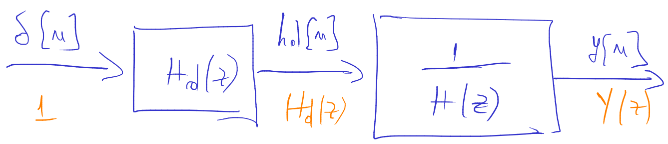
Lecture 02

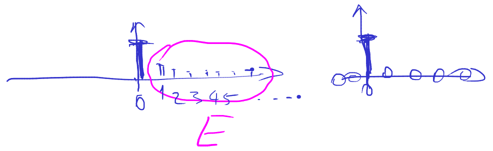
Inverse least-squares method

$$H(z) = \frac{b_0}{1 + a_1 z^{-1} + \dots + a_N z^{-N}}$$



$$H(z) = H_d(z) \Rightarrow y[n] = \delta[n]$$

$$H(z) \simeq H_d(z) \Rightarrow y[n] \simeq \delta[n]$$



$$E = \sum_{n=1}^{\infty} \left(y[n]\right)^2 = \text{want minimum}$$

$$= \sum_{n=0}^{\infty} \left(y[n] - \delta[n]\right)^2$$

$$\boxed{\sum_{\ell=1}^{N} a_\ell \cdot r_{dd}[k,\ell] = -r_{dd}[k]}$$

$k = 1, 2, 3, \dots N$

$$r_{dd}[k,\ell] = \sum_{n=1}^{\infty} h_d[n-k] \cdot h_d[n-\ell]$$

$$r_{dd}[k] = \sum_{n=1}^{\infty} h_d[n] \cdot h_d[n-k]$$

$k = 1$: $\sum_{\ell=1}^{N} a_\ell \cdot r_{dd}[1,\ell] = -r_{dd}[1]$

$(=)$

$$\begin{cases} a_1 \cdot r_{dd}[1,1] + a_2 \cdot r_{dd}[1,2] + \dots + a_N \cdot r_{dd}[1,N] = -r_{dd}[1] \\ a_1 \cdot r_{dd}[2,1] + a_2 \cdot r_{dd}[2,2] + \dots + a_N \cdot r_{dd}[2,N] = -r_{dd}[2] \\ \vdots \\ a_1 \cdot r_{dd}[N,1] + a_2 \cdot r_{dd}[N,2] + \dots + a_N \cdot r_{dd}[N,N] = -r_{dd}[N] \end{cases}$$

$k = 2$: $\rightarrow$ (second line); $k = N$ $\rightarrow$ (last line)

$$\begin{bmatrix} r_{dd}[1,1] & r_{dd}[1,2] & \dots & r_{dd}[1,N] \\ r_{dd}[2,1] & r_{dd}[2,2] & \dots & r_{dd}[2,N] \\ & \vdots & & \\ r_{dd}[N,1] & r_{dd}[N,2] & \dots & r_{dd}[N,N] \end{bmatrix} \cdot \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix} = \begin{bmatrix} -r_{dd}[1] \\ -r_{dd}[2] \\ \vdots \\ -r_{dd}[N] \end{bmatrix}$$

Inverse least-squares filter design

Same result obtained in a different way

$$h[n] \simeq h_d[n]$$

$$E = \sum_{n=0}^{\infty} \left(h[n] - h_d[n] \right)^2 = \text{want minimum}$$

If we have $H(z) = \dfrac{b_0}{1 + a_1 z^{-1} + \dots + a_N z^{-N}}$

$x = \delta$

$y = h$

$$y[n] = -a_1 y[n-1] - \dots - a_N y[n-N] + b_0 x[n]$$

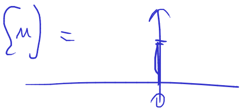
$$\boxed{h[n] = -a_1 h[n-1] - \dots - a_N \cdot h[n-N] + b_0 \delta[n]}$$

for $n \geq 1$:

$$\boxed{h[n] = -a_1 h[n-1] - \dots - a_N \cdot h[n-N]}$$

$$= - \sum_{l=1}^{N} a_l h[n-l]$$

Prony's method

$\delta[n] =$ 

$$H(z) = \frac{b_0 + b_1 z^{-1} + \dots + b_M z^{-M}}{1 + a_1 z^{-1} + \dots + a_N z^{-N}}$$

$$y[n] = -a_1 y[n-1] - \dots - a_N y[n-N] + b_0 x[n] + \dots + b_M x[n-M]$$

$x = \delta$

$y = h$

$$h[n] = -a_1 h[n-1] - \dots - a_N \cdot h[n-N] + \underbrace{b_0 \delta[n] + b_1 \delta[n-1] + \dots + b_M \delta[n-M]}_{\text{disappear for } n > M}$$

For $n > M$:

$$\boxed{h[n] = -a_1 \cdot h[n-1] - \dots - a_N \cdot h[n-N]}$$

$h[n]$ is computed only on the prev. values of $h[n-1] \dots h[n-N]$

For the desired $h_d[n]$:

$$\hat{h}_d[n] = -a_1 \cdot h_d[n-1] - \dots - a_N h_d[n-N] = \text{the predicted value of } h_d[n]$$

I want $h_d[n] \simeq \hat{h}_d[n]$

$$\boxed{h_d[n] \simeq \underbrace{-a_1 \cdot h_d[n-1] - \dots - a_N h_d[n-N]}_{\hat{h}_d[n]}}$$

What we want

$$E = \sum_{n=M+1}^{n=\infty} \left(h_d[n] - \hat{h}_d[n] \right)^2 \quad \text{want minimum}$$

$$= \sum_{n=M+1}^{\infty} \Big(h_d[n] + \underbrace{a_1 h_d[n-1] + \dots + a_N h_d[n-N]}_{\sum_{\ell=1}^{N} a_\ell \cdot h_d[n-\ell]} \Big)^2 \quad \text{want minimum}$$

$a_k h_d[n-k]$

$$\frac{\partial E}{\partial a_k} = \sum_{n=M+1}^{\infty} \cancel{2} \left(h_d[n] + \sum_{\ell=1}^{N} a_\ell \cdot h_d[n-\ell] \right) \cdot h_d[n-k] = 0 \quad \text{for } k = 1, 2, \dots N$$

System:

$$\underbrace{\sum_{n=M+1}^{\infty} h_d[n] \cdot h_d[n-k]}_{R_{dd}[k]} + \underbrace{\sum_{n=M+1}^{\infty} \sum_{\ell=1}^{N} a_\ell \cdot h_d[n-\ell] \cdot h_d[n-k]}_{\sum_{\ell=1}^{N} a_\ell \underbrace{\sum_{n=M+1}^{\infty} h_d[n-\ell] \cdot h_d[n-k]}_{R_{dd}[k,\ell]}} = 0$$

$$\boxed{\sum_{\ell=1}^{N} a_\ell \cdot R_{dd}[k,\ell] = -R_{dd}[k]} \quad \text{the same!}$$

for $k = 1, 2, 3, \dots N$

$$\begin{array}{c} k=1 \rightarrow \\ k=2 \rightarrow \\ \\ k=N \rightarrow \end{array} \begin{bmatrix} R_{dd}[1,1] & R_{dd}[1,2] & \dots & R_{dd}[1,N] \\ R_{dd}[2,1] & R_{dd}[2,2] & \dots & R_{dd}[2,N] \\ & \vdots & & \\ R_{dd}[N,1] & R_{dd}[N,2] & \dots & R_{dd}[N,N] \end{bmatrix} \cdot \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix} = \begin{bmatrix} -R_{dd}[1] \\ -R_{dd}[2] \\ \vdots \\ -R_{dd}[N] \end{bmatrix} \Rightarrow \text{Find } a_1, a_2 \dots a_N$$

Then find $b_0, b_1, \dots b_M$ just like for Padé

Example:

Prony, order 2, $h_d[n] = \{0 \dots \underset{\uparrow}{1} \; 2 \; 3 \; 2 \; 1 \; 2 \; 3 \; 0 \dots 0\}$

$h_d[0]$ $h_d[1]$ $h_d[2]$ $h_d[3]$ $[4]$ $[5]$ $h_d[6]$

$$H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}}$$

1) Find a_1, a_2:

$$\begin{bmatrix} R_{dd}[1,1] & R_{dd}[1,2] \\ R_{dd}[2,1] & R_{dd}[2,2] \end{bmatrix} \cdot \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} -R_{dd}[1] \\ -R_{dd}[2] \end{bmatrix}$$

$$R_{dd}[k,\ell] = \sum_{\substack{n=M+1 \\ 2+1 \\ n=3}} h_d[n-k] \, h_d[n-\ell]$$

$$R_{dd}[k] = \sum_{\substack{M=3 \\ M+1}}^{\infty} h_d[n] \cdot h_d[n-k]$$

$$R_{dd}[\underset{k}{1},\underset{l}{1}] = \sum_{m=3}^{\infty} h_d[m-1]\cdot h_d[m-1] = h_d[2]\cdot h_d[2] + h_d[3]\cdot h_d[3] + \dots$$

$$= 3^2 + 2^2 + 1^2 + 2^2 + 3^2 = 9+4+1+4+9 = 27$$

$$R_{dd}[\underset{k}{1},\underset{l}{2}] = \sum_{m=3}^{\infty} h_d[m-1]\cdot h_d[m-2] =$$

$$= h_d[2]\cdot h_d[1] + h_d[3]\cdot h_d[2] + h_d[4]\cdot h_d[3] + \dots$$

$$= \cancel{3\cdot 2} +$$

$$\begin{array}{l|cccccccc}
h_d: \; \overset{h_d[0]}{1} \;\; 2 & \overset{h_d[2]}{3} & 2 & 1 & 2 & 3 & 0 & \dots \\
\quad\quad\;\; 1 & 2 & 3 & 2 & 1 & 2 & 3 \\
\hline
 & 6 & +6 & +2 & +2 & +6 & +0 & = 22
\end{array}$$

$$R_{dd}[2,1] = \sum_{m=3}^{\infty} h_d[m-2]\cdot h_d[m-1]$$

$$= R_{dd}[1,2] = 22$$

$$R_{dd}[\underset{k}{2},\underset{l}{2}] = \sum_{m=3}^{\infty} h_d[m-2]\cdot h_d[m-2] = h_d[1]^2 + h_d[2]^2 + \dots$$

$$= 2^2 + 3^2 + 2^2 + 1^2 + 2^2 + 3^2 =$$

$$= 4+9+4+1+4+9 = 31$$

$$R_{dd}[\underset{k}{1}] = \sum_{m=3}^{\infty} h_d[m]\, h_d[m-1] = h_d[3]\cdot h_d[2] + h_d[4]\, h_d[3] + \dots$$

$$\begin{array}{ccc|cccccc}
\overset{h_d[0]}{1} & 2 & 3 & \overset{h_d[3]}{2} & 1 & 2 & 3 & \times \\
 & 1 & 2 & 3 & 2 & 1 & 2 & 3 \\
\hline
 & & & 6 & +2 & +2 & +6 & +0 \dots & = 16
\end{array}$$

$$R_{dd}[2] = \sum_{m=3}^{\infty} h_d[m]\cdot h_d[m-2]$$

$$\begin{array}{ccc|cccccc}
\overset{h_d[0]}{1} & 2 & 3 & \overset{h_d[3]}{2} & 1 & 2 & 3 & 0 & 0 \\
 & & 1 & 2 & 3 & 2 & 1 & 2 & 3 \\
\hline
 & & & 4 & 3 & 4 & 3 & \dots & = 14
\end{array}$$

$$\begin{bmatrix} 27 & 22 \\ 22 & 31 \end{bmatrix} \cdot \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} -16 \\ -14 \end{bmatrix}$$

$$\begin{cases} 27 a_1 + 22 a_2 = -16 \\ 22 a_1 + 31 a_2 = -14 \end{cases} \Rightarrow a_2 = \frac{-16 - 27 a_1}{22}$$

$$22 a_1 + 31 \cdot \frac{-16 - 27 a_1}{22} = -14 \Rightarrow a_1 = -0.53$$

$$a_2 = -1.38$$

$$a_1 = \frac{\begin{vmatrix} -16 & 22 \\ -14 & 31 \end{vmatrix}}{\begin{vmatrix} 27 & 22 \\ 22 & 31 \end{vmatrix}} = \frac{-16\cdot 32 + 14\cdot 22}{27\cdot 31 - 22\cdot 22} = -0.53$$

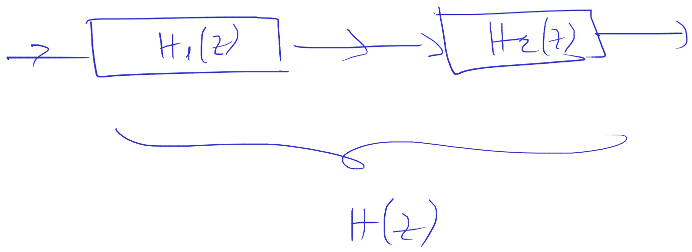
For b_0, b_1, b_2: use Padé:

$$
\begin{cases}
b_0 = h_d[0] \Rightarrow b_0 = 1 \\
-\underbrace{a_1}_{-0.53}\underbrace{h_d[0]}_{1} + b_1 = \underbrace{h_d[1]}_{2} \Rightarrow b_1 = 2 - 0.53 = 1.47 \\
-\underbrace{a_1}_{-0.53}\underbrace{h_d[1]}_{2} - \underbrace{a_2}_{-1.38}\underbrace{h_d[0]}_{1} + b_2 = \underbrace{h_d[2]}_{3} \Rightarrow b_2 = \dots
\end{cases}
$$

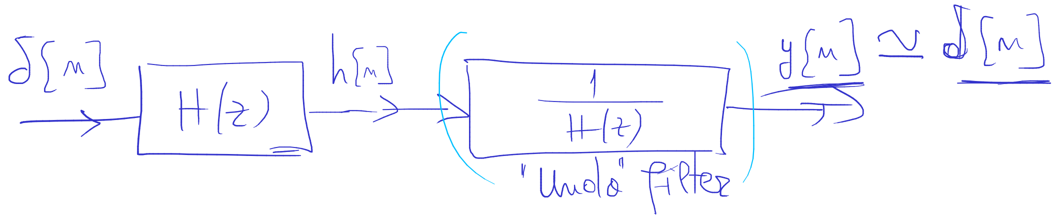
Shanks:

$$
H(z) = \frac{b_0 + b_1 z^{-1} + \dots + b_M z^{-M}}{1 + a_1 z^{-1} + \dots + a_N z^{-N}} = \underbrace{\frac{\cancel{b_0}\ 1}{1 + a_1 z^{-1} + \dots + a_N z^{-N}}}_{H_1(z)} \cdot \underbrace{\frac{\left(b_0 + b_1 z^{-1} + \dots + b_M z^{-M}\right)}{\cancel{b_0}\ 1}}_{H_2(z)}
$$

$h_2[n] = \{b_0, b_1, \dots b_M\}$



FIR inverse filter found by least-sq. methods



Want an FIR filter to approximate $\frac{1}{H(z)}$