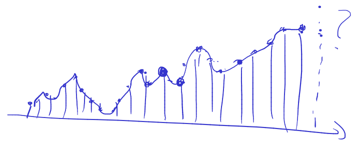
Lecture 14

Linear prediction (in short)



Know: $x[n]$ from $n = -\infty$ up to the current moment $(n-1)$

Can we predict the next value, $x[n]$?

Find these

Linear model: Assume $x[n] \simeq \underbrace{a_1 x[n-1] + a_2 x[n-2] + \dots a_p x[n-p]}_{\hat{x}[n]}$ linear combination of the last p samples

$\hat{x}[n]$ = the predicted value

Want: Find a_p, based on the signal values that you know already

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$$x[n] = \underbrace{a_1 x[n-1] + a_2 x[n-2] + \dots a_p x[n-p]}_{\hat{x}[n]} + e[n]$$

$e[n]$ prediction error $\Leftrightarrow$ white noise $w[n]$

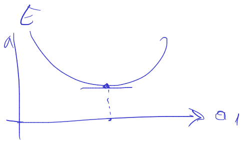
$\Leftrightarrow$

$$\boxed{x[n] - a_1 x[n-1] - a_2 x[n-2] - \dots - a_p x[n-p] = w[n]}$$

$\Leftrightarrow$ equation of an AR process of order p, (just with $-a_p$ instead of a_p)

Find $a_1 \dots a_p$ in order to make $\hat{x}[n] \simeq x[n]$

$\Leftrightarrow$ want to minimize $\overline{\left(x[n] - \hat{x}[n]\right)^2} = E$



$$\Leftrightarrow \begin{cases} \frac{\partial E}{\partial a_1} = 0 \\ \frac{\partial E}{\partial a_2} = 0 \\ \vdots \\ \frac{\partial E}{\partial a_p} = 0 \end{cases} \Leftrightarrow \text{Yule-Walker eq. system}$$

$$E = \overline{\left(x[n] - \hat{x}[n]\right)^2} = \overline{\left(x[n] - a_1 x[n-1] - a_2 x[n-2] - \dots - a_p x[n-p]\right)^2}$$

Want minimum

$$\frac{\partial E}{\partial a_1} = \frac{\partial}{\partial a_1}\overline{\left(\quad\right)^2} = \overline{\frac{\partial}{\partial a_1}\left(\quad\right)^2} = \overline{\cancel{2}\cdot(\dots \text{same} \dots)\cdot(\cancel{-}\, x[n-1])} = \cancel{0}$$

$$\Leftrightarrow \overline{\left(x[n]x[n-1] - a_1 \cdot x[n-1]\cdot x[n-1] - a_2 x[n-2]\, x[n-1] - \dots - a_p\, x[n-p]\, x[n-1]\right)} = 0$$

$$\Leftrightarrow \underbrace{\overline{x[n]\cdot x[n-1]}}_{\gamma_{xx}[1]} - a_1 \cdot \underbrace{\overline{x[n-1]\, x[n-1]}}_{\gamma_{xx}[0]} - a_2 \underbrace{\overline{x[n-2]\, x[n-1]}}_{\gamma_{xx}[-1]} - \dots - a_p \underbrace{\overline{x[n-p]\, x[n-1]}}_{\gamma_{xx}[-p+1]} = 0$$

$$\Leftrightarrow \boxed{\gamma_{xx}[1] - a_1 \gamma_{xx}[0] - a_2 \gamma_{xx}[-1] - \dots - a_p \gamma_{xx}[-p+1] = 0}$$

Same as eq 2 from Yule-Walker system, but with a_p with minus sign

$\frac{\partial E}{\partial a_2} = 0 \implies$ next eq. in Y-W. system

$\vdots$

$\frac{\partial E}{\partial a_p} = 0 \implies$ last eq. — " —

Find the best coef. $a_1, \dots a_p$ for linear prediction by solving the Yule-Walker system (and take a_k with opposite sign)

Exercise 1 / Lab 12

$$x[n] = \frac{1}{2} x[n-1] + w[n] + w[n-1] \qquad \gamma_{xx}[m] = ??$$

$\mu_x = ?$ (average value)

white noise

$$w[n] \longrightarrow \boxed{H(z)} \longrightarrow x[n]$$

$$\sigma_w^2 \cdot \delta[m] = \gamma_{ww}[m] \qquad \gamma_{xx}[m] = ?$$

$$\boxed{\underbrace{\Gamma_{ww}(z)}_{\text{a constant } \sigma_w^2} \cdot H(z) \cdot H(z^{-1}) = \Gamma_{xx}(z)}$$

inverse Z transform

$$x[n] = \underbrace{\frac{1}{2}}_{-a_1} x[n-1] + \underbrace{1}_{b_0} \cdot w[n] + \underbrace{1}_{b_1} \cdot w[n-1] \implies H(z) = \frac{1 + 1 \cdot z^{-1}}{1 + \frac{-1}{2} z^{-1}} = \frac{1 - z^{-1}}{1 - \frac{1}{2} z^{-1}}$$

$\leftarrow$ degree 1

$\leftarrow$ degree 1

ARMA(1,1)

$$H(z^{-1}) = \frac{1 - z}{1 - \frac{1}{2} z}$$

$$\Gamma_{xx}(z) = \sigma_w^2 \cdot \frac{1 - z^{-1}}{1 - \frac{1}{2} z^{-1}} \cdot \frac{1 - z}{1 - \frac{1}{2} z}$$

~~Do~~ the inverse Z transform and find $\gamma_{xx}[m]$

$$= \sigma_w^2 \cdot \frac{(z-1)}{(z - \frac{1}{2})} \cdot \frac{(1-z)}{(1 - \frac{1}{2} z)} = \sigma_w^2 \cdot \frac{(-1)(z-1)(z-1)}{(-\frac{1}{2})(z - \frac{1}{2})(z-2)} = 2 \cdot \sigma_w^2 \cdot \frac{(z-1)(z-1)}{(z - \frac{1}{2})(z-2)}$$

$$\frac{z}{z-a} \longleftrightarrow a^n u[n], \; |z| > |a|$$

$$-a^n u[-n-1], \; |z| < |a|$$

$$A(z) = \frac{(z-1)(z-1)}{(z-\frac{1}{2})(z-2)}$$

$$\frac{A(z)}{z} = \frac{(z-1)(z-1)}{z(z-\frac{1}{2})(z-2)} = \frac{A}{z} + \frac{B}{z - \frac{1}{2}} + \frac{C}{z-2}$$

$$A(z) = A + B \cdot \frac{z}{z - \frac{1}{2}} + C \cdot \frac{z}{z - 2} \longrightarrow a[n] = A\,\delta[n] + B \cdot \left(\frac{1}{2}\right)^n u[n] - C \cdot (2)^n u[-n-1]$$

> 1

$$A = \frac{(-1)(-1)}{(-\frac{1}{2})(-2)} = \frac{1}{1} = 1$$

$$B = \frac{(-\frac{1}{2}) \cdot (-\frac{1}{2})}{\frac{1}{2} \cdot (-\frac{3}{2})} = \frac{1/4}{-3/4} = -\frac{1}{3}$$

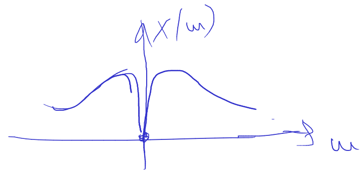
$$C = \frac{1 \cdot 1}{2 \cdot \frac{3}{2}} = \frac{1}{3}$$

$$\gamma_{xx}[m] = 2 \cdot \sigma_w^2 \left(A \cdot \delta[m] + B \cdot \left(\frac{1}{2}\right)^m u[m] - C \cdot 2^m u[-m-1] \right)$$

$$= 2 \cdot \sigma_w^2 \left(\delta[m] - \frac{1}{3} \left(\frac{1}{2}\right)^m u[m] - \frac{1}{3} \cdot 2^m u[-m-1] \right)$$

Average value of output $x[n] = 0$ because average value of input $= 0$.

Average value $= DC$ component $X(\omega)\big|_{\omega=0}$



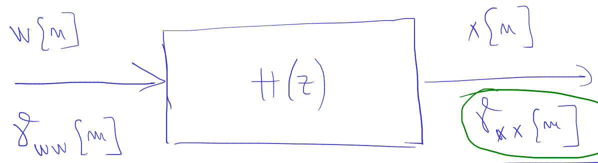
DC component of input $\cdot H(0) =$ DC component of output

$\underbrace{X(0)}_{=0} \cdot H(0) = \underbrace{Y(0)}_{=0}$

Exercise 2 / Lab 12

$$\gamma_{xx}[m] = \frac{1}{4}^{|m|}$$

Find system equation.



$$\boxed{\underbrace{\Gamma_{ww}(z)}_{ct = \sigma_w^2 \text{ a constant}} \cdot H(z) \cdot H(z^{-1}) = \Gamma_{xx}(z)}$$

Z transform

$$\gamma_{xx}[m] = \frac{1}{4}^{|m|} = \begin{cases} \left(\frac{1}{4}\right)^m, & m \geq 0 \\ \left(\frac{1}{4}\right)^{-m}, & m < 0 \end{cases}$$

$$\Gamma_{xx}(z) = \mathcal{Z}\{\gamma_{xx}[m]\} = \sum_{-\infty}^{\infty} \gamma_{xx}[m] \cdot z^{-m} = \sum_{m=0}^{\infty} \left(\frac{1}{4}\right)^m z^{-m} + \sum_{m=-\infty}^{-1} \left(\frac{1}{4}\right)^{-m} \cdot z^{-m}$$

$m \to -m$

$$= \underbrace{\sum_{m=0}^{\infty} \Big(\underbrace{\frac{1}{4z}}_{q}\Big)^m}_{\frac{1}{1-\frac{1}{4z}}} + \underbrace{\sum_{m=1}^{\infty} \left(\frac{1}{4}\right)^m \cdot z^m}_{\underbrace{\sum_{m=1}^{\infty} \Big(\underbrace{\frac{z}{4}}_{q}\Big)^m}_{= \frac{1}{1-\frac{z}{4}} - 1}}$$

$$\boxed{\underbrace{q^0}_{1} + q^1 + q^2 + \dots q^\infty = \frac{1}{1-q}}$$

$$= \frac{1}{1-\frac{1}{4z}} + \frac{1}{1-\frac{z}{4}} - 1 = \frac{1}{\left(1 - \frac{1}{4} z^{-1}\right)} + \frac{1}{\left(1 - \frac{1}{4} z\right)} - 1$$

$$= \frac{\left(1-\frac{1}{4}z\right) + \left(1-\frac{1}{4}z^{-1}\right) - \left(1-\frac{1}{4}z^{-1}\right)\left(1-\frac{1}{4}z\right)}{\left(1-\frac{1}{4}z^{-1}\right)\left(1-\frac{1}{4}z\right)}$$

$$= \frac{1-\cancel{\frac{1}{4}z}+\cancel{1}-\cancel{\frac{1}{4}z^{-1}}-\cancel{1}+\cancel{\frac{1}{4}z}+\cancel{\frac{1}{4}z^{-1}}-\frac{1}{16}}{\left(1-\frac{1}{4}z^{-1}\right)\left(1-\frac{1}{4}z\right)} = \frac{15/16}{(\quad)\cdot(\quad)}$$

$$\Gamma_{xx}(z) = \frac{15/16}{\left(1-\frac{1}{4}z^{-1}\right)\left(1-\frac{1}{4}z\right)}$$

$p = \frac{1}{4}$ $\qquad$ $p = 4$

Pick the terms which have poles & zeros in unit circle for $H(z)$

$$\underbrace{\Gamma_{ww}(z)}_{\text{constant}} \cdot H(z) \cdot H(z^{-1}) = \Gamma_{xx}(z)$$

$$\Rightarrow H(z) = \frac{\overset{b_0}{1}}{1\underbrace{-\frac{1}{4}}_{a_1}z^{-1}} \Longrightarrow x[n] = \underbrace{+\frac{1}{4}}_{-a_1}x[n-1] + \underbrace{1}_{b_0}\,w[n]$$

END OF 2020-2021 LECTURES