

Assignment SCOA-1

Title: Implementation of Fuzzy Operations, Fuzzy Relations.

Problem Statement:

Implement Union, Intersection, Complement and Difference operations on fuzzy sets. Also create fuzzy relation by Cartesian product of any two fuzzy sets and perform max-min composition on any two fuzzy relations.

Objectives

- Provide an understanding of the basic mathematical elements of fuzzy sets.
- Understand and analyse concepts of fuzzy set.
- To use fuzzy set operations to implement current computing techniques used in fuzzy computing.

Outcomes

The students will be able to

- Learn mathematical basis as well as the general principles of various soft computing techniques.
- To analyze the applications using fuzzy set which uses current techniques, skills, and tools necessary for computing.
- To identify and solves the engineering problems using the fuzzy set theory and identify the differences and similarities between fuzzy sets and classical sets theories.

Software and Hardware Requirements:

4 GB RAM, Processor i3 and above 500 GB HDD. OS Fedora / Ubuntu Text Editor, Jupyter Notebook, Python 2.7 or 3.6

Theory:

Fuzzy Logic:

Fuzzy logic is an organized method for dealing with imprecise data. It is a multivalued logic that allows intermediate values to be defined between conventional solutions. In classical set theory, the membership of elements in a set is assessed in binary terms according to a bivalent condition — an element either belongs or does not belong to the set. By contrast, fuzzy set theory permits the gradual assessment of the membership of elements in a set; this is described with the aid of a membership function valued in the real unit interval $[0, 1]$.

Bivalent Set Theory can be somewhat limiting if we wish to describe a 'humanistic' problem mathematically. For example, Fig 1 below illustrates bivalent sets to characterize the temperature of a room. The most obvious limiting feature of bivalent sets that can be seen clearly from the diagram is that they are mutually exclusive – it is not possible to have membership of more than one set. Clearly, it is not accurate to define a transition from a quantity such as 'warm' to 'hot' by the application of one degree Fahrenheit of heat. In the real world a smooth (unnoticeable) drift from warm to hot would occur.

Fuzzy Sets:

If “X is collection of objects” (named universe of discourse) “denoted generically by x, then a fuzzy set

$$\tilde{A} = \{(\mu_{\tilde{A}}(x)) \mid x \in X\}$$

Where: $\mu_{\tilde{A}}(x) = X \rightarrow [0, 1]$ is called membership function or degree of membership (also, degree of compatibility or degree of truth) of x in A”

If the interval of real numbers $[0, 1]$ is replaced by discrete set $\{0, 1\}$, then fuzzy set A becomes classic or crisp set.

Example:

A set X in which each element y has a grade of membership $\mu_X(y)$ in the range 0 to 1, i.e. set membership may be partial.

e.g. if cold is a fuzzy set, exact temperature values might be mapped to the fuzzy set as follows:

15 degrees \rightarrow 0.2 (slightly cold)

10 degrees \rightarrow 0.5 (quite cold)

0 degrees \rightarrow 1 (totally cold)

$$\tilde{A} = \{(x_1, 0.1), (x_2, 0.7), (x_3, 1), (x_4, 0)\}$$

Fuzzy Set Operations

Having two fuzzy sets A and B, the universe of information U and an element of the universe, the following relations express the union, intersection and complement operation on fuzzy sets.

Union/Fuzzy ‘OR’

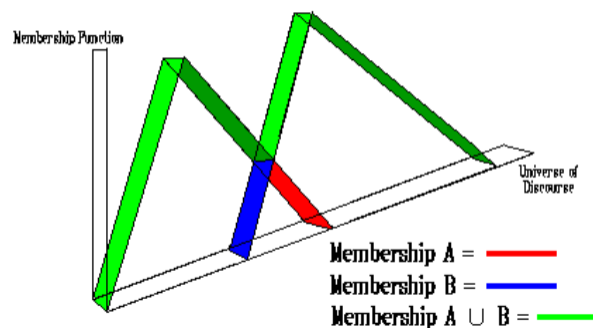
Let us consider the following representation to understand how the Union/Fuzzy ‘OR’ relation works –

$$\mu_{\tilde{A} \cup \tilde{B}}(y) = \mu_{\tilde{A}} \vee \mu_{\tilde{B}} \quad \forall y \in U$$

Here \vee represents the ‘max’ operation

Example

$$\tilde{A} \cup \tilde{B} = \{(x_1, 0.4), (x_2, 0.7), (x_3, 1), (x_4, 0.2)\}$$



Intersection/Fuzzy ‘AND’

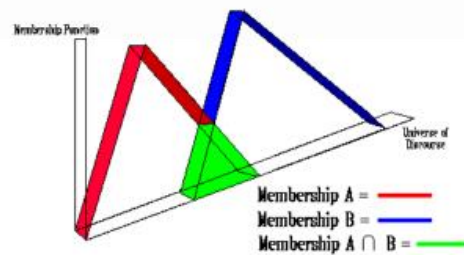
Let us consider the following representation to understand how the Intersection/Fuzzy ‘AND’ relation works –

$$\mu_{\tilde{A} \cap \tilde{B}}(y) = \mu_{\tilde{A}} \wedge \mu_{\tilde{B}} \quad \forall y \in U$$

Here \wedge represents the ‘min’ operation.

Example

$$\tilde{A} \cap \tilde{B} = \{(x_1, 0.1), (x_2, 0.3), (x_3, 1), (x_4, 0)\}$$



Complement/Fuzzy 'NOT'

Let us consider the following representation to understand how the Complement/Fuzzy 'NOT' relation works. example

$$\mu_{\tilde{A}} = 1 - \mu_A(y) \quad y \in U$$

$$\tilde{A}^c = \{(x_1, 0.9), (x_2, 0.3), (x_3, 0), (x_4, 1)\}$$



Difference/ Relative Complement

The set difference of sets A and B (denoted by $A - B$) is the set of elements which are only in A but not in

Example - If $A = \{10, 11, 12, 13\}$ and $B = \{13, 14, 15\}$, then $(A - B) = \{10, 11, 12\}$ and $(B - A) = \{14, 15\}$. Here, we can see $(A - B) \neq (B - A)$



Fuzzy Cartesian product

The totally ordered set $I = [0, 1]$ is a distributive but not complemented lattice under the operations of infimum \wedge and supremum \vee . On $L = I \times I$ we define a partial order, in terms of the partial order on I , as follows:

- (i) $(r_1, r_2) \leq (s_1, s_2)$ iff $r_1 \leq s_1, r_2 \leq s_2$, whenever $s_1 \neq 0$ and $s_2 \neq 0$,
- (ii) $(0, 0) \leq (s_1, s_2)$ whenever $s_1 = 0$ or $s_2 = 0$,

For every $(r_1, r_2), (s_1, s_2) \in L$. The Cartesian product $L = I \times I$ is then a distributive but not complemented vector lattice. The operations of minimum and maximum in L are given respectively by

$$(r_1, r_2) \wedge (s_1, s_2) = (r_1 \wedge s_1, r_2 \wedge s_2)$$

$$\text{And } (r_1, r_2) \vee (s_1, s_2) = (r_1 \vee s_1, r_2 \vee s_2)$$

For every $(r_1, r_2), (s_1, s_2) \in L$ where the equality holds in the last relation when $r_i \leq s_i$. When we speak of an L -fuzzy subset we mean that the associated membership function takes its values from the lattice $L = I \times I$. An L -fuzzy subset of X is thus a function from X to L [4]. The notation $\{(x, A(x)) : x \in X\}$ or, simply, $\{(x, r)\}$, where $r = A(x)$, will be used to denote a fuzzy subset A of X . Similarly, an L -fuzzy subset of X , a fuzzy subset of $X \times Y$ and an L -fuzzy subset of $X \times Y$ will be denoted respectively by $\{(x, (r_1, r_2))\}$, $\{((x, y), r)\}$ and $\{((x, y), (r_1, r_2))\}$. To each fuzzy subset $\{(x, r_1)\}$ of X and fuzzy subset $\{(y, r_2)\}$ of Y there corresponds an L -fuzzy subset $\{((x, y), (r_1, r_2))\}$ of $X \times Y$. Throughout this paper the notation $(x, r) \in A$; where $A \in I^x$, will mean that $A(x) = r$.

Definition.

The fuzzy Cartesian product of two ordinary sets X and Y , symbolically $X \boxtimes Y$, is the collection of all L -fuzzy subsets of $X \times Y$.

$$X \boxtimes Y = L^{X \times Y}$$

An element of $X \boxtimes Y$ is then a function

$$C : X \times Y \rightarrow L,$$

or

$$C = \{((x, y), (r_1, r_2)) : (x, y) \in X \times Y, (r_1, r_2) = C(x, y) \in L\}.$$

The fuzzy Cartesian \boxtimes product of a fuzzy subset $A = \{(x, r)\}$ of X and a fuzzy subset $B = \{(y, s)\}$ of Y is the

L^{\boxtimes} -fuzzy subset $A \boxtimes B$ of XY defined by:

$$A \boxtimes B = \{((x, y), \boxtimes(A(x), B(y))) : x \in X, y \in Y\} = \{((x, y), (r, s))\}.$$

It is clear that $A \boxtimes B$ is an element of $X \boxtimes Y$, for every $A \in I^X$ and $B \in I^Y$. The above definitions can be generalized for any finite number of sets.

fuzzy relations (Max-Min Composition)

Max -Min Composition of fuzzy Relations Fuzzy relation in different product space can be combined with each other by the operation called \boxtimes -Composition \parallel . There are many composition methods in use, e.g. max-product method, max-average method and max-min method. But max-min composition method is best known in fuzzy logic applications.

Definition

Composition of Fuzzy Relations

Consider two fuzzy relation; $R (X \times Y)$ and $S (Y \times Z)$, then a relation $T (X \times Z)$, can be expressed as max-min composition

$$T = R \circ S$$

$$\mu T (x, z) = \max\text{-min} [\mu R (x, y), \mu S (y, z)] = V [\mu R (x, y) \wedge \mu S (y, z)]$$

If algebraic product is adopted, then max-product composition is adopted:

$$T = R \circ S$$

$$\mu_T(x, z) = \max [\mu_R(x, y) \cdot \mu_S(y, z)] = V [\mu_R(x, y) \cdot \mu_S(y, z)]$$

The max-min composition can be interpreted as indicating the strength of the existence of relation between the elements of X and Z.

Calculations of $(R \circ S)$ are almost similar to matrix multiplication.

Crisp relation:

Crisp relation is defined on the Cartesian product of two sets.

Consider

$$X \times Y = \{(x, y) | x \in X, y \in Y\}$$

The relation on this Cartesian product will be,

$$\mu_R = \begin{cases} 1, & (x, y) \in R \\ 0, & (x, y) \notin R \end{cases}$$

Example: Let $X = \{1, 4, 5\}$ and $Y = \{3, 6, 7\}$ then for relation $R = x < y$

$$R = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Fuzzy relation:

Let X, Y belongs to R
be universal sets then,

$$R = \{((x, y), \mu_R(x, y)) \mid (x, y) \in X \times Y\}$$

Is called a fuzzy relation in
 $X \times Y \subseteq R$

Example: Let $X = \{1, 2, 3\}$ and $Y = \{1, 2\}$

If, $\mu_R(x, y) = e^{-(x-y)^2}$ then

$$R = \left\{ \frac{e^{-(1-1)^2}}{(1,1)}, \frac{e^{-(1-2)^2}}{(1,2)}, \frac{e^{-(2-1)^2}}{(2,1)}, \frac{e^{-(2-2)^2}}{(2,2)}, \frac{e^{-(3-1)^2}}{(3,1)}, \frac{e^{-(3-2)^2}}{(3,2)} \right\} \quad R = \begin{bmatrix} 1 & 0.37 \\ 0.37 & 1 \\ 0.02 & 0.37 \end{bmatrix}$$

Max-Min Composition:

Let X, Y and Z be universal sets and let R and Q be relations that relate them as,

$$R = \{ (x, y) | x \in X, y \in Y, R \subset X \times Y \}$$

$$Q = \{ (y, z) | y \in Y, z \in Z, Q \subset Y \times Z \}$$

Then S will be a relation that relates elements of X with elements of Z as,

$$S = R \circ Q$$

$$S = \{ (x, z) | x \in X, z \in Z, S \subset X \times Z \}$$

Max min composition is then defined as,

$$\mu_S(x, z) = \max \left(\min \left(\mu_R(x, y), \mu_Q(y, z) \right) \right)$$

Example:

$$R = \begin{bmatrix} 0.6 & 0.5 & 0.4 \\ 0.2 & 0.1 & 0.2 \end{bmatrix} \quad Q = \begin{bmatrix} 0.2 & 0.6 \\ 0.1 & 0.3 \\ 0.7 & 0.5 \end{bmatrix} \quad S = \begin{bmatrix} 0.4 & 0.6 \\ 0.2 & 0.2 \end{bmatrix}$$

Algorithms / Pseudocodes:

Union

```
for obj1, obj 2 in zip (set1, set 2)
    new obj = Fuzzy (obj1.val, max (obj1.mem, obj2.mem))
    Union.append (new obj)
```

Intersection

```
for obj1,obj 2 in zip (set1,set 2)
    new obj = Fuzzy (obj1.val, min(obj1.mem, obj2.mem))
    intersection.append (new obj)
```

Complement

```
for obj1,obj 2 in zip (set1,set 2)
    new obj = Fuzzy (obj.val, 1 - obj. mem)
    NOT set1.append (new abj)
```

Cartesian Product

```
for ka in A
    for kb in B:
        Y. append ([ka kb, min (A[ka] B[kb])])
```

Max - Min Composition

N is the dimension of both Rand S

```
for i in range (N):
    List=[]
    for j in range(N):
        T= R[i]
        new =[]
        for k in range(N):
            new append (min (I[k], S[K][j]))
        List, append (max (new))
    min max.append (list)
```

Conclusion:

The concepts of union, intersection and complement are implemented using fuzzy sets which helped to understand the differences and similarities between fuzzy set and classical set theories. It provides the basic mathematical foundations to use the fuzzy set operations. With the use of fuzzy logic principles max min composition of fuzzy set is calculated which describes the relationship between two or more fuzzy sets.

Test Cases

Function	Input	Output	Expected Output	Result
1) Union	$A = \{0.2, 0.3, 0.6, 0.6\}$ $B = \{0.9, 0.9, 0.4, 0.5\}$	$Y = \{0.9, 0.9, 0.6, 0.6\}$	$Y = \{0.9, 0.9, 0.6, 0.6\}$	Success
2) Intersection	$A = \{''\}$ $B = \{''\}$	$Y = \{0.2, 0.3, 0.4, 0.5\}$	$Y = \{0.2, 0.3, 0.4, 0.5\}$	Success
3) Complement	$A = \{''\}$ $B = \{''\}$	$Y = \{0.8, 0.7, 0.4, 0.4\}$	$Y = \{0.8, 0.7, 0.4, 0.4\}$	Success
4) Difference	$A = \{''\}$ $B = \{''\}$	$Y = \{0.1, 0.1, 0.6, 0.5\}$	$Y = \{0.1, 0.1, 0.6, 0.5\}$	Success
5) Cartesian product	$A = \{''\}$ $B = \{''\}$	$Y = \{0.2, 0.2, 0.2, 0.2, 0.3, 0.3, 0.3, 0.6, 0.6, 0.4, 0.5, 0.6, 0.6, 0.4, 0.5\}$	$Y = \{''\}$	Success
6) Max-Min composition	$R = \begin{bmatrix} 0.6 & 0.5 & 0.4 \\ 0.2 & 0.1 & 0.2 \end{bmatrix}$ $Q = \begin{bmatrix} 0.2 & 0.6 \\ 0.1 & 0.3 \\ 0.7 & 0.5 \end{bmatrix}$	$Y = \begin{bmatrix} 0.4 & 0.6 \\ 0.2 & 0.2 \end{bmatrix}$	$Y = \{''\}$	Success