

In order to avoid problems with integers, we consider slightly modified version of the algorithm from the “para”.

Algorithm 1

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1: procedure INIT
2:   Choose a random hash function  $h : [n] \rightarrow [0, 1]$   ▷ Yes, we can't, but
   it is not important
3:    $z \leftarrow 1$ 
4: procedure PROCESS( $j$ )
5:    $z \leftarrow \min(z, h(j))$ 
6: procedure OUTPUT
7:   return  $\frac{1}{z} - 1$ 

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We will run k (exact value will be defined later) copies of our algorithm and then output mean value of their answers. Let \hat{d}_k denote the output of our algorithm and d denote the true answer. We will also use \hat{d} to denote \hat{d}_1 i.e. the answer of one copy of Algorithm 1.

Claim 1. *For any ε there exists such k that*

$$P\left(\left|d - \hat{d}_k\right| \leq \varepsilon d\right) \geq 1 - \delta$$

where δ is some constant less than $\frac{1}{2}$.

Let X be a random variable uniformly distributed on $[0, 1]$. Fix d . Then, \hat{d} is equal to the minimum of d i.i.d. copies of X . The cumulative distribution function for \hat{d} is

$$F_{\hat{d}}(x) = \begin{cases} 0, & x < 0 \\ 1 - (1 - x)^d, & 0 \leq x \leq 1 \\ 1, & x > 1 \end{cases}$$

And, therefore, its density is

$$\rho_{\hat{d}}(x) = \begin{cases} d(1 - x)^{d-1}, & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

First, we need to compute $\mathbb{E}[\hat{d}]$.

$$\mathbb{E}[\hat{d}] = \int_0^1 \rho_{\hat{d}}(x) \cdot x = \int_0^1 d(1 - x)^{d-1} x = \frac{1}{d+1}$$

Note that it matches the formula in the output procedure of Algorithm 1. Let Y_d be $\hat{d} - \frac{1}{x+1}$.

Theorem 1 (Berry–Esseen). *If X_1, X_2, \dots , are i.i.d. random variables such that*

- $\mathbb{E}[X] = 0$
- $\mathbb{E}[X^2] = \sigma^2 > 0$
- $\mathbb{E}[|X|^3] = \rho < \infty$

and F_n be the cumulative distribution function of $\frac{1}{\sigma\sqrt{n}} \sum_{i \leq n} X_i$, then, for all x and n

$$|F_n(x) - \Phi(x)| \leq \frac{C\rho}{\sigma^3\sqrt{n}}$$

where C is some positive constant and $\Phi(x)$ denotes the cumulative distribution function of the standard normal distribution.

Thus, we want to compute $\mathbb{E}[Y_d^2]$ and $\mathbb{E}[|Y_d|^3]$.

$$\mathbb{E}[Y_d^2] = \int_0^1 d(1-x)^{d-1} \left(x - \frac{1}{d+1}\right)^2 = \frac{d}{(1+d)^2(2+d)}$$

Calculation of $\mathbb{E}[|Y_d|^3]$ can cause psychological trauma¹, so instead of doing that we bound ρ with $\mathcal{O}(d^{-3})$, see Appendix A for more. Thus, our constants are

$$\begin{aligned}\sigma &= \frac{1}{d+1} \sqrt{\frac{d}{d+2}} \\ \rho &= \mathcal{O}(d^{-3})\end{aligned}$$

Now it is time to recall our main goal.

By definition $|d - \hat{d}_k|$ is equal to arithmetic mean of k i.i.d. copies of Y_d .

Therefore, condition in Claim 1 is equivalent to $F_k\left(\varepsilon d \frac{\sqrt{k}}{\sigma}\right) - F_k\left(-\varepsilon d \frac{\sqrt{k}}{\sigma}\right) \leq 1 - \delta$. After we apply Theorem 1 we have to choose such k that the following condition is satisfied.

$$2\Phi\left(\varepsilon d \frac{\sqrt{k}}{\sigma}\right) - 1 - 2\frac{C\rho}{\sigma^3\sqrt{k}} \geq 1 - \delta \tag{1}$$

Now it is obvious that such k exists. Moreover,

¹Its true value is $\frac{2d(6d^{2+d} + (1+d)^d - d^2(1+d)^d)}{(1+d)^{4+d}(2+d)(3+d)}$. Check it if you are brave enough.

$$\begin{aligned}\frac{d}{\sigma} &= d(d+1)\sqrt{\frac{d+2}{d}} = \Omega(d^2) \\ \frac{\rho}{\sigma^3} &= \mathcal{O}(1)\end{aligned}$$

So, k not only exists, but also does not depend on d . Now, it remains to see that once we found some suitable $k = k_0$ for some constant $\varepsilon = \varepsilon_0$, we can choose $k(\varepsilon) = \left(\frac{\varepsilon_0}{\varepsilon}\right)^2 k_0$ for all $\varepsilon \leq \varepsilon_0$ and $k(\varepsilon) = k_0$ for all $\varepsilon > \varepsilon_0$. Thus, $k = \mathcal{O}\left(\frac{1}{\varepsilon^2}\right)$.

A Appendix

We want to prove that $\mathbb{E}[|X|^3] = \mathcal{O}(d^{-3})$. By definition,

$$\mathbb{E}[|X|^3] = \int_0^1 d(1-x)^{d-1} \left| x - \frac{1}{d+1} \right|^3$$

We split it into the following two parts

$$\int_0^{\frac{1}{d+1}} d(1-x)^{d-1} \left(\frac{1}{d+1} - x \right)^3 \quad (2)$$

$$\int_{\frac{1}{d+1}}^1 d(1-x)^{d-1} \left(x - \frac{1}{d+1} \right)^3 \quad (3)$$

First, it is easy to see that integral 3 is the beta function.

$$(3) = d \frac{d}{d+1} \int_0^1 (1-x)^{d-1} x^3 = \frac{d^2}{d+1} B(4, d)$$

Since d is a positive integer, the factor $B(4, d)$ can be replaced with $\frac{3!(d-1)!}{(d+3)!}$ which is $\mathcal{O}(d^{-4})$.

Thus, it remains to bound integral 2.

$$(2) \leq d \int_0^{\frac{1}{d-1}} \left(\frac{1}{d-1} - x \right)^3 = d \int_0^{\frac{1}{d-1}} x^3 = \frac{d}{4} \left(\frac{1}{d-1} \right)^4 = \mathcal{O}(d^{-3})$$