自觉遵守考场纪律 如考试作弊 此答卷无效

姓名________ 学号________

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东 南 大 学 考 试 卷 (A)

课程名称 数学物理方法 考试学期 17-18-3 得分________

适用专业 面上 考试形式 闭卷 考试时间长度 120分钟

题目	一	二	三	四	五
得分					

注意：本份试卷可能会用到以下公式：

1、$\mathscr{L}[\sin\alpha t](p)=\dfrac{\alpha}{p^2+\alpha^2}$，$\mathscr{L}[\cos\alpha t](p)=\dfrac{p}{p^2+\alpha^2}$，$\mathscr{L}[t^n \mathrm{e}^{at}](p)=\dfrac{n!}{(p-a)^{n+1}}$；

2、$\mathscr{L}[f(t-t_0)H(t-t_0)](p)=\tilde{f}(p)\mathrm{e}^{-t_0p},\ t_0\geq 0$;

3、$\mathscr{L}[\delta(t-t_0)](p)=\mathrm{e}^{-t_0p},\ t_0\geq 0$;

4、$\mathscr{F}[f(x-b)](\lambda)=\mathrm{e}^{-i\lambda b}\hat{f}(\lambda)$; $\mathscr{F}[\mathrm{e}^{-Ax^2}](\lambda)=\sqrt{\dfrac{\pi}{A}}\mathrm{e}^{-\lambda^2/(4A)}, A>0$;

5、$(x^\nu J_\nu(x))'=x^\nu J_{\nu-1}(x),\ (x^{-\nu}J_\nu(x))'=-x^{-\nu}J_{\nu+1}(x)$.

一 填空题(每题5分，共30分)

1. 有长为l的细弦做微小横振动，弦的一端固定，另一端无外力作用可自由滑动，则此弦振动方程的边界条件为 $\underline{u(0,t)=u_x(l,t)=0}$.

2. 特征值问题
$$\begin{cases} X''(x)+\lambda X(x)=0, & 0<x<l,\\ X'(0)=0,\ X'(l)=0 \end{cases}$$
的所有特征值及其对应的特征函数是 $\underline{\lambda_n=(\frac{n\pi}{l})^2,\ X_n(x)=\cos\frac{n\pi x}{l},\ n=0,1,\cdots}$.

3. 给定初边值问题
$$\begin{cases} u_t-a^2u_{xx}=\cos x, & 0<x<\pi, t>0,\\ u_x(0,t)=0,\ u(\pi,t)=A, & t>0,\\ u(x,0)=0, & 0\leq x\leq\pi, \end{cases}$$
其中$A\neq 0$为常数，则当$w(x)=\underline{\frac{1}{a^2}(1+\cos x)+A}$ 时, 利用变换$u(x,t)=v(x,t)+w(x)$, 可把此问题化为齐次方程齐次边界条件的初边值问题.

4. 像函数$\dfrac{1}{(p^2+1)(p+1)}$ 的Laplace逆变换为 $\underline{\frac{1}{2}\mathrm{e}^{-t}+\frac{1}{2}\sin t-\frac{1}{2}\cos t}$.

5. 对于一维波动方程$u_{tt}-u_{xx}=0,-\infty<x<\infty,t>0$, 区间$[1,2]$的决定区域是 $\underline{\{(x,t)\mid t+1\le x\le 2-t,\ t\ge 0\}}$.

6. 给定一维波动方程初值问题

$$\begin{cases} u_{tt}-a^2u_{xx}=0, & x\in R,t>0,\\ u|_{t=0}=\sin x,\ u_t|_{t=0}=\cos x, & x\in R,\end{cases}$$

它的解为 $\underline{u(x,t)=\sin x\cos at+\frac{1}{a}\cos x\sin at}$.

二 简单计算题(32分)

1. 求函数$f(x)=\begin{cases} 1-|x|, & |x|\le 1,\\ 0, & |x|>1\end{cases}$ 的Fourier变换.

解：由定义

$$\begin{aligned}\mathscr{F}[f(x)](\lambda) &= \int_{-\infty}^{\infty} f(x)\mathrm{e}^{-i\lambda x}\mathrm{d}x\\ &= \int_{-1}^{1}(1-|x|)\mathrm{e}^{-i\lambda x}\mathrm{d}x \qquad \cdots\cdots\cdots\cdots 3分\\ &= 2\int_0^1(1-x)\cos\lambda x\mathrm{d}x\\ &= \frac{2}{\lambda^2}(1-\cos\lambda). \qquad \cdots\cdots\cdots\cdots 8分\end{aligned}$$

2. 用Laplace变换法求解下列方程

$$\begin{cases} y''+y=-\delta(t-\pi)+\delta(t-2\pi), & t>0,\\ y(0)=0,\ y'(0)=1.\end{cases}$$

解：记$\tilde{y}(p)=\mathscr{L}[y(t)]$, 对方程两边做Laplace变换，得

$$p^2\tilde{y}+\tilde{y}-py(0)-y'(0)=-\mathrm{e}^{-\pi p}+\mathrm{e}^{-2\pi p}.$$

即

$$\tilde{y}(p)=\frac{1}{p^2+1}(1-\mathrm{e}^{-\pi p}+\mathrm{e}^{-2\pi p}). \qquad \cdots\cdots\cdots\cdots 4分$$

做Laplace逆变换，得

$$\begin{aligned} y(t) &= \sin t-\sin(t-\pi)H(t-\pi)+\sin(t-2\pi)H(t-2\pi)\\ &= \sin t(1+H(t-\pi)+H(t-2\pi)) \qquad \cdots\cdots\cdots\cdots 8分\\ &= \begin{cases}\sin t, & 0\le t<\pi,\\ 2\sin t, & \pi\le t<2\pi,\\ 3\sin t, & t\ge 2\pi.\end{cases}\end{aligned}$$

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3. 利用特征线方法求解下列问题

$$\begin{cases} u_{xx}+4u_{xy}+3u_{yy}=0, & x\in R, y>0, \\ u|_{y=0}=\cos x,\ u_y|_{y=0}=x, & x\in R. \end{cases}$$

解: 特征方程为$\mathrm{d}^2y-4\mathrm{d}x\mathrm{d}y+3\mathrm{d}^2x=(\mathrm{d}y-\mathrm{d}x)(\mathrm{d}y-3\mathrm{d}x)=0.$ ············2分
求得特征线$x-y=C_1$, $3x-y=C_2$, 作特征变换$\xi=x-y$, $\eta=3x-y$, 则由链式法则得

$$u_{xx}=u_{\xi\xi}+6u_{\xi\eta}+9u_{\eta\eta},\ u_{yy}=u_{\xi\xi}+2u_{\xi\eta}+u_{\eta\eta}, u_{xy}=-u_{\xi\xi}-4u_{\xi\eta}-3u_{\eta\eta}.$$

于是方程化为$u_{\xi\eta}=0$, 从而得方程的通解

$$u(x,y)=f(x-y)+g(3x-y).$$ ············5分

利用初始条件，得

$$f(x)+g(3x)=\cos x,\ -f'(x)-g'(3x)=x.$$

求得

$$f(x)=-\frac{1}{2}\cos x-\frac{3}{4}x^2-\frac{3}{2}C,\ g(3x)=\frac{3}{2}\cos x+\frac{3}{4}x^2+\frac{3}{2}C.$$

所以，解为

$$u(x,y) = \frac{3}{2}\cos(x-\frac{y}{3})-\frac{1}{2}\cos(x-y)+\frac{1}{12}(3x-y)^2-\frac{3}{4}(x-y)^2.$$ ·········8分

4. 给定边值问题

$$\begin{cases} u_{rr}+\frac{1}{r}u_r+k^2u=0, & 0<r<1, \\ |u(0)|<\infty,\ u(1)=A, \end{cases}$$

其中k, A是常数，且$k>0, A\neq 0$. 记α_n为$J_0(x)$的第n个正零点.

利用Bessel函数理论证明：(1) 若$k=\alpha_n$, $n=1,2,\cdots$, 则上述边值问题无解；(2)若$k\neq\alpha_n$, $n=1,2,\cdots$, 则上述边值问题有唯一解，并求此解.

证明: (1) 假设问题有解. 因为Bessel方程的通解为$u(r)=C_1J_0(kr)+C_2Y_0(kr)$，由$Y_0(0)=\infty$和$|u(0)|<\infty$, 得$C_2=0$，所以解$u(r)=C_1J_0(kr)$. 又由边界条件$u(1)=A$, 得

$$C_1J_0(k)=A\neq 0.$$

因为$k=\alpha_n, =1,2,\cdots$, 所以$J_0(k)=0$, 从而得到矛盾, 故边值问题无解. ······4分

(2) 由(1)的证明知，当$k\neq\alpha_n$, $n=1,2,\cdots$时$J_0(k)\neq 0$, 从而得

$$C_1=\frac{A}{J_0(k)}.$$

于是边值问题有唯一解, 并且解为$u(r)=A\dfrac{J_0(kr)}{J_0(k)}$. ·········8分

三 (13分) 用分离变量法求解下列在弹性力作用下的弦振动模型

$$\begin{cases} u_{tt} - a^2 u_{xx} + bu = 0, & 0 < x < l, t > 0, \\ u(0,t) = 0,\ u(l,t) = 0, & t > 0, \\ u(x,0) = \varphi(x),\ u_t(x,0) = \psi(x), & 0 \le x \le l, \end{cases}$$

其中a, b是正常数.

解: 设$U(x,t) = X(x)T(t)$为非零特解，则

$$(T''(t) + bT(t))X(x) = a^2 T(t) X''(x) \Longleftrightarrow \frac{T''(t) + bT(t)}{a^2 T(t)} = \frac{X''(x)}{X(x)} = -\lambda.$$

于是得常微分方程$X''(x) + \lambda X(x) = 0,\ T''(t) + (b + a^2\lambda)T(t) = 0$. 由齐次边界条件，得$X(0) = X(l) = 0$. ······························4 分

解特征值问题$\begin{cases} X''(x) + \lambda X(x) = 0, & 0 < x < l, \\ X(0) = X(l) = 0, \end{cases}$ 得

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2,\ X_n(x) = \sin\frac{n\pi x}{l},\ n = 1, 2, \cdots. \qquad \cdots\cdots\cdots\cdots 6 \text{ 分}$$

把$\lambda = \lambda_n$代入$T(t)$所满足的方程，记$\omega_n^2 = b + (\frac{n\pi a}{l})^2$, 得

$$T_n''(t) + \omega_n^2 T_n(t) = 0.$$

求得通解

$$T_n(t) = C_n \cos\omega_n t + D_n \sin\omega_n t,\ n = 1, 2, \cdots. \qquad \cdots\cdots\cdots\cdots\cdots\cdots 8\text{分}$$

于是得到形式解

$$u(x,t) = \sum_{n=1}^{\infty} [C_n \cos\omega_n t + D_n \sin\omega_n t] \sin\frac{n\pi x}{l}. \qquad \cdots\cdots\cdots\cdots\cdots\cdots 10 \text{ 分}$$

由初始条件，得

$$\begin{aligned} \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{l} &= u(x,0) = \varphi(x),\ 0 \le x \le l, \\ \sum_{n=1}^{\infty} D_n \omega_n \sin\frac{n\pi x}{l} &= u_t(x,0) = \psi(x),\ 0 \le x \le l, \end{aligned}$$

由此确定系数C_n, D_n:

$$C_n = \frac{2}{l}\int_0^l \varphi(x) \sin\frac{n\pi x}{l} \mathrm{d}x,\ D_n = \frac{2}{l\omega_n}\int_0^l \psi(x) \sin\frac{n\pi x}{l} \mathrm{d}x,\ n = 1, 2, \cdots. \qquad \cdots\cdots 13\text{分}$$

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四 (12分) 用Fourier变换法推导出定解问题

$$\begin{cases} u_t + au_x + bu = h(x,t), & -\infty < x < \infty,\ t > 0, \\ u(x,0) = g(x), & -\infty < x < \infty \end{cases}$$

的求解公式, 其中a, b是常数.

解：记$\hat{u}(\lambda,t) = \mathscr{F}[u(x,t)], \hat{h}(\lambda,t) = \mathscr{F}[h(x,t)], \hat{g}(\lambda) = \mathscr{F}[g(x)]$, 对方程和定解条件作Fourier变换，得

$$\frac{\mathrm{d}\hat{u}}{\mathrm{d}t} + (b + a\lambda i)\hat{u} = \hat{h}(\lambda,t),\ t > 0;\quad \hat{u}|_{t=0} = \hat{g}(\lambda). \qquad \cdots\cdots\cdots\cdots\cdots\cdots 4分$$

解此像函数所满足的初值问题，得

$$\hat{u}(\lambda) = \hat{g}(\lambda)\mathrm{e}^{-(b+a\lambda i)t} + \int_0^t \hat{h}(\lambda,s)\mathrm{e}^{-(b+a\lambda i)(t-s)}\mathrm{d}s. \qquad \cdots\cdots\cdots\cdots\cdots\cdots 8分$$

因为

$$\mathscr{F}^{-1}[\hat{g}(\lambda)\mathrm{e}^{-(b+a\lambda i)t}] = \mathrm{e}^{-bt}g(x-at),$$
$$\mathscr{F}^{-1}[\hat{h}(\lambda,s)\mathrm{e}^{-(b+a\lambda i)(t-s)}] = \mathrm{e}^{-b(t-s)}h(x-a(t-s),s),$$

所以作Fourier逆变换，得到解

$$\begin{aligned} u(x,t) &= \mathscr{F}^{-1}[\hat{g}(\lambda)\mathrm{e}^{-(b+a\lambda i)t}] + \int_0^t \mathscr{F}^{-1}[\hat{h}(\lambda,s)\mathrm{e}^{-(b+a\lambda i)(t-s)}]\mathrm{d}s \\ &= \mathrm{e}^{-bt}g(x-at) + \int_0^t \mathrm{e}^{-b(t-s)}h(x-a(t-s),s)\mathrm{d}s. \qquad \cdots\cdots\cdots\cdots 12分 \end{aligned}$$

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五 (13分) 利用Bessel级数及分离变量法理论求解下列半圆形薄膜的振动问题

$$\begin{cases} u_{tt} - a^2\Big(u_{rr} + \dfrac{1}{r}u_r + \dfrac{1}{r^2}u_{\theta\theta}\Big) = 0, & 0 < r < b,\ 0 < \theta < \pi, t > 0 \\ |u(0,\theta,t)| < \infty,\ u(b,\theta,t) = 0, & 0 < \theta < \pi, t > 0, \\ u(r,0,t) = u(r,\pi,t) = 0, & 0 \le r \le b, t > 0 \\ u(r,\theta,0) = 0, u_t(r,\theta,0) = r\sin\theta, & 0 \le r \le b, t \ge 0. \end{cases}$$

记$N_{nk}^2 = \int_0^b xJ_n^2(\alpha_k^{(n)}x/b)\mathrm{d}x = \dfrac{b^2}{2}J_{n+1}^2(\alpha_k^{(n)})$, $\alpha_k^{(n)}$是Bessel函数$J_n(x)$ 的第k个正零点.

解：设$U(r,\theta,t) = R(r)\Phi(\theta)T(t)$是非零特解，将其代入方程得

$$\frac{T''(t)}{a^2T(t)} = \frac{R''(r) + \frac{1}{r}R'(r)}{R(r)} + \frac{1}{r^2}\frac{\Phi''(\theta)}{\Phi(\theta)} = -\lambda.$$

从而有

$$T''(t) + a^2\lambda T(t) = 0,\ \Phi''(\theta) + \nu\Phi(\theta) = 0,\ r^2R''(r) + rR'(r) + (\lambda r^2 - \nu)R(r) = 0.$$

代入边界条件，得$\Phi(0) = \Phi(\pi) = 0,\ |R(0)| < \infty, R(b) = 0.$ ············4分

先求解特征值问题

$$\Phi''(\theta) + \nu\Phi(\theta) = 0, 0 < \theta < \pi;\ \Phi(0) = \Phi(\pi) = 0,$$

得

$$\nu_n = n^2,\ \Phi_n(\theta) = \sin n\pi\theta,\ n = 1, 2, \cdots.$$ ············6分

把$\nu = n^2$代入$R(r)$所满足的方程，得特征值问题

$$\begin{cases} r^2R''(r) + rR'(r) + (\lambda r^2 - n^2)R(r) = 0,\ 0 < r < b, \\ |R(0)| < \infty, R(b) = 0, \end{cases}$$

解此特征值问题，得

$$\lambda_{nk} = \Big(\frac{\alpha_k^{(n)}}{b}\Big)^2,\ R_{nk}(r) = J_n(\alpha_k^{(n)}r/b),\ k = 1, 2, \cdots.$$ ············8分

把$\lambda = \lambda_{nk}$代入$T(t)$所满足的方程，得$T''_{nk}(t) + \Big(\frac{\alpha_k^{(n)}a}{b}\Big)^2 T_{nk}(t) = 0, t > 0$,求得通解

$$T_{nk}(t) = C_{nk}\cos\frac{\alpha_k^{(n)}at}{b} + D_{nk}\sin\frac{\alpha_k^{(n)}at}{b},\ n, k = 1, 2, \cdots.$$ ············10分

于是得一般解

$$u(r,\theta,t) = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\Big[C_{nk}\cos\frac{\alpha_k^{(n)}at}{b} + D_{nk}\sin\frac{\alpha_k^{(n)}at}{b}\Big]J_n(\alpha_k^{(n)}r/b)\sin n\theta.$$

由初始条件，得

$$u(r,\theta,0)=\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}C_{nk}J_n(\alpha_k^{(n)}r/b)\sin n\theta=0,$$

$$u_t(r,\theta,0)=\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}D_{nk}\frac{\alpha_k^{(n)}a}{b}J_n(\alpha_k^{(n)}r/b)\sin n\theta=r\sin\theta.$$

由$\{\sin n\theta\}$在$[0,\pi]$上的正交性及$\{J_n(\alpha_k^{(n)}r/b)\}$在$[0,b]$上带权正交性，得

$$C_{nk}=0,\ n,k=1,2,\cdots;\ D_{nk}=0,\ n\neq 1.$$

当$n=1$时

$$\sum_{k=1}^{\infty}D_{1k}\frac{\alpha_k^{(1)}a}{b}J_1(\alpha_k^{(1)}r/b)=r,$$

由此得

$$D_{1k}=\frac{b}{\alpha_k^{(1)}N_{1k}^2}\int_0^b r^2J_1(\alpha_k^{(1)}r/b)\mathrm{d}r=\frac{2b^2}{(\alpha_k^{(1)})^2J_2(\alpha_k^{(1)})}.$$

于是求得解

$$u(r,\theta,t)=\sin\theta\sum_{k=1}^{\infty}\frac{2b^2}{(\alpha_k^{(1)})^2J_2(\alpha_k^{(1)})}J_1(\alpha_k^{(1)}r/b)\sin\frac{\alpha_k^{(1)}at}{b}.\qquad\cdots\cdots\cdots\cdots 13\text{分}$$

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解法II (前部分同) 因为$\{\sin n\theta\}$在$[0,\pi]$上正交，且初始条件仅与$\sin\theta$有关，所以只需取$n=1$，解特征值问题

$$\begin{cases} r^2R''(r)+rR'(r)+(\lambda r^2-1^2)R(r)=0,\ 0<r<b,\\ |R(0)|<\infty, R(b)=0,\end{cases}$$

得

$$\lambda_k=\left(\frac{\alpha_k^{(1)}}{b}\right)^2,\ R_k(r)=J_1(\alpha_k^{(1)}r/b),\ k=1,2,\cdots.\qquad\cdots\cdots\cdots\cdots 8\text{分}$$

再把$\lambda=\lambda_k$代入$T(t)$所满足的方程，得$T_k''(t)+\left(\frac{\alpha_k^{(1)}a}{b}\right)^2T_k(t)=0,t>0$,求得通解

$$T_k(t)=C_k\cos\frac{\alpha_k^{(1)}at}{b}+D_k\sin\frac{\alpha_k^{(1)}at}{b},\ k=1,2,\cdots.\qquad\cdots\cdots\cdots\cdots 10\text{分}$$

于是得一般解

$$u(r,\theta,t)=\sum_{k=1}^{\infty}\left[C_k\cos\frac{\alpha_k^{(1)}at}{b}+D_k\sin\frac{\alpha_k^{(1)}at}{b}\right]J_1(\alpha_k^{(1)}r/b)\sin\theta.$$

由初始条件，得

$$\sum_{k=1}^{\infty}C_kJ_1(\alpha_k^{(1)}r/b)\sin\theta=0,\quad \sum_{k=1}^{\infty}D_k\frac{\alpha_k^{(1)}a}{b}J_1(\alpha_k^{(1)}r/b)\sin\theta=r\sin\theta.$$

求得$C_k=0,\ D_k=\frac{b}{\alpha_k^{(1)}N_{1k}^2}\int_0^b r^2J_1(\alpha_k^{(1)}r/b)\mathrm{d}r=\frac{2b^2}{(\alpha_k^{(1)})^2J_2(\alpha_k^{(1)})}$. 所以解为

$$u(r,\theta,t)=\sin\theta\sum_{k=1}^{\infty}\frac{2b^2}{(\alpha_k^{(1)})^2J_2(\alpha_k^{(1)})}J_1(\alpha_k^{(1)}r/b)\sin\frac{\alpha_k^{(1)}at}{b}.\qquad\cdots\cdots\cdots\cdots 13\text{分}$$