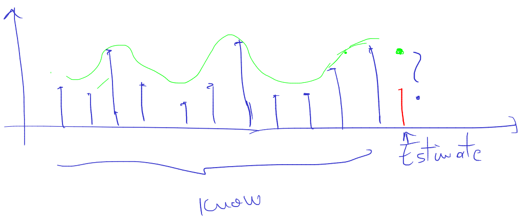
Lecture 13

$x[n] \leftrightarrow X(z)$



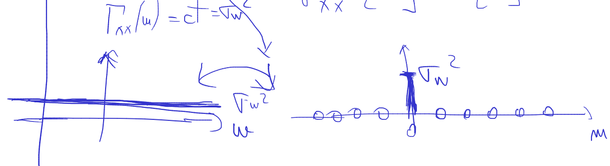
White noise

$\Gamma_{xx}(z) = ct = \sigma_w^2$

$x[n]$

$\gamma_{xx}[m] = \delta[m]$ = Dirac

$\Gamma_{xx}(\omega) = ct = \sigma_w^2$



$x[n] \rightarrow$ · μ, σ^2, χ^2

Autocorrelation function : $\gamma_{xx}[m] = E\{x[n]\, x[n-m]\} = \overline{x[n] \cdot x[n-m]}$

$\gamma_{xx}[m] \xleftrightarrow{\mathcal{Z}} \Gamma_{xx}(z) = \mathcal{Z}\{\gamma_{xx}[m]\}$

Autocorrelation sequence

It's Z transform

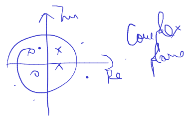
$z = e^{j\omega}$

$\gamma_{xx}[m] \xleftrightarrow{\mathcal{F}} \Gamma_{xx}(\omega) = \mathcal{F}\{\gamma_{xx}[m]\}$ = the power spectral density of $x[n]$

Prove that $\Gamma_{xx}(z)$ can be written as $\boxed{\Gamma_{xx}(z) = \underbrace{\sigma_w^2}_{\text{constant}} \cdot H(z) \cdot H^*\left(\frac{1}{z^*}\right)}$ complex conjugated *

Z transf. of the autcorr

$H(z^{-1})$ if all coeffs. are $\in \mathbb{R}$



$z = 0.5$, $p = -3$

$$H(z) = \frac{z - 0.5}{z + 3}$$

$$H^*\left(\frac{1}{z^*}\right) = ? = \left(\frac{\frac{1}{z^*} - 0.5}{\frac{1}{z^*} + 3}\right)^* = \underbrace{\frac{\frac{1}{z} - 0.5}{\frac{1}{z} + 3}}$$

$\in \mathbb{R}$

$z/$

$$H\left(\frac{1}{z}\right) = H(z^{-1})$$

$$(a+b)^* = a^* + b^*$$
$$(a \cdot b)^* = a^* \cdot b^*$$
$$\left(\frac{a}{b}\right)^* = \frac{a^*}{b^*}$$

$$(a+jb)^* = a - jb$$

$$= \frac{1 - 0.5z}{1 + 3z} =$$

$$= \frac{-0.5(z-2)}{3\left(z + \frac{1}{3}\right)} = \frac{-0.5}{3} \cdot \frac{z-2}{z+\frac{1}{3}}$$

$z = 2$, $p = -\frac{1}{3}$

$H(z^{-1})$ has the same zeros and poles as $H(z)$, but inversed $\frac{1}{z}$, $\frac{1}{p}$
(z, p)

$p = 1/4$ $\quad$ $p = 4$

$z = 2.75$ $\quad$ $z = \frac{1}{2.75}$

$x \longleftrightarrow \frac{1}{x}$
Reciprocal

$$\underbrace{\Gamma_{xx}(z)}_{\text{output autocorr.}} = \underbrace{\sigma_w^2}_{\Gamma_{xx}(z) = \text{constant}} \cdot H(z) \cdot H(z^{-1})$$

$x[n] \to \boxed{H(z)} \to y[n]$

$\gamma_{xx}[m] \qquad \gamma_{yy}[m]$

$$\boxed{\Gamma_{xx}(z) \cdot H(z) \cdot H(z^{-1}) = \Gamma_{yy}(z)} \quad \Big|_{z = e^{j\omega}}$$

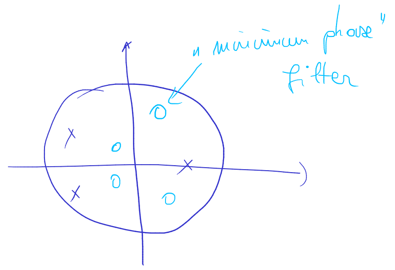
$$\Gamma_{xx}(\omega) \cdot |H(\omega)|^2 = \Gamma_{yy}(\omega)$$

$w[n] \xrightarrow{\text{white noise}} \boxed{H(z)} \to x[n]$

$$\Gamma_{ww}(z) = \sigma_w^2 \cdot H(z) \cdot H(z^{-1}) = \Gamma_{xx}(z)$$
$$= ct$$

$x[n] \xrightarrow{\text{non-white noise}} \boxed{H'(z) = \frac{1}{H(z)}} \to w[n]$ white noise

Whitening filter



$$\Gamma_{xx}(z) = 2 \cdot \frac{(z-0.5)(z-2)(z-\frac{1}{3})(z-3)}{(z-0.2)(z-5)(z-0.1)(z-10)}$$

$\sigma_w^2 \cdot H(z) \cdot H(z^{-1})$

$H(z) =$ stable

$\frac{1}{H(z)} =$ also stable

$H(z^{-1}) = H^*\left(\frac{1}{z^*}\right)$

$w[n] \to \boxed{H(z)} \to x[n]$

Model types (for random processes):

I AutoRegressive with Moving Average $\boxed{ARMA(p, q)}$

$$H(z) = \frac{B(z) \text{ (degree } q)}{A(z) \text{ (degree } p)} = \frac{b_0 + b_1 z^{-1} + \dots + b_q z^{-q}}{1 + a_1 z^{-1} + \dots a_p z^{-p}}$$

II AutoRegressive process $AR(p)$

$$H(z) = \frac{1}{A(z)} = \frac{1}{1 + a_1 z^{-1} + \dots + a_p z^{-p}}$$

III Moving Average process $MA(q)$:

$$H(z) = B(z) = b_0 + b_1 z^{-1} + \dots + b_q z^{-q}$$

In general (for ARMA process):

$w[n]$ for an AR process

$$x[n] = -\sum_{k=1}^{p} a_k\, x[n-k] + \sum_{k=0}^{q} b_k\, w[n-k]$$

$E\{a+b\} = E\{a\} + E\{b\}$

$E\{c \cdot a_n\} = c \cdot E\{a_n\}$

$$\gamma_{xx}[m] = \underset{\text{over } n}{E}\{x[n] \cdot x[n-m]\}$$

$$= \underset{\text{over } n}{E}\left\{-\sum_{k=1}^{p} a_k\, \underbrace{x[n-k] \cdot x[n-m]} + \sum_{k=0}^{q} b_k\, \underbrace{w[n-k] \cdot x[n-m]}\right\}$$

$$= -\sum_{k=1}^{p} a_k\, \underbrace{E\{x[n-k] \cdot x[n-m]\}}_{\gamma_{xx}[m-k]} + \sum_{k=0}^{q} b_k\, \underbrace{E\{w[n-k] \cdot x[n-m]\}}_{\gamma_{wx}[m-k]}$$

$$\gamma_{xx}[m] = -\sum_{k=1}^{p} a_k \cdot \gamma_{xx}[m-k] + \sum_{k=0}^{q} b_k \cdot \gamma_{wx}[m-k]$$

$\gamma_{wx}[m]$ for an AR process

$$\sum_{k=0}^{q} b_k \cdot \gamma_{wx}[m-k] = \ldots = \sigma_w^2 \cdot \sum_{k=0}^{q-m} b_{k+m} \cdot h[k]$$

for an AR process:
- when $m = 0$, σ_w^2
- when $m > 0$, 0

for AR: $\sigma_w^2 \cdot h[0] = \sigma_w^2$

↳ ~~for~~ for $m = 0$: $\sigma_w^2 \cdot \sum_{k=0}^{q} b_k\, h[k]$

for $m > 0$: $\sigma_w^2 \cdot \sum_{k=0}^{q-m} b_{k+m}\, h[k] = 0$

$k + m > 0 \Rightarrow b_{k+m} = 0$

For an AR process:

$$\gamma_{xx}[m] = -\sum_{k=1}^{p} a_k\, \gamma_{xx}[m-k] + \begin{cases} \sigma_w^2, & \text{if } m = 0 \\ 0, & \text{if } m > 0 \end{cases}$$

(when $m < 0$, $\gamma_{xx}[m] = \gamma_{xx}[-m]$)

$\Leftrightarrow$

$$\boxed{\gamma_{xx}[m] + \sum_{k=1}^{p} a_k\, \gamma_{xx}[m-k] = \begin{cases} \sigma_w^2, & \text{if } m = 0 \\ 0, & \text{if } m > 0 \end{cases}}$$

$$\begin{cases}
m = 0: & \gamma_{xx}[0] + a_1 \gamma_{xx}[-1] + a_2 \gamma_{xx}[-2] + \ldots + a_p \gamma_{xx}[-p] = \sigma_w^2 \\
m = 1: & \gamma_{xx}[1] + a_1 \gamma_{xx}[0] + a_2 \gamma_{xx}[-1] + \ldots\ a_p \gamma_{xx}[1-p] = 0 \\
 & \vdots \\
m = p-1: & \gamma_{xx}[p-1] + a_1 \cdot \gamma_{xx}[p-2] + a_2 \gamma_{xx}[p-3] + \ldots\ a_p \gamma_{xx}[-1] = 0 \\
m = p: & \gamma_{xx}[p] + a_1 \gamma_{xx}[p-1] + a_2 \gamma_{xx}[p-2] + \ldots + a_p \gamma_{xx}[0] = 0
\end{cases}$$

$\Leftrightarrow$

$(\Longleftrightarrow)$

$$\begin{bmatrix} \gamma_{xx}[0] & \gamma_{xx}[-1] & \cdots & \gamma_{xx}[-p] \\ \gamma_{xx}[1] & \gamma_{xx}[0] & \cdots & \gamma_{xx}[-p+1] \\ & \vdots & & \\ \gamma_{xx}[p] & \gamma_{xx}[p-1] & \cdots & \gamma_{xx}[0] \end{bmatrix} \cdot \begin{bmatrix} 1 \\ a_1 \\ \vdots \\ a_p \end{bmatrix} = \begin{bmatrix} \sigma_w^2 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$(p+1)$