SEM-II M-2

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CSD/AIDS/AIML - 2151

Course Name : Probability & statistics.

UNIT: NO: 1 : Random Variables & Probability Distributions.

Conditional probability, Baye's Theorem. Random variables: Discrete & cts random variables, probability f^n & distributⁿ f^n, Joint distributions. Independent random variables, Conditional distribution.

* Sample space & Events:

Sample Space :- The set 'S' of all possible outcomes of g^n experiments is called sample space.

Ex If two coins are tossed, then sample space ; $S = \{HH, HT, TH, TT\}$ (is a sample space)

A particular outcome ie an element of 'S' is called sample point.

Event: The subset 'A' of a sample space S is called an event.

Ex Let A be an event of getting head on first coin. $A = \{HH, HT\}$.

~~Ex~~ ~~If a one die~~

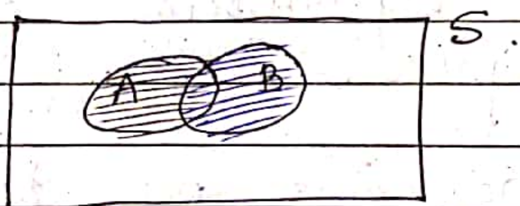
Ex A die is thrown once. Then
(i) Find sample space
(ii) An event of getting prime number

Soln.
Sample Sp. $= \{1, 2, 3, 4, 5, 6\}$; $n(S) = 6$

Event; $A = \{2, 3, 5\}$; $n(A) = 3$

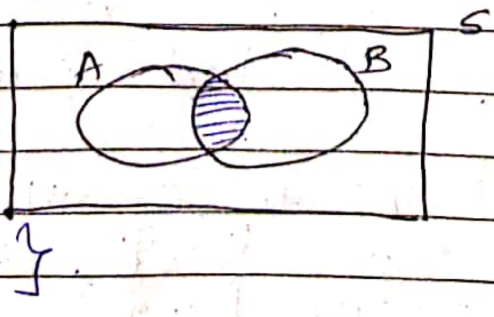
* If A & B be two events, then

(i) $A \cup B$ is an event that either A occurs or B occurs or both.



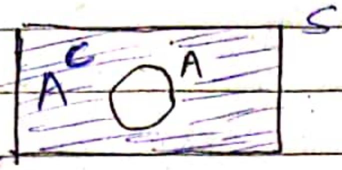
$A = \{2, 4, 6\}$ & $B = \{2, 3, 5\}$
then
$A \cup B = \{2, 3, 4, 5, 6\}$

(ii) $A \cap B$ is an event that both A & B occur simultaneously.



$A \cap B = \{2\}$

(iii) A^c / $\bar{A}$ (ie Complement of A) is an event that A does not occurs.



$A^c = \{1, 3, 5\}$.

* Mutually exclusive Events: Two events A & B are said ~~twoo~~ to be mutually exclusive, if they are disjoint

ie $A \cap B = \phi$

ie A & B do not occurs simultaneously.

Ex If two coins are tossed, then

S.Sp, S = {HH, HT, TH, TT}

let A be an event of two head on coin

A = {HH}

& B be an event of two tail on coin

B = {TT}

then $A \cap B = \phi$

⇒ A & B are mutually exclusive events.

* Probability of an event *

let 'n' be the no. of sample point in sample space 'S' & 'm' be the no. of sample point in an event $A \subseteq S$. Then probability of an event A is denoted & defined by

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$$P(A) = \frac{\text{No. of sample pt in event } A}{\text{No. of sample point in sample space's'}} = \frac{m}{n}$$

Ex If two coins are tossed, Let A be an event of getting both heads, B be an event of getting both tails. Find probability of A & B ie $P(A)$ & $P(B) = ?$

⇒ $S = \{HH, TH, HT, TT\}$

$n(S) = 4$

$A = \{HH\}$, $B = \{TT\}$

$n(A) = 1$ & $n(B) = 1$

$\therefore P(A) = \frac{n(A)}{n(S)} = \frac{1}{4}$, $P(B) = \frac{1}{4}$

* Axioms of Probability :- Let 'S' be a sample space & 'A' be the event in S.

If $P(A)$ is prob. of event A, then foll. axioms holds.

(i) For every event A, $0 \leq P(A) \leq 1$

(ii) $P(S) = 1$

(iii) If A & B are mutually exclusive events, then

$$P(A \cup B) = P(A) + P(B)$$

$$\{(i.e)\ P(A \cap B) = 0\}.$$

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Note:

(i) The null set $\phi \subseteq S$ & hence ϕ is also an event, called impossible event.
$P(\phi) = 0$

(ii) If A^c is complement of A, then
$P(A^c) = 1 - P(A)$ $\{\because P(S) = 1\}$

(iii) If events A & B are such that $A \subset B$ then $P(A) \leq P(B)$

Ex $S = \{1, 2, 3, 4, 5, 6\}$; $P(S) = 1$

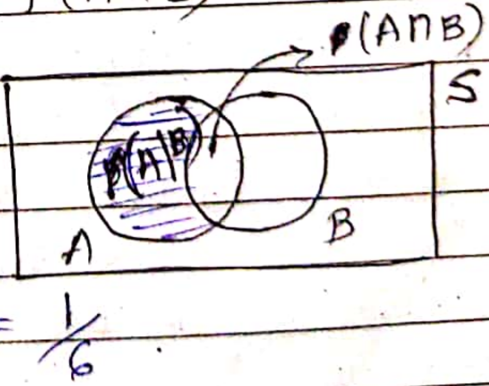
$A = \{2\}$, $B = \{2, 4, 6\}$ & $A \subset B$

$P(A) = \frac{n(A)}{n(S)} = \frac{1}{6}$; $P(B) = \frac{n(B)}{n(S)} = \frac{3}{6}$ =

$P(A) < P(B)$ ie $\frac{1}{6} < \frac{3}{6}$

$P(A^c) = 1 - P(A) = 1 - \frac{1}{6} = \frac{5}{6}$

(iv) If A & B are two events, then
$P(A|B) = P(A) - P(A \cap B)$



$P(A) = \frac{1}{6}$

$A \cap B = \{2\}$;

$P(A \cap B) = \frac{n(A \cap B)}{n(S)} = \frac{1}{6}$

$\therefore P(A|B) = \frac{1}{6} - \frac{1}{6} = 0$

(V) Additn Thm. of prob.

(a) If A & B are any two events, then
$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

(b) If A, B & C are 3-events, then
$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$

Q Two Cards are drawn from a pack of 52 cards. Find probability that
(i) Both are hearts
(ii) one is heart & other is spade.

→ Two cards are drawn from 52 cards
The sample space becomes

$S = {}^{52}C_2 = \frac{52!}{2!\,50!}$ $\qquad {}^nC_r = \frac{n!}{r!(n-r)!}$

$= \frac{52 \times 51}{2 \times 1} = 1326$ ways.

$\therefore n(S) = 1326$

Event → (i) Both are hearts can be drawn from 13 hearts

$A = {}^{13}C_2 = \frac{13!}{2!\,11!} = \frac{13 \times 12}{2 \times 1}$

$= 78$ ways. $\therefore n(A) = 78$

$\therefore P(A) = P(\text{Both hearts}) = \frac{n(A)}{n(S)}$

$= \frac{78}{1326} = \frac{1}{17}$

And = $\times$
or = $+$

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Event → (ii) One is heart & other is spade

There are 13 - hearts & 13 - spade

$B = {}^{13}C_1 \times {}^{13}C_1 = \frac{13!}{1!\ 12!} \times \frac{13!}{1!\ 12!} = 13 \times 13$

$= 169$

$\therefore n(B) = 169$

$\therefore P(B) = P(\text{one is heart \& other is spade})$

$= \frac{n(B)}{n(S)} = \frac{169}{1326} = \frac{13}{102}$

Ex Three horse A, B, C are in a race. A is thrice as likely to win as B & B is twice as likely to win as C. Find their respective prob. of winning. What is prob. that A or B wins.

Solⁿ.

Let 'p' be prob. of winning of horse C

ie $P(C) = p$.

Then $P(B) = 2p$ & $P(A) = 3 \cdot P(B) = 6p$

But $P(A) + P(B) + P(C) = 1$

$6p + 2p + p = 1 \Rightarrow p = 1/9$

$\therefore P(C) = 1/9$, $P(B) = 2/9$ & $P(A) = 6/9$

$\therefore P(A \text{ or } B \text{ wins}) = P(A) + P(B)$

$= \frac{6}{9} + \frac{2}{9} = \frac{8}{9}$

$\because P(A \cap B) = 0$. A & B are mutually exclusive.

* Conditional Probability *

एक event होऊन जाते आणि दुसरा event find करावं लागते आणि जो event होऊन जात तो Sample space बनते दुसऱ्या event साठी

Let A & B be any two events in sample space 'S'. Then prob. of happening of an event A, when another event B is known to have already happened is called conditional prob. of A given B.

It is denoted by $P(A|B)$ & given by $P(A|B) = \dfrac{P(A\cap B)}{P(B)} = \dfrac{n(A\cap B)}{n(B)} = \dfrac{n(A\cap B)/n(S)}{n(B)/n(S)}$

$\because P(B) > 0.$

Ex 3- Coins are tossed.

$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}.$

$n(S) = 8.$

Let:

A : Atleast two head appears

B : last coin shows tail.

$A = \{HHH, HHT, HTH, THH\}$; $n(A) = 4$

$B = \{HHT, HTT, THT, TTT\}$; $n(B) = 4.$

~~P(A)~~ =

Find the probability of event A, if B is occurred

or

Find the prob. when atleast 2-heads appeared, if last coin shows tail.

ie $P(A|B) = \dfrac{P(A\cap B)}{P(B)} = \dfrac{n(A\cap B)}{n(B)} = \dfrac{1}{4}.$

$P(A\cap B) = \{HHT\}$, $n(A\cap B) = 1$

$P(A\cap B) = \dfrac{n(A\cap B)}{n(S)} = \dfrac{1}{8}$

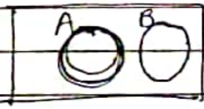
$P(B) = \dfrac{n(B)}{n(S)} = \dfrac{4}{8}$

$P(A|B) = \dfrac{1/8}{4/8} = \dfrac{1}{4}$

* If prob. of happening of A is independent of happening of B. ie $A\cap B = \phi$ then

$P(A|B) = P(A)$

& $P(A\cap B) = P(A)\cdot P(B).$



$\because P(A|B) = \dfrac{P(A\cap B)}{P(B)}$ ← $P(A)$

In general, if $A_1, A_2, A_3, \dots A_n$ are independent events then

$P(A_1 \cap A_2 \cap A_3 \cap \dots A_n) = P(A_1)\cdot P(A_2) \dots P(A_n)$

* Multiplication Thm of prob.

For any two events A given B,

$P(A\cap B) = P(A)\cdot P(B|A)$

$P(A_i|B) = \frac{P(A_i)\cdot P(B|A_i)}{\sum P(A_i)\, P(B|A_i)}$

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Baye's Th^m. statement

Suppose that events $A_1, A_2, \ldots, A_n$ form a disjoint partition of a sample space 'S' (i.e. events A_i are mutually exclusive & their union is S) & B be any event. Then for any i,

$$P(A_i|B) = \frac{P(A_i)\cdot P(B|A_i)}{P(A_1)\cdot P(B|A_1) + P(A_2)\cdot P(B|A_2) + \cdots + P(A_n)\cdot P(B|A_n)}.$$

Ex(1) The chances of solving a problem by three students A, B, C independently are $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$ resply. What is the prob. that prob will Solved?

Sol^n: G^n; $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{3}$, $P(C) = \frac{1}{4}$

P(prob. is solved) = P(prob. is solved by at least one student)

= 1 − P(prob. is not solved by any of them) → (1)

∴ Probabilities of A, B, C ~~are~~ not solving problems are

$P(A^c) = 1 - P(A) = 1 - \frac{1}{2} = \frac{1}{2}$

$P(B^c) = 1 - P(B) = 1 - \frac{1}{3} = \frac{2}{3}$

$P(C^c) = 1 - P(C) = 1 - \frac{1}{4} = \frac{3}{4}$.

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∴ P(Prob. is not solved by any students)

$= P(A^c \cap B^c \cap C^c)$

$= P(A^c)\cdot P(B^c)\cdot P(C^c)$ {∵ A, B, C are independent}

$= \frac{1}{2}\cdot\frac{2}{3}\cdot\frac{3}{4} = \frac{1}{4}$.

∴ Eq^n (1) ⇒ P(prob. is solved) $= 1 - \frac{1}{4} = \frac{3}{4}$.

Or

By Additional Th^m,

P(prob. is solved) $= P(A \cup B \cup C)$

$= P(A) + P(B) + P(C) - P(A\cap B) - P(A\cap C) - P(B\cap C) + P(A\cap B\cap C)$

$= P(A) + P(B) + P(C) - P(A)\cdot P(B) - P(A)\cdot P(C) - P(B)\cdot P(C) + P(A)\cdot P(B)\cdot P(C)$

(∵ A, B, C have tried independently)

$= \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \left(\frac{1}{2}\cdot\frac{1}{3}\right) - \left(\frac{1}{4}\cdot\frac{1}{2}\right) - \left(\frac{1}{3}\cdot\frac{1}{4}\right) + \left(\frac{1}{2}\cdot\frac{1}{3}\cdot\frac{1}{4}\right)$

$= \frac{3}{4}$.

H.W. In a certain college, 25% of students failed in maths, 15% of students failed in physics & 10% of students failed in both subjects. A student is selected at random.

(i) If he failed in mathematics what is prob. that he failed in physics.

(ii) What is prob. that he failed in maths or physics?

⇒ Let A & B be events that students failed in maths & physics resp.

$P(A) = \frac{25}{100}$, $P(B) = \frac{15}{100}$ & $P(A \cap B) = \frac{10}{100}$

(i) Prob. that student failed in phy, gⁿ that he/she has failed in maths

$$P(B|A) = \frac{P(A \cap B)}{P(A)} = \frac{10/100}{25/100} = \frac{10}{25} = \frac{2}{5}.$$

(ii) Prob. that he/she failed in maths or physics is

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$= \frac{25}{100} + \frac{15}{100} - \frac{10}{100}$$
$$= 3/10.$$

$P(A) = \frac{{}^{2}C_1 \times {}^{3}C_1}{{}^{6}C_2} = \frac{3}{3 \times 5} = 1/5$

Ex We are gⁿ three urns as follows:

Urn A contains 1 white, 2 black & 3 red
Urn B contains 2 white, 1 black & 1 red → marbles,
Urn C " 4 white, 5 black & 3 red marbles.

One urn is chosen at random & two marbles are drawn. If one is white & other is red, then what are probabilities that they come from urns A, B, or C?

$n(S) = 6 + 4 + 12 = 22$

⇒ Let D be the event that one marble is white & other is red.

$P(A) = \frac{6}{22}$

To find :- $P(A|D)$, $P(B|D)$ & $P(C|D)$

Now; P(one white & one red are drawn from urn A)

$$= P(D|A)$$
$$= \frac{{}^{1}C_1 \times {}^{3}C_1}{{}^{6}C_2} = \frac{1 \times 3}{15} = 1/5$$

P(one white & one red are drawn from urn B)

$$= P(D|B) = \frac{{}^{2}C_1 \times {}^{1}C_1}{{}^{4}C_2} = \frac{\frac{2!}{1!\,1!} \times 1}{\frac{4!}{2!\cdot 2!}} = \frac{2 \times 1}{6}$$
$$= 1/3$$

lly; $P(D|C) = \frac{{}^{4}C_1 \times {}^{3}C_1}{{}^{12}C_2} = \frac{2}{11}$

To choose urn A from A, B, C urn
$P(A) = 1/3$
lly $P(B) = 1/3 = P(C)$

Now, By Baye's Thᵐ;

$$P(A|D) = \frac{P(A) \cdot P(D|A)}{P(A) \cdot P(D|A) + P(B) \cdot P(D|B) + P(C) \cdot P(D|C)}$$

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$$P(A|D) = \frac{\left(\frac{1}{3}\cdot\frac{1}{5}\right)}{\left(\frac{1}{3}\cdot\frac{1}{5}\right)+\left(\frac{1}{3}\cdot\frac{1}{3}\right)+\left(\frac{1}{3}\cdot\frac{2}{11}\right)}$$

$$= \frac{33}{118}$$

$$P(B|D) = \frac{P(B)\cdot P(D|B)}{P(A)\cdot P(D|A) + P(B)\cdot P(D|B) + P(C)\, P(D|C)}$$

$$= \frac{55}{118}$$

$$P(C|D) = \frac{P(C)\cdot P(D|C)}{P(A)\cdot P(D|A) + P(B)\cdot P(D|B) + P(C)\cdot P(D|C)}$$

$$= \frac{15}{94}$$

H.W. Ex: A factory manufacturing television has four units A, B, C & D. The units manufacture 15%, 20%, 30% & 35% of the total outputs resply. It was found that out of their outputs, 1%, 2%, 2% & 3% are defective. A television is chosen at random from total output & was found to be defective. What is probability that it was manufactured by unit C?

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Sol^n. Here; $P(A) = \frac{15}{100}$, $P(B) = \frac{20}{100}$, $P(C) = \frac{30}{100}$ & $P(D) = \frac{35}{100}$

Let E be an event that televisions are defective.

$\therefore P(E|A) = \frac{1}{100}$, $P(E|B) = \frac{2}{100} = P(E|C)$

$P(E|D) = \frac{3}{100}$.

By Baye's Th^m, the req. prob. is

$$P(C|E) = \frac{P(C)\cdot P(E|C)}{P(A)\cdot P(E|A) + P(B)\cdot P(E|B) + P(C)\cdot P(E|C) + P(D)\cdot P(E|D)}$$

$$P(C|E) = \frac{3/5}{\frac{3}{20}+\frac{2}{5}+\frac{3}{5}+\frac{21}{20}} = \frac{3}{11}.$$

$P_1 = P(A) = \frac{15}{100}$; $P_2 = P(B) = \frac{20}{100}$; $P(C) = P_3 = \frac{30}{100}$

$P_4 = P(D) = 35/100$; $P_1' = P(E|A) = 1/100$

$P_2' = P(E|B) = 2/100$

$P_3' = P(E|C) = 2/100$; $P_4' = P(E|D) = 3/100$.

$$P(C|E) = \frac{P_3 \cdot P_3'}{P_1 P_1' + P_2 P_2' + P_3 P_3' + P_4 P_4'} = \frac{P(C)\cdot P(E|C)}{P(A)\cdot P(E|A) + \ldots}$$

* Random Variables *

If we assign a real no. 'x' to each sample point of sample space S, then we have a fn defined on that sample sp. This functn is called random variable (R.V). It is denoted by X / Y.

Ex: Suppose that coin is tossed twice. Then the sample sp is gn by $S = \{HH, HT, TH, TT\}$ ie $n(S) = 4$

Let the random variable X is gn by $X = (\text{no. of heads})^2 - (\text{no. of tail})$

S	HH	HT	TH	TT
X.	$2^2 - 0 = 4$	$1^2 - 1 = 0$	$1^2 - 1 = 0$	$0^2 - 2 = -2$

ie $X = \{-2, 0, 4\}$

R.V. →

→ Discrete Random Variable (d.r.v):-
R.V. takes finite no. or countably infinite no. of values
Ex $X = \{x_1, x_2, \ldots, x_n\}$ or $X = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ are discrete r.v.

→ Continuous random variable (c.r.v):-
R.V. takes non-countably ∞-no. of values
Ex: $X = \{\ldots, -2, \ldots, -1, \ldots, 0, \ldots 1, \ldots, 2, \ldots\}$
X takes all real values from $-\infty$ to ∞
$\therefore$ X is cts. r.v.

(4) 2-times coin tossed – Discrete.

Ex (1) No. of children in a family – Discrete
(2) Time of running 5 km – cts
(3) Amount of sugar in a coke – cts
wt of persons in one city → cts.

* Prob. function / Prob. distribution for discrete random variable

Defn. Discrete r.v. Let $X : S \to R$ be discrete r.v with prob. function $f(x)$.
A Probability mass function $f(x)$ for X is defn. by

discrete PMF असतं

(i) $f(x) \geq 0, \ \forall x$
(ii) $\sum_x f(x) = 1, \ \forall x \in X$

Note: (Ex on next pg)

* Distribution function / cumulative distributn functn for discrete random variables.

Let discrete r.v. X takes values $x_1, x_2, x_3, \ldots, x_n$ with probabilities $f(x_1), f(x_2), \ldots, f(x_n)$ resply. then distributn function $F(x)$ for discrete r.v. X defd as

$$F(x) = P(X < x) = P(-\infty < X \leq x) = \sum_{u \leq x} f(u)$$

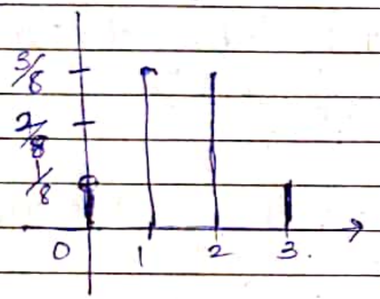
ie

$$F(x) = \begin{cases} 0 & 0 < x < x_1 \\ f(x_1) & x_1 \leq x < x_2 \\ f(x_1) + f(x_2) & x_2 \leq x < x_3 \\ \vdots & \\ f(x_1) + f(x_2) + \cdots + f(x_n) & ; x_n \leq x < \infty \end{cases}$$

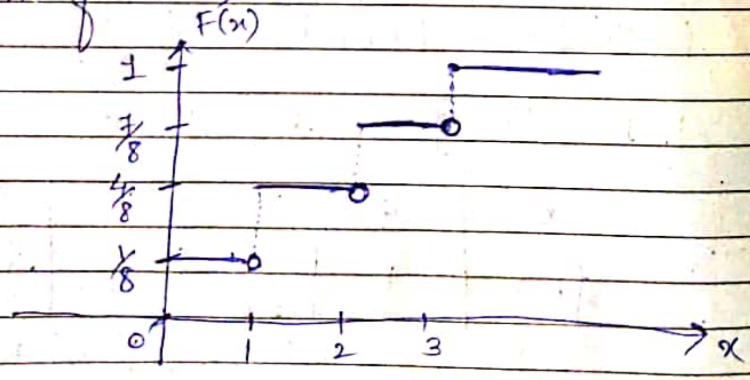
(ii) Distribution f^n $F(x)$ is g^n by

$$F(x) = \begin{cases} 0 & -\infty < x < 0 \\ 0 + \frac{1}{8} = \frac{1}{8} & 0 \le x < 1 \\ 0 + \frac{1}{8} + \frac{3}{8} = \frac{4}{8} = \frac{1}{2} & 1 \le x < 2 \\ \frac{1}{8} + \frac{3}{8} + \frac{3}{8} = \frac{7}{8} & 2 \le x < 3 \\ \frac{1}{8} + \frac{3}{8} + \frac{3}{8} + \frac{1}{8} = 1 & 3 \le x < \infty \end{cases}$$

(iii) Graph of $f(x)$



Graph of $F(x)$



aces card - A.

Q. Let X be a random variable giving number of aces in random draw of four cards from pack of 52 cards. Find probability f^n of distribution f^n for X.

Sol^n. X denotes no. of aces in random.

Here $X = \{0, 1, 2, 3, 4\}$.

1st to find sample space

Now, 4 cards can be drawn out of 52 cards in

$$^{52}C_4 = \frac{52!}{4!\,48!} = \frac{52 \times 51 \times 50 \times 49}{4 \cdot 3 \cdot 2 \cdot 1}$$

$$^{52}C_4 = 270725 \text{ ways}$$

$\therefore$ Sample sp. S contains 2,70,725 sample points

(i) Zero ace card (non-ace card) can be drawn

$$^{48}C_4 = \frac{48 \cdot 47 \cdot 46 \cdot 45}{1 \cdot 2 \cdot 3 \cdot 4} = 1,94,580 \text{ ways}$$

$$P(X=0) = \frac{1,94,580}{2,70,725}$$

(ii) one face card (1 ace + 3 non-ace) can be drawn

$$^4C_1 \times {}^{48}C_3 = \frac{4!}{1!\,3!} \times \frac{48!}{3!\,45!}$$

$$= 69,184 \text{ ways}$$

$$P(X=1) = \frac{69184}{270725}$$

(ii) Two ace cards (2-ace & 2-non-ace) can be drawn

$^4C_2 \times {}^{48}C_2 = \frac{4!}{2!\,2!} \times \frac{48!}{2!\,46!} = 6768$ ways

$P(X=2) = \frac{6768}{270725}$

(iii) Three ace cards (3-ace & 1-non-ace) can be drawn

$^4C_3 \times {}^{48}C_1 = \frac{4!}{3!\,1!} \times \frac{48!}{1!\,47!} = 192$ ways.

$P(X=3) = \frac{192}{270725}$

(iv) All four ace cards can be drawn.

$^4C_4 = \frac{4!}{4!\,0!} = 1$ ways.

$P(X=4) = \frac{1}{270725}$

∴ Prob. funct^n $f(x)$ is given by

x	0	1	2	3	4
$f(x)$	$\frac{194580}{270725}$	$\frac{69184}{270725}$	$\frac{6768}{270725}$	$\frac{192}{270725}$	$\frac{1}{270725}$

∴ $\sum f(x) = 1$ & $f(x) > 0$

Distribution funct^n $F(x)$ is given by

~~F(x) =~~

$$F(x) = \begin{cases} 0 & , -\infty < x < 0 \\ 194580/270725 & ; 0 \le x < 1 \\ = \frac{194580+69184}{270725} = 263764/270725 & 1 \le x < 2 \\ \frac{270532}{270725} & 2 \le x < 3 \\ 270724/270725 & 3 \le x < 4 \\ 1 & 4 \le x < 1 \end{cases}$$

H.W A discrete random variable X has foll. probability f^n:

x	0	1	2	3	4	5	6	7
$f(x)$	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2+k$

Find value of k & $P(x > 6)$.

Sol^n

If $f(x)$ is prob. f^n, then

(i) $f(x) \ge 0 \Rightarrow k \ge 0$

(ii) $\sum_x f(x) = 1 \Rightarrow 0 + k + 2k + 2k + 3k + k^2 + 2k^2 + (7k^2 + k) = 1$

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$\because 10k^2 + 9k = 1 \Rightarrow 10k^2 + 9k - 1 = 0$

$\Rightarrow (10k-1)(k+1) = 0 \Rightarrow k = \frac{1}{10}$ or $k = -1$

$\because k \geq 0$, $\therefore k = \frac{1}{10}$ is req^n. value of k.

(ii) $P(x > 6)$ i.e. $P(x=7) = 7k^2 + k$

$= 7\left(\frac{1}{10}\right)^2 + \frac{1}{10} = \frac{17}{100}$.

Q A discrete random variable x having foll. prob. function.

x	0	1	2	3	4	5	6	7	8
$f(x)$	a	$3a$	$5a$	$7a$	$9a$	$11a$	$13a$	$15a$	$17a$

Find (i) Constant 'a' (ii) $P(x<3)$
(iii) $P(1 < x < 6)$ (iv) Distribution function

Sol^n: Since $f(x)$ is probability mass function (PMF),

$f(x) \geq 0 \Rightarrow a \geq 0 \; ; \forall x$

& $\sum_x f(x) = 1 \Rightarrow a + 3a + 5a + 7a + 9a + 11a + 13a + 15a + 17a = 1$

$\Rightarrow \boxed{a = \frac{1}{81}}$.

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(ii) $P(x<3) = P(x=0) + P(x=1) + P(x=2)$
$= a + 3a + 5a = 9a$
$= 9 \times \frac{1}{81} = \frac{1}{9}$

(iii) $P(1<x<6) = P(x=2) + P(x=3) + P(x=4) + P(x=5)$
$= 5a + 7a + 9a + 11a = 32a$
$= 32 \times \frac{1}{81} = \frac{32}{81}$

(iv) Distribution f^n; $F(x)$

$$F(x) = \begin{cases} 0 & , -\infty < x < 0 \\ a = \frac{1}{81} & 0 \leq x < 1 \\ a + 3a = 4a = \frac{4}{81} & 1 \leq x < 2 \\ 4a + 5a = 9a = \frac{9}{81} & 2 \leq x < 3 \\ 9a + 7a = 16a = \frac{16}{81} & 3 \leq x < 4 \\ 16a + 9a = \frac{25}{81} & 4 \leq x < 5 \\ 25a + 11a = \frac{36}{81} & 5 \leq x < 6 \\ 36a + 13a = \frac{49}{81} & 6 \leq x < 7 \\ 49a + 15a = \frac{64}{81} & 7 \leq x < 8 \\ 64a + 17a = \frac{81}{81} = 1 & 8 \leq x < \infty \end{cases}$$

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* Probability function or Probability distribution of continuous Random Variable.

Let x be continuous random variable. Then $f(x)$ is called probability funct^n or probability distribution f^n (p.d.f) or probability density f^n (p.d.f), if $f(x)$ satisfies (i) $f(x) \geq 0$

(ii) $\int_{-\infty}^{\infty} f(x)\, dx = 1$.

Note. Here we find probability that x lies between a & b. It defined as

$$P(a \leq X \leq b) = \int_a^b f(x)\, dx$$

or

$$P(a < x < b) = \int_a^b f(x) \cdot dx$$

Ex Let X be a random variable having density function

$$f(x) = \begin{cases} kx, & 0 \leq x \leq 2 \\ 0, & \text{otherwise.} \end{cases}$$

Find (i) Constant 'k' (ii) $P\left(\frac{1}{2} < X < \frac{3}{2}\right)$ (iii) $P(X > 1)$

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Sol^n. Since $f(x)$ is prob. density function, $f(x) \geq 0 \Rightarrow k \geq 0$, $\forall x$.

and $\int_{-\infty}^{\infty} f(x)\, dx = 1$

ie $\int_0^2 f(x) \cdot dx = 1$

$\Rightarrow \int_0^2 kx \cdot dx = 1 \Rightarrow k \cdot \left[\frac{x^2}{2}\right]_0^2 = 1$

$\Rightarrow \boxed{k = \frac{1}{2}}$

(ii) $P\left(\frac{1}{2} < X < \frac{3}{2}\right) = \int_{1/2}^{3/2} f(x) \cdot dx$

$= \int_{1/2}^{3/2} k \cdot x \cdot dx = \frac{1}{2} \cdot \int_{1/2}^{3/2} x \cdot dx$

$= \frac{1}{2} \left[\frac{x^2}{2}\right]_{1/2}^{3/2} = \frac{1}{4} \cdot \left[\left(\frac{3}{2}\right)^2 - \left(\frac{1}{2}\right)^2\right]$

$= \frac{1}{4} \cdot \frac{8}{4} = \frac{1}{2}$.

(iii) $P(X > 1) = P(1 < X < \infty) = P(1 < X < 2)$

$= \int_1^2 kx \cdot dx = \frac{1}{2} \cdot \left[\frac{x^2}{2}\right]_1^2 = \frac{3}{4}$

Q. Let X be a continuous random variable with probability function

$$f(x) = \begin{cases} \left(\frac{x}{6} + k\right), & 0 \le x \le 3 \\ 0 ; & \text{otherwise.} \end{cases}$$

Find (i) Constant k, (ii) $P(1 \le X \le 2)$.

Sol^n: Given: X be a continuous random variable
$f(x)$ is a probability density function (p.d.f)

(i) $\Rightarrow f(x) \ge 0 \Rightarrow k \ge 0 \; ; \forall x$

& $\int_{-\infty}^{\infty} f(x) \cdot dx = 1$

ie $\int_0^3 f(x) \cdot dx = 1 \Rightarrow \int_0^3 \left(\frac{x}{6} + k\right) dx = 1$

$\Rightarrow \left[\frac{x^2}{2} \times \frac{1}{6} + kx\right]_0^3 = 1$

$\Rightarrow \boxed{k = \frac{1}{12}}$

(ii) $P(1 \le X \le 2) = \int_1^2 \left(\frac{x}{6} + k\right) \cdot dx = \int_1^2 \left(\frac{x}{6} + \frac{1}{12}\right) dx$

$= \left[\frac{1}{6} \cdot \frac{x^2}{2} + \frac{1}{12} \cdot x\right]_1^2 = \frac{1}{3}$

11.10 A random variable X has density f^n

$$f(x) = \begin{cases} k \cdot e^{-3x} & ; x > 0 \\ 0 & ; x \le 0 \end{cases}$$

Find.
(i) Constant k $(k=3)$ (ii) $P(1 < X < 3)$
(iii) $P(X \ge 2)$ (iv) $P(X < 1)$

Ans

$k = 3$ (ii) $\int_1^3 3e^{-3x} dx = e^{-3} - e^{-9}$

(iii) $\int_2^{\infty} 3e^{-3x} dx = e^{-6}$

(iv) $P(-\infty < X < 1) = \int_{-\infty}^{1} f(x) \, dx = \int_{-\infty}^{0} f(x) \cdot dx + \int_0^1 f(x) \, dx$

$= \int_{-\infty}^{0} 0 \cdot dx + \int_0^1 k \cdot e^{-3x} dx = 1 - e^{-3}$

* Distribution function of Continuous Random Variable. *

The distribution function $F(x)$ of continuous random variable X is defined as

$F(x) = P(X \le x) = P(-\infty < X \le x) = \int_{-\infty}^{x} f(v) \, dv$

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Note (i) $P(a \le X \le b) = P(X \le b) - P(X \le a)$
$= F(b) - F(a)$ (||ly for < sign)

(ii) $\frac{d}{dx}[F(x)] = f(x)$

using (ii), we can find density f^n $f(x)$ from distribution f^n $F(x)$.

Q: Let $f(x) = \frac{c}{x^2+1}$, $-\infty < X < \infty$ is prob. f^n for random variable X.

Find: (i) constant 'c', (ii) Distribution f^n of X

Sol^n: Since, $f(x)$ is a probability density f^n (p.d.f)

ie $f(x) \ge 0 \Rightarrow c \ge 0 \quad \forall x$

&

$$\int_{-\infty}^{\infty} f(x) \cdot dx = 1 = \int_{-\infty}^{\infty} \frac{c}{x^2+1} dx = 1$$

$$\Rightarrow c \cdot \int_{-\infty}^{\infty} \frac{1}{x^2+1} \cdot dx = 1$$

$$\Rightarrow c \cdot \left[\tan^{-1} x\right]_{-\infty}^{\infty} = 1$$

$$\Rightarrow c \cdot \left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right] = 1 \Rightarrow \boxed{c = \frac{1}{\pi}}$$

$\therefore$ Prob. density f^n; $f(x) = \frac{1}{\pi \cdot (x^2+1)}$

(ii) Distribution function of random variable X,

$$F(x) = P(X \le x) = P(-\infty < X \le x) = \int_{-\infty}^{x} f(v) \cdot dv$$

$$F(x) = \int_{-\infty}^{x} \frac{1}{\pi(v^2+1)} \cdot dv$$

$$= \frac{1}{\pi} \cdot \left[\tan^{-1} v\right]_{-\infty}^{x} = \frac{1}{\pi}\left\{\tan^{-1} x + \frac{\pi}{2}\right\}$$

$$\Rightarrow F(x) = \frac{1}{\pi}\left\{\tan^{-1} x + \frac{\pi}{2}\right\}, \quad -\infty < x < \infty$$

Q Prove that for suitable constant c

$$F(x) = \begin{cases} c\,(1-e^{-x})^2 & ; x > 0 \\ 0 & ; \text{otherwise} \end{cases}$$

is a distribution function.

Sol^n: Let $F(x)$ is a distribution function
Let $f(x)$ is the prob. density f^n

By def^n; $\frac{d}{dx}[F(x)] = f(x)$

$$\Rightarrow f(x) = \begin{cases} 2c \cdot (1-e^{-x}) \cdot e^{-x} & ; x > 0 \\ 0 & ; x \le 0 \end{cases}$$

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$\because f(x) \geq 0 \Rightarrow c \geq 0 \; ; \forall x.$

& $\int_{-\infty}^{\infty} f(x).dx = 1$

$\Rightarrow \int_{0}^{\infty} f(x).dx = 1 \Rightarrow \int_{0}^{\infty} 2c\,e^{-x}.(1-e^{-x})dx = 1.$

$\Rightarrow 2c.\int_{0}^{\infty} e^{-x}(1-e^{-x}).dx = 1$

$\Rightarrow 2c.\left\{\int_{0}^{\infty} e^{-x}.dx - \int_{0}^{\infty} e^{-2x}.dx\right\} = 1$

$\Rightarrow 2c.\left\{\left[-e^{-x}\right]_{0}^{\infty} - \left(\frac{e^{-2x}}{-2}\right)_{0}^{\infty}\right\} = 1$

$\Rightarrow 2c\left\{(-0+1) + \frac{1}{2}[0-1]\right\} = 1$

$\Rightarrow 2c.\left\{\frac{1}{2}\right\} = 1 \Rightarrow \boxed{c=1}$

• $F(x)$ is distribut^n f^n g^n by

$$F(x) = \begin{cases} (1-e^{-x})^2 & ; x > 0 \\ 0 & ; \text{otherwise} \end{cases}$$

4 Prob. density f^n, $f(x) = \begin{cases} 2e^{-x}(1-e^{-x}) & ; x > 0 \\ 0 & ; x \leq 0 \end{cases}$

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H.W. A random variable X has density function

$$f(x) = \begin{cases} kx^2 & ; 1 \leq x \leq 2 \\ kx & ; 2 < x < 3 \\ 0 & ; \text{otherwise} \end{cases}$$

Find (i) constant k (ii) $P\left(\frac{1}{2} < x < \frac{3}{2}\right)$
(iii) Distribution function $F(x)$

Sol^n.

Since $f(x)$ is a prob. density f^n

$\therefore f(x) \geq 0 \Rightarrow k \geq 0$

& $\int_{-\infty}^{\infty} f(x).dx = 1$

$\Rightarrow \int_{-\infty}^{1} f(x)dx + \int_{1}^{2} f(x)dx + \int_{2}^{3} f(x)dx + \int_{3}^{\infty} f(x)dx = 1$

$\Rightarrow 0 + \int_{1}^{2} kx^2 dx + \int_{2}^{3} kx\,dx + 0 + 0 = 1$

$\Rightarrow \frac{k}{3}.\left[x^3\right]_{1}^{2} + \frac{k}{2}\left[x^2\right]_{2}^{3} = 1$

$\Rightarrow \frac{29k}{6} = 1 \Rightarrow \boxed{k = 6/29}$

(ii) $P\left(\frac{1}{2} < x < \frac{3}{2}\right) = \int_{1}^{2} f(x).dx = \int_{1}^{2} k.x^2 dx$

$= \frac{6}{29}.\frac{1}{3}.\left[x^3\right]_{1}^{2} = \frac{2\cancel{6} \times 7}{29 \times \cancel{3}}$

$= 14/29$

To find distribution function $F(x)$.

For $x < 1$; $F(x) = 0$

For $2 \le x < 3$; $F(x) = \int_{-\infty}^{x} f(x)\,dx$

$\Rightarrow F(x) = \int_{-\infty}^{1} f(x)\,dx + \int_{1}^{2} f(x)\,dx + \int_{2}^{x} f(x)\,dx$

$= 0 + \int_{1}^{2} k x^2\,dx + \int_{2}^{x} k x\,dx$

$= k \left[\frac{x^3}{3}\right]_1^2 + \frac{k}{2}\left[x^2\right]_2^x$

$= \frac{k}{3}(8-1) + \frac{k}{2}(x^2-4)$

$= \frac{(3x^2+2)}{29}$ $\left\{\because k = \frac{6}{29}\right\}$

For $1 \le x \le 2$; $F(x) = \int_{-\infty}^{x} f(x)\,dx$

$\Rightarrow F(x) = \int_{-\infty}^{1} f(x)\,dx + \int_{1}^{x} f(x)\,dx$

$= 0 + \int_{1}^{x} k x^2\,dx = \frac{6}{29}\left[\frac{x^3}{3}\right]_1^x$

$= \frac{2}{29}(x^3-1)$.

For $x \ge 3$; $F(x) = \int_{-\infty}^{\infty} f(x)\,dx$

$F(x) = \int_{-\infty}^{1} f(x)\,dx + \int_{1}^{2} f(x)\,dx + \int_{2}^{3} f(x)\,dx + \int_{3}^{\infty} f(x)\,dx$

$= 0 + \int_{1}^{2} k x^2\,dx + \int_{2}^{3} k x\,dx + 0$

$= \frac{k}{3}\left[x^3\right]_1^2 + \frac{k}{2}\left[x^2\right]_2^3$

$= \frac{k}{3}\cdot(7) + \frac{k}{2}[5] = \frac{14}{29} + \frac{15}{29} = 1$

$=$

$$\therefore F(x) = \begin{cases} 0 & ; -\infty < x < 1 \\ \frac{2}{29}(x^3-1) & ; 1 \le x \le 2 \\ \frac{(3x^2+2)}{29} & ; 2 < x < 3 \\ 1 & ; x \ge 3 \end{cases}$$

4.10 A random variable X has foll. prob. distribution:

X :	0	1	2	3	4	5	6	7	8
$f(x)$:	a	3a	5a	7a	9a	11a	13a	15a	17a

(i) Find value of a (ii) Find $P(X<3)$, $P(0<X<$
(iii) Distribution f^n

Ans (i) $a = 1/81$ (ii) $1/9$, $24/81$

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X	0	1	2	3	4	5	6	7	8
F(x)	a	4a	9a	16a	25a	36a	49a	64a	81a

H.W.

Ex Find distribution f^n for random variable X having density function

$$f(x) = \begin{cases} x^2/9 & ; \ 0 < x < 3 \\ 0 & ; \ \text{otherwise} \end{cases}$$

& hence find $P(1 \le X \le 2)$.

Ans
$$F(x) = \begin{cases} 0 & ; \ x < 0 \\ x^3/27 & ; \ 0 \le x \le 3 \\ 1 & ; \ x \ge 3 \end{cases}$$

& $P(1 \le X \le 2) = 7/27$

* Joint Distribution of Discrete Random Variables and Joint Prob. function.

Let X & Y be two discrete random variables. Then $f(x,y) = P(X=x, Y=y)$ is called joint probability function if

(i) $f(x,y) \ge 0$

(ii) $\sum_x \sum_y f(x,y) = 1$.

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Marginal Probability f^n.

1) Marginal Prob. function of r.v. X is g^n by

$$f_1(x_i) = P(X = x_i) = \sum_{j=1}^{n} f(x_i, y_j) \ ; \text{ where } i = 1, 2, \dots, m$$

2) Marginal Prob. f^n of r.v. Y is g^n by

$$f_2(y_j) = P(Y = y_j) = \sum_{i=1}^{m} f(x_i, y_j) \ ; \text{ where } j = 1, 2, \dots, n$$

Joint Probability Table.

X \ Y	y_1	y_2	y_3	...	y_n	Marginal Prob. f^n of 'X' → Total
x_1	$f(x_1, y_1)$	$f(x_1, y_2)$	$f(x_1, y_3)$	....	$f(x_1, y_n)$	$f_1(x_1)$
x_2	$f(x_2, y_1)$	$f(x_2, y_2)$	$f(x_2, y_3)$	...	$f(x_2, y_n)$	$f_1(x_2)$
x_3	$f(x_3, y_1)$	$f(x_3, y_2)$	...	...	$f(x_3, y_n)$	$f_1(x_3)$
⋮	⋮					⋮
x_m	$f(x_m, y_1)$	...		..	$f(x_m, y_n)$	$f_1(x_m)$
Marginal Prob. f^n of Y	$f_2(y_1)$	$f_2(y_2)$	$f_2(y_3)$	..	$f_2(y_n)$	Grand Total = 1