# Resolution with Small Symmetries

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# Definitions

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$\varphi \colon \{\text{Lit}\} \to \{\text{Lit}\}$ is a $k$-**symmetry function** iff:

1. It is an automorphism of literals.
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**Resolution with** $k$**-symmetries** has two rules:

1. Resolution rule.

$$\frac{A \vee x \qquad B \vee \neg x}{A \vee B}$$

2. Symmetry rule.

$$\frac{\varphi(B_1), \varphi(B_2), \ldots, \varphi(B_l)}{\varphi(A)}$$

where $\varphi$ is a $k$-symmetry and $B_1, B_2, \ldots, B_l \vdash^* A$ was inferred before.

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- Resolution with global, local and dynamic symmetries are called SRC-I, SRC-II, and SRC-III, correspondingly.

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- Any formula that is hard for Resolution can be transformed into a formula that is hard for SRC-II (Szeider 2005).
- Superpolynomial lower bounds for SRC-III is a long-standing open problem.

# Example

$$(x_1 \vee x_7) \wedge (x_2 \vee \neg x_7) \wedge (x_3 \vee \neg x_2) \wedge (\neg x_1 \vee x_7) \wedge (x_4 \vee \neg x_7) \wedge (x_5 \vee \neg x_4) \wedge (x_4 \vee x_6)$$

- Use the resolution rule:

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- Use a local 5-symmetry that negates $x_1$ and swaps $x_2 \leftrightarrow x_4$ and $x_3 \leftrightarrow x_5$:

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- Infer $\dfrac{x_3 \vee \neg x_2 \qquad x_1 \vee x_2 \qquad \neg x_1 \vee x_5}{x_3 \vee x_5}$ by applying to the inference of $\neg x_1 \vee x_5$ a dynamic 5-symmetry $\varphi$ defined as follows:

$$\varphi(x_1) = \neg x_3 \qquad \varphi(x_3) = x_4$$
$$\varphi(x_7) = \neg x_2 \qquad \varphi(x_2) = \neg x_7$$
$$\varphi(x_4) = x_1$$

# Connection to the practice

- Many sets of practical formulas have symmetries. One example is formulas that emerge from bounded model checking.
- Two main approaches to exploiting those symmetries in CDCL-based solvers are:
  - Symmetry breaking (Puget 1993; Sakallah 2009).
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| Symmetry breaking | Symmetry handling |
|---|---|
| Adds clauses that “break” symmetries | Adds clauses that are symmetric to the inferred ones |
| Maintains equisatisfiability, but removes satisfying assignments | Preserves the set of satisfying assignments |
| Restricted to global symmetries | Works with local and dynamic symmetries |
| Usually performs better when suitable | Essential for the best results in some specialized cases |

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- Resolution with local 3-symmetries (or global 4-symmetries) polynomially refutes Tseitin formulas on a rectangle grid, which are hard for constant-depth Frege.

# Syntactic version of Resolution with symmetries

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  - Each vertex of the graph is labeled with a symmetry.
  - There is a source (every other vertex is reachable from it).
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- Going along an edge in a symmetry graph corresponds to applying the symmetry function of the head of this edge to the clause.

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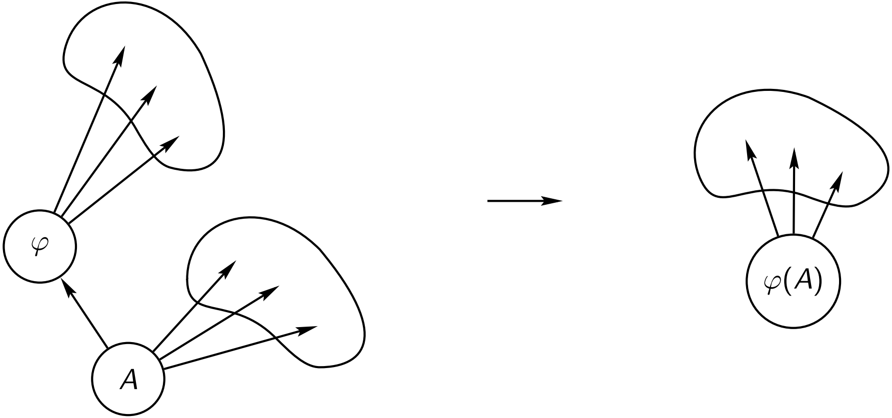
  - A set of rules that modify the symmetry graphs.
- Proof size measures (equivalent):
  - The number of entries.
  - The total bit length.

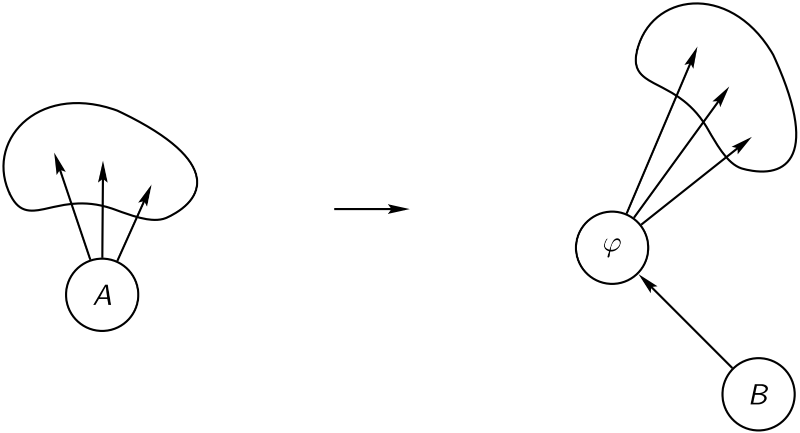
# Symmetry rule

Let $A$ be a clause, and $\varphi$ be a symmetry function.



# Edge addition

Let $A$ and $B$ be clauses, and $\varphi$ be a symmetry function such that $\varphi(B) = A$.



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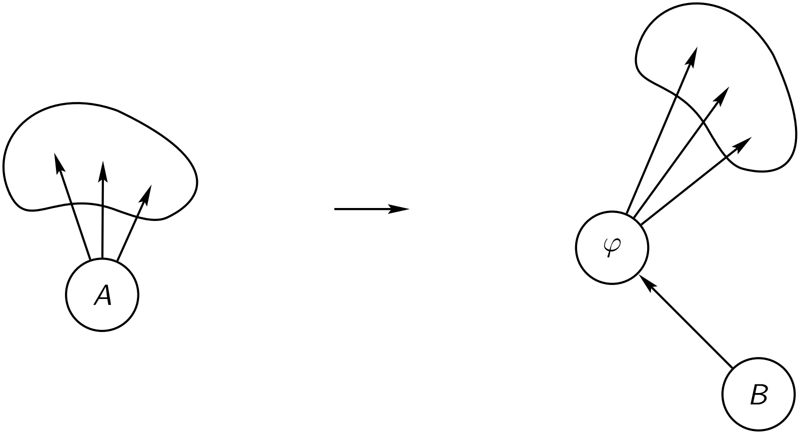
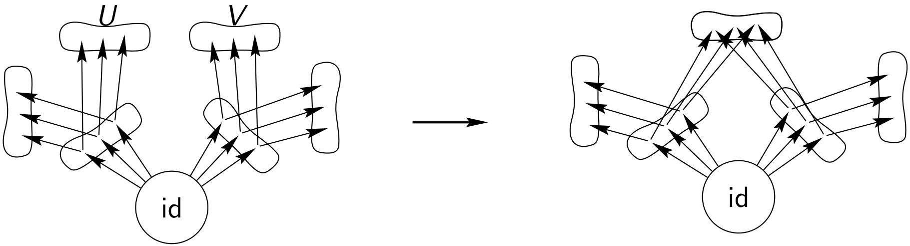



To make this rule the inverse of the symmetry rule, we add the graph combining rule:

$$\frac{\langle C, G\rangle \qquad \langle C, H\rangle}{\langle C, (G+H)/_{\mathrm{src}(G)=\mathrm{src}(H)}\rangle}$$

# Gluing rule

Assume that $V$ and $U$ are isomorphic subgraphs of the symmetry graph.



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- For example, we can restrict us to layered graphs (i.e., graphs with a rank-function $\rho$ on vertices such that an edge $v \to u$ exists iff $\rho(v) < \rho(u)$).
- Resolution with 1-symmetries with this additional restriction forms* a new proof system that we call $G_2$-Frege.

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- An alternative definition of $G_2$-Frege:
  - $G_2(x_1, x_2, \ldots, x_k) := \sum_j x_j \geq 2$.
  - Proof lines are disjunctions of literals and $G_2$-functions with the additional property that each literal can participate in each lemma only once.
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- Examples of the inference rules:

$$\frac{G_2(x, y) \vee z \vee t \qquad G_2(x, y) \vee \neg t}{G_2(x, y) \vee z} \qquad \frac{G_2(x, y) \vee G_2(z, t, s)}{G_2(x, y) \vee z \vee t \vee \neg s}$$

$$\frac{G_2(x, y) \vee z \vee t \qquad G_2(x, y) \vee \neg z \vee t \qquad G_2(x, y) \vee z \vee \neg t}{G_2(x, y) \vee G_2(z, t)}$$

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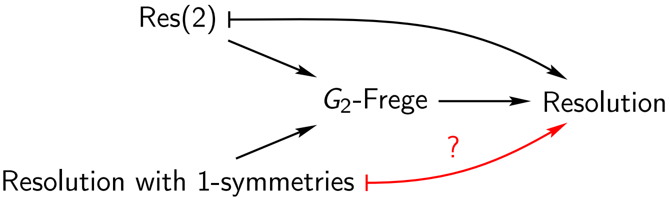
- $G_2$-Frege is p-simulated by Res(2).



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3. Constructing practical symmetry handling heuristics for the special classes of formulas with suitable structures of symmetries.