

# Information Theory

## Lecture notes 2015-2016

## Introduction

# Organization

Professors:

- ▶ Lectures: Nicolae Cleju
- ▶ Laboratories: Daniel Matasaru

# Grades

$$\text{Final grade} = 0.75 \text{ Exam} + 0.25 \text{ Lab}$$

# Time schedule

- ▶ 14 weeks of lectures (3h each)
- ▶ 14 weeks of laboratories (2h each)
- ▶ Office hours: by appointment

# Course structure

1. Chapter I: Discrete Information Sources
2. Chapter II: Discrete Transmission Channels
3. Chapter III: Source Coding
4. Chapter IV: Channel Coding

1. ***Elements of Information Theory*, Valeriu Munteanu, Daniela Tarniceriu, Ed. CERM I 2007**
2. *Elements of Information Theory*, Thomas M. Cover, Joy A. Thomas, 2nd Edition, Wiley 2006
3. *Information and Coding Theory*, Gareth A. Jones, J. Mary Jones, Springer 2000
4. *Transmisia si codarea informatiei*, lectures at ETTI (Romanian)



## Introduction to probabilities

# Basic notions of probability

- ▶ Random variable = the outcome of an experiment
- ▶ Distribution (probability mass function)
- ▶ Discrete distribution
- ▶ Alphabet
- ▶ Logarithm function
- ▶ Exponential function
- ▶ Average of some values

# Basic properties

- ▶ Two independent events:

$$p(A \cap B) = p(A) \cdot p(B)$$

## Chapter I: Discrete information sources

# Block diagram of a communication system

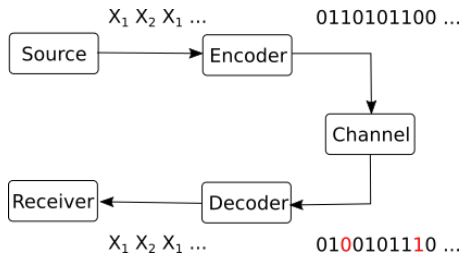


Figure 1: Block diagram of a communication system

# What is information?

## Example:

- ▶ I tell you the following sentence: “your favorite football team lost the last match”.
- ▶ Does this message carry information? How, why, how much?
- ▶ Consider the following facts:
  - ▶ the message carries information only because you didn't already know the result.
  - ▶ if you already known the result, the message is useless (brings no information)
  - ▶ since you didn't know the result, there were multiple results possible (win, equal or lose)
  - ▶ the actual information in the message is that *lost* happened, and not *win* or *equal*
  - ▶ if the result was to be expected, there is little information. If the result is highly unusual, there is more information in this message

# Information source

- ▶ We will always consider information in a context similar to the above example.
- ▶ We will use terminology from probability theory to define information:
  - ▶ there is a *probabilistic source* that can produce a number of different *events*.
  - ▶ each event has a certain probability. We know all the probabilities beforehand.
  - ▶ at one time, an event is randomly selected according to its probability.
  - ▶ afterwards, a new message can be selected, and so on ==> a stream of messages is produced.
- ▶ The source is called an *information source* and the selected event is a *message*.
- ▶ A message carries the information that **it** happened, and not the other possible message events that could have been selected.
- ▶ The quantity of information is dependent in its probability.

# Discrete memoryless source

- ▶ A discrete memoryless source (DMS) is an information source where the messages are **independent** , i.e. the choice of a message at one time does not depend on what were the previous messages
- ▶ Each message has a fixed probability. The set of probabilities is the *distribution* of the source.

$$S : \begin{pmatrix} s_1 & s_2 & s_3 \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}$$

- ▶ Properties:
  - ▶ Discrete: it can take a value from a discrete set (alphabet)
  - ▶ Complete:  $\sum p(s_i) = 1$
  - ▶ Memoryless: successive values are independent of previous values (e.g. successive throws of a coin)
- ▶ A message from a DMS is also called a *random variable* in probabilistics.



# Examples

- ▶ A coin is a discrete memoryless source (DMS) with two messages:

$$S : \begin{pmatrix} heads & tails \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

- ▶ A dice is a discrete memoryless source (DMS) with six messages:

$$S : \begin{pmatrix} s_1 & s_2 & s_3 & s_4 & s_5 & s_6 \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \end{pmatrix}$$

- ▶ Playing the lottery can be modeled as DMS:

$$S : \begin{pmatrix} s_1 & s_2 \\ 0.9999 & 0.0001 \end{pmatrix}$$

# Examples

- ▶ An extreme type of DMS containing the certain event:

$$S : \begin{pmatrix} s_1 & s_2 \\ 1 & 0 \end{pmatrix}$$

- ▶ Receiving an unknown *bit* (0 or 1) with equal probabilities:

$$S : \begin{pmatrix} 0 & 1 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

- ▶ When a DMS provides a new message, it gives out some new information, i.e. the information that a particular message took place.
- ▶ The information attached to a particular event (message) is rigorously defined as:

$$i(s_i) = -\log_2(p(s_i))$$

- ▶ Properties:
  - ▶  $i(s_i) \geq 0$
  - ▶ lower probability (rare events) means higher information
  - ▶ higher probability (frequent events) means lower information
  - ▶ a certain event brings no information:  $-\log(1) = 0$
  - ▶ an event with probability 0 brings infinite information (but it never happens..)

# Entropy of a DMS

- ▶ We usually don't care about a single message. We are interested in a large number of them (think millions of bits of data).
- ▶ We are interested in the *average* information of a message from a DMS.
- ▶ Definition: the entropy of a DMS source  $S$  is **the average information of a message**:

$$H(S) = \sum_k p_k i(s_k) = - \sum_k p_k \log_2(p_k)$$

where  $p_k = p(s_k)$  is the probability of message  $k$ .

# The choice of logarithm

- ▶ Any base of logarithm can be used in the definition.
- ▶ Usual convention: use binary logarithm  $\log_2()$ .  $H(S)$  measured in *bits* (*bits / message*)
- ▶ If using natural logarithm  $\ln()$ ,  $H(S)$  is measured in *nats*.
- ▶ Logarithm bases can be converted to/from one another:

$$\log_b(x) = \frac{\log_a(x)}{\log_a(b)}$$

- ▶ Entropies using different logarithms differ only in scaling:

$$H_b(S) = \frac{H_a(S)}{\log_a(b)}$$

# Examples

- ▶ Coin:  $H(S) = 1 \text{ bit/message}$
- ▶ Dice:  $H(S) = \log(6) \text{ bits/message}$
- ▶ Lottery:  $H(S) = -0.9999 \log(0.9999) - 0.0001 \log(0.0001)$
- ▶ Receiving 1 bit:  $H(S) = 1 \text{ bit/message}$  (hence the name!)

# Interpretation of the entropy

All the following interpretations of entropy are true:

- ▶  $H(S)$  is the *average uncertainty* of the source  $S$
- ▶  $H(S)$  is the *average information* of messages from source  $S$
- ▶ A long sequence of  $N$  messages from  $S$  has total information  $\approx N \cdot H(S)$
- ▶  $H(S)$  is the minimum number of bits (0,1) required to uniquely represent an average message from source  $S$

# Properties of entropy

We prove the following **properties of entropy**:

1.  $H(S) \geq 0$  (non-negative)
2.  $H(S)$  is maximum when all  $n$  messages have equal probability  $\frac{1}{n}$ . The maximum value is  $\max H(S) = \log(n)$ .
3. *Diversification* of the source always increases the entropy



# The entropy of a binary source

- ▶ Consider a general DMS with two messages:

$$S : \begin{pmatrix} s_1 & s_2 \\ p & 1 - p \end{pmatrix}$$

- ▶ It's entropy is:

$$H(S) = -p \cdot \log(p) - (1 - p) \cdot \log(1 - p)$$

- ▶ Graphical plot. . .

## Example - Game

Game: I think of a number between 1 and 8. You have to guess it by asking yes/no questions.

- ▶ How much uncertainty does the problem have?
- ▶ How is the best way to ask questions? Why?
- ▶ What if the questions are not asked in the best way?
- ▶ On average, what is the number of questions required to find the number?

## Example - Game v2

- ▶ Suppose I choose a number according to the following distribution:

$$S : \begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{8} \end{pmatrix}$$

- ▶ On average, what is the number of questions required to find the number?
  - ▶ What questions would you ask?
- ▶ What if the distribution is:

$$S : \begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ 0.14 & 0.29 & 0.4 & 0.17 \end{pmatrix}$$

- ▶ In general:
  - ▶ What distribution makes guessing the number the most difficult?
  - ▶ What distribution makes guessing the number the easiest?

# Information flow of a DMS

- ▶ Suppose that message  $s_i$  takes time  $t_i$  to be transmitted via some channel.
- ▶ Definition: the information flow of a DMS  $S$  is **the average information transmitted per unit of time**:

$$H_\tau(S) = \frac{H(S)}{\bar{t}}$$

where  $\bar{t}$  is the average duration of transmitting a message:

$$\bar{t} = \sum_i p_i t_i$$

- ▶ Definition: the  $n$ -th order extension of a DMS  $S$ ,  $S^n$  is the source with messages has as messages all the combinations of  $n$  messages of  $S$ :

$$\sigma_i = \underbrace{s_j s_k \dots s_l}_n$$

- ▶ If  $S$  has  $k$  messages,  $S^n$  has  $k^n$  messages
- ▶ Since  $S$  is DMS

$$p(\sigma_i) = p(s_j) \cdot p(s_k) \cdot \dots \cdot p(s_l)$$

# Extended DMS - Example

► Examples:

$$S : \begin{pmatrix} s_1 & s_2 \\ \frac{1}{4} & \frac{3}{4} \end{pmatrix}$$

$$S^2 : \begin{pmatrix} \sigma_1 = s_1 s_1 & \sigma_2 = s_1 s_2 & \sigma_3 = s_2 s_1 & \sigma_4 = s_2 s_2 \\ \frac{1}{16} & \frac{3}{16} & \frac{3}{16} & \frac{9}{16} \end{pmatrix}$$

$$S^3 : \begin{pmatrix} s_1 s_1 s_1 & s_1 s_1 s_2 & s_1 s_2 s_1 & s_1 s_2 s_2 & s_2 s_1 s_1 & s_2 s_1 s_2 & s_2 s_2 s_1 & s_2 s_2 s_2 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

## Extended DMS - Another example

- ▶ Long sequence of binary messages:

010011001110010100...

- ▶ Can be grouped in bits, half-bytes, bytes, 16-bit words, 32-bit long words, and so on.

# Property of DMS

- ▶ Theorem: The entropy of a  $n$ -th order extension is  $n$  times larger than the entropy of the original DMS

$$H(S^n) = nH(S)$$

- ▶ Interpretation: grouping messages from a long sequence in blocks of  $n$  does not change total information (e.g. groups of 8 bits = 1 byte)



# An example [memoryless is not enough]

- The distribution (frequencies) of letters in English:

letter	probability	letter	probability
A	.082	N	.067
B	.015	O	.075
C	.028	P	.019
D	.043	Q	.001
E	.127	R	.060
F	.022	S	.063
G	.020	T	.091
H	.061	U	.028
I	.070	V	.010
J	.002	W	.023
K	.008	X	.001
L	.040	Y	.020
M	.024	Z	.001

- Text from a memoryless source with these probabilities:

OCRO HLI RGWR NMIELWIS EU LL NBNESEBYA TH EEI  
ALHENHTTPA OOBTTVA NAH BRL

*(taken from Elements of Information Theory, Cover, Thomas)*

- What's wrong? **Memoryless**

# Sources with memory

- ▶ **Definition:** A source has memory of order  $m$  if the probability of a message depends on the last  $m$  messages.
- ▶ The last  $m$  messages = the **state** of the source ( $S_i$ ).
- ▶ A source with  $n$  messages and memory  $m \Rightarrow n^m$  states in all.
- ▶ For every state, messages can have a different set of probabilities.  
Notation:  $p(s_i|S_k) = \text{"probability of } s_i \text{ in state } S_k \text{"}$ .
- ▶ Also known as *Markov sources*.

# Example

- ▶ A source with  $n = 4$  messages and memory  $m = 1$ 
  - ▶ if last message was  $s_1$ , choose next message with distribution

$$S_1 : \begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ 0.4 & 0.3 & 0.2 & 0.1 \end{pmatrix}$$

- ▶ if last message was  $s_2$ , choose next message with distribution

$$S_2 : \begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ 0.33 & 0.37 & 0.15 & 0.15 \end{pmatrix}$$

- ▶ if last message was  $s_3$ , choose next message with distribution

$$S_3 : \begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ 0.2 & 0.35 & 0.41 & 0.04 \end{pmatrix}$$

- ▶ if last message was  $s_4$ , choose next message with distribution

$$S_4 : \begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ 0.1 & 0.2 & 0.3 & 0.4 \end{pmatrix}$$

# Transitions

- ▶ When a new message is provided, the source **transitions** to a new state:

$$S_i S_j S_k S_l$$

old state

$$S_i S_j S_k S_l$$

new state

- ▶ The message probabilities = the probabilities of transitions from some state  $S_u$  to another state  $S_v$

# Transition matrix

- ▶ The transition probabilities are organized in a **transition matrix**  $[T]$

$$[T] = \begin{bmatrix} p_{11} & p_{12} & \dots & p_{1N} \\ p_{21} & p_{22} & \dots & p_{2N} \\ \dots & \dots & \dots & \dots \\ p_{N1} & p_{N2} & \dots & p_{NN} \end{bmatrix}$$

- ▶  $p_{ij}$  is the transition probability from state  $S_i$  to state  $S_j$
- ▶  $N$  is the total number of states

# Graphical representation

Example here

# Entropy of sources with memory

- ▶ Each state  $S_k$  has a different distribution  $\rightarrow$  each state has a different entropy  $H(S_k)$

$$H(S_k) = - \sum_i p(s_i|S_k) \cdot \log(p(s_i|S_k))$$

- ▶ Global entropy = average entropy

$$H(S) = \sum_k p_k H(S_k)$$

where  $p_k$  = probability that the source is in state  $S_i$  (i.e. after a very long sequence of messages, how many times the source was in state  $S_k$ )

# Ergodic sources

- ▶ Let  $p_i^{(t)}$  = the probability that source  $S$  is in state  $S_i$  at time  $t$ .
- ▶ In what state will it be at time  $t + 1$ ? (after one more message) (probabilities)

$$[p_1^{(t)}, p_2^{(t)}, \dots, p_N^{(t)}] \cdot [T] = [p_1^{(t+1)}, p_2^{(t+1)}, \dots, p_N^{(t+1)}]$$

- ▶ After one more message:

$$[p_1^{(t)}, p_2^{(t)}, \dots, p_N^{(t)}] \cdot [T] \cdot [T] = [p_1^{(t+2)}, p_2^{(t+2)}, \dots, p_N^{(t+2)}]$$

- ▶ In general, after  $n$  messages the probabilities that the source is in a certain state are:

$$[p_1^{(0)}, p_2^{(0)}, \dots, p_N^{(0)}] \cdot [T]^n = [p_1^{(n)}, p_2^{(n)}, \dots, p_N^{(n)}]$$



- ▶ A source is called **ergodic** if every state can be reached from every state, in a finite number of steps.

## Property of ergodic sources:

- ▶ After many messages, the probabilities of the states *become stationary* (converge to some fixed values), irrespective of the initial probabilities.

$$\lim_{n \rightarrow \infty} [p_1^{(n)}, p_2^{(n)}, \dots, p_N^{(n)}] = [p_1, p_2, \dots, p_N]$$

# Finding the stationary probabilities

- ▶ After  $n$  messages and after  $n + 1$  messages, the probabilities are the same:

$$[p_1, p_2, \dots, p_N] \cdot [T] = [p_1, p_2, \dots, p_N]$$

- ▶ Also  $p_1 + p_2 + \dots + p_N = 1$ .

$\Rightarrow$  solve system of equations, find values.

# Entropy of ergodic sources with memory

- ▶ The entropy of an ergodic source with memory is

$$H(S) = \sum_k p_k H(S_k) = - \sum_k p_k \sum_i p(s_i | S_k) \cdot \log(p(s_i | S_k))$$

# Example English text as sources with memory

(taken from *Elements of Information Theory*, Cover, Thomas)

- ▶ Memoryless source, equal probabilities:

XFOML RXKHRJFFJUJ ZLPWCFWKCYJ  
FFJEYVKCQSGXYD QPAAMKBZAACIBZLHJQD

- ▶ Memoryless source, probabilities of each letter as in English:

OCRO HLI RGWR NMIELWIS EU LL NBNSEBYA TH EEI  
ALHENHTTPA OOBTTVA NAH BRL

- ▶ Source with memory  $m = 1$ , frequency of pairs as in English:

ON IE ANTSOUTINYS ARE T INCTORE ST BE S DEAMY  
ACHIN D ILONASIVE TUCOOWE AT TEASONARE FUSO  
TIZIN ANDY TOBE SEACE CTISBE

- ▶ Source with memory  $m = 2$ , frequency of triplets as in English:

IN NO IST LAT WHEY CRATICT FROURE BERS GROCID  
PONDENOME OF DEMONSTURES OF THE REPTAGIN IS  
REGOACTIONA OF CRE

- ▶ Source with memory  $m = 3$ , frequency of 4-plets as in English:

THE GENERATED JOB PROVIDUAL BETTER TRAND THE DISPLAYED  
CODE, ABOVEY UPONDULTS WELL THE CODERST IN THESTICAL  
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# Chapter summary

- ▶ Information of a message:  $i(s_k) = -\log_2(p(s_k))$
- ▶ Entropy of a memoryless source:  
 $H(S) = \sum_k p_k i(s_k) = -\sum_k p_k \log_2(p_k)$
- ▶ Properties of entropy:
  1.  $H(S) \geq 0$
  2. Is maximum when all messages have equal probability  
( $H_{\max}(S) = \log(n)$ )
  3. *Diversification* of the source always increases the entropy
- ▶ Sources with memory: definition, transitions
- ▶ Stationary probabilities of ergodic sources with memory:  
 $[p_1, p_2, \dots, p_N] \cdot [T] = [p_1, p_2, \dots, p_N], \sum_i p_i = 1.$
- ▶ Entropy of sources with memory:

$$H(S) = \sum_k p_k H(S_k) = -\sum_k p_k \sum_i p(s_i|S_k) \cdot \log(p(s_i|S_k))$$

## Chapter II: Discrete Transmission Channels

# What are they?

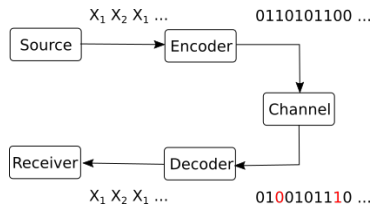


Figure 2: Communication system

- ▶ A system of two related random variables
- ▶ Input random variable  $X = x_1, x_2, \dots$ , output random variable  $Y = y_1, y_2, \dots$
- ▶  $X$  and  $Y$  are *related*, but still *random* (usually because of *noise*)

# What do we want

- ▶ Successful communication: receive  $Y$ , deduce what was sent  $X$
- ▶ We are interested in **deducing  $X$  when knowing  $Y$**
- ▶ How much does knowing  $Y$  tell us about  $X$ ?
  - ▶ Depends on the relation between them
  - ▶ Is the same as how much  $X$  tells us about  $Y$  (symmetrical)



# Nomenclature

- ▶ Discrete: the input alphabet and the output alphabet are finite
- ▶ Memoryless: the output symbol depends only on the current input symbol
- ▶ Stationary: the noise arising on the channel is time invariant (i.e. its statistics do not vary in time)

# Systems of two random variables

- ▶ Two random variables:  $X = x_1, x_2, \dots$ ,  $Y = y_1, y_2, \dots$
- ▶ Example: throw a dice (X) and a coin (Y) simultaneously
- ▶ How to describe this system?

A single joint information source:

$$X \cap Y : \begin{pmatrix} x_1 \cap y_1 & x_1 \cap y_2 & \dots & x_i \cap y_j \\ p(x_1 \cap y_1) & p(x_1 \cap y_2) & \dots & p(x_i \cap y_j) \end{pmatrix}$$

Arrange in a nicer form (table):

	$y_1$	$y_2$	$y_3$
$x_1$	...	...	...
$x_2$	...	...	...
$x_3$	...	...	...

- ▶ Elements of the table:  $p(x_i \cap y_j)$

# Joint probability matrix

The table constitutes the **joint probability matrix**:

$$P(X, Y) = \begin{bmatrix} p(x_1 \cap y_1) & p(x_1 \cap y_2) & \cdots & p(x_1 \cap y_M) \\ p(x_2 \cap y_1) & p(x_2 \cap y_2) & \cdots & p(x_2 \cap y_M) \\ \vdots & \vdots & \cdots & \vdots \\ p(x_N \cap y_1) & p(x_N \cap y_2) & \cdots & p(x_N \cap y_M) \end{bmatrix}$$

$$\sum_i \sum_j p(x_i \cap y_j) = 1$$

- ▶ This matrix completely defines the two-variable system
- ▶ This matrix completely defines the communication process

# Joint entropy

- ▶ The distribution  $X \cap Y$  determines the **joint entropy**:

$$H(X, Y) = - \sum_i \sum_j p(x_i \cap y_j) \cdot \log(p(x_i \cap y_j))$$

- ▶ This is the global entropy of the system (knowing the input and the output)

# Marginal distributions

- ▶  $p(x_i) = \sum_j p(x_i \cap y_j)$  = sum of row  $i$  from  $P(X,Y)$
- ▶  $p(y_j) = \sum_i p(x_i \cap y_j)$  = sum of column  $j$  from  $P(X,Y)$
- ▶ The distributions  $p(x)$  and  $p(y)$  are called **marginal distributions** (“summed along the margins”)

## Examples [marginal distributions not enough]

- ▶ Example 1:

$$P(X, Y) = \begin{bmatrix} 0.3 & 0 \\ 0 & 0.7 \end{bmatrix}$$

- ▶ Example 2:

$$P(X, Y) = \begin{bmatrix} 0.15 & 15 \\ 0.15 & 0.55 \end{bmatrix}$$

- ▶ Both have identical  $p(x)$  and  $p(y)$ , but are completely different
- ▶ Which one is better for a transmission?
- ▶ Marginal distribution are useful, but not enough. Essential is the *relation* between  $X$  and  $Y$ .

$$p(A \cap B) = p(A) \cdot p(B|A)$$

$$p(B|A) = \frac{p(A \cap B)}{p(A)}$$

- ▶ “The conditional probability of B **given A**” (i.e. given that event A happened)
- ▶ Examples. . .
- ▶ Independence:

$$p(A \cap B) = p(A)p(B)$$

$$p(B|A) = p(B)$$

# Channel matrix

Noise (or channel) matrix:

$$P(Y|X) = \begin{bmatrix} p(y_1|x_1) & p(y_2|x_1) & \cdots & p(y_M|x_1) \\ p(y_1|x_2) & p(y_2|x_2) & \cdots & p(y_M|x_2) \\ \vdots & \vdots & \cdots & \vdots \\ p(y_1|x_N) & p(y_2|x_N) & \cdots & p(y_M|x_N) \end{bmatrix}$$

- ▶ Defines the probability of an output **given an input**
- ▶ Each row = a separate distribution that indicates the probability of the outputs **if the input is**  $x_i$ )
- ▶ The sum of each row is 1 (there must be some output if the input is  $x_i$ )



# Relation of channel matrix and joint probability matrix

- ▶  $P(Y|X)$  is obtained from  $P(X, Y)$  by dividing every row to its sum ( $p(x_i)$ )
- ▶ This is known as *normalization* of rows
- ▶  $P(X, Y)$  can be obtained back from  $P(Y|X)$  by multiplying each row with  $p(x_i)$
- ▶  $P(Y|X)$  contains less information than  $P(X, Y)$

# Definition of a discrete transmission channel

**Definition:** A discrete transmission channel is defined by three items:

1. The input alphabet  $X = \{x_1, x_2, \dots\}$
2. The output alphabet  $Y = \{y_1, y_2, \dots\}$
3. The noise (channel) matrix  $P(Y|X)$  which defines the conditional probabilities of the outputs  $y_j$  for every possible input  $x_i$

# Graphical representation of a channel

- ▶ Nice picture with arrows :)

# Three examples

Three examples to help you remember conditional probabilities

- ▶ Play and win the lottery
- ▶ Gambler's paradox
- ▶ CNN: Crippled cruise ship returns; passengers happy to be back

## Conditional entropy $H(Y|X)$ (mean error)

- ▶ Since each row is a distribution, each row has an entropy
- ▶ Entropy of row  $x_i$ :

$$H(Y|x_i) = - \sum_j p(y_j|x_i) \log(p(y_j|x_i))$$

- ▶ *“The uncertainty of the output message when the input message is  $x_i$ ”*
- ▶ Example: lottery

## Conditional entropy $H(Y|X)$ (mean error)

- ▶ A different  $H(Y|x_i)$  for every  $x_i$
- ▶ Compute the average over all  $x_i$ :

$$\begin{aligned} H(Y|X) &= \sum_i p(x_i) H(Y|x_i) \\ &= - \sum_i \sum_j p(x_i) p(y_j|x_i) \log(p(y_j|x_i)) \\ &= - \sum_i \sum_j p(x_i \cap y_j) \log(p(y_j|x_i)) \end{aligned}$$

- ▶ **“The uncertainty of the output message when we know the input message”** (any input, in general)

# Equivocation matrix

Equivocation matrix:

$$P(X|Y) = \begin{bmatrix} p(x_1|y_1) & p(x_1|y_2) & \cdots & p(x_1|y_M) \\ p(x_2|y_1) & p(x_2|y_2) & \cdots & p(x_2|y_M) \\ \vdots & \vdots & \cdots & \vdots \\ p(x_N|y_1) & p(x_N|y_2) & \cdots & p(x_N|y_M) \end{bmatrix}$$

- ▶ Defines the probability of an input **given an output**
- ▶ Each column = a separate distribution that indicates the probability of the inputs **if the output is  $y_j$**
- ▶ The sum of each column is 1 (there must be some input if the output is  $y_j$ )

# Relation of equivocation matrix and joint probability matrix

- ▶  $P(X|Y)$  is obtained from  $P(X, Y)$  by dividing every column to its sum ( $p(y_j)$ )
- ▶ This is known as *normalization* of columns
- ▶  $P(X, Y)$  can be obtained back from  $P(X|Y)$  by multiplying each column with  $p(y_j)$
- ▶  $P(X|Y)$  contains less information than  $P(X, Y)$



## Conditional entropy $H(X|Y)$ (equivocation)

- ▶ Since each column is a distribution, each column has an entropy
- ▶ Entropy of column  $y_j$ :

$$H(X|y_j) = - \sum_i p(x_i|y_j) \log(p(x_i|y_j))$$

- ▶ *“The uncertainty of the input message when the output message is  $y_j$ ”*
- ▶ Example: ...

## Conditional entropy $H(X|Y)$ (equivocation)

- ▶ A different  $H(X|y_j)$  for every  $y_j$
- ▶ Compute the average over all  $y_j$ :

$$\begin{aligned}H(X|Y) &= \sum_j p(y_j) H(X|y_j) \\&= - \sum_i \sum_j p(y_j) p(x_i|y_j) \log(p(x_i|y_j)) \\&= - \sum_i \sum_j p(x_i \cap y_j) \log(p(x_i|y_j))\end{aligned}$$

- ▶ **“The uncertainty of the input message when we know the output message”** (any output, in general)
- ▶ Should be small for a good communication

# Properties of conditional entropies

For a general system with two random variables  $X$  and  $Y$ :

- ▶ Conditioning always reduces entropy:

$$H(X|Y) \leq H(X)$$

$$H(Y|X) \leq H(Y)$$

(knowing something cannot harm)

- ▶ If the variables are independent:

$$H(X|Y) = H(X)$$

$$H(Y|X) = H(Y)$$

(knowing the second variable does not help at all)

# Mutual information $I(X,Y)$

- ▶ Mutual information  $I(X,Y)$  = the average information that one variable has about the other
- ▶ Mutual information  $I(X,Y)$  = the average information that is transmitted on the channel
- ▶ Consider a communication channel with  $X$  as input and  $Y$  as output:
  - ▶ We are the receiver and we want to find out the  $X$
  - ▶ When we don't know the output:  $H(X)$
  - ▶ When we know the output:  $H(X|Y)$
- ▶ How much information was transmitted?
  - ▶ Reduction of uncertainty:

$$I(X, Y) = H(X) - H(X|Y)$$

## Mutual information $I(X,Y)$

$$\begin{aligned}I(X, Y) &= H(X) - H(X|Y) \\&= -\sum_i p(x_i) \log(p(x_i)) + \sum_i \sum_j p(x_i \cap y_j) \log(p(x_i|y_j)) \\&= -\sum_i \sum_j p(x_i \cap y_j) \log(p(x_i)) + \sum_i \sum_j p(x_i \cap y_j) \log(p(x_i|y_j)) \\&= \sum_i \sum_j p(x_i \cap y_j) \log\left(\frac{p(x_i|y_j)}{p(x_i)}\right) \\&= \sum_i \sum_j p(x_i \cap y_j) \log\left(\frac{p(x_i \cap y_j)}{p(x_i)p(y_j)}\right)\end{aligned}$$

# Properties of mutual information

Mutual information  $I(X, Y)$  is:

- ▶ commutative:  $I(X, Y) = I(Y, X)$
- ▶ non-negative:  $I(X, Y) \geq 0$
- ▶ a special case of the Kullback–Leibler distance (relative entropy distance)

**Definition:** the Kullback–Leibler distance of two distributions is

$$D_{KL}(P, Q) = \sum_i P(i) \log\left(\frac{P(i)}{Q(i)}\right)$$

- ▶ In our case, the distributions are:
  - ▶  $P = p(x_i \cap y_j)$  (distribution of our system)
  - ▶  $Q = p(x_i) \cdot p(y_j)$  (distribution of two independent variables)

$$I(X, Y) = D(p(x_i \cap y_j), p(x_i) \cdot p(y_j))$$

# Relations between the informational measures

- ▶ Nice picture with two circles :)
- ▶ All six:  $H(X)$ ,  $H(Y)$ ,  $H(X, Y)$ ,  $H(X|Y)$ ,  $H(Y|X)$ ,  $I(X, Y)$
- ▶ All relations on the picture are valid relations:

$$H(X, Y) = H(X) + H(Y) - I(X, Y)$$

$$H(X, Y) = H(X) + H(Y|X) = H(Y) + H(X|Y)$$

$$I(X, Y) = H(X) - H(X|Y) = H(Y) - H(Y|X)$$

...

- ▶ If know three, can find the other three
- ▶ Simplest to find first  $H(X)$ ,  $H(Y)$ ,  $H(X, Y)$   $\longrightarrow$  then find others

# Types of communication channels

## 1. Channels with zero equivocation

$$H(X|Y) = 0$$

- ▶ Each column of the noise (channel) matrix contains only one non-zero value
- ▶ No doubts on the input messages when the output messages are known
- ▶ All input information is transmitted

$$I(X, Y) = H(X)$$

- ▶ Example: codewords. . .



# Types of communication channels

## 2. Channels with zero mean error

$$H(Y|X) = 0$$

- ▶ Each row of the noise (channel) matrix contains only one non-zero value
  - ▶ No doubts on the output messages when the input messages are known
  - ▶ *The converse is not necessary true!*
  - ▶ All input information is transmitted
- 
- ▶ Example: AND gate

# Types of communication channels

## 3. Channels uniform with respect to the input

$$H(Y|x_i) = \textit{same}$$

- ▶ Each row of noise matrix contains the same values, possibly in different order
- ▶  $H(Y|x_i) = \textit{same} = H(Y|X)$
- ▶  $H(Y|X)$  does not depend on the actual probabilities  $p(x_i)$

# Types of communication channels

## 4. Channels uniform with respect to the output

$$H(X|y_j) = \textit{same}$$

- ▶ Each column of noise matrix contains the same values, possibly in different order
- ▶ If the input messages are equiprobable, the output messages are also equiprobable

# Types of communication channels

## 5. Symmetric channels

- ▶ Uniform with respect to the input and to the output
- ▶ Example: binary symmetric channel

# Channel capacity

- ▶ What is the maximum information we can transmit on a certain channel?
- ▶ **Definition:** the information capacity of a channel is the maximum value of the mutual information, where the maximization is done over the input probabilities  $p(x_i)$

$$C = \max_{p(x_i)} I(X, Y)$$

- ▶ i.e. the maximum mutual information we can obtain if we are allowed to choose  $p(x_i)$  as we want
- ▶ Useful alternative expression:

$$C = \max_{p(x_i)} (H(X) - H(X|Y))$$

# What channel capacity means

- ▶ Channel capacity is the maximum information we can transmit on a channel, on average, with one message
- ▶ One of the most important notions in information theory
- ▶ Its importance comes from Shannon's second theorem (noisy channel theorem)

# Preview of the channel coding theorem

- ▶ Even though some information  $I(X, Y)$  is transmitted on the channel, there still is the  $H(X|Y)$  uncertainty on the input
- ▶ We want error-free transmission, with no uncertainty
- ▶ Solution: use error coding (see chapter IV)
- ▶ How coding works:
  - ▶ coder receives  $k$  symbols (bits, usually) that we want to transmit
  - ▶ coder appends additional  $m$  symbols computed via some coding algorithm
  - ▶ the total  $k + m$  bits are transmitted over a noisy channel
  - ▶ the decoding algorithm tries to detect and correct errors, based on the additional  $m$  bits that were appended
- ▶ Coding rate:

$$R = \frac{k}{k + m}$$

- ▶ stronger protection = bigger  $m$  = less efficient
- ▶ weaker protection = smaller  $m$  = more efficient

# Preview of the channel coding theorem

- ▶ A rate is called **achievable** for a channel if, for that rate, there exists a coding and decoding algorithm guaranteed to correct all possible errors on the channel

## Shannon's noisy channel coding theorem (second theorem)

For a given channel, all rates below capacity  $R < C$  are achievable. All rates above capacity,  $R > C$ , are not achievable.



# Channel coding theorem explained

In layman terms:

- ▶ For all coding rates  $R < C$ , there is a way to recover the transmitted data perfectly (de/coding algorithm will detect and correct all errors)
- ▶ For all coding rates  $R > C$ , there is no way to recover the transmitted data perfectly

Example:

- ▶ Send binary digits (0,1) on a channel with capacity 0.7 bits/message
- ▶ There exists coding schemes with  $R < 0.7$  that allow perfect recovery
  - ▶ i.e. for every 7 bits of data coding adds 3 or more bits, on average  $\Rightarrow$   
$$R = \frac{7}{7+3}$$
- ▶ With less than 3 bits for every 7 bits of data  $\Rightarrow$  impossible to recover all the data

# Efficiency and redundancy

- ▶ Efficiency of a channel:

$$\eta_C = \frac{I(X, Y)}{C}$$

- ▶ Absolute redundancy of a channel:

$$R_C = C - I(X, Y)$$

- ▶ Relative redundancy of a channel:

$$\rho_C = \frac{R_C}{C} = 1 - \frac{I(X, Y)}{C} = 1 - \eta_C$$

# Examples of channels and their capacity

0  $\longrightarrow$  0

1  $\longrightarrow$  1

Figure 3: Noiseless binary channel

- Capacity = 1 bit/message, when  $p(x_1) = p(x_2) = \frac{1}{2}$

# Noisy binary non-overlapping channel

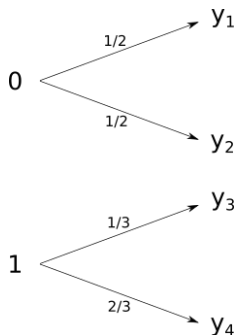


Figure 4: Noisy binary non-overlapping

- ▶ There is noise ( $H(Y|X) > 0$ ), but can deduce the input ( $H(X|Y) = 0$ )
- ▶ Capacity = 1 bit/message, when  $p(x_1) = p(x_2) = \frac{1}{2}$

# Noisy typewriter

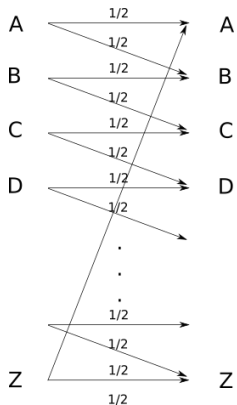


Figure 5: Noisy typewriter

$$\begin{aligned}\max I(X, Y) &= \max (H(Y) - H(Y|X)) = \max H(Y) - 1 \\ &= \log(26) - 1 = \log(13)\end{aligned}$$

# Noisy typewriter

- ▶ Capacity =  $\log(13)$  bit/message, when input probabilities are uniform
- ▶ Can transmit 13 letters with no errors (A, C, E, G, ...)

# Binary symmetric channel

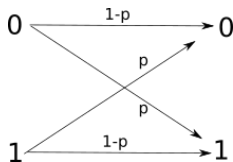


Figure 6: Binary symmetric channel (BSC)

- ▶ Capacity  $= 1 - H_p = 1 + p \log(p) + (1 - p) \log(1 - p)$
- ▶ Capacity is reached when input distribution is uniform

# Binary erasure channel

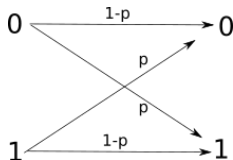


Figure 7: Binary erasure channel

- ▶ Different from BSC: here we know when errors happened
- ▶ Capacity =  $1 - p$
- ▶ Intuitive meaning: lose  $p$  bits, remaining bits = capacity =  $1 - p$