Designing IIR filters with the Pade method

Laboratory 2, SDP

4 Theoretical exercises

1. Use the Pade method to find out the parameters of the system with the following system function

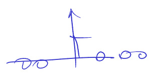
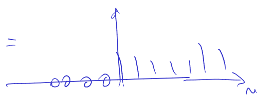
$$H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}},$$

considering the desired impulse response:

$$h_d[n] = \left(\frac{1}{3}\right)^n \cos(n\pi)\, u[n] + u[n-3]$$

System equation:

$$y[n] = -a_1 y[n-1] - a_2 y[n-2] + b_0 x[n] + b_1 x[n-1] + b_2 x[n-2]$$

If $x[n] = \delta[n]$ $\quad \delta[n] =$  $\quad h[n] =$ 

$y[n] = h[n]$

$$h[n] = -a_1 h[n-1] - a_2 h[n-2] + b_0 \delta[n] + b_1 \delta[n-1] + b_2 \delta[n-2]$$

$$\begin{cases}
n=0: & h[0] = -a_1 \underbrace{h[-1]}_{=0} - a_2 \underbrace{h[-2]}_{=0} + b_0 \underbrace{\delta[0]}_{=1} + b_1 \underbrace{\delta[-1]}_{=0} + b_2 \underbrace{\delta[-2]}_{=0} = \underbrace{h_d[0]}_{1} \\
n=1: & h[1] = -a_1 \underbrace{h[0]}_{1} - a_2 \underbrace{h[-1]}_{=0} + b_0 \underbrace{\delta[1]}_{=0} + b_1 \underbrace{\delta[0]}_{1} + b_2 \underbrace{\delta[-1]}_{=0} = \underbrace{h_d[1]}_{-1/3} \\
n=2: & h[2] = -a_1 \underbrace{h[1]}_{-1/3} - a_2 \underbrace{h[0]}_{1} + b_0 \underbrace{\delta[2]}_{=0} + b_1 \underbrace{\delta[1]}_{=0} + b_0 \underbrace{\delta[0]}_{=1} = \underbrace{h_d[2]}_{=1/9} \\
n=3: & h[3] = -a_1 \underbrace{h[2]}_{1/9} - a_2 \underbrace{h[1]}_{-1/3} + b_0 \underbrace{\delta[3]}_{=0} + b_1 \underbrace{\delta[2]}_{0} + b_2 \underbrace{\delta[1]}_{0} = \underbrace{h_d[3]}_{26/27} \\
n=4: & h[4] = -a_1 \underbrace{h[3]}_{26/27} - a_2 \underbrace{h[2]}_{1/9} + b_0 \underbrace{\delta[4]}_{0} + b_1 \underbrace{\delta[3]}_{0} + b_2 \underbrace{\delta[2]}_{0} = \underbrace{h_d[4]}_{82/81}
\end{cases}$$

solve first, simpler!

$h_d[0] = 1 \cdot 1 \cdot 1 + 0 = 1$

$h_d[1] = \frac{1}{3} \cdot (-1) = -\frac{1}{3}$

$h_d[2] = \frac{1}{9} \cdot 1 = \frac{1}{9}$

$h_d[3] = \frac{1}{27} \cdot (-1) + 1 = \frac{26}{27}$

$h_d[4] = \frac{1}{81} \cdot 1 + 1 = \frac{82}{81}$

$$\begin{bmatrix} 1 \\ -1/3 \\ 1/9 \\ 26/27 \\ 82/81 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ 1/3 & -1 & 0 & 0 & 1 \\ -1/9 & 1/3 & 0 & 0 & 0 \\ -26/27 & -1/9 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} a_1 \\ a_2 \\ b_0 \\ b_1 \\ b_2 \end{bmatrix}$$

$$\begin{cases} \frac{26}{27} = -\frac{1}{9} a_1 + \frac{1}{3} a_2 & \Big| \cdot 27 \\ \frac{82}{81} = \frac{-26}{27} a_1 - \frac{1}{9} a_2 & \Big| \cdot 81 \end{cases} \Leftrightarrow \begin{cases} 26 = -3 a_1 + 9 a_2 \\ 82 = -26 \cdot 3\, a_1 - 9 a_2 \end{cases}$$

$$56 = -3a_1 - 26 \cdot 3 a_1 = -81 a_1 \Rightarrow a_1 = \frac{-56}{81} = \dots$$

$$a_2 = 3\left(\frac{26}{27} + \frac{1}{9} a_1\right) = \frac{26}{9} + \frac{1}{3} \cdot \frac{-56}{81} = \dots$$

$$\begin{cases} 1 = b_0 \\ -\frac{1}{3} = -a_1 + b_1 \\ \frac{1}{9} = \frac{1}{3} a_1 - a_2 + b_2 \end{cases} \Rightarrow \begin{aligned} b_0 &= 1 \\ b_1 &= -\frac{1}{3} + a_1 = \dots \\ b_2 &= \frac{1}{9} - \frac{1}{3} a_1 + a_2 = \dots \end{aligned}$$

$$\Rightarrow \text{Designed } H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}} = \dots$$