PSS Curs 13

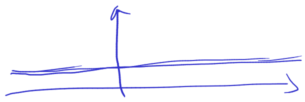
$x[n]$ — $\gamma_{xx}[m]$ — D.s.p. (în z) $\Gamma_{xx}(z)$ $\xleftrightarrow{z=e^{j\omega}}$ Densitate spectr. de putere $\Gamma_{xx}(\omega)$

Exemplu: zg. alb $w[n]$

$\gamma_{ww}[m] = \sigma_w^2 \delta[m]$

$\Gamma_{ww}(z) = \sigma_w^2 = ct$

$\Gamma_{ww}(\omega) = \sigma_w^2 = ct$

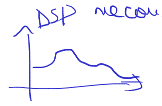


$x[n] \rightarrow \boxed{H(z)} \rightarrow y[n]$

$\gamma_{xx}[m]$ $\gamma_{yy}[m]$

$$\boxed{\Gamma_{xx}(z) \cdot H(z) \cdot H(z^{-1}) = \Gamma_{yy}(z)}$$

DSP neconstantă



Zg. alb — Aleator, ne-alb

$w[n] \rightarrow \boxed{H(z)} \rightarrow x[n]$

$\gamma_{xx}[m]$

$$\Gamma_{xx}(z) = \sigma_w^2 \cdot H(z) \cdot H^*\left(\frac{1}{z^*}\right) \iff \underbrace{\sigma_w^2}_{\Gamma_{ww}(z)} \cdot H(z) \cdot H(z^{-1}) = \Gamma_{xx}(z)$$

$H^*\left(\frac{1}{z^*}\right) = H(z^{-1})$

- $H(z)$ = stabil (și polii și zerourile în cercul unitate)

$$\Gamma_{xx}(z) = 3 \cdot \frac{(z-0.5)(z-2)(z+0.79)\left(z+\frac{1}{0.79}\right)}{(z-2.1)\left(z-\frac{1}{2.1}\right)(z-1.8)\left(z-\frac{1}{1.8}\right)}$$

$3 = \sigma_w^2$; $H(z)$; $H(z^{-1}) = H^*\left(\frac{1}{z^*}\right)$

- $x[n] \rightarrow \boxed{H_{I}(z) = \frac{1}{H(z)}} \rightarrow w[n]$

Filtru de albire

$w[n] \rightarrow \boxed{H(z) = \frac{B(z)}{A(z)}} \rightarrow x[n]$

Ec. cu diferențe: $x[n] = -a_1 \cdot x[n-1] - a_2 x[n-2] - \ldots - a_p x[n-p] + b_0 \cdot w[n] + b_1 w[n-1] + \ldots + b_q w[n-q]$

$$\boxed{x[n] + \sum_{k=1}^{p} a_k \cdot x[n-k] = \sum_{k=0}^{q} b_k \cdot w[n-k]}$$

ARMA(q, p)

$\boxed{\text{Autoregresiv } AR(p)}$:

$$H(z) = \frac{1^{\,b_0=1}}{A(z)} = \frac{1}{1+a_1 z^{-1}+\ldots+a_p z^{-p}}$$

Ec. cu dif. : $x[n] + \sum_{k=1}^{p} a_k x[n-k] = w[n] \iff \boxed{x[n]+a_1 x[n-1]+\ldots+a_p x[n-p] = w[n]}$

Leg. între param. modelului (a_k, b_k) și $\gamma_{xx}[m]$

$$E\{a+b\} = E\{a\}+E\{b\}$$

$$E\{ct \cdot x\} = ct \cdot E\{x\}$$

$$x[n] = -\sum_{k=1}^{p} a_k \cdot x[n-k] + \sum_{k=0}^{q} b_k \cdot w[n-k]$$

$$\gamma_{xx}[m] = \overline{x[n]\cdot x[n-m]} = E\{x[n]\cdot x[n-m]\} = E\left\{-\sum_{k=1}^{p} a_k \cdot \{x[n-k]\cdot x[n-m]\} + \sum_{k=0}^{q} b_k \{w[n-k]\cdot x[n-m]\}\right\}$$

$$\gamma_{xx}[m] = -\sum_{k=1}^{p} a_k \cdot \underbrace{E\{x[n-k]\cdot x[n-m]\}}_{\gamma_{xx}[m-k]} + \sum_{k=0}^{q} b_k \cdot \underbrace{E\{w[n-k]\cdot x[n-m]\}}_{\gamma_{wx}[m-k]}$$

$$\gamma_{xx}[m] = -\sum_{k=1}^{p} a_k \gamma_{xx}[m-k] + \sum_{k=0}^{q} b_k \cdot \gamma_{wx}[m-k]$$

← în general, pt ARMA

Pentru AR : $b_0 = 1$, $b_{1,2,3} = 0$

$\gamma_{wx}[m]$

$$\gamma_{xx}[m] = -\sum_{k=1}^{p} a_k \cdot \gamma_{xx}[m-k] + \begin{cases} 0, & m > q \\ \sigma_w^2 \cdot \sum_{k=0}^{q-m} h[k]\, b_{k+m} & 0 \le m < q \end{cases}$$

$$\gamma_{xx}[-m] = \gamma_{xx}[m]$$

Pentru procese AR:

$$\boxed{\gamma_{xx}[m] = \begin{cases} -\sum_{k=1}^{p} a_k \gamma_{xx}[m-k], m > 0 \\ -\sum_{k=1}^{p} a_k \gamma_{xx}[m-k] + \sigma_w^2, m = 0 \\ \gamma_{xx}^*[-m], m < 0 \end{cases}}$$

$m = 0$: $\gamma_{xx}[0] = -\sum_{k=1}^{p} a_k \gamma_{xx}[0-k] + \sigma_w^2$

Ecuațiile Yule-Walker

(ec. normale)

$$\begin{cases} \gamma_{xx}[0] + a_1 \gamma_{xx}[-1] + a_2 \gamma_{xx}[-2] + \ldots + a_p \gamma_{xx}[-p] = \sigma_w^2 \\ m=1: \ \gamma_{xx}[1] + a_1 \gamma_{xx}[0] + a_2 \gamma_{xx}[-1] + \ldots a_p \gamma_{xx}[-p+1] = 0 \\ m=2: \ \gamma_{xx}[2] + a_1 \gamma_{xx}[1] + a_2 \gamma_{xx}[0] + \ldots a_p \gamma_{xx}[-p+2] = 0 \\ \vdots \\ m=p: \ \gamma_{xx}[p] + a_1 \gamma_{xx}[p-1] + a_2 \gamma_{xx}[p-2] + \ldots a_p \gamma_{xx}[0] = 0 \end{cases}$$

$$\begin{bmatrix} \gamma_{xx}[0] & \gamma_{xx}[-1] & \gamma_{xx}[-2] \dots & \gamma_{xx}[-p] \\ \gamma_{xx}[1] & \gamma_{xx}[0] & \dots & \gamma_{xx}[-p+1] \\ \vdots & & & \\ \gamma_{xx}[p] & \gamma_{xx}[p-1] & \dots & \gamma_{xx}[0] \end{bmatrix} \cdot \begin{bmatrix} 1 \\ a_1 \\ a_2 \\ \vdots \\ a_p \end{bmatrix} = \begin{bmatrix} \sigma_w^2 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$A \cdot X = B$$

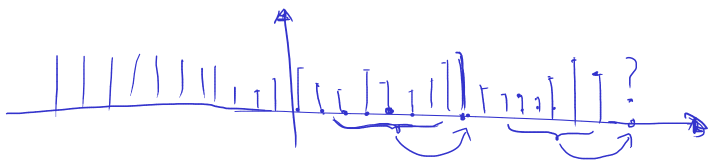
$$w[n] \longrightarrow \boxed{H(z)} \longrightarrow x[n]$$

$$H(z) = \frac{1}{1 + a_1 z^{-1} + \dots + a_p z^{-p}}$$

Predicție liniară (f.f.f. scurt)

$x[n] = \{ 1, 2, 3, 2, 2, 2, 3, 1, 2, 3$?

ce val. urmează?



$$x[n] \simeq \underbrace{a_1 \cdot x[n-1] + a_2 \cdot x[n-2] + \dots + a_p \cdot x[n-p]}$$

$\hat{x}[n] = x[n]$ prezis

$$x[n] = \underbrace{\boxed{a_1 \cdot x[n-1] + a_2 x[n-2] + \dots a_p x[n-p]}}_{\hat{x}[n]} + e[n]$$

$e[n]$ eroare de predicție

$$\boxed{x[n] - a_1 x[n-1] - a_2 x[n-2] - \dots - a_p x[n-p] = e[n]}$$

eroare de predicție $\Leftrightarrow$ zgomot alb

Cum calculez a_1, a_p?

Vreau $(x[n] - \hat{x}[n])$ cât mai mici

adev.   prezis

Vreau Eroarea $\boxed{\varepsilon = \sum_n (x[n] - \hat{x}[n])^2}$ cât mai mic $\Leftrightarrow$

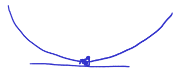
Vreau ca media $\overline{(x[n] - \hat{x}[n])^2}$ cât mai mică

Vom găsi cei mai buni coef. $\{a_p\}$ din condiția : $\overline{\left(x[n]-\hat{x}[n]\right)^2}$ să fie minim

$E\left\{\left(x[n]-\hat{x}[n]\right)^2\right\}$ vreau minim

$\Leftrightarrow$

Vreau ca $\underbrace{E\left\{\left(x[n]-a_1 x[n-1]-a_2 x[n-2]-\dots-a_p x[n-p]\right)^2\right\}}_{A}$ minim



Vreau $a_1, a_2, \dots a_p$ astfel încât A = minim :

$$\begin{cases} \dfrac{\partial A}{\partial a_1}=0 \\ \dfrac{\partial A}{\partial a_2}=0 \\ \vdots \\ \dfrac{\partial A}{\partial a_p}=0 \end{cases}$$

$\cancel{\dfrac{\partial E\{x\}}{\partial x}}$ $\Leftrightarrow$

$$\frac{\partial A}{\partial a_1}=0 \Leftrightarrow E\left\{\frac{\partial}{\partial a_1}\left(\qquad\right)^2\right\}=0$$

$$\frac{\partial}{\partial x}E\{f(x)\} = E\left\{\frac{\partial f}{\partial x}\right\}$$

$$E\left\{\cancel{2}\cdot\left(x[n]-a_1x[n-1]-\dots a_p x[n-p]\right)\cdot\left(\cancel{-}\,x[n-1]\right)\right\}=\cancel{0}$$

$$E\left\{x[n]\cdot x[n-1]-a_1 x[n-1]x[n-1]-\dots a_p x[n-p][n-1]\right\}=0$$

$$\underbrace{E\left\{x[n]\cdot x[n-1]\right\}}_{\gamma_{xx}[1]} - a_1\underbrace{E\left\{x[n-1]x[n-1]\right\}}_{\gamma_{xx}[0]} - \dots \cdot a_p\cdot \underbrace{E\left\{x[n-p]\,x[n-1]\right\}}_{\gamma_{xx}[-p+1]}=0$$

$$\boxed{\gamma_{xx}[1]-a_1\gamma_{xx}[0]-a_2\gamma_{xx}[-1]-\dots-a_p\cdot\gamma_{xx}[-p+1]=0}$$

$$\frac{\partial A}{\partial a_2}=0 \rightarrow \gamma_{xx}[2]-a_1\gamma_{xx}[1]-a_2\gamma_{xx}[0]-\dots-a_p\gamma_{xx}[-p+2]=0$$

$$\frac{\partial A}{\partial a_p}=0 \rightarrow$$

Ec. Yule-Walker :

$$\begin{cases} \gamma_{xx}[0]+a_1\gamma_{xx}[-1]+a_2\gamma_{xx}[-2]+\dots+a_p\gamma_{xx}[-p]=\sigma_w^2 \\ n=1 : \rightarrow \gamma_{xx}[1]+a_1\gamma_{xx}[0]+a_2\gamma_{xx}[-1]+\dots a_p\gamma_{xx}[-p+1]=0 \\ n=2 : \rightarrow \gamma_{xx}[2]+a_1\gamma_{xx}[1]+a_2\gamma_{xx}[0]+\dots a_p\gamma_{xx}[-p+2]=0 \\ \vdots \\ n=p \rightarrow \gamma_{xx}[p]+a_1\gamma_{xx}[p-1]+a_2\gamma_{xx}[p-2]+\dots a_p\gamma_{xx}[0]=0 \end{cases}$$