every state, messages can have a different set of probabilities. Notation: $p(s_i|S_k) =$ "*probability of* s_i *in state* S_k".
- Also known as *Markov sources*.

Example

- A source with $n = 4$ messages and memory $m = 1$
  - if last message was s_1, choose next message with distribution
$$S_1 : \begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ 0.4 & 0.3 & 0.2 & 0.1 \end{pmatrix}$$
  - if last message was s_2, choose next message with distribution
$$S_2 : \begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ 0.33 & 0.37 & 0.15 & 0.15 \end{pmatrix}$$
  - if last message was s_3, choose next message with distribution
$$S_3 : \begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ 0.2 & 0.35 & 0.41 & 0.04 \end{pmatrix}$$
  - if last message was s_4, choose next message with distribution
$$S_4 : \begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ 0.1 & 0.2 & 0.3 & 0.4 \end{pmatrix}$$

Transitions

- When a new message is provided, the source **transitions** to a new state:

$$\ldots \underbrace{s_i s_j s_k}_{\text{old state}} s_l$$

$$\ldots s_i \underbrace{s_j s_k s_l}_{\text{new state}}$$

- The message probabilities = the probabilities of transitions from some state S_u to another state S_v

Transition matrix

- The transition probabilities are organized in a **transition matrix** $[T]$

$$[T] = \begin{bmatrix} p_{11} & p_{12} & \dots & p_{1N} \\ p_{21} & p_{22} & \dots & p_{2N} \\ \dots & \dots & \dots & \dots \\ p_{N1} & p_{N2} & \dots & p_{NN} \end{bmatrix}$$

- p_{ij} is the transition probability from state S_i to state S_j
- N is the total number of states

Graphical representation

At whiteboard: draw states and transitions for previous example (source with $n = 4$ messages and memory $m = 1$)

Entropy of sources with memory

- What entropy does s source with memory have?
- Each state S_k has a different distribution –> each state has a different entropy $H(S_k)$

$$H(S_k) = -\sum_i p(s_i|S_k) \cdot \log(p(s_i|S_k))$$

- Global entropy = average entropy

$$H(S) = \sum_k p_k H(S_k)$$

where p_k = probability that the source is in state S_i

  - (i.e. after a very long sequence of messages, the fraction of time when the source was in state S_k)

Ergodic sources

- Let $p_i^{(n)}$ = the probability that source S is in state S_i at time n.
- In what state will it be at time $n+1$? (after one more message)
  - i.e. what are the probabilities of the states at time $n+1$?

$$[p_1^{(n)}, p_2^{(n)}, ..., p_N^{(n)}] \cdot [T] = [p_1^{(n+1)}, p_2^{(n+1)}, ..., p_N^{(n+1)}]$$

- After one more message:

$$[p_1^{(n)}, p_2^{(n)}, ..., p_N^{(n)}] \cdot [T] \cdot [T] = [p_1^{(n+2)}, p_2^{(n+2)}, ..., p_N^{(n+2)}]$$

- In general, starting from time 0, after n messages the probabilities that the source is in a certain state are:

$$[p_1^{(0)}, p_2^{(0)}, ..., p_N^{(0)}] \cdot [T]^n = [p_1^{(n)}, p_2^{(n)}, ..., p_N^{(n)}]$$

Ergodicity

- A source is called **ergodic** if every state can be reached from every state, in a finite number of steps.

Property of ergodic sources:

- After many messages, the probabilities of the states *become stationary* (converge to some fixed values), irrespective of the initial probabilities.

$$\lim_{n \to \infty} [p_1^{(n)}, p_2^{(n)}, ... p_N^{(n)}] = [p_1, p_2, ... p_N]$$

- These are the probabilities to be used in the entropy formula for memory sources

Finding the stationary probabilties

- How to find the stationary probabilities?
- When n is very large, after n messages and after $n+1$ messages the probabilities are the same:

$$[p_1, p_2, ...p_N] \cdot [T] = [p_1, p_2, ...p_N]$$

- Also $p_1 + p_2 + ... + p_N = 1$.

$=>$ solve system of equations, find values.

Entropy of ergodic sources with memory

- The entropy of an ergodic source with memory is

$$H(S) = \sum_k p_k H(S_k) = -\sum_k p_k \sum_i p(s_i|S_k) \cdot \log(p(s_i|S_k)$$

Example English text as sources with memory

(taken from Elements of Information Theory, Cover, Thomas)

- Memoryless source, equal probabilities:

  XFOML RXKHRJFFJUJ ZLPWCFWKCYJ
  FFJEYVKCQSGXYD QPAAMKBZAACIBZLHJQD

- Memoryless source, probabilities of each letter as in English:

  OCRO HLI RGWR NMIELWIS EU LL NBNESEBYA TH EEI
  ALHENHTTPA OOBTTVA NAH BRL

- Source with memory $m = 1$, frequency of pairs as in English:

  ON IE ANTSOUTINYS ARE T INCTORE ST BE S DEAMY
  ACHIN D ILONASIVE TUCOOWE AT TEASONARE FUSO
  TIZIN ANDY TOBE SEACE CTISBE

- Source with memory $m = 2$, frequency of triplets as in English:

  IN NO IST LAT WHEY CRATICT FROURE BERS GROCID
  PONDENOME OF DEMONSTURES OF THE REPTAGIN IS
  REGOACTIONA OF CRE

- Source with memory $m = 3$, frequency of 4-plets as in English:

  THE GENERATED JOB PROVIDUAL BETTER TRAND THE DISPLAYED
  CODE, ABOVERY UPONDULTS WELL THE CODERST IN THESTICAL
  IT DO HOCK BOTHE MERG. (INSTATES CONS ERATION. NEVER
  ANY OF PUBLE AND TO THEORY. EVENTIAL CALLEGAND TO ELAST
  BENERATED IN WITH PIES AS IS WITH THE)

Chapter summary

- Information of a message: $i(s_k) = -\log_2(p(s_k))$
- Entropy of a memoryless source:
  $H(S) = \sum_k p_k i(s_k) = -\sum_k p_k \log_2(p_k)$
- Properties of entropy:
  1. $H(S) \geq 0$
  2. Is maximum when all messages have equal probability ($H_{max}(S) = \log(n)$)
  3. *Diversfication* of the source always increases the entropy
- Sources with memory: definition, transitions
- Stationary probabilities of ergodic sources with memory:
  $[p_1, p_2, ...p_N] \cdot [T] = [p_1, p_2, ...p_N]$, $\sum_i p_i = 1$.
- Entropy of sources with memory:

$$H(S) = \sum_k p_k H(S_k) = -\sum_k p_k \sum_i p(s_i|S_k) \cdot \log(p(s_i|S_k)$$