(1) $y[n] = 0.8\, y[n-1] + x[n] + x[n-1]$

$x[n]$ = Random process with $\gamma_{xx}[m] = \left(\frac{1}{2}\right)^{|m|}$

a) $\Gamma_{yy}(z) = ?$

b) $\gamma_{yy}[m] = ?$

c) $\sigma_y^2 = ?$

$x[n] \rightarrow \boxed{H(z)} \rightarrow y[n]$

$\gamma_{xx}[m] \qquad \gamma_{yy}[m]$

$$\underbrace{\Gamma_{xx}(z)}_{X(z)\cdot X(z^{-1})} \cdot H(z) \cdot H(z^{-1}) = \underbrace{\Gamma_{yy}(z)}_{Y(z)\cdot Y(z^{-1})}$$

$$H(z) = \frac{1 + z^{-1}}{1 - 0.8\, z^{-1}}$$

$$\Gamma_{xx}(z) = \mathcal{Z}\{\gamma_{xx}[m]\} = \sum_{m=-\infty}^{\infty} \gamma_{xx}[m] \cdot z^{-m} = \sum_{m=-\infty}^{\infty} \left(\frac{1}{2}\right)^{|m|} \cdot z^{-m} =$$

Same as in L11 $\left(\frac{1}{4}^{|m|}\right) \Rightarrow$ $= \dfrac{1}{1 - \frac{1}{2} z^{-1}} + \dfrac{1}{1 - \frac{1}{2} z} - 1$

$$= \frac{\left(1 - \frac{1}{2}z\right)\left(1 - \frac{1}{2}z^{-1}\right) - \left(1 - \frac{1}{2}z\right)\left(1 - \frac{1}{2}z^{-1}\right)}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 - \frac{1}{2}z\right)} = \frac{2 - \frac{1}{2}z - \frac{1}{2}z^{-1} - 1 + \frac{1}{2}z + \frac{1}{2}z^{-1} - \frac{1}{4}}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 - \frac{1}{2}z\right)}$$

$$= \frac{3/4}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 - \frac{1}{2}z\right)} = \frac{3/4}{\left(1 - \frac{1}{2}z^{-1}\right) \cdot \frac{1}{2}\left(2 - z\right)} = \frac{-3/2}{\left(1 - \frac{1}{2}z^{-1}\right)(z - 2)} = \frac{-3/2 \cdot z}{\left(z - \frac{1}{2}\right)(z - 2)}$$

$$\Gamma_{yy}(z) = \Gamma_{xx}(z) \cdot H(z) \cdot H(z^{-1}) = \frac{-3/2 \cdot z}{\left(z - \frac{1}{2}\right)(z - 2)} \cdot \frac{1 + z^{-1}}{1 - 0.8 z^{-1}} \cdot \frac{1 + z}{1 - 0.8 z} =$$

$$= \frac{-3/2 \cdot z\,(z+1)(z+1)}{\left(z - \frac{1}{2}\right)(z - 2)(z - 0.8)(1 - 0.8z)} = \frac{-3/2}{-0.8} \cdot \frac{z\,(z+1)^2}{\left(z - \frac{1}{2}\right)(z - 2)(z - 0.8)\left(z - \frac{1}{0.8}\right)}$$

$\Gamma_{yy}(\omega) \overset{z = e^{j\omega}}{=} \ldots$

b) $\gamma_{yy}[m] = \mathcal{Z}^{-1}\{\Gamma_{yy}(z)\}$

$$\frac{\Gamma_{yy}(z)}{z} = \frac{-3/2}{-0.8} \cdot \frac{(z+1)^2}{\left(z - \frac{1}{2}\right)\ldots\ldots(\,)} = \frac{1.5}{0.8}\left(\frac{A}{z - \frac{1}{2}} + \frac{B}{z - 2} + \frac{C}{z - 0.8} + \frac{D}{z - \frac{1}{0.8}}\right)$$

$$A = \left(z = \tfrac{1}{2}\right) = \frac{(1/2 + 1)^2}{\left(\frac{1}{2} - 2\right)\left(\frac{1}{2} - 0.8\right)\left(\frac{1}{2} - \frac{1}{0.8}\right)} = \ldots\ldots \qquad B, C, D = \ldots\ldots$$

$$\gamma_{yy}[m] = \frac{1.5}{0.8}\left(A \cdot \left(\frac{1}{2}\right)^{m} u[m] + B \cdot 2^{-m} u[-m-1] + C \cdot 0.8^{m} u[m] + D \cdot \left(\frac{1}{0.8}\right)^{-m} u[-m-1]\right)$$

$\left(\frac{1}{2} < 1\right)$ $\quad (2 > 1)$ $\quad (0.8 < 1)$ $\quad \left(\frac{1}{0.8} > 1\right)$

c). $\gamma_{yy}[0] = \overline{(y[m])^2} = \sigma_y^2 + \mu^2$

$\Rightarrow \sigma_y^2 = \gamma_{yy}[0] - \mu^2 = \gamma_{yy}[0] - 0^2 = \gamma_{yy}[0] = \cdots\cdots$