Sensitivity Conjecture

Theorem 1 (Not the Sensitivity Conjecture itself, but equivalent). *Any set H of $2^{n-1}+1$ vertices of the n-cube contains a vertex with at least $\sqrt{n}$ neighbors in H.*

The proof can be found here: `https://www.cs.stanford.edu/~knuth/papers/huang.pdf`. I suggest you to read the Knuth's version, because I can't make it shorter or simpler. As I remember, there was some hard equation in the end, but can be easily proven if we change it to ... $\leq$ ... inequality. The original equality is also true (we don't need its full strength), but it uses some observations about eigenvalues of the matrix A, which can be found in the original paper.

Connection to Boolean functions

Let $e_i \in \{0,1\}^n$ denote an n-bit Boolean string whose i^{th} bit is 1 and the rest of the bits are 0. Consider $f : \{0,1\}^n \to \{0,1\}$ be a Boolean functions. On an input $x \in \{0,1\}^n$, the i^{th} bit is said to be sensitive for f if $f(x \oplus e_i) \neq f(x)$, i.e., flipping the i^{th} bit results in flipping the output of f. The sensitivity of f on input x, denoted by $s(f,x)$, is the number of bits that are sensitive for f on input x.

Definition 1. *The sensitivity of a Boolean function f, denoted by $s(f)$, is the maximum value of $s(f,x)$ over all choices of x.*

For $B \subseteq \{1,2,...,n\}$ let $e_B \in \{0,1\}^n$ denote the characteristic vector of B. We say that a "block" B is sensitive for f on x if $f(x \oplus e_B) \neq f(x)$. The block sensitivity of f on x, denoted by $bs(f,x)$, is the maximum number of pairwise disjoint sensitive blocks of f on x.

Definition 2. *The block sensitivity of a Boolean function f, denoted by $bs(f)$, is the maximum possible value of $bs(f,x)$ over all choices of x.*

Block sensitivity seems to be somehow unnatural at first glance, but it is polynomially related to many others measures.

Theorem 2 (Sensitivity Conjecture). $bs(f) \leq \mathsf{poly}(s(f))$

Theorems 1 and 2 are known to be equivalent.

Definition 3. *The deterministic decision-tree complexity of a Boolean function f, denoted by $D(f)$, is the depth of a minimum-depth decision tree that computes f.*

Definition 4. *A polynomial $p : \mathbb{R}^n \to \mathbb{R}$ represents f if*

$$\forall x \in \{0,1\}^n \; p(x) = f(x)$$

The degree of a Boolean function f, denoted by $deg(f)$, is the degree of the unique multilinear polynomial that represents f.

	$bs(f)$	$D(f)$	$deg(f)$
$bs(f)$	1(1)	1(1)	$2(\log_3 6)$
$D(f)$	3(2)	1(1)	$2(\log_3 6)$
$deg(f)$	$2(\log_4 5)$	1(1)	1(1)

The table above should be read in the following way:

$$\begin{aligned} bs(f) &= \mathcal{O}((deg(f))^2) \\ bs(f) &= \Omega((deg(f))^{\log_3 6}) \end{aligned}$$