Min-Max Programming Problem Subject to Addition-Min Fuzzy Relation Inequalities

Xiao-Peng Yang, Xue-Gang Zhou, and Bing-Yuan Cao

***Abstract*—A BitTorrent-like peer-to-peer file-sharing system can be reduced into a system of addition-min fuzzy relation inequalities. Addition-min is a new composition, and the solution set of addition-min fuzzy relation inequalities differs much from that of general max-t-norm fuzzy relation inequalities or equations. In order to avoid network congestion and improve the stability of data transmission, a min-max programming problem is proposed subject to addition-min fuzzy relation inequalities in this paper. Based on some relevant theorems on resolution, a novel algorithm is developed step by step to find an optimal solution of the proposed problem. Moreover, an application example is given to illustrate the feasibility and efficiency of the algorithm. Finally, some further discussions are made concerning the optimal solution of the proposed problem.**

***Index Terms*—Addition-min composition, bitTorrent-like P2P file-sharing system, fuzzy relation equation, fuzzy relation inequalities, min-max programming.**

I. Introduction

FUZZY relation equations with max-min composition were introduced by Sanchez [3], [4] and investigated by many fuzzy mathematics researchers. As an extension, fuzzy relation equations or inequalities with other compositions were studied. In fact, the composition could be replaced by the general max-t-norm composition, although max-min and max-product compositions were the most frequently and commonly used t-norms. Properties of the solution set and novel solution method were proposed and investigated [5]–[17]. When the solution set of a system of fuzzy relation equations with max-t-norm composition is nonempty, it is fully determined by a unique maximum solution and a finite number of lower (or minimal) solutions. It is easy to compute the maximum solution, while finding all of the lower solutions becomes much more difficult. However, some researchers have proposed effective solution methods and studied some relevant properties of lower solution [17]–[20].

Meanwhile, optimization problems were introduced and studied subject to fuzzy relation equations or inequalities. First, Wang *et al.* [21] investigated the latticized linear programming with max-min fuzzy relation inequalities constraint. The authors minimized the latticized linear objective function based on the lower solutions set obtained by conservative-path method. In [22], Li and Fang made some further studies on a latticized linear optimization problem. The similar optimization problem was also investigated by Li and Wang [40]. For solving the proposed problem, they introduced concept of semitensor product. A long-time linear programming problem with fuzzy relation constraints was a hot researching topic, and the solution method kept on improving [23]–[27]. Optimizing a general nonlinear function with fuzzy relation constraints was usually handled by a genetic algorithm [28]–[30]. In recent years, Yang and Cao proposed the geometric programming subject to fuzzy relation equations. The objective function in geometric programming is a special nonlinear one. A specific method was developed to the proposed problem [31]–[36]. After then, further research on fuzzy relation geometric programming problems was done by some other scholars [37]–[39].

Recently, Li and Yang [1] introduced a system of fuzzy relation inequalities with addition-min composition operator (addition-min fuzzy relation inequalities) for the first time. It was shown in [1] that the data transmission in BitTorrent-like peer-to-peer (P2P) file-sharing systems might be reduced into a system of addition-min fuzzy relation inequalities, which was described as

$$\begin{cases} a_{11} \wedge x_1 + a_{12} \wedge x_2 + \cdots + a_{1n} \wedge x_n \geq b_1 \\ a_{21} \wedge x_1 + a_{22} \wedge x_2 + \cdots + a_{2n} \wedge x_n \geq b_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1} \wedge x_1 + a_{m2} \wedge x_2 + \cdots + a_{mn} \wedge x_n \geq b_m. \end{cases} \quad (1)$$

In system (1), $a_{ij}, x_j \in [0,1]$, $b_i > 0$, $(i = 1, 2, \ldots, m, j = 1, 2, \ldots, n)$, and the operation "$+$" represents ordinary addition, $a_{ij} \wedge x_j = \min\{a_{ij}, x_j\}$. Furthermore, a_{ij} represents the bandwidth between ith user and jth user, x_j is the quality level on which the file data are sent from jth user, and b_i is the quality requirement of download traffic of ith user. Here, when $b_i = 0$, the ith inequality $a_{i1} \wedge x_1 + a_{i2} \wedge x_2 + \cdots + a_{\text{in}} \wedge x_n \geq b_i$ holds for any $x = (x_1, x_2, \ldots, x_n) \in [0,1]^n$. That is to say, when $b_i = 0$, the ith inequality can be deleted from system (1) without changing its solution set. Therefore, we always assume that $b_i > 0$, $i = 1, 2, \ldots, m$. Based on some properties and discussion of the addition-min fuzzy relation inequalities, an

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algorithm is proposed to find a lower solution, although the lower solution may not be unique.

In [2], Yang investigated the related optimization problem. A minimization problem with positive-coefficient linear objective function, i.e.,

$$z(x) = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n \tag{2}$$

and addition-min fuzzy relation inequality constraints was set up to describe some optimal management models in P2P file-sharing system. It is obvious that one of the lower solutions of feasible domain should be an optimal solution to the minimization problem. However, obtaining all of them is very difficult. After finding all the pseudominimal (or pseudolower) indexes of system of addition-min fuzzy relation inequalities by a so-called PMI algorithm, the minimization problem was decomposed into t subproblems (if there are t pseudominimal indexes). Each of the subproblems was then converted into a linear programming and solved. Optimal solutions of the subproblems were obtained, and the optimal solution of the minimization problem was selected from them by pairwise comparison.

Considering the important applications of addition-min fuzzy relation inequalities in BitTorrent-like P2P file-sharing systems, we study a relevant optimization problem in this paper. The rest of the paper is organized as follows. In Section II, we present some basic definitions and existing results of the addition-min fuzzy inequalities. An optimization problem subject to addition-min fuzzy relation inequalities and the related analysis are given in Section III. The resolution method and a step-by-step algorithm for the proposed problem are developed in Section IV. Section V provides an application example. Further discussions and simple conclusions are arranged in Sections VI and VII, respectively.

II. Preliminaries

Let $I = \{1, 2, \ldots, m\}$ and $J = \{1, 2, \ldots, n\}$ be two index sets; then, system (1) can be written as

$$\sum_{j\in J} a_{ij} \wedge x_j \geq b_i, \quad \forall i \in I \tag{1}$$

or

$$A \odot x^T \geq b^T \tag{1}$$

where $A = (a_{ij})_{m\times n}$, $x = (x_1, x_2, \ldots, x_n)$, $b = (b_1, b_2, \ldots, b_m)$, and

$$(a_{i1}, a_{i2}, \ldots, a_{\text{in}}) \odot (x_1, x_2, \ldots, x_n)^T$$
$$= a_{i1} \wedge x_1 + a_{i2} \wedge x_2 + \cdots + a_{\text{in}} \wedge x_n.$$

Definition 1 (See [1] and [2]): Denote $X = [0, 1]^n$. Let $x^1 = (x_1^1, x_2^1, \ldots, x_n^1)$, $x^2 = (x_1^2, x_2^2, \ldots, x_n^2) \in X$, we define:

i) $x^1 \leq x^2$ if $x_j^1 \leq x_j^2, \forall j \in J$.

ii) $x^1 < x^2$ if $x^1 \leq x^2$ and there are some $j \in J$ such that $x_j^1 < x_j^2$.

In what follows, we shall denote the dual of order relation "$<$" and "$\leq$" by the symbol "$>$" and "$\geq$," respectively. Obviously, the operator "$\leq$" forms a partial order relation on X and $(X, \leq)$ becomes a lattice.

We denote the solution set of system (1) by $X(A, b) = \{x \in X | A \odot x^T \geq b^T\}$.

Definition 2: A solution $\hat{x} \in X(A, b)$ is said to be the maximum (or greatest) solution of system (1) if and only if $x \leq \hat{x}$ for all $x \in X(A, b)$. A solution $\check{x} \in X(A, b)$ is said to be a lower (or minimal) solution of system (1) if and only if $x \leq \check{x}$ implies $x = \check{x}$ for any $x \in X(A, b)$. A solution $\dot{\check{x}} \in X(A, b)$ is said to be the minimum solution of system (1) if and only if $x \geq \dot{\check{x}}$ for all $x \in X(A, b)$.

Definition 3: System (1) is said to be consistent if $X(A, b) \neq \emptyset$. Otherwise, it is said to be inconsistent.

Obviously, when system (1) is consistent, $\hat{x} = (1, 1, \ldots, 1)$ is the maximum solution. As shown in [2], if $X(A, b) \neq \emptyset$, then $X(A, b) = \bigcup_{\check{x}\in \check{X}(A,b)} \{x | \check{x} \leq x \leq \hat{x}\}$, where $\check{X}(A, b)$ is the set of all lower solutions of system (1). Now, we introduce some properties and existing results on system (1).

Theorem 1 (See [1] and [2]): For system (1), we have the following.

i) (1) is consistent if and only if $\sum_{j\in J} a_{ij} \geq b_i$ for arbitrary $i \in I$.

ii) Let $x^* \in X(A, b)$, $x \in X$. $x^* \leq x$ implies $x \in X(A, b)$.

iii) Let $x', x \in X$ and $x \leq x'$. $x' \notin X(A, b)$ implies $x \notin X(A, b)$.

iv) Let $x \in X(A, b)$, if $\sum_{j\in J} a_{ij} = b_i$ for some $i \in I$, then $(a_{i1}, a_{i2}, \ldots, a_{\text{in}}) \leq x$.

Theorem 2 (See [1] and [2]): Let $x \in X(A, b)$ be a solution of system (1); then, we have the following.

i) $x > 0$.

ii) For arbitrary $i \in I$, $j \in J$,

$$x_j \geq b_i - \sum_{k\in J-\{j\}} a_{ik} \wedge x_k \geq b_i - \sum_{k\in J-\{j\}} a_{ik}.$$

iii) For arbitrary $i \in I$, $j \in J$,

$$a_{ij} \geq b_i - \sum_{k\in J-\{j\}} a_{ik} \wedge x_k \geq b_i - \sum_{k\in J-\{j\}} a_{ik}.$$

Let $\dot{\check{x}} = (\dot{\check{x}}_1, \dot{\check{x}}_2, \ldots, \dot{\check{x}}_n)$, where

$$\dot{\check{x}}_j = \max_{i\in I} \left\{0, b_i - \sum_{k\in J-\{j\}} a_{ik}\right\} \tag{3}$$

$j = 1, 2, \ldots, n$. Then, the uniqueness of the lower solution of system (1) can be checked by the following theorem.

Theorem 3 (See [2]): System (1) has the unique lower solution if and only if $\dot{\check{x}}$ is a solution of (1), i.e., $\dot{\check{x}} \in X(A, b)$. In particular, when (1) has the unique lower solution, $\dot{\check{x}}$ is the unique lower solution of system (1).

Corollary 1: $\dot{\check{x}}$ is the minimum solution if and only if $\dot{\check{x}}$ is a solution of system (1).

From Theorem 3 and Corollary 1, we know that $\dot{\check{x}}$ is the potential minimum solution of system (1). The minimum solution does not always exist, but if it does so, it must be $\dot{\check{x}}$.

In the rest of the paper, we always assume that $X(A, b) \neq \emptyset$, unless it is pointed out in a special case.

III. Min-Max Programming Problem Subject to Addition-Min Fuzzy Relation Inequalities

Although the BitTorrent-like P2P transmission mechanism is an effective transmission mode, network congestion may appear when it is used to conduct a large-scale data transmission, such as a live broadcast of the Olympic tournament [2]. Lower data transmission quality levels will keep the data transmission more stable and avoid the network congestion. In order to avoid network congestion and keep the data transmitting stable, in the reduced BitTorrent-like P2P file-sharing system, the management operator would like to minimize the quality levels $x_1, x_2, \dots, x_n$, with the constraint that satisfying the users' quality requirements of download traffic. In [2], the authors aimed to minimize $x_1 + x_2 + \cdots + x_n$ as well as its general form $c_1x_1 + c_2x_2 + \cdots + c_nx_n$. However, although the total quality levels, i.e., $x_1^* + x_2^* + \cdots + x_n^*$, reach the minimum value, some of the quality levels, such as $x_{j_0}^*$ ($j_0 \in J$), might be much bigger than the other ones. In this case, network congestion will appear to the j_0th user. Under this consideration, minimizing the linear objective function mentioned above is not always the best choice. In order to avoid such situation, we replace the objective by a min-max function. That is to say, we will minimize the biggest quality levels, i.e., $g(x) = x_1 \vee x_2 \vee \cdots \vee x_n$, in this paper. Hence, we establish the following optimization model:

$$\begin{aligned} &\min g(x) = x_1 \vee x_2 \vee \cdots \vee x_n \\ &\text{s.t. } A \odot x^T \geq b^T. \end{aligned} \tag{4}$$

This is a min-max programming problem subject to addition-min fuzzy relation inequalities. It will minimize the biggest quality level.

The set of all optimal solutions of problem (4) is denoted by $X^*(A, b)$.

The feasible domain of problem (4) is $X(A, b) \neq \emptyset$. Since $X(A, b)$ is a bounded set and the objective function is $g(x) = x_1 \vee x_2 \vee \cdots \vee x_n$, the optimal solution of (4) must exist. Next, we aim to find an optimal solution of problem (4).

Theorem 4: There exists a lower solution $\check{x}$ of system (1) such that $\check{x}$ is an optimal solution of (4).

Proof: Take an arbitrary optimal solution $x^* = (x_1^*, x_2^*, \dots, x_n^*) \in X^*(A, b)$ of (4). If $x^* \in \check{X}(A, b)$, the proof is completed. Otherwise, $x^* \notin \check{X}(A, b)$. Since $x^* \in X(A, b) = \bigcup_{\check{x} \in \check{X}(A,b)} \{x | \check{x} \leq x \leq \hat{x}\}$, there exists $\check{x} = (\check{x}_1, \check{x}_2, \dots, \check{x}_n) \in \check{X}(A, b)$ such that $\check{x} \leq x^* \leq \hat{x}$. Hence

$$g(\check{x}) = \check{x}_1 \vee \check{x}_2 \vee \cdots \vee \check{x}_n \leq x_1^* \vee x_2^* \vee \cdots \vee x_n^* = g(x^*). \tag{5}$$

If $g(\check{x}) = g(x^*)$, then $\check{x}$ is an optimal solution of (4), and the proof is complete. Otherwise, $g(\check{x}) < g(x^*)$. However, $\check{x}$ is a feasible solution and x^* is an optimal solution of (4). Therefore, we get $g(x^*) \leq g(\check{x})$. This causes conflicts and the proof is complete. □

According to Theorem 4, it is clear that one of the optimal solutions of (4) can be selected from the lower solutions of system (1). However, the objective function, i.e., $g(x) = x_1 \vee x_2 \vee \cdots \vee x_n$, is nonlinear, and there are probably infinite lower solutions of system (1) (see Example 1). Therefore, it is very difficult to find an optimal solution by selecting it from the lower solutions. In the following, we will provide a novel method to deal with problem (4).

Example 1: Consider the following system of addition-min fuzzy relation inequalities:

$$\begin{cases} 0.7 \wedge x_1 + 0.9 \wedge x_2 \geq 1.4 \\ 0.8 \wedge x_1 + 0.5 \wedge x_2 \geq 1.1. \end{cases} \tag{6}$$

By (3), we obtain $\dot{\check{x}} = (0.6, 0.7)$. $0.7 \wedge 0.6 + 0.9 \wedge 0.7 = 1.3 \leq 1.4$; system (6) has more than one lower solution. In fact, the lower solution set of system (6) is $\check{X}(A, b) = \{x = (x_1, x_2) | x_1 + x_2 = 1.4, 0.6 \leq x_1 \leq 0.7\}$. It is an infinite set.

Theorem 5: If $\dot{\check{x}}$ is a solution of system (1), then

i) $\dot{\check{x}}$ is an optimal solution of problem (4);

ii) the optimal solution set of problem (4) is $X^*(A, b) = \{x \in X | \dot{\check{x}} \leq x \leq \overline{x}^*\}$, where $\overline{x}^* = (\overline{x}_1^*, \overline{x}_2^*, \cdots, \overline{x}_n^*)$, $\overline{x}_1^* = \overline{x}_2^* = \cdots = \overline{x}_n^* = \bigvee_{j \in J} \dot{\check{x}}_j$.

Proof: i) According to Theorem 3, if $\dot{\check{x}}$ is a solution of system (1), then $\dot{\check{x}}$ is the unique lower solution of system (1). Furthermore, $\dot{\check{x}}$ is an optimal solution of problem (4) following Theorem 4.

ii) First, let x be an arbitrary element in $X^*(A, b)$. Since $\dot{\check{x}}$ is a solution of system (1), x is also a solution of system (1) according to Theorem 1. Besides, $g(\dot{\check{x}}) \leq g(x) \leq g(\overline{x}^*)$. However $g(\dot{\check{x}}) = g(\overline{x}^*) = \bigvee_{j \in J} \dot{\check{x}}_j$. So we have $g(x) = \bigvee_{j \in J} \dot{\check{x}}_j = g(\dot{\check{x}})$. Therefore, x is an optimal solution of problem (4).

Second, let x be an optimal solution of problem (4). According to Corollary 1, $\dot{\check{x}}$ the minimum solution, which indicates $\dot{\check{x}} \leq x$. In addition, $x_j \leq \bigvee_{j \in J} x_j = g(x) = g(\dot{\check{x}}) = \bigvee_{j \in J} \dot{\check{x}}_j$, for any $j \in J$, i.e. $x \leq \dot{\check{x}}$. Hence, we have $x \in X^*(A, b)$. □

For solving problem (4) in case that $\dot{\check{x}}$ is not a solution of system (1), we convert it into a single-variable optimization problem, i.e.,

$$\begin{aligned} &(\mathrm{P}_0) \min z(x_0) = x_0 \\ &\text{s.t.} \begin{cases} a_{11} \wedge x_0 + a_{12} \wedge x_0 + \cdots + a_{1n} \wedge x_0 \geq b_1 \\ a_{21} \wedge x_0 + a_{22} \wedge x_0 + \cdots + a_{2n} \wedge x_0 \geq b_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1} \wedge x_0 + a_{m2} \wedge x_0 + \cdots + a_{mn} \wedge x_0 \geq b_m \end{cases} \end{aligned} \tag{7}$$

where $x_0 \in [0, 1]$. Furthermore, problem (P_0) is decomposed into m subproblems as follows:

$$\begin{aligned} &(\mathrm{P}_0^i) \min \ z(x_0) = x_0 \\ &\quad \text{s.t. } a_{i1} \wedge x_0 + a_{i2} \wedge x_0 + \cdots + a_{in} \wedge x_0 \geq b_i \end{aligned} \tag{8}$$

where $x_0 \in [0, 1]$, $i = 1, 2, \dots, m$.

IV. Resolution of Problem (4)

A. Solving Subproblem (P_0^i)

In this section, we discuss the existence and uniqueness of the optimal solution to problem (P_0^i) in Theorem 6. Furthermore, a solution formula is given to this problem in Theorem 7. At last, based on Theorems 6 and 7, we develop an algorithm to find the optimal solution of (P_0^i), with an illustrative numerical example.

Theorem 6: In subproblem (P_0^i), we have the following.

i) (P_0^i) has no optimal solution if and only if $a_{i1} + a_{i2} + \cdots + a_{in} < b_i$.

ii) (P_0^i) has a unique optimal solution if and only if $a_{i1} + a_{i2} + \cdots + a_{in} \geq b_i$.

Proof: i) (Necessity). Suppose (P_0^i) has no optimal solution. (Proof by contradiction) If $a_{i1} + a_{i2} + \cdots + a_{in} < b_i$ does not hold, then $a_{i1} + a_{i2} + \cdots + a_{in} \geq b_i$. We have

$$\begin{aligned} &a_{i1} \wedge 1 + a_{i2} \wedge 1 + \cdots + a_{in} \wedge 1 \\ &= a_{i1} + a_{i2} + \cdots + a_{in} \geq b_i. \end{aligned} \tag{9}$$

This indicates that 1 is a feasible solution of problem (P_0^i). Denote the feasible domain of (P_0^i) by D_0^i. Then, $1 \in D_0^i$, $D_0^i \neq \emptyset$. Obviously, $D_0^i \subseteq [0,1]$ is a bounded set. There exists a greatest lower bound $x_0^{i*} \in D_0^i$ such that $x_0^{i*} \leq y$ for any $y \in D_0^i$. Thus, x_0^{i*} is an optimal solution of (P_0^i). This brings about a contradiction.

(Sufficiency). Suppose $a_{i1} + a_{i2} + \cdots + a_{in} < b_i$. Then, for any $x_0 \in [0,1]$,

$$\begin{aligned} &a_{i1} \wedge x_0 + a_{i2} \wedge x_0 + \cdots + a_{in} \wedge x_0 \\ &\leq a_{i1} + a_{i2} + \cdots + a_{in} < b_i. \end{aligned} \tag{10}$$

Hence, $D_0^i = \emptyset$ and (P_0^i) has no optimal solution.

ii) The converse-negative proposition of (i) is "(P_0^i) has at least one optimal solution if and only if $a_{i1} + a_{i2} + \cdots + a_{in} \geq b_i$." In order to complete the proof of (ii), we just need to verify that the optimal solution of (P_0^i) is unique when it exists. Suppose both x_0^{i*} and y_0^{i*} are optimal solutions of (P_0^i). According to its definition, we have $z(x_0^{i*}) = x_0^{i*} \leq y_0^{i*} = z(y_0^{i*})$ and $z(y_0^{i*}) = y_0^{i*} \leq x_0^{i*} = z(x_0^{i*})$. Hence, $y_0^{i*} = x_0^{i*}$. This verifies the uniqueness of the optimal solution. □

For convenience in solving (P_0^i), we assume that $a_{i1} \leq a_{i2} \leq \cdots \leq a_{in}$. In fact, if the inequality $a_{i1} \leq a_{i2} \leq \cdots \leq a_{in}$ does not hold, we can exchange the positions of $a_{i1}, a_{i2}, \ldots, a_{in}$ in the constraint $a_{i1} \wedge x_0 + a_{i2} \wedge x_0 + \cdots + a_{in} \wedge x_0 \geq b_i$.

If $a_{i1} + a_{i2} + \cdots + a_{in} \geq b_i$, then $b_i \in (0, a_{i1} + a_{i2} + \cdots + a_{in}]$. Since $a_{i1} \leq a_{i2} \leq \cdots \leq a_{in}$, it is easy to verify that $0 \leq na_{i1} \leq a_{i1} + (n-1)a_{i2} \leq a_{i1} + a_{i2} + (n-2)a_{i3} \leq \cdots \leq a_{i1} + a_{i2} + \cdots + a_{in}$. Rewrite $(0, a_{i1} + a_{i2} + \cdots + a_{in}]$ as the following form:

$$\begin{aligned} &(0, a_{i1} + a_{i2} + \cdots + a_{in}] \\ =\ &(0, na_{i1}] \cup (na_{i1}, a_{i1} + (n-1)a_{i2}] \\ &\cup (a_{i1} + (n-1)a_{i2}, a_{i1} + a_{i2} + (n-2)a_{i3}] \cup \cdots \\ &\cup (a_{i1} + \cdots + a_{i(n-2)} + 2a_{i(n-1)}, a_{i1} + \cdots + a_{in}]. \end{aligned} \tag{11}$$

Denote

$$\Delta_k = \begin{cases} (a_{i1} + \cdots + a_{i(n-k-1)} + (k+1)a_{i(n-k)} \\ a_{i1} + \cdots + a_{i(n-k)} + ka_{i(n-k+1)}], & k \neq n, \\ (0, na_{i1}], & k = n \end{cases} \tag{12}$$

where $k \in J = \{1, 2, \ldots, n\}$. Then, $b_i \in (0, a_{i1} + a_{i2} + \cdots + a_{in}] = \Delta_n \cup \Delta_{n-1} \cup \cdots \cup \Delta_1$. Furthermore, there exists a unique $n_0 \in J$ such that $b_i \in \Delta_{n_0}$.

Theorem 7: Suppose $a_{i1} + a_{i2} + \cdots + a_{in} \geq b_i$ and $b_i \in \Delta_{n_0}$, $n_0 \in J$. Then

$$x_0^{i*} = \begin{cases} \dfrac{b_i - a_{i1} - \cdots - a_{i(n-n_0)}}{n_0}, & n_0 = 1, 2, \ldots, n-1 \\ \dfrac{b_i}{n}, & n_0 = n \end{cases} \tag{13}$$

is the unique optimal solution of subproblem (P_0^i).

Proof (Feasibility): We will verify the feasibility of x_0^{i*}.

Case 1: If $n_0 = n$, then $b_i \in \Delta_{n_0} = \Delta_n = (0, na_{i1}]$, i.e., $0 < b_i \leq na_{i1}$, and $x_0^{i*} = \frac{b_i}{n}$. Therefore, we have

$$0 < x_0^{i*} = \frac{b_i}{n} \leq a_{i1} \leq a_{i2} \leq \cdots \leq a_{in} \leq 1. \tag{14}$$

Based on Inequality (14), it is easy to check that

$$\begin{aligned} &a_{i1} \wedge x_0^{i*} + a_{i2} \wedge x_0^{i*} + \cdots + a_{in} \wedge x_0^{i*} \\ =\ &x_0^{i*} + x_0^{i*} + \cdots + x_0^{i*} \\ =\ &nx_0^{i*} \\ =\ &b_i. \end{aligned} \tag{15}$$

Hence, x_0^{i*} is a feasible solution of subproblem (P_0^i).

Case 2: If $n_0 \neq n$, then $x_0^{i*} = \frac{b_i - a_{i1} - \cdots - a_{i(n-n_0)}}{n_0}$ and

$$\begin{aligned} b_i \in \Delta_{n_0} = &(a_{i1} + \cdots + a_{i(n-n_0-1)} + (n_0+1)a_{i(n-k)}, \\ &a_{i1} + \cdots + a_{i(n-n_0)} + ka_{i(n-n_0+1)}]. \end{aligned} \tag{16}$$

Following (16), it is obvious that

$$b_i - (a_{i1} + \cdots + a_{i(n-n_0-1)} + (n_0+1)a_{i(n-n_0)}) > 0 \tag{17}$$

and

$$b_i - (a_{i1} + \cdots + a_{i(n-n_0)} + n_0 a_{i(n-n_0+1)}) \leq 0. \tag{18}$$

For $j = 1, 2, \ldots, n - n_0$, following the assumption "$a_{i1} \leq a_{i2} \leq \cdots \leq a_{in}$," we have $a_{ij} \leq a_{i(n-n_0)}$. According to Inequality (17)

$$\begin{aligned} x_0^{i*} - a_{ij} &= \frac{b_i - a_{i1} - \cdots - a_{i(n-n_0)}}{n_0} - a_{ij} \\ &\geq \frac{b_i - a_{i1} - \cdots - a_{i(n-n_0)}}{n_0} - a_{i(n-n_0)} \\ &= \frac{b_i - a_{i1} - \cdots - a_{i(n-n_0)} - n_0 a_{i(n-n_0)}}{n_0} \\ &= \frac{b_i - (a_{i1} + \cdots + a_{i(n-n_0-1)} + (n_0+1)a_{i(n-n_0)})}{n_0} \\ &> 0. \end{aligned} \tag{19}$$

Similarly, for $j = n - n_0 + 1, n - n_0 + 2, \ldots, n$, we have $a_{ij} \geq a_{i(n-n_0+1)}$. According to Inequality (18),

$$\begin{aligned} x_0^{i*} - a_{ij} &= \frac{b_i - a_{i1} - \cdots - a_{i(n-n_0)}}{n_0} - a_{ij} \\ &\leq \frac{b_i - a_{i1} - \cdots - a_{i(n-n_0)}}{n_0} - a_{i(n-n_0+1)} \\ &= \frac{b_i - a_{i1} - \cdots - a_{i(n-n_0)} - n_0 a_{i(n-n_0+1)}}{n_0} \\ &= \frac{b_i - (a_{i1} + \cdots + a_{i(n-n_0)} + n_0 a_{i(n-n_0+1)})}{n_0} \\ &\leq 0. \end{aligned} \tag{20}$$

That is to say, $x_0^{i*} > a_{ij}$ holds for $j = 1, 2, \ldots, n - n_0$ and $x_0^{i*} \leq a_{ij}$ holds for $j = n - n_0 + 1, n - n_0 + 2, \ldots, n$. Therefore, we get

$$\begin{aligned} & a_{i1} \wedge x_0^{i*} + a_{i2} \wedge x_0^{i*} + \cdots + a_{in} \wedge x_0^{i*} \\ &= a_{i1} + a_{i2} + \cdots + a_{i(n-n_0)} + x_0^{i*} + \cdots + x_0^{i*} \\ &= a_{i1} + a_{i2} + \cdots + a_{i(n-n_0)} + n_0 \cdot x_0^{i*} \\ &= a_{i1} + a_{i2} + \cdots + a_{i(n-n_0)} + n_0 \\ &\quad \cdot \frac{b_i - a_{i1} - \cdots - a_{i(n-n_0)}}{n_0} \\ &= b_i. \end{aligned} \tag{21}$$

Furthermore,

$$\begin{aligned} x_0^{i*} &= \frac{b_i - a_{i1} - \cdots - a_{i(n-n_0)}}{n_0} \\ &\leq \frac{a_{i(n-n_0+1)} + \cdots + a_{in}}{n_0} \leq \frac{n_0 \cdot a_{in}}{n_0} = a_{in} \leq 1 \\ x_0^{i*} &= \frac{b_i - a_{i1} - \cdots - a_{i(n-n_0)}}{n_0} \\ &\geq \frac{n_0 \cdot a_{i(n-n_0)}}{n_0} = a_{i(n-n_0)} \geq 0. \end{aligned} \tag{22}$$

Hence, x_0^{i*} is a feasible solution of subproblem (P_0^i).

(Optimality). Suppose y is an arbitrary feasible solution of (P_0^i). Then, $a_{i1} \wedge y + a_{i2} \wedge y + \cdots + a_{in} \wedge y \geq b_i$. In order to complete the proof, we have to verify that $z(x_0^{i*}) = x_0^{i*} \leq y = z(y)$. In fact, this is true. Otherwise, if $y < x_0^{i*}$, then

$$\begin{aligned} & a_{i1} \wedge y + a_{i2} \wedge y + \cdots + a_{in} \wedge y \\ &\leq a_{i1} + \cdots + a_{i(n-n_0)} + y \cdots + y \\ &< a_{i1} + \cdots + a_{i(n-n_0)} + x_0^{i*} \cdots + x_0^{i*} \\ &= b_i \end{aligned} \tag{23}$$

which leads to a contradiction.

(Uniqueness). This is due to Theorem 6. □

Based on Theorems 6 and 7, we develop the following Algorithm 1 for obtaining the unique optimal solution of (P_0^i).

Algorithm 1 (For solving subproblem (P_0^i))

Step 1: Check the feasibility of (P_0^i). If $a_{i1} + a_{i2} + \cdots + a_{in} < b_i$, then (P_0^i) has no optimal solution and stop. Otherwise, $b_i \in (0, a_{i1} + a_{i2} + \cdots + a_{in}]$, go to Step 2.

Step 2: Reorder $\{a_{i1}, a_{i2}, \ldots, a_{in}\}$ such that $a_{i1} \leq a_{i2} \leq \cdots \leq a_{in}$.

Step 3: Rewrite the interval

$$(0, a_{i1} + a_{i2} + \cdots + a_{in}] = \Delta_n \cup \Delta_{n-1} \cup \cdots \cup \Delta_1$$

according to (11) and (12), and find n_0 such that $b_i \in \Delta_{n_0}$, $n_0 \in J$.

Step 4: Take

$$x_0^{i*} = \begin{cases} \dfrac{b_i - a_{i1} - \cdots - a_{i(n-n_0)}}{n_0}, & n_0 = 1, 2, \ldots, n-1 \\ \dfrac{b_i}{n}, & n_0 = n. \end{cases}$$

Then, x_0^{i*} is the unique optimal solution of subproblem (P_0^i).

Example 2: Find the unique optimal solution of the following problem:

$$\begin{aligned} \min \quad & z(x_0) = x_0 \\ \text{s.t.} \quad & 0.5 \wedge x_0 + 0 \wedge x_0 + 0.7 \wedge x_0 + 0.9 \wedge x_0 \\ & + 0.8 \wedge x_0 + 0.5 \wedge x_0 \geq 3.0, \end{aligned} \tag{24}$$

where $x_0 \in [0, 1]$.

Solution:

Step 1: Check the feasibility of problem (24). Since $0.5 + 0 + 0.7 + 0.9 + 0.8 + 0.5 = 3.4 > 3.0$, problem (24) is feasible and go to Step 2.

Step 2: Reorder $\{0.5, 0, 0.7, 0.9, 0.8, 0.5\}$. We get the equivalent problem as follows:

$$\begin{aligned} \min \quad & z(x_0) = x_0 \\ \text{s.t.} \quad & 0 \wedge x_0 + 0.5 \wedge x_0 + 0.5 \wedge x_0 + 0.7 \wedge x_0 \\ & + 0.8 \wedge x_0 + 0.9 \wedge x_0 \geq 3.0. \end{aligned} \tag{25}$$

Let $(a_{11}, a_{12}, a_{13}, a_{14}, a_{15}, a_{16}) = (0, 0.5, 0.5, 0.7, 0.8, 0.9)$, $b_1 = 3.0$.

Step 3: Rewrite the interval $(0, 3.4]$ by (11) and (12). $(0, 3.4] = (0, 0] \cup (0, 2.5] \cup (2.5, 2.5] \cup (2.5, 3.1] \cup (3.1, 3.3] \cup (3.3, 3.4]$. Obviously, $b_1 = 3.0 \in (2.5, 3.1] = (a_{11} + a_{12} + 4a_{13}, a_{11} + a_{12} + a_{13} + 3a_{14}]$.

Step 4: The unique optimal solution of problem (24) is

$$x_0^* = \frac{b_1 - a_{11} - a_{12} - a_{13}}{3} = \frac{3.0 - 0 - 0.5 - 0.5}{3} = 0.6667.$$

B. Solving Problem (P_0)

In this section, for solving the single-variable problem (P_0), we provide two theorems in the following.

Theorem 8: In subproblem (P_0), we have the following.

i) (P_0) has no optimal solution if and only if there exists $i_0 \in I$, such that $a_{i_0 1} + a_{i_0 2} + \cdots + a_{i_0 n} < b_{i_0}$.

ii) (P_0) has a unique optimal solution if and only if $a_{i1} + a_{i2} + \cdots + a_{in} \geq b_i$ for all $i \in I$.

Proof: Since (P_0) is a single variable optimization problem, the uniqueness is self-evident. Based on the conclusion in Theorem 9 below and the proof of Theorem 6, the rest of the proof is trivial. □

Theorem 9: Suppose $a_{i1} + a_{i2} + \cdots + a_{in} \geq b_i$ for all $i \in I$, and the optimal solution of (P_0^i) is $x_0^{i*}, i \in I$. Then the optimal solution of (P_0) is $x_0^* = \bigvee_{i \in I} x_0^{i*}$.

Proof (Feasibility): Obviously, $x_0^* \geq x_0^{i*}$ holds for any $i \in I$. Since x_0^{i*} is the optimal solution of (P_0^i), we have

$$\begin{aligned} & a_{i1} \wedge x_0^* + a_{i2} \wedge x_0^* + \cdots + a_{in} \wedge x_0^* \\ \geq\ & a_{i1} \wedge x_0^{i*} + a_{i2} \wedge x_0^{i*} + \cdots + a_{in} \wedge x_0^{i*} \\ \geq\ & b_i \end{aligned} \tag{26}$$

$\forall i \in I$. Hence, x_0^* is a feasible solution of (P_0).

(Optimality): Suppose y is an arbitrary feasible solution of (P_0), then

$$a_{i1} \wedge y + a_{i2} \wedge y + \cdots + a_{in} \wedge y \geq b_i \quad \forall i \in I. \tag{27}$$

Thus, y is a feasible solution of (P_0^i), $\forall i \in I$. However, x_0^{i*} is the optimal solution of (P_0^i), $\forall i \in I$. Therefore, we have $z(y) = y \geq x_0^{i*} = z(x_0^{i*})$, $\forall i \in I$. Consequently, $y \geq \bigvee_{i \in I} x_0^{i*} = x_0^*$. $\square$

C. Solving Problem (4)

The relationship between problem (P_0) and problem (4) is shown in Theorem 10, which contributes to the resolution of problem (4). Besides, a novel algorithm for obtaining an optimal solution of problem (4) is given in this section.

Theorem 10: Let $x^* = (x_0^*, x_0^*, \ldots, x_0^*)$. Then, x_0^* is the unique optimal solution of (P_0) if and only if x^* is an optimal solution of problem (4).

Proof: $(\Rightarrow)$ Obviously, x_0^* satisfies the constraints of (P_0). Thus, $x^* = (x_0^*, x_0^*, \ldots, x_0^*)$ satisfies the constraints of problem (4), and it is a feasible solution of problem (4). Suppose $y = (y_1, y_2, \ldots, y_n)$ is an arbitrary feasible solution of problem (4), i.e., $y \in X(A, b)$. If we can prove that

$$y_1 \vee y_2 \vee \cdots \vee y_n \geq x_0^{i*} \quad \forall i \in I$$

then

$$y_1 \vee y_2 \vee \cdots \vee y_n \geq \bigvee_{i \in I} x_0^{i*} = x_0^*$$

i.e., $g(y) \geq g(x^*)$, and the proof is complete. Next, we aim to verify that $y_1 \vee y_2 \vee \cdots \vee y_n \geq x_0^{i*}, \forall i \in I$.

(Proof by contradiction) Otherwise, assume that $y_1 \vee y_2 \vee \cdots \vee y_n < x_0^{i*}$ for some $i \in I$. Let $\bar{y} = y_1 \vee y_2 \vee \cdots \vee y_n$. Then, $\bar{y} \geq y_j, \forall j \in J$. Thus

$$\begin{aligned} & a_{i1} \wedge \bar{y} + a_{i2} \wedge \bar{y} + \cdots + a_{in} \wedge \bar{y} \\ & \geq a_{i1} \wedge y_1 + a_{i2} \wedge y_2 + \cdots + a_{in} \wedge y_n. \end{aligned} \tag{28}$$

Furthermore, $y \in X(A, b)$ implies

$$a_{i1} \wedge y_1 + a_{i2} \wedge y_2 + \cdots + a_{in} \wedge y_n \geq b_i. \tag{29}$$

Inequalities (28) and (29) imply that

$$a_{i1} \wedge \bar{y} + a_{i2} \wedge \bar{y} + \cdots + a_{in} \wedge \bar{y} \geq b_i. \tag{30}$$

This indicates $\bar{y}$ is a feasible solution of problem (P_0). Hence, $z(\bar{y}) \geq z(x_0^{i*})$, i.e., $y_1 \vee y_2 \vee \cdots \vee y_n \geq x_0^{i*}$, which leads to a contradiction.

$(\Leftarrow)$ If $x^* = (x_0^*, x_0^*, \ldots, x_0^*)$ is an optimal solution of problem (4), then $(x_0^*, x_0^*, \ldots, x_0^*)$ satisfies the constraints of problem (4), i.e., x_0^* satisfies the constraints of problem (P_0). Thus, x_0^* is a feasible solution of problem (P_0).

Suppose y_0 is an arbitrary feasible solution of problem (P_0). Similarly, $y = (y_0, y_0, \ldots, y_0)$ is a feasible solution of problem (4). Since x^* is an optimal solution of problem (4), we have $g(y) \geq g(x^*)$, i.e., $y_0 \geq x_0^*$. Hence, $z(y_0) \geq z(x_0^*)$. x_0^* is an optimal solution of problem (P_0). $\square$

For checking the feasibility of problems (P_0) and (4), we denote

$$a^{\Sigma} = \left(\sum_{j=1}^{n} a_{1j}, \sum_{j=1}^{n} a_{2j}, \ldots, \sum_{j=1}^{n} a_{mj} \right). \tag{31}$$

It is easy to verify that problem (4) (or problem (P_0)) has an optimal solution if and only if $a^{\Sigma} \geq b$. Based on Theorems 8–10 and Algorithm 1, we develop the following Algorithm 2 to find an optimal solution of problem (4).

Algorithm 2 [For solving problem (4)]

Step 1: Compute a^{Σ} by (31).

Step 2: Check the feasibility of (4). If $a^{\Sigma} \geq b$, then (4) is solvable, go to Step 3. Otherwise, (4) has no optimal solution and stop.

Step 3: Compute $\dot{\bar{x}} = (\dot{\bar{x}}_1, \dot{\bar{x}}_2, \ldots, \dot{\bar{x}}_2)$ by (3).

Step 4: Check the feasibility of $\dot{\bar{x}}$. If $A \odot \dot{\bar{x}}^T \geq b^T$, then $\dot{\bar{x}}$ is an optimal solution of problem (4). Otherwise, $\dot{\bar{x}}$ is not a solution of system (1), and we go to Step 5.

Step 5: Convert problem (4) into problem (P_0).

Step 6: Decompose (P_0) into m subproblems, i.e., (P_0^i), $i \in I$.

Step 7: For every $i \in I$, find the optimal solution of (P_0^i) by Algorithm 1. Let x^{i*} be the unique optimal solution of subproblem (P_0^i).

Step 8: Let $x_0^* = \bigvee_{i \in I} x_0^{i*}$ and $x^* = (x_0^*, x_0^*, \ldots, x_0^*)$. Then, x_0^* is the unique optimal solution of problem (P_0) and x^* is an optimal solution of problem (4).

V. Application Example

Example 3: A six-user BitTorrent-like P2P file-sharing system is reduced into the following addition-min fuzzy relation inequalities:

$$A \odot x^T \geq b^T \tag{32}$$

where

$$A = (a_{ij}) = \begin{bmatrix} 0 & 0.6 & 0.8 & 0.5 & 0.6 & 0.9 \\ 0.5 & 0 & 0.7 & 0.9 & 0.8 & 0.5 \\ 0.8 & 0.7 & 0 & 0.4 & 0.7 & 0.8 \\ 0.7 & 0.5 & 0.6 & 0 & 0.8 & 0.6 \\ 0.8 & 0.6 & 0.9 & 0.7 & 0 & 0.7 \\ 0.6 & 0.7 & 0.7 & 0.9 & 0.5 & 0 \end{bmatrix},$$

$$\begin{aligned} b &= (b_1, b_2, \ldots, b_6) = (2.8, 3.0, 2.9, 2.5, 3.2, 3.0), \\ x &= (x_1, x_2, \ldots, x_6) \in [0, 1]^6 \end{aligned}$$

and $\odot$ is the addition-min composition. Here, a_{ij} represents the bandwidth between ith user and jth user, x_j is the quality level on which the file data are sent from jth user, and b_i is the quality requirement of download traffic of ith user. Now, we aim to find an optimal solution minimizing $g(x) = x_1 \vee x_2 \vee \cdots \vee x_6$.

Solution:

Our target is to find an optimal solution to the following optimization problem:

$$\begin{aligned} &\min \;\; g(x) = x_1 \vee x_2 \vee \cdots \vee x_6 \\ &\text{s.t. } A \odot x^T \geq b^T \end{aligned} \tag{33}$$

where A, b, x are as shown in Example 3.

Step 1: Compute a^{Σ} by (31)

$$\begin{aligned} a^{\Sigma} &= \left(\sum_{j=1}^{6} a_{1j}, \sum_{j=1}^{6} a_{2j}, \cdots, \sum_{j=1}^{6} a_{6j} \right) \\ &= (3.4, 3.4, 3.4, 3.2, 3.7, 3.4). \end{aligned}$$

Step 2: Check the feasibility of (33). Obviously

$$\begin{aligned} a^{\Sigma} &= (3.4, 3.4, 3.4, 3.2, 3.7, 3.4) \\ &\geq (2.8, 3.0, 2.9, 2.5, 3.2, 3.0) = b \end{aligned}$$

then (33) is solvable, go to Step 3.

Step 3: Compute $\dot{\tilde{x}} = (\dot{\tilde{x}}_1, \dot{\tilde{x}}_2, \ldots, \dot{\tilde{x}}_6)$ by (3). We get

$$\dot{\tilde{x}} = (0.3, 0.3, 0.4, 0.5, 0.4, 0.3).$$

Step 4: Check the feasibility of $\dot{\tilde{x}}$:

$$\begin{aligned} A \odot \dot{\tilde{x}}^T &= (1.9, 1.9, 1.7, 1.7, 1.8, 1.9,) \\ &< (2.8, 3.0, 2.9, 2.5, 3.2, 3.0) = b^T. \end{aligned}$$

$\dot{\tilde{x}}$ is not a solution of system (32); therefore, we continue to Step 5.

Step 5: Convert problem (33) into problem (P'_0) as follows:

$$\begin{aligned} (\mathrm{P}'_0) &\min \;\; z(x_0) = x_0 \\ &\text{s.t. } A \odot x^T \geq b^T \end{aligned} \tag{34}$$

where $x_0 \in [0, 1]$.

Step 6: Decompose (P_0) into m subproblems

$$(\mathrm{P}'^1_0) \; \min \;\; z(x_0) = x_0$$

$$\text{s.t.} \begin{cases} 0 \wedge x_0 + 0.6 \wedge x_0 + 0.8 \wedge x_0 + \\ 0.5 \wedge x_0 + 0.6 \wedge x_0 + 0.9 \wedge x_0 \geq 2.8, \\ x_0 \in [0, 1]. \end{cases} \tag{35}$$

$$(\mathrm{P}'^2_0) \; \min \;\; z(x_0) = x_0$$

$$\text{s.t.} \begin{cases} 0.5 \wedge x_0 + 0 \wedge x_0 + 0.7 \wedge x_0 + \\ 0.9 \wedge x_0 + 0.8 \wedge x_0 + 0.5 \wedge x_0 \geq 3.0, \\ x_0 \in [0, 1]. \end{cases} \tag{36}$$

$$(\mathrm{P}'^3_0) \; \min \;\; z(x_0) = x_0$$

$$\text{s.t.} \begin{cases} 0.8 \wedge x_0 + 0.7 \wedge x_0 + 0 \wedge x_0 + \\ 0.4 \wedge x_0 + 0.7 \wedge x_0 + 0.8 \wedge x_0 \geq 2.9, \\ x_0 \in [0, 1]. \end{cases} \tag{37}$$

$$(\mathrm{P}'^4_0) \; \min \;\; z(x_0) = x_0$$

$$\text{s.t.} \begin{cases} 0.7 \wedge x_0 + 0.5 \wedge x_0 + 0.6 \wedge x_0 + \\ 0 \wedge x_0 + 0.8 \wedge x_0 + 0.6 \wedge x_0 \geq 2.5, \\ x_0 \in [0, 1]. \end{cases} \tag{38}$$

$$(\mathrm{P}'^5_0) \; \min \;\; z(x_0) = x_0 \tag{39}$$

$$\text{s.t.} \begin{cases} 0.8 \wedge x_0 + 0.6 \wedge x_0 + 0.9 \wedge x_0 + \\ 0.7 \wedge x_0 + 0 \wedge x_0 + 0.7 \wedge x_0 \geq 3.2, \\ x_0 \in [0, 1]. \end{cases} \tag{40}$$

$$(\mathrm{P}'^6_0) \; \min \;\; z(x_0) = x_0$$

$$\text{s.t.} \begin{cases} 0.6 \wedge x_0 + 0.7 \wedge x_0 + 0.7 \wedge x_0 + \\ 0.9 \wedge x_0 + 0.5 \wedge x_0 + 0 \wedge x_0 \geq 3.0. \\ x_0 \in [0, 1]. \end{cases} \tag{41}$$

Step 7: Solving subproblem (P'^i_0) by Algorithm 1, $i = 1, 2, \ldots, 6$, we get their optimal solutions as follows:

$$\begin{aligned} x_0^{1*} = 0.5750, x_0^{2*} = 0.6667, x_0^{3*} = 0.6250, \\ x_0^{4*} = 0.5000, x_0^{5*} = 0.6500, x_0^{6*} = 0.4750. \end{aligned}$$

Step 8: Let $x_0^* = \bigvee_{i=1}^{6} x_0^{i*} = 0.6667$. Hence

$$x^* = (0.6667, 0.6667, 0.6667, 0.6667, 0.6667, 0.6667)$$

is an optimal solution of problem (33), with corresponding objective value $g(x^*) = 0.6667$.

VI. Discussions

In Section IV, we proposed a novel method to find an optimal solution of problem (4). In fact, the optimal solution may be not unique in some cases. Now, we will make some further discussion on the optimal solution of problem (4). In this section, we always assume that x^* is an optimal solution of problem (4) obtained by Algorithm 2.

Theorem 11: The optimal solution of problem (4) is unique if and only if x^* is a lower solution of system (1).

Proof: ($\Rightarrow$) If the optimal solution of problem (4) is unique, then x^* is the unique optimal solution. Following Theorem 4, x^* is a lower solution of system (1).

($\Leftarrow$) Suppose x^* is a lower solution of system (1) and y^* is an arbitrary optimal solution of problem (4). Then, $g(y^*) = g(x^*)$, i.e.,

$$y_1^* \vee y_2^* \vee \cdots \vee y_n^* = x_0^* \vee x_0^* \vee \cdots \vee x_0^* = x_0^*.$$

Thus

$$y_j^* \leq x_0^* \quad \forall j \in J$$

i.e. $y^* \leq x^*$. However, x^* is a lower solution of system (1); therefore, we have $y^* = x^*$. Consequently, the optimal solution of problem (4) is unique. $\square$

Here, we can verify whether x^* is a lower solution by Theorem 12.

Theorem 12 (see [1] and [2]): Let $x = (x_1, x_2, \ldots, x_n)$ be a solution of (1); then, x is a lower solution of (1) if and only if $I(x) \neq \emptyset$ and

$$\bigcap_{i \in I(x)} J_i(x) = \emptyset,$$

where

$$I(x) = \{i \in I | \sum_{j \in J} a_{ij} \wedge x_j = b_i\}$$

and $J_i(x) = \{j \in J | a_{ij} < x_j\}$, $i \in I(x)$.

Since

$$x^* \in X(A, b) = \bigcup_{\check{x} \in \check{X}(A,b)} \{x | \check{x} \leq x \leq \hat{x}\},$$

there exists at least one $\check{x} \in \check{X}(A, b)$ such that $\check{x} \leq x^*$. Therefore, if we denote $\check{X}^*(A, b) = \{\check{x} \in \check{X}(A, b) | \check{x} \leq x^*\}$, then $\check{X}^*(A, b) \neq \emptyset$. The set $\check{X}^*(A, b)$ is said to be the lower optimal solution set, in which the element is said to be lower optimal solution.

Theorem 13: The optimal solution set of problem (4) is

$$X^*(A, b) = \bigcup_{\check{x}^* \in \check{X}^*(A,b)} \{x | \check{x}^* \leq x \leq x^*\}.$$

Proof: i)

$$X^*(A, b) \subseteq \bigcup_{\check{x}^* \in \check{X}^*(A,b)} \{x | \check{x}^* \leq x \leq x^*\}$$

Let y^* be an arbitrary element in $X^*(A, b)$. Similar to the proof of Theorem 11, we have $y^* \leq x^*$. Additionally, there exists $\check{x}^* \in \check{X}(A, b)$ such that $\check{x}^* \leq y^* \leq \hat{x}$, since y^* is a feasible solution of (4). Hence, $\check{x}^* \leq y^* \leq x^* \leq \hat{x}$ and then $\check{x}^* \in \check{X}^*(A, b)$. It is concluded that

$$y^* \in \bigcup_{\check{x}^* \in \check{X}^*(A,b)} \{x | \check{x}^* \leq x \leq x^*\}.$$

ii)

$$X^*(A, b) \supseteq \bigcup_{\check{x}^* \in \check{X}^*(A,b)} \{x | \check{x}^* \leq x \leq x^*\}$$

Let y be an arbitrary element in

$$\bigcup_{\check{x}^* \in \check{X}^*(A,b)} \{x | \check{x}^* \leq x \leq x^*\}.$$

There exists $\check{x}^* \in \check{X}^*(A, b)$ such that $\check{x}^* \leq y \leq x^*$. It is clear that both y and $\check{x}^*$ are feasible solutions of problem (4). In addition, $g(\check{x}^*) \leq g(y) \leq g(x^*)$. Based on the optimality of x^*, we have $g(x^*) \leq g(\check{x}^*)$. Consequently, $g(y) = g(x^*)$. Hence, y is also an optimal solution of problem (4), i.e., $y \in X^*(A, b)$. $\square$

It is shown in Theorem 13 that the optimal solution set of problem (4) is fully determined by one maximum optimal solution and a finite number of lower optimal solution(s). Obtaining the optimal solution set depends on finding all the lower optimal solution(s).

VII. Conclusion

Recently, Li and Yang [1], [2] introduced addition-min fuzzy relation equalities and its related optimization problem with application background in the data transmission mechanism in BitTorrent-like P2P file-sharing systems. As shown in the problem statement in Section III, for avoiding the network congestion and improve the stability of data transmission, minimizing the biggest quality level, i.e., $\min g(x) = x_1 \vee x_2 \vee \cdots \vee x_n$ would be better than minimizing the total quality levels, i.e., $\min z(x) = x_1 + x_2 + \cdots + x_n$. Hence, we proposed the min-max optimization problem with addition-min fuzzy relation equalities constraint. Due to the distinguishing characteristic of the feasible domain, the existing solution methods to the general max-t-norm fuzzy relation optimization problems are useless to our proposed problem. Moreover, it is difficult to solve the proposed problem by using the method presented in [2] since the objective function is nonlinear and the lower solutions of the constraints, i.e., a system of addition-min fuzzy relation equalities, may be infinite. To overcome this difficulty, we develop a novel algorithm step by step for obtaining an optimal solution of the proposed problem. In addition, an application example that describes a six-user BitTorrent-like P2P file-sharing system is given to illustrate the feasibility and efficiency of the algorithm.

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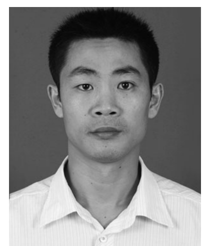

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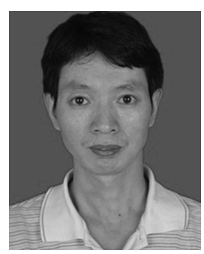

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