Engineering Mathematics IV

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Contents

1 Set Theory

A Set is a collection of well defined objects which is denoted by a capital letter and it's elements are described by small letters or numbers.

Types of Sets

- Universal Set (ξ or U)
- Null Set (ϕ)
- Subset ($\subset$)
- Superset ($\supset$)
- Compliment of a set (A^c or $\bar{A}$)
- Equal Sets ($=$)

Operations on Sets

- Union ($\cap$)
- Intersection ($\cup$)
- De Morgans
- Laws - Associative, Distributive

1.1 Random Experiments, Events and more

If the repetition of an experiment under identical condition results in different possible outcomes, then such an experiment is called Randome Experiment or Stochastic Experiment.

Sample Space (S) is a set of all possible outcomes of a random experiment.

Event (E) is a subset of Sample Space **S**

$\boxed{\text{Example}}$ Tossing of coin: **S** = {H,T}

Types of Events

- Mutually Exclusive
- Equally Likely

NOTE

Mutually Exclusive Events: are events that cannot occur at the same time like tossing of 1 coin can never give both heads and tails.

Independent Events: are events are completely independent of one another like outcome of second toss is independent of the first toss.

2 Probability

Let **A** be an event of **S**. If **A** occurs m different ways out of a total of n, then probability of **A** is denoted by

$$P(A) = \frac{\text{Favorable Cases}}{\text{Total Outcomes}} = \frac{m}{n}$$

Similarly we have a thing called odds in favor of A which is defined as the ratio of favorable cases to unfavorable cases

$$\text{Odds in favor of } A = \frac{\text{Favorable Cases}}{\text{Unfavorable Cases}} = \frac{m}{n-m}$$

2.1 Kalmogorov's Axioms

Let **E** be an experiment with sample space **S**. Let **A** be an event of **S**, then:

- $0 \leq P(A) \leq 1$
- $P(S) = 1$
- Given A & B are mutually exclusive then, $P(A \cup B) = P(A) + P(B)$
- If $A_1, A_2, A_3 ... A_n$ are mutually exclusive then, $P(\cup_{i=1}^{n} A_i) = \sum_{i=1}^{n} P(A_i)$

Theorem 2.1. *If A is an event of S then,*

i $P(\phi) = 0$

ii $P(A) + P(\bar{A}) = 1$

Proof. i) Let $A \cup \phi = \phi$

$$A \cap \phi = \phi \tag{1a}$$
$$P(A \cap \phi) = P(\phi)$$

$$A \cup \phi = \phi \tag{1b}$$
$$P(A \cup \phi) = P(\phi)$$

Using axiom from 2.1 & equation.(1b) we get,

$$P(A) + P(\phi) = P(A)$$
$$P(\phi) = 0$$

ii) Let $S = A \cup \bar{A}$

$$P(S) = P(A \cup \bar{A}) \qquad \text{[Mutually Exclusive]}$$

$$1 = P(A) + P(\bar{A})$$

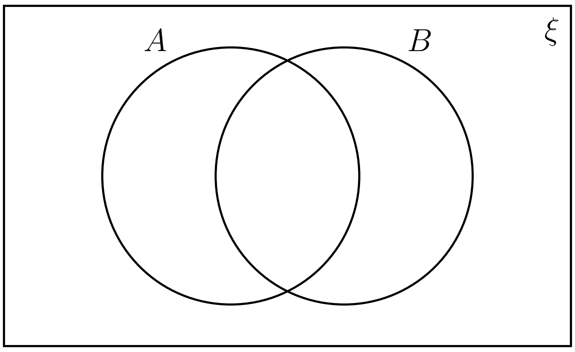
$$P(A) + P(\bar{A}) = 1$$

□

2.2 Addition Rule

If A & B are two events then $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ by addition rule.

Proof. Consider the following venn diagram having sets A and B.



$$A \cup B = (A \cap \bar{B}) \cup (A \cap B) \cup (\bar{A} \cap B)$$

Consider, $\quad B = (A \cap B) \cup (\bar{A} \cap B)$

$$P(B) = P((A \cap B) \cup (\bar{A} \cap B)) \qquad \text{[Mutually Exclusive]}$$

$$P(B) = P(A \cap B) \cup P(\bar{A} \cap B) \tag{3a}$$

Consider, $\quad A = (A \cap B) \cup (A \cap \bar{B})$

$$P(B) = P((A \cap B) \cup (A \cap \bar{B})) \qquad \text{[Mutually Exclusive]}$$

$$P(B) = P(A \cap B) \cup P(A \cap \bar{B}) \tag{3b}$$

Thus from (3a) and (3b) we get,

$$\boxed{P(A \cup B) = P(A) + P(B) - P(A \cap B)}$$

$\square$

Generalised Addition Rule

If $A_1, A_2, A_3 \ldots A_n$ are n events in a given sample space S.

$$\boxed{P(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} P(A_i) \; - \; \sum_{i=1}^{n} P(A_i \bigcap A_j) \; \ldots \; (-1)^n \, P(\bigcap \sum_{i=1}^{n} A_i)}$$

2.3 Condtional Probability

Conditional Probaility defines the probability of an event $\mathbb{A}$ under a given circumstance say $\mathbb{B}$ as follows:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

If the event is independent of the circumstance, then:

$$P(A) \cap B) = P(A) \times P(B)$$

Total Probability Theorem

If $B_1, B_2, B_3 \ldots B_k$ are partitions of S with $P(B_i) \neq 0$ & A is an arbitrary event of S, then

$$\boxed{P(A) = \sum_{i=1}^{k} P(A|B_i) \times P(B_i)}$$

2.4 Bayes' Theorem

Let $B_1, B_2, B_3 \ldots B_k$ be events of S and are said to be partitions of S if:

- $\bigcup_{i=1}^{k} B_i = S$
- $B_i \cup B_j = \phi$

Bayes' Theorem:

If $B_1, B_2, B_3 \ldots B_k$ are partitions of S with $P(B_i) \neq 0$ & A is an arbitrary event of S, then

$$P(B_i|A) = \frac{P(A|B_i) \times P(B_i)}{\sum\limits_{i=1}^{k} P(A|B_i) \times P(B_i)}$$

3 Random Variables

If a real variable X is associated with an outcome of a random experiment, it is called a *random variable* or a *stochastic variable* or simply a *variate*.

Types of Random Variables:

- Discrete Random Variables
- Continuous Random Variables

3.1 Probability Distribution Function (pdf)

This is a function that denotes the probability of a given event as a continuous/discrete function of $f(x)$ where $x \in \mathbb{R}$.

3.2 Cumulative Distribution Function (cdf)

This is a function that denotes the sum of probability of a given event as a continuous/discrete function of $F(X)$ where X will be $\leq x$.

3.3 Statistical Terminologies

- Mean (Expectation of x): Denoted by $\boxed{E(x)}$
- Variance: Denoted by $\boxed{V(x) \text{ or } \sigma^2}$
- Standard Deviation: Denoted by $\boxed{\sigma}$

	Discrete Random Variables	Continuous Random Variables
μ or $E(x)$	$\sum\limits_{i=1}^{n} x_i P(x_i)$	$\int\limits_{-\infty}^{\infty} xP(x)\,dx$
$E(x^2)$	$\sum\limits_{i=1}^{n} x_i^2 P(x_i)$	$\int\limits_{-\infty}^{\infty} x^2 P(x)\,dx$
$E(x-\mu)^2$ or σ^2	$V(x) = E(x^2) - E(x)^2$	

3.4 Chebyshev's Inequality

Let x be a random variable with $E(x) = \mu$ and c be any real number, then if $E(x-c)^2$ is finite and is any positive number,

$$\boxed{P\{|x-c| \geq \varepsilon\} \leq \frac{E(x-c)^2}{\varepsilon^2}}$$

OR

$$\boxed{P\{|x-c| \leq \varepsilon\} \geq 1 - \frac{E(x-c)^2}{\varepsilon^2}}$$

If $\boxed{c = \mu}$ then,

$$P\{|x-c| \geq \varepsilon\} \leq \frac{V(x)}{\varepsilon^2}$$

If $\boxed{c = \mu}$ & $\boxed{\varepsilon = k\sigma}$ then,

$$P\{|x-c| \geq \varepsilon\} \leq \frac{1}{k^2}$$

3.5 Markov's Inequality

For $a > 0$,

$$\boxed{P\{x \geq a\} \leq \frac{E(x)}{a}}$$

3.6 Uniform Distribution

If X is a continuous random variable defined over an interval $[a, b]$ and having probability distribution function

$$f(x) = \begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & \text{elsewhere} \end{cases}$$

then we say X has uniform distribution. Denoted as follows: $X \sim \mathbf{U}(a, b)$

We define the mean, variance as follows:

- $E(x) = \frac{a+b}{2}$
- $E(x^2) = \frac{1}{3}(a^2 + b^2 + ab)$
- $V(x) = \frac{(b-a)^2}{12}$

3.7 Two Dimensional Random Variables

Let x, y be 2 random variables distributed in a 2 dimensional space S.

$$x, y \to \text{random variable}$$

$$x(S) = x_1, x_2 \ldots x_n \qquad y(S) = y_1, y_2 \ldots y_m$$

then we define P(x=x_i,y=y_j) = P_{ij} such that,

- $P_{ij} \geq 0$
- $\sum_{i=1}^{n} \sum_{j=1}^{m} P_{ij} = 1$

3.7.1 Joint Probability Function

also known as Joint Probability Mass Function is function on the set (x_i, y_j, P_{ij}).

x_i \ y_j	y_1	y_2	$\ldots$	y_m	
x_1	P_{11}	P_{12}	$\ldots$	P_{1m}	$f(x_1)$
x_2	P_{21}	P_{22}	$\ldots$	P_{2m}	$f(x_2)$
$\vdots$	$\vdots$	$\vdots$	$\ldots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\ldots$	$\vdots$	$\vdots$
x_n	P_{n1}	P_{n2}	$\ldots$	P_{nm}	$f(x_n)$
	$g(y_1)$	$g(y_2)$	$\ldots$	$g(y_m)$	1

We define a few terms such as $f(x_i)$ and $g(y_j)$, known as marginal distribution of x and y, for the probability function of two variables $f(x, y)$.

$$\boxed{f(x_i) = \sum_{j=1}^{m} P_{ij} \quad ; \quad f(y_j) = \sum_{i=1}^{n} P_{ij}}$$

Based on the terms mentioned above, we have following formulae,

	Discrete Random Variables	Continuous Random Variables
$E(x)$	$\sum_{i=1}^{n} x_i f(x_i)$	$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x f(x,y)\, dx\, dy$
$E(y)$	$\sum_{j=1}^{m} y_j g(y_j)$	$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y f(x,y)\, dx\, dy$
$E(xy)$	$\sum_{1 \le i \le n, 1 \le j \le m}^{n} x_i\, y_i P_{ij}$	$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy f(x,y)\, dx\, dy$
$E(x^2)$	$\sum_{i=1}^{n} x_i^2 f(x_i)$	$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^2 f(x,y)\, dx\, dy$
$E(y^2)$	$\sum_{j=1}^{m} y_j^2 g(y_j)$	$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y^2 f(x,y)\, dx\, dy$

For a crv, (x, y) is associated with function $f(x, y)$ such that,

- $f(x, y) \ge 0$
- $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y)\, dx\, dy = 1$

$f(x, y)$ is known as the joint probability density function.

3.7.2 Covariance and Correlation Coefficient

The relation of the two variables x and y can be defined by covariance which when $+ve$ means that they are directly proportional and when $-ve$ means inversely proportional. When the covariance is 0, it means that the 2 variables are completely unrelated.

$$Cov(x, y) = \rho_{x\,y} = E(xy) - E(x)\, E(y)$$

This is called the Measure of Correlation.

Correlation Coefficient

The numerical measure of correlation is called the coefficient of correlation and is defined by the relation:

$$r(x, y) = r_{xy} = \frac{Cov(x, y)}{\sigma_x \sigma_y} = \frac{E(xy) - E(x)\, E(y)}{\sqrt{V(x)\, V(y)}}$$

3.7.3 Uniform Distribution of 2 random variables

Let x,y be 2 random variables uniformly distributed over the region **R** in the xy plane then the joint pdf will be as follows

$$f(x,y) = \begin{cases} \frac{1}{\text{Area of Region } \mathbf{R}} & (x,y) \in \mathbf{R} \\ 0 & \text{elsewhere} \end{cases}$$

3.8 Correlation Coefficient

Properties of ρ

- $\boxed{-1 \leq \rho \leq 1}$

  Proof: Let x, y be 2 random variables, then:

$$E\left(\left(\frac{x - E(x)}{\sqrt{V(x)}}\right) \pm \left(\frac{y - E(y)}{\sqrt{V(y)}}\right)\right)^2 \geq 0$$

$$E\left(\left(\frac{x - E(x)}{\sqrt{V(x)}}\right)^2 + \left(\frac{y - E(y)}{\sqrt{V(y)}}\right)^2 \pm 2 \times \underbrace{\left(\frac{(x - E(x))(y - E(y))}{\sqrt{V(x)V(y)}}\right)}_{\text{correlation coefficient } \rho}\right) \geq 0$$

  On simplification the equation becomes as follows,

$$2 \pm 2\rho_{xy} \geq 0 \Longrightarrow -1 \leq \rho \leq 1$$

  □

- $\boxed{\text{Y=AX+B}}$, A & B are constants

$$\rho^2 = 1 \text{ then } \begin{cases} A > 0, \rho = +1 \\ A < 0, \rho = -1 \end{cases}$$

- $\boxed{\text{V=AX+B,W=CY+D}}$

$$\rho_{vw} = \frac{AC}{|AC|}\rho_{xy}$$

3.9 Moment Generating Function

$$
\begin{aligned}
\mathsf{M}_x(t) &= E(e^{tx}) \\
&= \sum x e^{tx} P(x) \quad \text{if x is discreet} \\
&= \int e^{tx} f(x)\, dx \quad \text{if x is continuous}
\end{aligned}
$$

Properties

- $\mathsf{M}_x(t) = 1 + tR(x) + \frac{t^2}{2!}E(x^2) \cdots + \frac{t^n}{n!}E(x^n)$
- At $t = 0$, $\boxed{M_x'(0) = E(x)}$ & $\boxed{M_x''(t) = E(x^2)}$
- $V(x) = M_x''(0) - M_x'(0)^2$

4 Distributions

Types of Distributions:

- Binomial
- Poisson
- Normal
- Gamma
- Exponential
- Chisquare
- Uniform

4.1 Binomial Distribution

Here there are 2 events such that if probability of one is p, then the probability of other is $1-p$. Probability is in terms of a random variable X that denotes number of favorable cases.

$$\boxed{P(X=k) = \binom{n}{k} p^k \, (1-p)^{n-k}}$$

It is denoted by $\boxed{X \sim B(n,p)}$.

$X:$	0	1	$\ldots$	n-1	n
$P(X=k):$	$1-p)^n$	$\binom{n}{1} p \, (1-p)^{n-1}$	$\ldots$	$\binom{n}{n-1} p^n - 1 \, (1-p)$	$\binom{n}{n} p^n$

- $E(x) = np$
- $E(x^2) = n(n-1)p^2 + np$
- $V(x) = np(p-1) = npq$