# Final Project: Parallelizing a code solving the fully nonlinear shallow water equations with rotation

Mohamed Moustaoui

School of Mathematical and Statistical Sciences
Arizona State University

# The shallow water equations

- Consider the two-dimensional nonlinear shallow water equations with rotation on a flat terrain:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + g\frac{\partial h}{\partial x} - fv = 0$$
$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + g\frac{\partial h}{\partial y} + fu = 0$$
$$\frac{\partial h}{\partial t} + \frac{\partial (uh)}{\partial x} + \frac{\partial (vh)}{\partial y} = 0$$

- $u(x,y,t)$ and $v(x,y,t)$ are the horizontal velocities, $h(x,y,t)$ is the elevation
- $f$ is the Coriolis parameter that depends on the rotating frame
- For atmospheric and ocean equations, $f = 2\Omega \sin\phi$, where $\phi$ is the latitude and $\Omega$ is the angular velocity of Earth

# *Beta*-plane approximation

- The *beta*-plane approximation consists of approximating the sphere by the tangent plane at a given latitude $\phi_0$
- Using Taylor series, the Coriolis parameter becomes:

$$
\begin{aligned}
f &= 2\Omega sin\phi \\
&\approx 2\Omega sin\phi_0 + 2\Omega cos\phi_0(\phi - \phi_0) \\
&= 2\Omega sin\phi_0 + 2a^{-1}\Omega cos\phi_0 a(\phi - \phi_0) \\
&= f_o + \beta y
\end{aligned}
$$

- $\Omega$ and $a$ are the angular velocity and the radius of Earth
- $y = a(\phi - \phi_0)$ is the distance in the northward direction
- $f_o = 2\Omega sin\phi_0$ and $\beta = 2a^{-1}\Omega cos\phi_0$

# Perturbation equation for $h$

- If we let $h(x,y,t) = H + h'(x,y,t)$, where $H$ is the undisturbed elevation of the fluid layer, the shallow water equations become

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + g\frac{\partial h'}{\partial x} - fv = 0$$
$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + g\frac{\partial h'}{\partial y} + fu = 0$$
$$\frac{\partial h'}{\partial t} + u\frac{\partial h'}{\partial x} + v\frac{\partial h'}{\partial y} + (H + h')\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

- $H$ is calculated such that $c = \sqrt{gH} = 60ms^{-1}$, where $g = 9.81ms^{-2}$

# Domain of integration

- The shallow water equations are integrated in the domain

$$-\frac{L}{2} \leq x < \frac{L}{2}$$
$$-\frac{L}{2} \leq y < \frac{L}{2}$$

- The domain uses periodic boundary conditions

$$h(x+L,y,t) = h(x,y,t) \ \text{ and } \ h(x,y+L,t) = h(x,y,t)$$
$$u(x+L,y,t) = u(x,y,t) \ \text{ and } \ u(x,y+L,t) = u(x,y,t)$$
$$v(x+L,y,t) = v(x,y,t) \ \text{ and } \ v(x,y+L,t) = v(x,y,t)$$

where $L = 15,000km$

- The time integration combines the RK3 time stepping scheme and a semi-implicit scheme
- The space discretization uses the fourth-order finite difference approximations

# Diffusion and absorbing layer

- The equations are modified by adding extra terms to handle some numerical errors:
- A numerical smother in the form of a hyper-diffusion which selectively filters out small scales and reduces aliasing problems due to nonlinearity
- An absorbing layer surrounding the domain that reduces the perturbations exiting the domain at one boundary and re-entering at the opposite boundary due to periodic boundary conditions

# The full equations

- The modified equations are:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+g\frac{\partial h}{\partial x}-fv=-\sigma u-k\left(\frac{\partial^4 u}{\partial x^4}+\frac{\partial^4 u}{\partial y^4}\right)$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+g\frac{\partial h}{\partial y}+fu=-\sigma v-k\left(\frac{\partial^4 v}{\partial x^4}+\frac{\partial^4 v}{\partial y^4}\right)$$

$$\frac{\partial h}{\partial t}+u\frac{\partial h}{\partial x}+v\frac{\partial h}{\partial y}+(H+h)\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)$$

$$=-\sigma h-k\left(\frac{\partial^4 h}{\partial x^4}+\frac{\partial^4 h}{\partial y^4}\right)$$

- For simplicity, $h'$ is replaced by $h$. It should be understood that $h$ now represents the perturbation of elevation

# Combined time stepping schemes

- The terms in the equations are separated into the absorbing terms which will be treated by a semi-implicit scheme and the remaining terms that will be treated by the RK3 scheme:

$$
\begin{aligned}
\frac{\partial u}{\partial t} + F &= -\sigma u \\
\frac{\partial v}{\partial t} + G &= -\sigma v \\
\frac{\partial h}{\partial t} + R &= -\sigma h
\end{aligned}
$$

# Terms treated with RK3

- The terms in $F$, $G$ and $R$ include the nonlinear advection, pressure, Coriolis and diffusion terms :

$$F = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + g\frac{\partial h}{\partial x} - fv + k\left(\frac{\partial^4 u}{\partial x^4} + \frac{\partial^4 u}{\partial y^4}\right)$$

$$G = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + g\frac{\partial h}{\partial y} + fu + k\left(\frac{\partial^4 v}{\partial x^4} + \frac{\partial^4 v}{\partial y^4}\right)$$

$$R = u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y} + (H+h)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + k\left(\frac{\partial^4 h}{\partial x^4} + \frac{\partial^4 h}{\partial y^4}\right)$$

# Domain of integration

- The discretization of the domain uses $N_x = N_y = M = 600$ points with a grid spacing of $\Delta x = \Delta y = 25km$

$$x_1 = -L/2,\ x_2 = -L/2 + \Delta x, \cdots x_M = -L/2 + (M-1)\Delta x$$

$$y_1 = -L/2,\ y_2 = -L/2 + \Delta y, \cdots y_M = -L/2 + (M-1)\Delta y$$

- The domain uses a staggered grid where the locations of $u$, $v$ and $h$ relative to a given location $(x_i, y_j)$ on the grid are defined as follows:

$$u(x_i, y_i + \Delta y/2) = u_{i,j+1/2} = u_{i,q}$$
$$v(x_i + \Delta x/2, y_i) = v_{i+1/2,j} = v_{p,j}$$
$$h(x_i + \Delta x/2, y_i + \Delta y/2) = h_{i+1/2,j+1/2} = h_{p,q}$$

- Here, $p = i + 1/2$ and $q = j + 1/2$

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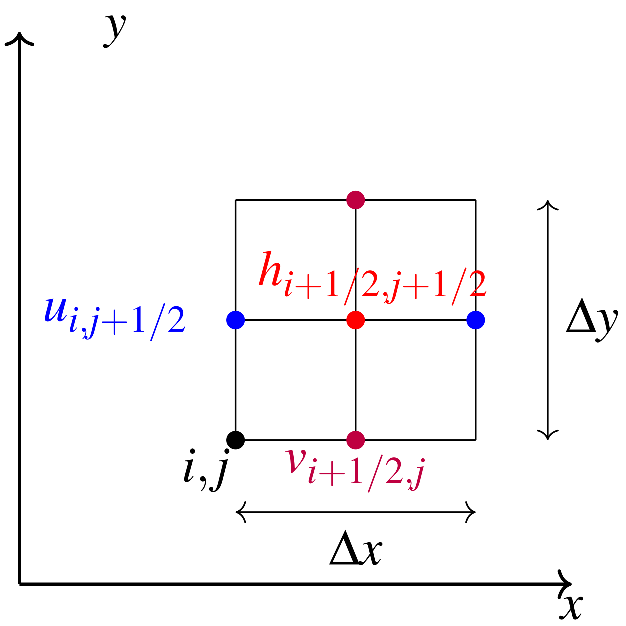


- The discretized periodic boundary conditions for $\psi = u, v, h$ are

$$\psi_{i(p)+M,j(q)} = \psi_{i(p),j(q)} \text{ and } \psi_{i(p),j(q)+M} = \psi_{i(p),j(q)}$$

# Discretized equations

- The discretized equations in space are

$$\begin{aligned}
\left.\frac{\partial u}{\partial t}\right|_{i,q} &= -F_{i,q} - \sigma u_{i,q} \\
\left.\frac{\partial v}{\partial t}\right|_{p,j} &= -G_{p,j} - \sigma v_{p,j} \\
\left.\frac{\partial h}{\partial t}\right|_{p,q} &= -R_{p,q} - \sigma h_{p,q}
\end{aligned}$$

# Time integration

- $u_i^{n+1}$, $v_i^{n+1}$ and $h_p^{n+1}$ are updated in time by combining the RK3 and the semi-implicit schemes as follows:
- The first RK3 substep: $\Delta\tau = \Delta t/3$

$$
\begin{aligned}
\frac{u_{i,q}^{*} - u_{i,q}^{n}}{\Delta\tau} &= -F_{i,q}^{n} - \sigma\frac{u_{i,q}^{*} + u_{i,q}^{n}}{2} \\
\frac{v_{p,j}^{*} - v_{p,j}^{n}}{\Delta\tau} &= -G_{p,j}^{n} - \sigma\frac{v_{p,j}^{*} + v_{p,j}^{n}}{2} \\
\frac{h_{p,q}^{*} - h_{p,q}^{n}}{\Delta\tau} &= -R_{pq}^{n} - \sigma\frac{h_{p,q}^{*} + h_{p,q}^{n}}{2}
\end{aligned}
$$

# Time integration

- The second RK3 substep: $\Delta\tau = \Delta t/2$

$$
\begin{aligned}
\frac{u^{**}_{i,q} - u^{n}_{i,q}}{\Delta\tau} &= -F^{*}_{i,q} - \sigma\frac{u^{**}_{i,q} + u^{n}_{i,q}}{2} \\
\frac{v^{**}_{p,j} - v^{n}_{p,j}}{\Delta\tau} &= -G^{*}_{p,j} - \sigma\frac{v^{**}_{p,j} + v^{n}_{p,j}}{2} \\
\frac{h^{**}_{p,q} - h^{n}_{p,q}}{\Delta\tau} &= -R^{*}_{pq} - \sigma\frac{h^{**}_{p,q} + h^{n}_{p,q}}{2}
\end{aligned}
$$

# Time integration

- The third and final RK3 substep: $\Delta\tau = \Delta t$

$$
\begin{aligned}
\frac{u_{i,q}^{n+1} - u_{i,q}^{n}}{\Delta\tau} &= -F_{i,q}^{**} - \sigma\frac{u_{i,q}^{n+1} + u_{i,q}^{n}}{2} \\
\frac{v_{p,j}^{n+1} - v_{p,j}^{n}}{\Delta\tau} &= -G_{p,j}^{**} - \sigma\frac{v_{p,j}^{n+1} + v_{p,j}^{n}}{2} \\
\frac{h_{p,q}^{n+1} - h_{p,q}^{n}}{\Delta\tau} &= -R_{pq}^{**} - \sigma\frac{h_{p,q}^{n+1} + h_{p,q}^{n}}{2}
\end{aligned}
$$

- Begin the time step loop for $it = 1, 2, \cdots, n_t$
  1. Begin the RK3 substep loop for $ir = 1, 2, 3$
     a. RK3 substep1: $ir = 1 \implies \Delta\tau = \Delta t/3,\ s = n,\ \ s+1 = *$
     b. RK3 substep2: $ir = 2 \implies \Delta\tau = \Delta t/2,\ s = *,\ \ s+1 = **$
     c. RK3 substep3: $ir = 3 \implies \Delta\tau = \Delta t,\ s = **,\ \ s+1 = n+1$
     d. Compute the tendency terms $F^s_{i,q}$, $G^s_{p,j}$ and $R^s_{p,q}$
     e. Advance $u$, $v$ and $h$ for the substep $s+1$:
        $$u^{s+1}_{i,q} = \left[(1-\sigma\Delta\tau/2)u^n_{i,q} - \Delta\tau F^s_{i,q}\right]/(1+\sigma\Delta\tau/2)$$
        $$v^{s+1}_{p,j} = \left[(1-\sigma\Delta\tau/2)v^n_{p,j} - \Delta\tau G^s_{p,j}\right]/(1+\sigma\Delta\tau/2)$$
        $$h^{s+1}_{p,q} = \left[(1-\sigma\Delta\tau/2)h^n_{p,q} - \Delta\tau G^s_{p,q}\right]/(1+\sigma\Delta\tau/2)$$
     f. Repeat steps a, b, c, d and e for the next substep $ir = ir + 1$
  2. End the RK3 substep loop

# Algorithm

3 Overwrite $u^n$, $v^n$ and $v^n$:

$$u^n_{i,q} = u^{n+1}_{i,q}$$
$$v^n_{p,j} = v^{n+1}_{p,j}$$
$$h^n_{p,q} = h^{n+1}_{p,q}$$

5 Repeat steps 1 , 2 and 3 for the next time step $it = it + 1$

- End the time step loop

# Linear contribution

- The "linear" contributions in $F_{i,q}$, $G_{p,j}$ and $R_{p,q}$:

$$g\frac{\partial h}{\partial x}\Big|_{i,q},\ g\frac{\partial h}{\partial y}\Big|_{p,j}\ \text{and}(H+h_{p,q})\left(\frac{\partial u}{\partial x}\Big|_{p,q}+\frac{\partial v}{\partial y}\Big|_{p,q}\right)$$

  are formulated in flux forms that promote fourth-order approximations for space derivatives
- These fluxes have the same forms as the one we used for the 2D linear shallow water equations

# Advection contribution

- The advection terms in $F_{i,q}$, $G_{p,j}$ and $R_{p,q}$ are computed with fourth-order approximations
- For example, the advection term for $h$ has the form

$$
\begin{aligned}
u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y}\bigg|_{p,q} &= \frac{U}{12\Delta x}\Big(8(h_{p+1,q} - h_{p-1,q}) - (h_{p+2,q} - h_{p-2,q})\Big) \\
&+ \frac{V}{12\Delta y}\Big(8(h_{p,q+1} - h_{p,q-1}) - (h_{p,q+2} - h_{p,q-2})\Big)
\end{aligned}
$$

- $U$ are $V$ are the $x-$ and $y-$ velocities interpolated to the $h$ point:

$$U = \frac{u_{i+1,q} + u_{i,q}}{2}$$

$$V = \frac{v_{p,j+1} + v_{p,j}}{2}$$

# Diffusion contribution

- The diffusion terms in $F_{i,q}$, $G_{p,j}$ and $R_{p,q}$ are computed with fourth-order approximations
- For example, the diffusion term for $h$ has the form

$$
\begin{aligned}
k \quad & \left(\frac{\partial^4 h}{\partial x^4} + v\frac{\partial^4 h}{\partial y^4}\right)\Big|_{p,q} = \\
& \frac{k}{\Delta x^4}\left(h_{p-2,q} - 4h_{p-1,q} + 6h_{p,q} - 4h_{p+1,q} + h_{p+2,q}\right) \\
& \frac{k}{\Delta y^4}\left(h_{p,q-2} - 4h_{p,q-1} + 6h_{p,q} - 4h_{p,q+1} + h_{p,q+2}\right)
\end{aligned}
$$

- The difusion coefficient has the form:

$$k = \gamma\frac{\Delta x^4}{\Delta t}$$

where $\gamma = 0.02$

# Coriolis contribution

- The beta plane approximation is centered at the equator $\phi_0 = 0$:

$$f_o = 2\Omega sin\phi_0 = 0 \text{ and } \beta = 2a^{-1}\Omega cos\phi_0 = 2a^{-1}\Omega$$

- This gives the Coriolis parameter:

$$f = f_o + \beta y = \beta y$$

- The discretized form of $f$ is

$$f_j = \beta y_j$$

- $y_j$ is defined at the same $y$-location as $v$

$$y_j = -L/2 + (j-1)\Delta y; \quad j = 1, \cdots M$$

# Coriolis contribution

- The Coriolis terms in $F_{i,q}$ and $G_{p,j}$ are computed as follows:

$$fv\big|_{i,q} = f_u V \quad \text{and} \quad fu\big|_{p,j} = f_v U$$

- Here, $f_u$ and $V$ are the Coriolis parameter and the $y-$velocity interpolated to the $u$ point:

$$\begin{aligned} f_u &= \frac{f_{j+1} + f_j}{2} \\ V &= 0.25\left(v_{p,j} + v_{p,j+1} + v_{p-1,j} + v_{p-1,j+1}\right) \end{aligned}$$

- $f_v = f_j$ is the Coriolis parameter and $U$ is the $x-$velocity interpolated to the $v$ point:

$$U = 0.25\left(u_{i,q} + u_{i,q-1} + u_{i+1,q} + u_{i+1,q-1}\right)$$

# Absorbing layer

- The coefficients $\sigma$ used to absorb perturbations near the boundary is defined as

$$\sigma = \frac{0.8}{\Delta t}\max(\sigma_x, \sigma_y)$$

- Here, $\sigma_x$ and $\sigma_y$ are equal to zero in the entire domain, except within the 10-points width layer adjacent to the boundaries
- In this layer, $\sigma_x$ and $\sigma_y$ decrease smoothly from 1 at the boundary to 0 at the 11th grid point away from the boundary
- For example, at the left $x$ boundary, $\sigma_x$ has the form

$$\sigma_{xi} = \left(\frac{10 - i + 1}{10}\right)^3$$

# Initial conditions for $u$ and $v$

- The initial conditions consists of two cyclones placed at $y = \pm y_0$, where $y_0 = 650km$
- The initial fields of $u$ and $v$ are computed from a prescribed stream function $\psi$ that have the form:

$$\begin{aligned}
\psi(x,y) &= v_{max} d_{max} \left[ (1 + d_1/d_{max}) e^{1 - d_1/d_{max}} \right] \\
&- v_{max} d_{max} \left[ (1 + d_2/d_{max}) e^{1 - d_2/d_{max}} \right]
\end{aligned}$$

- Here, $d_1 = \sqrt{x^2 + (y + y_0)^2}$ and $d_2 = \sqrt{x^2 + (y - y_0)^2}$
- $v_{max} = 30ms^{-1}$ and $d_{max} = 200km$
- The velocities $u$ and $v$ defined by the stream function $\psi$ are:

$$u = -\frac{\partial \psi}{\partial y} \quad \text{and} \quad v = \frac{\partial \psi}{\partial x}$$

# Initial conditions for $h$

- The initial conditions for $h$ are computed by solving the nonlinear balance equations:

$$\begin{aligned} g\frac{\partial h}{\partial x} &= -u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} + fv = Q_x \\ g\frac{\partial h}{\partial y} &= -u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} - fu = Q_y \end{aligned}$$

- These can be combined to derive an elliptical equations for $h$:

$$g\Delta h = g\left(\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2}\right) = \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y}$$

- This equation is solved in a separate code by using periodic boundary condition

## Input files

- The initial conditions for $u$, $v$ and $h$ are computed by a separate program and archived in files that we be used as input files in the final project:
- The input files are :

  shallow_init_u.bin
  shallow_init_v.bin
  shallow_init_h.bin

# Initial conditions for $u$, $v$ and $h$