Izpeljava tipov

Problem, ki ga rešujemo: (izpeljava tipa)
Če imam izraz e, ali obstaja A, da velja $\emptyset \vdash e : A$
Enostavnejši problem: (preverjanje tipa)
Če imam izraz e in tip A, ali velja $\emptyset \vdash e : A$.

Mi bomo reševali prvega s Hindley-Milnerjevim algoritmom.

Ideja: ko moramo ugoniti tip v izpeljavi, ustvarimo spremenljivko $\alpha, \beta, \ldots$
Če se morajo tipi ujemati, dodamo enačbo

$$
\dfrac{
\dfrac{
\dfrac{}{f:\alpha \vdash f:\alpha} \quad \dfrac{}{f:\alpha \vdash 10 : int}
}{f:\alpha \vdash f\ 10 : \beta}
}{\emptyset \vdash \lambda f. f\ 10 : \alpha \to \beta}
\quad
\dfrac{
\dfrac{
\dfrac{}{x:\gamma \vdash x:\gamma} \quad \dfrac{}{x:\gamma \vdash 3 : int}
}{x:\gamma \vdash x > 3 : \delta}
}{\emptyset \vdash \lambda x. x > 3 : \alpha}
}{\emptyset \vdash (\lambda f. f\ 10)(\lambda x. x>3) : \beta}
$$

$$
\begin{array}{l} \alpha = int \to \beta \\ \alpha = \gamma \to \delta \\ \gamma = int \\ \delta = bool \end{array}
\rightsquigarrow
\begin{array}{l} \alpha = int \to \beta \\ \alpha = int \to bool \\ \gamma = int \\ \delta = bool \end{array}
\rightsquigarrow
\begin{array}{l} \alpha = int \to bool \\ int \to bool = int \to \beta \\ \gamma = int \\ \delta = bool \end{array}
$$

$$
\rightsquigarrow
\begin{array}{l} \alpha = int \to bool \\ int = int \\ bool = \beta \\ \gamma = int \\ \delta = bool \end{array}
\rightsquigarrow
\begin{array}{l} \alpha = int \to bool \\ \beta = bool \\ \gamma = int \\ \delta = bool \end{array}
$$

$A ::= \alpha \mid \cdots$

$\Gamma \vdash e : A \mid \mathcal{E}$ ... v kontekstu Γ za e izpeljemo tip A ob omejitvah $\mathcal{E}$.

$$\frac{}{\Gamma \vdash n : \mathrm{int} \mid \emptyset} \qquad \frac{\Gamma \vdash e_1 : A_1 \mid \mathcal{E}_1 \qquad \Gamma \vdash e_2 : A_2 \mid \mathcal{E}_2}{\Gamma \vdash e_1 + e_2 : \mathrm{int} \mid \mathcal{E}_1, \mathcal{E}_2, A_1 = \mathrm{int}, A_2 = \mathrm{int}}$$

pod. za $-, *$

$$\frac{}{\Gamma \vdash \mathrm{true} : \mathrm{bool} \mid \emptyset} \qquad \frac{}{\Gamma \vdash \mathrm{false} : \mathrm{bool} \mid \emptyset}$$

$=, <, >$: D.N.

$$\frac{\Gamma \vdash e : A \mid \mathcal{E} \qquad \Gamma \vdash e_1 : A_1 \mid \mathcal{E}_1 \qquad \Gamma \vdash e_2 : A_2 \mid \mathcal{E}_2}{\Gamma \vdash \mathrm{if}\ e\ \mathrm{then}\ e_1\ \mathrm{else}\ e_2 : A_1 \mid \mathcal{E}, \mathcal{E}_1, \mathcal{E}_2, A = \mathrm{bool}, A_1 = A_2} \qquad \frac{(x : A \in \Gamma)}{\Gamma \vdash x : A \mid \emptyset}$$

(se ne pojavlja v $\Gamma, B, \mathcal{E}$)

$$\frac{\Gamma, x : \alpha \vdash e : B \mid \mathcal{E} \qquad \alpha \text{ svež}}{\Gamma \vdash \lambda x. e : \alpha \to B \mid \mathcal{E}} \qquad \frac{\Gamma \vdash e_1 : A_1 \mid \mathcal{E}_1 \qquad \Gamma \vdash e_2 : A_2 \mid \mathcal{E}_2 \qquad \alpha \text{ svež}}{\Gamma \vdash e_1\, e_2 : \alpha \mid \mathcal{E}_1, \mathcal{E}_2, A_1 = A_2 \to \alpha}$$

Primer: $\lambda f. \lambda x.\ \mathrm{if}\ x\ \mathrm{then}\ f\,x\ \mathrm{else}\ f(f\,x)$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash x : \beta} \qquad \dfrac{\dfrac{}{\Gamma \vdash f : \alpha} \quad \dfrac{}{\Gamma \vdash x : \beta}}{\Gamma \vdash f\,x : \gamma} \qquad \dfrac{\dfrac{}{\Gamma \vdash f : \alpha} \quad \dfrac{\dfrac{}{\Gamma \vdash f : \alpha} \quad \dfrac{}{\Gamma \vdash x : \beta}}{\Gamma \vdash f\,x : \sigma}}{\Gamma \vdash f(f\,x) : \varphi}}{\overbrace{f : \alpha, x : \beta}^{\Gamma} \vdash \mathrm{if}\ x\ \mathrm{then}\ f\,x\ \mathrm{else}\ f(f\,x) : \gamma}}{f : \alpha \vdash \lambda x. \mathrm{if}\ x\ \mathrm{then}\ f\,x\ \mathrm{else}\ f(f\,x) : \beta \to \gamma}}{\emptyset \vdash \lambda f. \lambda x.\ \mathrm{if}\ x\ \mathrm{then}\ f\,x\ \mathrm{else}\ f(f\,x) : \alpha \to (\beta \to \gamma)}$$

$(\mathrm{bool} \to \mathrm{bool}) \to (\mathrm{bool} \to \mathrm{bool})$

$\alpha = \beta \to \gamma$
$\alpha = \beta \to \sigma$
$\alpha = \sigma \to \varphi$
$\varphi = \gamma$
$\beta = \mathrm{bool}$

rešimo $\rightsquigarrow$

$\beta = \sigma = \gamma = \varphi = \mathrm{bool}$
$\alpha = \mathrm{bool} \to \mathrm{bool}$

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma\vdash f:\alpha\mid\emptyset}\quad\dfrac{}{\Gamma\vdash x:\beta\mid\emptyset}}{\Gamma\vdash f\,x:\gamma\mid\alpha=\beta\to\gamma}\qquad\dfrac{}{\Gamma\vdash 3:\mathrm{int}\mid\emptyset}}{\overbrace{f:\alpha,x:\beta}^{\Gamma}\vdash f\,x+3:\mathrm{int}\mid \alpha=\beta\to\gamma,\gamma=\mathrm{int},\mathrm{int}=\mathrm{int}}}{f:\alpha\vdash\lambda x.\,f\,x+3:\beta\to\mathrm{int}\mid\dots}}{\emptyset\vdash\lambda f.\lambda x.\,f\,x+3:\alpha\to(\beta\to\mathrm{int})\mid\dots}$$

$$\begin{matrix}\alpha=\beta\to\gamma\\ \gamma=\mathrm{int}\\ \cancel{\mathrm{int}=\mathrm{int}}\end{matrix}\quad\rightsquigarrow\quad\begin{matrix}\gamma\mapsto\mathrm{int}\\ \alpha=\beta\to\mathrm{int}\end{matrix}\quad\rightsquigarrow\quad\begin{matrix}\alpha\mapsto\beta\to\mathrm{int}\\ \gamma\mapsto\mathrm{int}\end{matrix}$$

$$\emptyset\vdash\lambda f.\lambda x.\,f\,x+3:(\beta\to\mathrm{int})\to(\beta\to\mathrm{int})$$

S σ označujemo substitucije, t.j. končne preslikave $\alpha_1\mapsto A_1,\dots,\alpha_n\mapsto A_n$

Def.
$$\sigma(\mathrm{int})=\mathrm{int}$$
$$\sigma(\mathrm{bool})=\mathrm{bool}$$
$$\sigma(A\to B)=\sigma(A)\to\sigma(B)$$
$$\sigma(\alpha)=\begin{cases}A & \alpha\mapsto A\in\sigma\\ \alpha & \text{sicer}\end{cases}$$

Def $\sigma\vDash\mathcal{E}$ … σ reši enačbe $\mathcal{E}$

$$\sigma\vDash A_1=A_1',\dots,A_n=A_n'\iff\sigma(A_1)=\sigma(A_1')\land\dots\land\sigma(A_n)=\sigma(A_n').$$

Trditev 5 Če velja $\Gamma\vdash e:A\mid\mathcal{E}$, potem za poljubno $\sigma\vDash\mathcal{E}$ velja $\sigma(\Gamma)\vdash e:\sigma(A)$.

Trditev 6 Če velja $\Gamma\vdash e:A$ in $\Gamma\vdash e:A'\mid\mathcal{E}$, potem obstaja $\sigma\vDash\mathcal{E}$, da je $\sigma(A')=A$.

Def $\mathcal{E}\rightsquigarrow\sigma$ … $\mathcal{E}$ ima najbolj splošno rešitev σ $\mathrm{unify}(\mathcal{E})=\sigma$

$$\dfrac{}{\emptyset\rightsquigarrow\emptyset}\qquad\dfrac{\mathcal{E}\rightsquigarrow\sigma}{\mathrm{int}=\mathrm{int},\mathcal{E}\rightsquigarrow\sigma}$$

podobno za $\mathrm{bool}=\mathrm{bool}$, $\alpha=\alpha$

$\alpha\notin FV(A)$ ← vse proste spremenljivke v A

$$\dfrac{A_1=A_2,B_1=B_2,\mathcal{E}\rightsquigarrow\sigma}{A_1\to B_1=A_2\to B_2,\mathcal{E}\rightsquigarrow\sigma}\qquad\dfrac{(\alpha\mapsto A)(\mathcal{E})\rightsquigarrow\sigma}{\alpha=A,\mathcal{E}\rightsquigarrow\sigma\circ(\alpha\mapsto A)}$$

podobno za $A=\alpha$

$$(\underbrace{\alpha_1 \mapsto A_1, \dots, \alpha_n \mapsto A_n}_{\sigma}) \circ (\alpha_1' \mapsto A_1', \dots, \alpha_m' \mapsto A_m')$$

$$= \alpha_1' \mapsto \sigma(A_1'), \dots, \alpha_m' \mapsto \sigma(A_m'), \underbrace{\alpha_1 \mapsto A_1, \dots, \alpha_n \mapsto A_n}_{\text{BREZ } \alpha_1', \dots, \alpha_m'}$$

Trditev 7 Če $\mathcal{E} \succcurlyeq \sigma$, potem $\sigma \vDash \mathcal{E}$.

Trditev 8 Če $\sigma \vDash \mathcal{E}$, potem za poljubno σ' velja $\sigma' \circ \sigma \vDash \mathcal{E}$.

Trditev 9 Če $\mathcal{E} \succcurlyeq \sigma$ in velja $\sigma' \vDash \mathcal{E}$, potem obstaja σ'', da je $\sigma' = \sigma'' \circ \sigma$.

Trditev 10 Če $\sigma' \vDash \mathcal{E}$, potem obstaja σ, da velja $\mathcal{E} \succcurlyeq \sigma$.

Dokaz 5

$$\frac{}{\Gamma \vdash n : \mathrm{int} \mid \emptyset} \qquad \frac{\Gamma \vdash e_1 : A_1 \mid \mathcal{E}_1 \qquad \Gamma \vdash e_2 : A_2 \mid \mathcal{E}_2}{\Gamma \vdash e_1 + e_2 : \mathrm{int} \mid \mathcal{E}_1, \mathcal{E}_2, A_1 = \mathrm{int}, A_2 = \mathrm{int}}$$ pod. za $-, *$

$$\frac{}{\Gamma \vdash \mathrm{true} : \mathrm{bool} \mid \emptyset} \qquad \frac{}{\Gamma \vdash \mathrm{false} : \mathrm{bool} \mid \emptyset}$$ $=, <, >$: D.N.

$$\frac{\Gamma \vdash e : A \mid \mathcal{E} \qquad \Gamma \vdash e_1 : A_1 \mid \mathcal{E}_1 \qquad \Gamma \vdash e_2 : A_2 \mid \mathcal{E}_2}{\Gamma \vdash \text{if } e \text{ then } e_1 \text{ else } e_2 : A_1 \mid \mathcal{E}, \mathcal{E}_1, \mathcal{E}_2, A = \mathrm{bool}, A_1 = A_2} \qquad \frac{(x : A \in \Gamma)}{\Gamma \vdash x : A \mid \emptyset}$$

(se ne pojavlja v $\Gamma, B, \mathcal{E}$)

$$\frac{\Gamma, x : \alpha \vdash e : B \mid \mathcal{E} \qquad \alpha \text{ svež}}{\Gamma \vdash \lambda x. e : \alpha \to B \mid \mathcal{E}} \qquad \frac{\Gamma \vdash e_1 : A_1 \mid \mathcal{E}_1 \qquad \Gamma \vdash e_2 : A_2 \mid \mathcal{E}_2 \qquad \alpha \text{ svež}}{\Gamma \vdash e_1 e_2 : \alpha \mid \mathcal{E}_1, \mathcal{E}_2, A_1 = A_2 \to \alpha}$$

Indukcija na $\Gamma \vdash e : A \mid \mathcal{E}$.

- aplikacije:

  Imamo $\Gamma \vdash e_1 e_2 : \alpha \mid \mathcal{E}_1, \mathcal{E}_2, A_1 = A_2 \to \alpha$.

  in $\sigma \vDash \mathcal{E}_1, \mathcal{E}_2, A_1 = A_2 \to \alpha$.

  Po inverziji velja $\Gamma \vdash e_1 : A_1 \mid \mathcal{E}_1$ ter $\Gamma \vdash e_2 : A_2 \mid \mathcal{E}_2$.

  Ker $\sigma \vDash \mathcal{E}_1$ in $\sigma \vDash \mathcal{E}_2$, lahko uporabimo I.P. in dobimo

  $\sigma(\Gamma) \vdash e_1 : \sigma(A_1)$ in $\sigma(\Gamma) \vdash e_2 : \sigma(A_2)$. Ker $\sigma(A_1) = \sigma(A_2) \to \sigma(\alpha)$,

  velja $$\frac{\sigma(\Gamma) \vdash e_1 : \sigma(A_2) \to \sigma(\alpha) \qquad \sigma(\Gamma) \vdash e_2 : \sigma(A_2)}{\sigma(\Gamma) \vdash e_1 e_2 : \sigma(\alpha)}$$ $\square$

- ostalo podobno.

Dokaz 6 TAPL

Dokaz 7

- vse trivialno razen parametrov

- $$\frac{(\alpha \mapsto A)(\mathcal{E}) \succcurlyeq \sigma \qquad \alpha \notin FV(A)}{\alpha = A, \mathcal{E} \succcurlyeq \sigma \circ (\alpha \mapsto A)}$$

  $\underline{\sigma \circ (\alpha \mapsto A)} \vDash \underline{\alpha = A}, \mathcal{E}$

1) L.S. $(\sigma \circ (\alpha \mapsto A))(\alpha) = \sigma(A)$

   D.S. $(\sigma \circ (\alpha \mapsto A))(A) = \sigma(A)$ (ker $\alpha \notin FV(A)$).

2) $(\sigma \circ (\alpha \mapsto A)) \vDash \mathcal{E} \iff \sigma \vDash (\alpha \mapsto A)(\mathcal{E}) \leftarrow$ I.P.

Trditev 8 Očitno

Trditev 9 Indukcija na $\mathcal{E} \rightsquigarrow \sigma$.

- $\emptyset \rightsquigarrow \emptyset$ in $\sigma \models \emptyset$, potem je $\sigma = \sigma \circ \emptyset$

- $\dfrac{A_1 = A_2, B_1 = B_2, \mathcal{E} \rightsquigarrow \sigma}{A_1 \to B_1 = A_2 \to B_2, \mathcal{E} \rightsquigarrow \sigma}$. Če $\sigma' \models A_1 \to B_1 = A_2 \to B_2, \mathcal{E}$, potem $\sigma' \models A_1 = A_2, B_1 = B_2, \mathcal{E}$. Po I.P. $\exists \sigma''. \sigma' = \sigma'' \circ \sigma$.

- ostalo podobno.