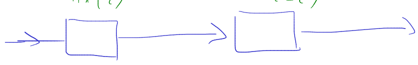
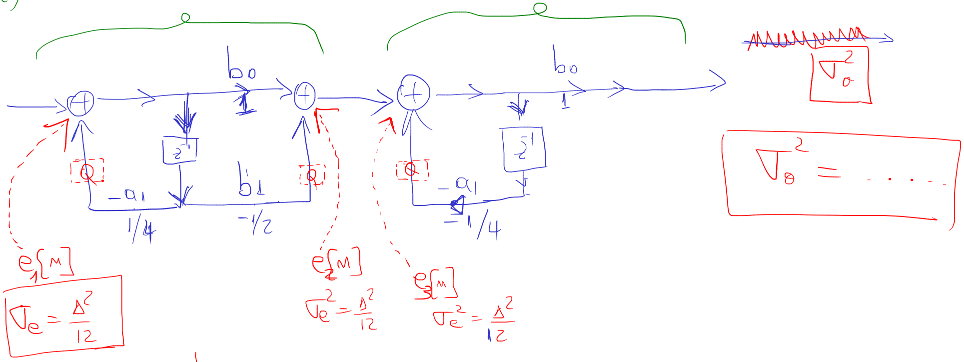
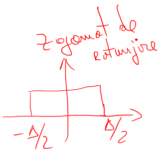
Ex. 1 / Lab 10

a) $$H(z) = \frac{1-\frac{1}{2}z^{-1}}{\underbrace{\left(1-\frac{1}{4}z^{-1}\right)}_{H_1(z)}\underbrace{\left(1+\frac{1}{4}z^{-1}\right)}_{H_2(z)}} = \underbrace{\frac{1-\frac{1}{2}z^{-1}}{1-\frac{1}{4}z^{-1}}}_{H_1(z)} \cdot \underbrace{\frac{1}{1+\frac{1}{4}z^{-1}}}_{H_2(z)}$$

$b_0 = 1,\ b_1 = -\frac{1}{2},\ a_1 = -\frac{1}{4}$ (pentru H_1); $b_0 = 1,\ a_1 = \frac{1}{4}$ (pentru H_2)





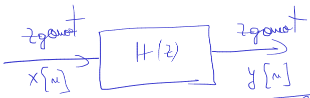


b).

Puterea $\frac{\Delta^2}{12} = \frac{2^{-2b}}{12}$

$\Delta = 2^{-b}$

$\sigma_o = ?$



$$\boxed{\sigma_{in}^2 \cdot \sum (h[n])^2 = \sigma_{out}^2}$$

Doar pt. $H_2(z)$ pt. că $e_2[n]$ trece doar prin $H_2(z)$

↓

$$\sigma_{out}^2 = \sigma_{out_1}^2 + \sigma_{out_2}^2 + \sigma_{out_3}^2 = \frac{\Delta^2}{12}\cdot\sum(h[n])^2 + \frac{\Delta^2}{12}\cdot\sum(h_2[n])^2 + \frac{\Delta^2}{12}\cdot\sum(h_2[n])^3$$

$$\rightarrow \boxed{\sigma_{out}^2 = \frac{2^{-2b}}{12}\cdot\left(\underbrace{\sum(h[n])^2}_{?=} + 2\cdot\underbrace{\sum(h_2[n])}_{\overset{?}{=}\ 16/15}\right)}$$

$$\sum_{n=-\infty}^{\infty}(h_2[n])^2 = \sum_{\substack{\text{poli din}\\\text{interiorul}\\\text{cercului}\\\text{unitate}}} \text{reziduurile funcției } H_2(z)\cdot H_2(z^{-1})\cdot z^{-1} = A_1 = \frac{16}{15}$$

Reziduuri

$$\frac{A_1}{z-\text{ceva}} + \frac{A_2}{z-\text{altceva}} + \dots$$

$$H_2(z)\cdot H_2(z^{-1})\cdot z^{-1} = \frac{1}{1+\frac{1}{4}z^{-1}} \cdot \frac{1}{1+\frac{1}{4}z}\cdot z^{-1} = \frac{\cancel{z}}{z+\frac{1}{4}}\cdot\frac{4}{1+\frac{1}{4}z}\cdot\frac{1}{\cancel{z}} =$$

$$= \frac{1}{z+\frac{1}{4}}\cdot\frac{4}{z+4} = \frac{4}{\underbrace{\left(z+\frac{1}{4}\right)}_{\text{pol}=-\frac{1}{4}}\underbrace{(z+4)}_{\text{pol}=-4}} = \frac{A_1}{z+\frac{1}{4}} + \frac{A_2}{z+4}$$

$\left(x \leftrightarrow \frac{1}{x}\right)$

$$A_1 = \frac{4}{-\frac{1}{4}+4} = \frac{16}{15}$$

$$\sum (h[n])^2 = \sum_{\substack{\text{în polii}\\ \text{int. cerc. un.}}} \text{rez.}\ H(z)H(z^{-1})z^{-1} = A_1 + A_2 =$$

$$H(z)\cdot H(z^{-1})\cdot z^{-1} = \frac{(1-\frac{1}{2}z^{-1})}{(1-\frac{1}{4}z^{-1})(1+\frac{1}{4}z^{-1})} \cdot \frac{(1-\frac{1}{2}z)}{(1-\frac{1}{4}z)(1+\frac{1}{4}z)} \cdot \frac{1}{z} = \frac{(z-\frac{1}{2})(1-\frac{1}{2}z)\cdot(-16)}{(z-\frac{1}{4})(z+\frac{1}{4})(z-4)(z+4)}$$

$$= \frac{A_1}{z-\frac{1}{4}} + \frac{A_2}{z+\frac{1}{4}} + \frac{A_3}{z-4} + \frac{A_4}{z+4}$$

Rez.

$$A_1 = \frac{(\frac{1}{4}-\frac{1}{2})(1-\frac{1}{2}\cdot\frac{1}{4})\cdot(-16)}{\frac{1}{2}\cdot(\frac{1}{4}-4)(\frac{1}{4}+4)} = \frac{-1/4\cdot 7/8\cdot(-16)}{\frac{1}{2}\cdot\frac{-15}{16}\cdot\frac{17}{16}} = \frac{3.5}{\frac{-15\cdot 17}{2\cdot 256}} = \frac{7\cdot 256}{-15\cdot 17} = \frac{-7\cdot 256}{15\cdot 17}$$

$$A_2 = \frac{(-1/4-1/2)(1+1/2\cdot 1/4)\cdot(-16)}{-1/2\cdot(-\frac{1}{4}-4)(-1/4+4)} = \frac{-3/4\cdot\frac{9}{8}\cdot(-16)}{-1/2\cdot\frac{-17}{16}\cdot\frac{15}{16}} = \frac{3\cdot 9\cdot \cancel{16}}{\cancel{4\cdot 8}\,2}\cdot\frac{2\cdot 16\cdot 16}{17\cdot 15} = \dots$$

$$= \frac{21}{2}\cdot\frac{21\cdot 256}{17\cdot 15}$$

$$A_1 + A_2 = \frac{14\cdot 256}{17\cdot 15}$$

$\boxed{\text{Factorizare spectrală}}$

~~x[n] proces~~

Fie un proces aleator $x[n]$

F. de autocorelație :

$$\gamma_{xx}[m] = E_n\{x[n]\cdot x[n-m]\}$$

$$z^2 \qquad \ln z^2$$

$$\ln(\Gamma_{xx}(z))$$

$$\updownarrow$$

$$\Gamma_{xx}(z) = \text{transf. } Z = \sum_m \gamma_{xx}[m]\cdot z^{-m}$$

Fie

$$V(z) = \ln(\Gamma_{xx}(z)) = \text{o funcție în } z = \text{transf. } Z \text{ a unui semnal } v[n] = \sum_n v[n]\cdot z^{-n}$$

(în anumite condiții)

Atunci:

$$\Gamma_{xx}(z) = e^{V(z)} = e^{\sum_{m=-\infty}^{\infty} v[m]\cdot z^{-m}} = e^{v[0] + \sum_{-\infty}^{-1} v[m] z^{-m} + \sum_{1}^{\infty} v[m] z^{-m}}$$

$$= \underbrace{e^{v[0]}}_{\sigma_w^2}\cdot e^{\sum_{-\infty}^{-1} v[m] z^{-m}} \cdot \underbrace{e^{\sum_{1}^{\infty} v[m] z^{-m}}}_{H(z)}$$

$\Rightarrow$ Termenul din mijloc:

$$e^{\sum_{m=-\infty}^{-1} v[m] z^{-m}} \underset{m \to -m}{\overset{\text{Sch. var.}}{=}} e^{\sum_{m=1}^{\infty} v[-m] z^{m}} = e^{\left(\left(\sum_{m=1}^{\infty} v[-m] z^{m}\right)^{*}\right)^{*}}$$

$(a^*)^* = a$

$(a+b)^* = a^* + b^*$

$(a \cdot b)^* = a^* \cdot b^*$

$$= e^{\left(\sum \left(v[-m] \cdot z^{m}\right)^{*}\right)^{*}} = e^{\left(\sum (v[-m])^{*} \cdot (z^{m})^{*}\right)^{*}} = e^{\left(\sum_{m=1}^{\infty} v[m] \cdot (z^{*})^{m}\right)^{*}}$$

$\boxed{v[m] = v^*[-m]}$ = simetrie pară

$(z^n)^* = (z^*)^n$

$e^{a^*} = (e^a)^*$

$$= \boxed{\left(e^{\sum_{m=1}^{\infty} v[m] \left(\frac{1}{z^*}\right)^{-m}}\right)^{*}} = \left(H\left(\frac{1}{z^*}\right)\right)^{*} = H^*\left(\frac{1}{z^*}\right)$$

Am notat $\boxed{H(z) = e^{\sum_{m=1}^{\infty} v[m] z^{-m}}}$

$\Rightarrow$ O transf. Z $\Gamma_{xx}(z)$ se poate descompune sub forma:

$$\Gamma_{xx}(z) = \sigma_w^2 \cdot H(z) \cdot H^*\left(\frac{1}{z^*}\right)$$

Exemplu:

$$H(z) = \frac{z+3}{z-7}$$

$$H^*\left(\frac{1}{z^*}\right) = \left(\frac{\frac{1}{z^*} + 3}{\frac{1}{z^*} - 7}\right)^{*} = \frac{\left(\frac{1}{z^*}\right)^{*} + \underbrace{3^*}_{=3}}{\left(\frac{1}{z^*}\right)^{*} - \underbrace{7^*}_{=7}} = \frac{\frac{1}{z} + 3}{\frac{1}{z} - 7} = \frac{z^{-1}+3}{z^{-1}-7} = H(z^{-1})$$

$$\boxed{H^*\left(\frac{1}{z^*}\right) = H(z^{-1}) \text{ dacă toți coef. lui } H(z) \text{ sunt reali}}$$

$$\Gamma_{xx}(z) = \sigma_w^2 \cdot H(z) \cdot H(z^{-1})$$

$$H(z) = \frac{z+3}{z-7} \rightarrow \begin{matrix} \text{zerou: } -3 \\ \text{pol : } 7 \end{matrix}$$

$$H(z^{-1}) = \frac{z^{-1}+3}{z^{-1}-7} = \frac{\frac{1}{z}+3}{\frac{1}{z}-7} = \frac{\frac{1+3z}{z}}{\frac{1-7z}{z}} = \frac{1+3z}{1-7z} = \frac{3}{-7} \cdot \frac{z+\frac{1}{3}}{z-\frac{1}{7}}$$

$$= \frac{-3}{7} \cdot \frac{z+1/3}{z-1/7} \rightarrow \begin{matrix} \text{zerou: } -1/3 \\ \text{pol : } 1/7 \end{matrix}$$

Dacă $H(z)$ are polii p_k și zerourile z_k, at. $H(z^{-1})$ are polii $\frac{1}{p_k}$ și zerourile $\frac{1}{z_k}$

$\Gamma_{xx}(z) = \sigma_w^2 \cdot H(z) \cdot H(z^{-1}) \Rightarrow$ polii și zer. lui $\Gamma_{xx}(z)$ tb. să fie în perechi p_k și $\frac{1}{p_k}$

z_k și $\frac{1}{z_k}$

Exemplu

$\frac{z-\frac{1}{2}}{z+3} \cdot \frac{z-2}{z+\frac{1}{3}}$ poate fi tr. Z a unei f. de autocorelație

$\frac{(z-\frac{1}{2})(z-7)}{(z-\frac{1}{4})(z-\frac{1}{5})}$ nu ~~este~~ este tr. Z a unei f. de autocorelație

---

Cepstrul $v[n]$ = acel semnal $v[n]$ a cărui transf. Z ($\mathcal{F}$) este logaritmul transf. Z ($\mathcal{F}$) a funcției de autocorelație a unui proces aleator

$$x[n] \longrightarrow \gamma_{xx}[m] \xrightarrow{Z} \Gamma_{xx}(z)$$

$$v[n] \longrightarrow V(z) = \ln(\Gamma_{xx}(z))$$

$x[n] \longrightarrow \gamma_{xx}[n] \overset{\mathcal{F}}{\longleftrightarrow} \Gamma_{xx}(\omega)$ (densitate spectrală de putere)

$\boxed{v[n]} \longleftrightarrow V(\omega) = \ln \Gamma_{xx}(\omega)$

$$\xrightarrow{w[n]} \boxed{H(z)} \xrightarrow{x[n]}$$

$$W(z) \cdot H(z) = X(z)$$

$\gamma_{ww}[n]$ $\qquad$ $\gamma_{xx}[n]$ $\qquad$ Legătura dintre autocorelația intrării și a ieșirii pentru sistem $H(z)$

$$\boxed{\Gamma_{ww}(z) \cdot H(z) \cdot H(z^{-1}) = \Gamma_{xx}(z)}$$

Dacă avem un proces aleator $x[n]$ cu $\Gamma_{xx}(z)$ factorizat ca:

$$\Gamma_{xx}(z) = \underbrace{\sigma_w^2}_{= \text{ o constantă}} \cdot H(z) \cdot H(z^{-1})$$

$$\xrightarrow{w[n]} \boxed{H(z)} \xrightarrow{x[n]}$$

$\Gamma_{ww}(z) = \sigma_w^2$ ← constant

$\Gamma_{ww}(\omega) = \sigma_w^2$

$=$

zgomot alb

$$\underset{w[n]}{\xrightarrow{\text{zg. alb}}} \boxed{H(z)} \underset{x[n]}{\xrightarrow{\text{zg.}}}$$

$\Gamma_{ww}(z) = \sigma_w^2$

$$\sigma_w^2 \cdot H(z) \cdot H(z^{-1}) = \Gamma_{xx}(z)$$