In order to avoid problems with integers, we consider slightly modified version of the algorithm from the "para".

Algorithm 1

---

procedure INIT
    Choose a random hash function $h : [n] \to [0,1]$ ▷ Yes, we can't, but it is not important
    $z \leftarrow 1$
procedure PROCESS(j)
    $z \leftarrow \min(z, h(j))$
procedure OUTPUT
    return $\frac{1}{z} - 1$

We will run k (exact value will be defined later) copies of our algorithm and then output mean value of their answers. Let $\hat{d}_k$ denote the output of our algorithm and d denote the true answer. We will also use $\hat{d}$ to denote $\hat{d}_1$ i.e. the answer of one copy of Algorithm 1.

Claim 1. *For any ε and δ there exists such k that*

$$P(|d - \hat{d}_k| \leq \varepsilon d) \geq 1 - \delta$$

Let X be a random variable uniformly distributed on $[0,1]$. Fix d. Then, $\hat{d}$ is equal to the minimum of d i.i.d. copies of X. The cumulative distribution function for $\hat{d}$ is

$$F_{\hat{d}}(x) = \begin{cases} 0, & x < 0 \\ 1 - (1-x)^d, & 0 \leq x \leq 1 \\ 1, & x > 1 \end{cases}$$

And, therefore, its density is

$$\rho_{\hat{d}}(x) = \begin{cases} d(1-x)^{d-1}, & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

First, we need to compute $\mathbb{E}[\hat{d}]$.

$$\mathbb{E}[\hat{d}] = \int_0^1 \rho_{\hat{d}}(x) \cdot x = \int_0^1 d(1-x)^{d-1} x = \frac{1}{d+1}$$

Note that it matches the formula in the output procedure of Algorithm 1. Let Y_d be $\hat{d} - \frac{1}{x+1}$.

Theorem 1 (Berry–Esseen)**.** *If $X_1, X_2, \dots,$ are i.i.d. random variables such that*

- $\mathbb{E}[X] = 0$
- $\mathbb{E}[X^2] = \sigma^2 > 0$
- $\mathbb{E}[|X|^3] = \rho < \infty$

and F_n be the cumulative distribution function of $\frac{1}{\sigma\sqrt{n}}\sum_{i\le n} X_i$, then, for all x and n

$$|F_n(x) - \Phi(x)| \le \frac{C\rho}{\sigma^3\sqrt{n}}$$

where C is some positive constant and $\Phi(x)$ denotes the cumulative distribution function of the standard normal distribution.

Thus, we want to compute $\mathbb{E}[Y_d^2]$ and $\mathbb{E}[|Y_d|^3]$.

$$\mathbb{E}[Y_d^2] = \int_0^1 d(1-x)^{d-1}(x - \frac{1}{d+1})^2 = \frac{d}{(1+d)^2(2+d)}$$

Calculation of $\mathbb{E}[|Y_d|^3]$ can cause psychological trauma[1], so instead of doing that we bound ρ with $\mathcal{O}(d^{-3})$[2]. Thus, our constants are

$$\sigma = \frac{1}{d+1}\sqrt{\frac{d}{d+2}}$$
$$\rho = \mathcal{O}(d^{-3})$$

Now it is time to recall our main goal.

By definition $|d - \hat{d}_k|$ is equal to arithmetic mean of k i.i.d. copies of Y_d. Therefore, condition in Claim 1 is equivalent to $F_k(\varepsilon d\frac{\sqrt{k}}{\sigma}) - F_k(-\varepsilon d\frac{\sqrt{k}}{\sigma}) \le 1 - \delta$. After we apply Theorem 1 we have to choose such k that the following condition is satisfied.

$$2\Phi(\varepsilon d\frac{\sqrt{k}}{\sigma}) - 1 - 2\frac{C\rho}{\sigma^3\sqrt{k}} \ge 1 - \delta \tag{1}$$

Now it is obvious that such k exists. Moreover,

[1]Its true value is $\frac{2d(6d^{2+d}+(1+d)^d - d^2(1+d)^d)}{(1+d)^{4+d}(2+d)(3+d)}$. Check it if you are brave enough.

[2]Need to find a proper way to do that. Now it can be obtained from the value of ρ.

$$\frac{d}{\sigma} = d(d+1)\sqrt{\frac{d+2}{d}} = \Omega(d^2)$$
$$\frac{\rho}{\sigma^3} = \mathcal{O}(1)$$

So, k not not only exists, it is also not very large. Further calculations I will do later.