IIR filter design with the Prony method

Lab 3, SDP

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Designing IIR filters with the Prony method.

2 Theoretical notions

2.1 Autocorrelation function

For a $x[n]$, the autocorrelation function is defined as:

$$r_{xx}[k] = \sum_{n=-\infty}^{\infty} x[n]x[n+k] \tag{1}$$

In Matlab, for a vector `x` of length `L` (elements going from `x[1]` to `x[L]`), the autocorrelation function is calculated with the `xcorr()` function, as in the following example:

```
x = [1,2,3,4];
rxx = xcorr(x) % Computes the autocorrelation of x
```

```
rxx =

    4.0000 11.0000 20.0000 30.0000 20.0000 11.0000 4.0000
```

In total there are $2L-1$ values (where L = length of `x`), starting from $r_{xx}[-(L-1)]$ and ending at $r_{xx}[L-1]$. So the value $r_{xx}[0]$ from theory is located in Matlab in the middle of the resulting vector, `rxx(L)`:

```
L = length(x);
rxx(L)   % Value of r_xx[0]
rxx(L+1) % Value of r_xx[1]
rxx(L-3) % Value of r_xx[-3]
```

2.2 Partial autocorrelation for the Prony method

For the Prony method we need the values of a **partial autocorrelation** function, defined as:

$$r_{xx}[k,l] = r_{xx}[k-l] = \sum_{n=M+1}^{\infty} h[n-k]h[n-l] = \sum_{n=M+1-k}^{\infty} h[n]h[n+(k-l)] \tag{2}$$

The difference is that the **sum doesn't start at** $n = 0$, but from a higher value, so some of the first elements in the sum are missing.

The partial autocorrelation can be calculated like the usual one, if the first $M+1-max(k,l)$ elements of the vector are transformed to 0.

Let the following be the example to calculate $r_{xx}[k=1, l=2]$, with $M = 2$:

```
M = 2;
x = [1,2,3,4];
k=1;
l=2;

x(1 : M+1-max(k,l)) = 0; % We set the first values to 0
x % Displays the changed x
```

```
rxx = xcorr(x) % Calculates the partial autocorrelation
```

2.3 The Prony method

In the Prony method, the coefficients $\{a_k\}$ are first calculated from a system of equations using the partial autocorrelation values:

$$\begin{bmatrix} r_{dd}[1,1] & r_{dd}[1,2] & \dots & r_{dd}[1,N] \\ r_{dd}[2,1] & r_{dd}[2,2] & \dots & r_{dd}[2,N] \\ \vdots & \dots & \dots & \vdots \\ r_{dd}[N,1] & r_{dd}[N,2] & \dots & r_{dd}[N,N] \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix} = \begin{bmatrix} -r_{dd}[1,0] \\ -r_{dd}[2,0] \\ \vdots \\ -r_{dd}[N,0] \end{bmatrix} \tag{3}$$

The b_k coefficients are obtained from the same equations as in the Pade method, replacing the $\{a_k\}$ values found above. The equations for b_k can be written as follows:

$$b_n = h_d[n] + \sum_{k=1}^{N} a_k h_d[n-k]$$

or, in matrix form:

$$\begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ \dots \\ b_M \end{bmatrix} = \begin{bmatrix} h_d[0] \\ h_d[1] \\ h_d[2] \\ \dots \\ h_d[M] \end{bmatrix} + \begin{bmatrix} 0 & 0 & \cdots & 0 \\ -h_d[0] & 0 & \cdots & 0 \\ -h_d[1] & -h_d[0] & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ -h_d[M-1] & -h_d[M-2] & \cdots & -h_d[M-N] \end{bmatrix} \cdot \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix} \tag{4}$$

3 Theoretical exercises

1. Use the Prony method to find the parameters of the 2nd-order system with the following system function:

   $$H(z) = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}}$$

   which approximates the desired impulse response

   $$h_d[n] = \{...0, \underset{\uparrow}{1}, 2, 3, 2, 1, 2, 3\}$$

   (the origin of time $n = 0$ is the first value of 1 in the sequence).

4 Practical exercises

1. Compute and display the autocorrelation function for the constant vector $\{3, 3, 3, 3, 3, 3, 3\}$. Use the `xcorr()` and `stem()` functions.

   Then indicate what are the values of $r_{xx}[0]$ and $r_{xx}[2]$.

2. Create a function `r = xcorr_prony(x, k, l, M)` to calculate the partial autocorrelation for a vector `x`. The function must return a single value, $r_{xx}[k-l]$, for the specified k and l.

   Note: remember that $r_{xx}[0] =$ `rxx(L)` in Matlab.

   Test the function by checking the following values for `x = [1,2,3,2,1,2,3]` and `M=2`:

   - $r_{xx}[1,1] = 27$
   - $r_{xx}[1,2] = 22$
   - $r_{xx}[2,1] = 22$
   - $r_{xx}[2,2] = 31$
   - $r_{xx}[1,0] = r_{xx}[1] = 16$
   - $r_{xx}[2,0] = r_{xx}[2] = 14$

   Template:

```
function r = xcorr_prony(x, k, l, M)
% Computes restricted autocorrelation for the Prony method
%Inputs:
% x = the input vector
% k,l = the element to compute
% M = the degree of the numerator polynomial B(z)
% Returns:
% r = rxx[k-l]

...

end
```

3. Use the Prony method to find the coefficients a_k and b_k, for a system of order 2 with $M = 2$ and $N = 2$, and a desired impulse response equal to $h_d[n] = \{1, 2, 3, 2, 1, 2, 3\}$.

   Use the `linsolve()` function to solve the system of equations of a_k.

```
hd = [1,2,3,2,1,2,3];
M = 2; % numerator degree
N = 2; % denominator degree
```

```
% Find coefficients a_k
A = ... % 2x2 matrix
y = ... % 2x1 column vector

a = linsolve(A,y) % solve for a_k

% Find coefficients b_k
M = ... % copy part of the matrix from the Pade method
b = ... % compute the b_k coefficients
```

4. Implement in Matlab a general function for the Prony method, for a system of any order and any signal $h_d[n]$.

   ```
   [b,a] = prony_method(order, hd)
   ```

   The function shall receive as arguments:

   - `order`: order of the desired filter
   - `hd`: a vector with the desired impulse response

   The function will return the coefficients of the designed filter:

   - `b`: coefficients of the numerator
   - `a`: coefficients of the denominator

5. Use the Prony method to find the parameters of the 2nd order filter which approximates the following higher order filter of order 3:

$$H(z) = \frac{0.0736 + 0.0762z^{-1} + 0.0762z^{-1} + 0.0736z^{-3}}{1 - 1.3969z^{-1} + 0.8778z^{-1} - 0.1812z^{-3}}$$

   a. Use the `impz()` function to generate a sufficiently long impulse response of the given filter (e.g. 100 samples);
   b. Use the `prony_method()` function to design the 2nd order filter;
   c. Plot the impulse response of the original filter on the same graph and of the projected one, for the first 50 samples.

6. Load an audio signal into Matlab and filter it with the filter designed above. Play the filtered signal to the audio output of the system.

5 Final questions

1. TBD