Formula Sheet

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Contents

1 Partial Derivatives

1.1 Euler's Theorem

Assuming z to be homogenous i.e. $\boxed{z = x^n f(y/x)}$,

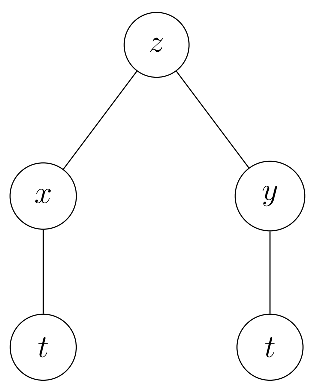
$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nu$$

On partially differentiating[1] above equation wrt x and y we get,

$$x^2\frac{\partial^2 z}{\partial x^2} + y^2\frac{\partial^2 z}{\partial y^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} = n(n-1)z$$

1.2 Total Differentiation

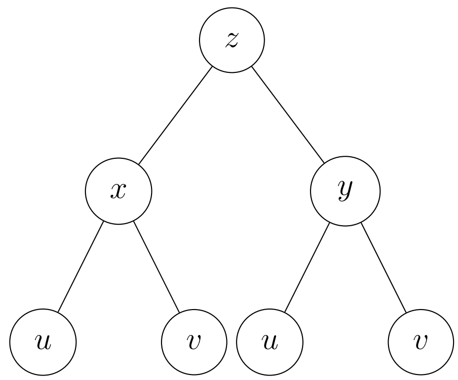
$z = f(x, y)$	$x = \phi(t)$	$y = \phi(t)$



Differentiation: $\dfrac{dz}{dt} = \dfrac{dz}{dx}\dfrac{dx}{dt} + \dfrac{dz}{dy}\dfrac{dy}{dt}$

[1] Important proof

- $\boxed{z = f(x,y)}$ $\boxed{x = \phi(u,v)}$ $\boxed{y = \phi(u,v)}$



$$\textbf{Differentiation:} \quad \begin{aligned} \frac{dz}{dt} &= \frac{dz}{dx}\frac{dx}{du} + \frac{dz}{dy}\frac{dy}{du} \\ \frac{dz}{dt} &= \frac{dz}{dx}\frac{dx}{dv} + \frac{dz}{dy}\frac{dy}{dv} \end{aligned}$$

- Implicit Equation $\boxed{f(x,y) = c}$

1.3 Mean Value Theorems

- Rolle's Mean Value Theorem
  - $f(x)$ is continuous in $[a,b]$
  - $f(x)$ is differentiable in (a,b)
  - $f(a) = f(b)$

  There exists a $c \in (a,b)$ such that $\boxed{f'(c) = 0}$
- Lagrange's Mean Value Theorem
  - $f(x)$ is continuous in $[a,b]$
  - $f(x)$ is differentiable in (a,b)

  There exists a $c \in (a,b)$ such that $\boxed{f'(c) = \frac{f(b)-f(a)}{b-a}}$

- Cauchy's Mean Value Theorem
  - $f(x)$ & $g(x)$ is continuous in $[a, b]$
  - $f(x)$ & $g(x)$ is differentiable in (a, b)
  - $g'(x) \neq 0$

  There exists a $c \in (a, b)$ such that $\boxed{\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}}$

1.4 Taylor's Theorem

Single Variable

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a)\dots$$

♢ Mclaurin's Series

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0)\dots$$

Two Variables

$$f(x,y) = f(a,b) + \big[(x-a)\,f'_x(a,b) + (y-b)\,f'_y(a,b)\big]$$
$$+\frac{1}{2!}\left[(x-a)^2\,f_{xx}(a,b) + 2(x-a)(y-b)\,f_{xy}(a,b) + (y-b)^2\,f_{yy}(a,b)\right]$$
$$+\frac{1}{3!}[(x-a)^3 f_{xxx}\,f(a,b) + 3(x-a)^2(y-b)\,f_{xxy}(a,b)$$
$$+3(x-a)(y-b)^2\,f_{xyy}(a,b) + (y-b)^3\,f_{yyy}(a,b)$$

♢ Mclaurin's Series

$$f(x,y) = f(0,0) + \big[x\,f'_x(0,0) + y\,f'_y(0,0)\big]$$
$$+\frac{1}{2!}\left[x^2\,f_{xx}(0,0) + 2xy\,f_{xy}(0,0) + y^2\,f_{yy}(0,0)\right]\dots$$

NOTE

Try to memorise all the expansions of $sin\,x, cos\,x, log\,x, e^x, a^x$, will be helpful a lot in solving limits and indeterminate forms

NOTE

Leibinitz Rule(for n^{th} differentiation)[a]

$$(UV)_n = \binom{n}{0} UV_n + \binom{n}{1} U_1 V_{n-1} + \binom{n}{2} U_2 V_{n-2} \cdots + \binom{n}{n} U_n V$$

[a] U_n denotes n^{th} derivative of U

1.5 Maxima & Minima

- $f_x = 0$
  $f_y = 0$
- $r = f_{xx}$
  $s = f_{xy}$
  $t = f_{yy}$

$$rs - t^2 = \begin{cases} > 0 & \text{Minima } (r > 0) \\ & \text{Maxima } (r < 0) \\ < 0 & \text{Saddle Point} \\ = 0 & \text{No conclusion} \end{cases}$$

1.6 Lagrange's Method of Undetermined Multipier

Let $f(x, y, z)$[2] be subject to the condition $\phi(x, y, z) = 0$ then using concept of linear combinations,

$$\boxed{F = f(x, y, z) + \lambda\, \phi(x, y, z)}$$

where $\lambda \in \mathbb{R}$. To get the minimum solve the following equations,

$$\frac{\partial F}{\partial x} = 0 \quad \frac{\partial F}{\partial y} = 0 \quad \frac{\partial F}{\partial z} = 0$$

[2] Here f is the function to be minimised

1.7 Errors and Approximations

Here δf denotes the error in f.

$$\delta f = \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial}\delta y$$

NOTE

A better way would be taking log and then differentiating as shown below.

$$V = \frac{4}{3}\pi r^3$$

[Taking log]
$\Rightarrow log\, V = log(\frac{4}{3}) + \log \pi + 3 \log r$

[Taking derivative]
$\Rightarrow \frac{1}{V}\partial V = 3\frac{1}{r}\partial r$
$\Rightarrow \frac{\partial V}{V} = 3\frac{\partial r}{r}$

[Considering $\partial x \sim \delta x$]
$\Rightarrow \frac{\delta V}{V} = 3\frac{\delta r}{r}$

2 3D Geometry

2.1 Sphere

- Basic Equation: $\boxed{x^2 + y^2 + z^2 + 2fx + 2gy + 2hz + d = 0}$
- Center: $\boxed{(-f, -g, -h)}$

  Radius: $\boxed{\sqrt{f^2 + g^2 + h^2 - d}}$
- Other forms of equations:
  - When x_o, y_o, z_o is the centre and r is radius.
    $\boxed{(x - x_o)^2 + (y - y_o)^2 + (z - z_o)^2 = r^2}$
  - When x_1, y_1, z_1 & x_2, y_2, z_2 are diametrically opposite ends.
    $\boxed{(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + (z - z_1)(z - z_2) = 0}$
- Intersection of Plane($T = 0$) with Sphere($C = 0$) the the equation of family of spheres is given by $\boxed{S + \lambda T = 0}$
  - Center: $\boxed{(x, y, z)}$ Radius: $\boxed{r^2 - p^2}$

NOTE

Direct formula for p where x_o, y_o, z_o is the centre of the circle and $ax + by + cz + d = 0$ is equation of the plane:

$$p = \left| \frac{ax_o + by_o + cz_o + d}{\sqrt{a^2 + b^2 + c^2}} \right|$$

Equation of line passing from center normal to the plane

$$\frac{x - x_p}{a} = \frac{y - y_o}{b} = \frac{z - z_o}{c}$$

For $k = 1$, on solving we get point on plane & $k = 2$, we get mirrored point in plane

$$\frac{x - x_p}{a} = \frac{y - y_o}{b} = \frac{z - z_o}{c} = \frac{-(ax_o + by_o + cz_o + d)}{a^2 + b^2 + c^2} \times k$$

- Orthogonal Spheres

  Let d be the distance between the centres and r_1 and r_2 be the radii of the two spheres. Then condition for orthogonal circles would be:

  $$\boxed{d^2 = r_1^2 + r_2^2} \Rightarrow 2f_1f_2 + 2g_1g_2 + 2h_1h_2 = d_1 + d_2$$

- Tangent to Sphere

  If the tangent touches the circle $x^2 + y^2 + z^2 + 2fx + 2gy + 2hz + d = 0$ at point (x_1, y_1, z_1) then the equation of the tangent would be:

  $$\boxed{xx_1 + yy_1 + zz_1 + f(x + x_1) + g(y + y_1) + h(z + z_1) = 0}$$