Rekurzivne definicije = fiksne točke

Kako bi interpretirali

$\mathrm{rec}\, f\, x . e : A \to B$

S preslikavo $g : [\![A]\!] \to [\![B]\!]$, ki bo zadoščala $g(a) = \dots g \dots$

$\mathrm{fact} := \mathrm{rec}\, f\, n .\ \mathrm{if}\ n = 0\ \mathrm{then}\ 1\ \mathrm{else}\ n * \mathrm{fact}(n-1)$

$[\![\mathrm{fact}]\!] : \mathbb{N} \to \mathbb{N} \qquad [\![\mathrm{fact}]\!](n) = n! = \varphi(n)$

tako danes označujem fakulteto

$[\![\mathrm{fact}]\!] = \varphi$

$$\varphi(n) = \begin{cases} 1 & \text{če je } n = 0 \\ n \cdot \varphi(n-1) & \text{sicer} \end{cases}$$

Namesto $g(a) = \dots g \dots$ lahko pišemo $g = \overbrace{a \mapsto \dots g \dots}^{\Psi}$ oz.

$g = \Psi(g)$

Npr.

$$\varphi = \Psi(\varphi), \text{ kjer je } \Psi(f) := n \mapsto \begin{cases} 1 & \text{če je } n = 0 \\ n \cdot f(n-1) & \text{sicer} \end{cases}$$

$$f_1 = \Psi(m \mapsto 0) = n \mapsto \begin{cases} 1 & \text{če je } n = 0 \\ 0 & \text{sicer} \end{cases}$$

$$\Psi(m \mapsto m^2) = n \mapsto \begin{cases} 1 & \text{če je } n = 0 \\ n(n-1)^2 & \text{sicer} \end{cases}$$

$$\Psi(\varphi) = n \mapsto \begin{cases} 1 & \text{če je } n = 0 \\ n \cdot \varphi(n-1) & \text{sicer} \end{cases} = \varphi$$

$$f_2 = \Psi(\Psi(m \mapsto 0)) = n \mapsto \begin{cases} 1 & \text{če je } n = 0 \\ n \cdot f_1(n-1) & \text{sicer} \end{cases} = \begin{cases} 1 & n = 0 \\ 1 & n = 1 \\ 0 & n \geq 2 \end{cases}$$

$$f_3 = n \mapsto \begin{cases} 1 & n = 0 \\ 1 & n = 1 \\ 2 & n = 2 \\ 0 & \text{sicer} \end{cases}$$

Vidimo, da zaporedje $f_1, f_2, f_3, \dots$ konvergira k φ.

Def Delno urejena množica $(D, \leq)$ je domena, če:

- obstaja najmanjši element $\bot_D \in D$, ki mu pravimo dno (bottom) $(\forall x \in D.\ \bot_D \leq x)$
- vsaka naraščajoča veriga $x_0 \leq x_1 \leq x_2 \leq \dots \in D$ ima supremum $\bigvee_i x_i$. $(\forall i.\ x_i \leq \bigvee_i x_i) \wedge (\forall y. (\forall i. x_i \leq y) \Rightarrow \bigvee_i x_i \leq y)$

Def Naj bosta $(D, \leq_D)$ in $(E, \leq_E)$ domeni.
Monotona preslikava $f : D \to E$ je zvezna,
$x \leq_D y \Rightarrow fx \leq_E fy$ če velja $f(\bigvee_i x_i) = \bigvee_i f x_i$.

Lema Če je f monotona, velja $\bigvee_i f x_i \leq f(\bigvee_i x_i)$.

Izrek (Tarski, Kleene, ..). Naj bo $(D, \leq)$ domena in $f : D \to D$ zvezna preslikava. Tedaj ima f najmanjšo fiksno točko.

Dokaz Definirajmo $x_i = f^i(\bot)$. Ker je $\bot$ dno, velja $x_0 = \bot \leq f(\bot) = x_1$
Ker je f monotona je $f(x_0) \leq f(x_1)$ oz. $x_1 \leq x_2$. Podobno $x_i \leq x_{i+1}$
Ker je D domena, ima veriga $x_0 \leq x_1 \leq x_2 \dots$ supremum $\bigvee_i x_i$.
Pokažimo, da je $x = \bigvee_i x_i$ fiksna točka.

$$f(x) = f(\bigvee_i x_i) = \bigvee_i f x_i = \bigvee_i x_{i+1} = x.$$

Pokažimo, da je najmanjša. Naj bo y fiksna točka f.

$$x_0 = \bot \leq y.$$
$$x_1 = f(x_0) \leq f(y) = y$$
$$x_2 = f(x_1) \leq f(y) = y \qquad \Longrightarrow \bigvee_i x_i \leq y$$
$$\vdots$$
$$x_i \leq y$$

∎

Interpretacije z domenami

Vsak tip A bomo interpretirali z domeno $[\![A]\!]$

Ideja: $x \leq y$ ... x vsebuje manj informacije kot y.

$$[\![\mathrm{bool}]\!] = \begin{array}{c} tt \quad ff \\ \diagdown \; \diagup \\ \bot \end{array} = \mathbb{B}_\bot$$

ploska ureditev

$D_\bot = (D \cup \{\bot\}, \; x \leq y \iff x = \bot \vee x = y)$

za množice, v splošnem

$(D \cup \{\bot\}, x \leq_{D_\bot} y \iff$
$x = \bot$ ali
$x \leq_D y$)

$$[\![\mathrm{int}]\!] = \begin{array}{c} \cdots -2 \; -1 \; 0 \; 1 \; 2 \; 3 \cdots \\ \bot \end{array} = \mathbb{Z}_\bot$$

$$[\![A \to B]\!] = [[\![A]\!] \to_\bot [\![B]\!]]_\bot$$ ← zaradi zaradi

let $f\,x = f\,x$ in
$(\lambda x. \lambda y. x)(f\,5)$

$$\bot_{[[\![A]\!] \to [\![B]\!]]_\bot} \neq a \in [\![A]\!] \mapsto \bot_{[\![B]\!]}$$

$[D \to E]$ je domena vseh zveznih preslikav $D \to E$.

$$f \leq_{[D \to E]} g \iff \forall x \in D.\; f x \leq_E g x$$

$[D \to_\bot E]$ je domena vseh strogih zveznih preslikav

$$f \text{ je } \underline{\text{stroga}} \iff f(\bot_D) = \bot_E$$

Opomba Za leno izvajanje bi izbrali $[\![A \to B]\!] = [[\![A]\!] \to \cancel{{}_\bot} [\![B]\!]]_\bot$

Trditev Če sta D in E domeni, je domena tudi $[D \to E]_\bot$ oz. $[D \to_\bot E]_\bot$.

Dokaz (za $[D \to E]_\bot$).

Vzemimo zvezne preslikave $f_0 \leq f_1 \leq \cdots$

Pokažimo $(\bigvee_i f_i)(x) = \bigvee_i f_i x$.

Definirajmo $g(x) = \bigvee_i f_i x$. in pokažimo, da je supremum verige $f_0 \leq f_1 \leq \cdots$

$f_i(x) \leq \bigvee_i f_i(x) = g(x)$. g je zgornja meja.

Naj bo $\forall i.\ f_i \leq h$ za neko zg. mejo h.

Tedaj je $f_i(x) \leq h(x)$ za vse $x \in D$. Torej je $g(x) = \bigvee_i f_i(x) \leq h(x)$.

Pokažimo, da je g zvezna.

$$g\left(\bigvee_j x_j\right) = \bigvee_i f_i\left(\bigvee_j x_j\right) = \bigvee_i \bigvee_j f_i(x_j) = \bigvee_j \bigvee_i f_i(x_j)$$
$$= \bigvee_j g(x_j)$$

■

Izraz $\Gamma \vdash e : A$ bomo interpretirali z zvezno preslikavo

$$[\![\Gamma \vdash e : A]\!] : [\![\Gamma]\!] \to [\![A]\!]$$

Pri čemer je $[\![x_1 : A_1, \dots, x_n : A_n]\!] = [\![A_1]\!] \times \dots \times [\![A_n]\!]$, kjer je urejenost podana z

$$(\eta_1, \eta_2, \dots, \eta_n) \leq (\eta_1', \eta_2', \dots, \eta_n') \iff \eta_1 \leq \eta_1' \wedge \dots \wedge \eta_n \leq \eta_n'.$$

<u>Def</u>

$$[\![\Gamma \vdash n : \mathrm{int}]\!](\eta) = n$$

$$[\![\Gamma \vdash e_1 + e_2 : \mathrm{int}]\!](\eta) = \begin{cases} n_1 + n_2 & \text{če je } [\![e_1]\!](\eta) = n_1 \neq \bot \\ & \text{in } [\![e_2]\!](\eta) = n_2 \neq \bot \\ \bot & \text{sicer} \end{cases}$$

$[\![\mathrm{true}, \mathrm{false}, *, -, <, >, =]\!]$ = podobno

$$[\![\text{if } e \text{ then } e_1 \text{ else } e_2]\!](\eta) = \begin{cases} [\![e_1]\!](\eta) & \text{če je } [\![e]\!](\eta) = tt \\ [\![e_2]\!](\eta) & \text{če je } [\![e]\!](\eta) = ff \\ \bot & \text{sicer} \end{cases}$$

$$[\![\Gamma \vdash \lambda x. e : A \to B]\!](\eta) = a \mapsto \begin{cases} [\![\Gamma, x : A \vdash e : B]\!](\eta, a) & a \neq \bot \\ \bot & \text{sicer} \end{cases}$$

$$[\![e_1 e_2]\!](\eta) = \begin{cases} [\![e_1]\!](\eta)([\![e_2]\!](\eta)) & \text{če je } [\![e_1]\!](\eta) \neq \bot \\ \bot & \text{sicer} \end{cases}$$

$$[\![x_1 : A_1, \dots, x_n : A_n \vdash x_i : A_i]\!](\eta_1, \dots, \eta_n) = \eta_i$$

$$[\![\Gamma \vdash \mathrm{rec}\, f\, x . e : A \to B]\!](\eta) = g \text{, kjer je } g \text{ najmanjša fiksna točka}$$

$$\Psi : [[\![A]\!] \to_\bot [\![B]\!]]_\bot \longrightarrow [[\![A]\!] \to_\bot [\![B]\!]]_\bot$$

$$\Psi(\underset{\in [[\![A]\!] \to_\bot [\![B]\!]]_\bot}{f}) = a \longmapsto \begin{cases} [\![\Gamma, f : A \to B, x : A \vdash e : B]\!](\eta, f, a) & a \neq \bot \\ \bot & \text{sicer} \end{cases}$$

Pokažimo, da zares dobimo zvezne preslikave

- $[\![\Gamma \vdash x_i : A_i]\!](\bigvee_j \eta) = [\![\Gamma \vdash x_i : A_i]\!](\bigvee_j \eta_1^j, \bigvee_j \eta_2^j, \dots, \bigvee_j \eta_n^j)$

$$= \bigvee_j \eta_i^j = \bigvee_j ([\![\Gamma \vdash x_i : A_i]\!](\eta_1^j, \eta_2^j, \dots, \eta_n^j))$$

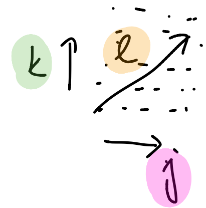
- $$[\![e_1 e_2]\!](\bigvee_j \eta_j) = \begin{cases} [\![e_1]\!](\bigvee_j \eta_j)([\![e_2]\!](\bigvee_j \eta_j)) & \text{če} \neq \bot \\ \bot & \text{sicer} \end{cases}$$

$$(\text{ker } [\![e_2]\!] \text{ zv.}) = \begin{cases} [\![e_1]\!](\bigvee_j \eta_j)(\bigvee_j [\![e_2]\!](\eta_j)) \\ \bot \end{cases}$$

$$(\text{ker so } A \to B \text{ zv.}) = \begin{cases} \bigvee_k [\![e_1]\!](\bigvee_j \eta_j)([\![e_2]\!](\eta_k)) \\ \bot \end{cases}$$

$$= \begin{cases} \bigvee_j \bigvee_k [\![e_1]\!](\eta_j)([\![e_2]\!](\eta_k)) \\ \bot \end{cases}$$

$$= \begin{cases} \bigvee_\ell [\![e_1]\!](\eta_\ell)([\![e_2]\!](\eta_\ell)) \\ \bot \end{cases} = \bigvee_\ell [\![e_1 e_2]\!](\eta_j).$$



- ostalo : DN.