Unit-2

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* Mathematical Expectation:

Syllabus: Mathematical Expectation, Variance & Standard deviation, moments, moment generating functⁿs, Skewness & kurtosis.

* **Mathematical Expectation or Expected value of discrete random variable** X having values $x_1, x_2, \dots, x_n$ with probabilities $P(X = x_i) = f(x_i)$; $i = 1, 2, \dots, n$, is defⁿ as

$$E(X) = x_1 f(x_1) + x_2 f(x_2) + \dots + x_n f(x_n) = \sum_{i=1}^{n} x_i f(x_i)$$

Or $\boxed{E(X) = \sum x \cdot f(x)}$ $\because X$ is discrete r.v.

* Mathematical Expectatⁿ of Continuous r. variable X having prob. function (or density fⁿ) $f(x)$ is defined as

$$\boxed{E(X) = \int_{-\infty}^{\infty} x \cdot f(x) \cdot dx}$$

$\because X$ is cts random variable.

The expected value of r.v. X is also called mean of X & denoted by μ or μ_x.

Or

If X be a random v. with prob. distribution $f(x)$ then mean or expected value of X is

1) $\mu_1 = \mu_x = E(X) = \sum_x x \cdot f(x)$; if X is discrete random variable

2) $\mu = \mu_x = E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\, dx$; if X cts random variable.

Ex 2 A random variable X has density f^n

p.d.f — $f(x) = \begin{cases} e^{-x} & ; x \geq 0 \\ 0 & ; x < 0 \end{cases}$

Find (i) $E(X)$ (ii) $E(X^2)$ (iii) $E(e^{2X/3})$

Sol^n.

Since X is continuous random variable

(i) $E(X) = \int_{-\infty}^{\infty} x \cdot f(x) dx = \int_{-\infty}^{0} x \cdot f(x) dx + \int_{0}^{\infty} x\, f(x) dx$

$= \int_{0}^{\infty} x \cdot e^{-x} . dx$

$= \left[x \cdot \left(\frac{e^{-x}}{-1} \right) - e^{-x} \right]_0^{\infty}$

$\mu_1 = E(X) = \left[0 - e^{-\infty} - (0 - 1) \right] = \underline{\underline{1}}$

(ii) $E(X^2) = \int_{-\infty}^{\infty} x^2 \cdot f(x) \cdot dx = \int_{-\infty}^{0} 0 \cdot dx + \int_{0}^{\infty} x^2 \cdot f(x) \cdot dx$

$= \int_{0}^{\infty} x^2 \cdot e^{-x} dx$

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$$E(x^2) = \left\{ \cancel{-x^2 \cdot e^{-x}} - \left[2x \cdot (-e^{-x}) \right] \right\}_0^{\infty}$$

$$= \{(0 - 0) - (0 - \infty$$

$$E(x^2) = \left\{ -x^2 \cdot e^{-x} \right\}_0^{\infty} - \int_0^{\infty} 2x(-e^{-x})\, dx$$

$$= \{0 - 0\} + 2 \cdot \int_0^{\infty} x \cdot e^{-x} \cdot dx = \underline{\underline{2}}.$$

(iii) $$E\left[e^{2x/3}\right] = \int_{-\infty}^{\infty} e^{2x/3} \cdot f(x)\, dx = \int_0^{\infty} e^{2x/3} \cdot e^{-x} \cdot dx$$

$$= \int_0^{\infty} e^{-\frac{1}{3}x} \cdot dx = \left[\frac{e^{-\frac{1}{3}x}}{\left(-\frac{1}{3}\right)} \right]_0^{\infty}$$

$$= (-3) \cdot [0 - 1] = 3 //.$$

Q1 Let X be discrete r.v. giving number of heads in three tosses of fair coin. Find $E(x)$

Sol^n

x	0	1	2	3
p.m.f.: $f(x)$	$1/8$	$3/8$	$3/8$	$1/8$

$S = \{HHH, HHT, HTH, THH, TTH, THT, HTT, TTT\}$

$n(S) = 8$

Expected value of X is $E(X) = \sum x \cdot f(x)$

$$\Rightarrow E(X) = 0 \cdot \frac{1}{8} + 1 \cdot \frac{3}{8} + 2 \cdot \frac{3}{8} + 3 \cdot \frac{1}{8} = \frac{3+6+3}{8}$$

$$E(x) = \frac{12}{8} = \frac{3}{2}$$

* Variance & Standard Deviation.

The variance of random variable X with mean $(\mu = \mu_x)$ with prob. distribution $f(x)$ is given by

1) $\sigma_x^2 = Var(X) = E[(X-\mu)^2] = \sum_x (x-\mu)^2 \cdot f(x)$; if X is discrete random variable

2) $\sigma_x^2 = Var(X) = E[(X-\mu)^2] = \int_{-\infty}^{\infty} (x-\mu)^2 \cdot f(x) \cdot dx$, if X is continuous random variable

* The +ve square root of variance is called standard deviation or standard error

ie standard deviation of X;

$\sigma_x = +\sqrt{Var(X)}$

Ex 1: Find $Var(X)$ & σ_x ($\because X$ is discrete)

$\Rightarrow$ $Var(X) = E[(X-\mu)^2] = \sum (x-\mu)^2 \cdot f(x)$

$$\sigma_x^2 = \left(0-\frac{3}{2}\right)^2 \cdot \frac{1}{8} + \left(1-\frac{3}{2}\right)^2 \cdot \frac{3}{8} + \left(2-\frac{3}{2}\right)^2 \cdot \frac{3}{8} + \left(3-\frac{3}{2}\right)^2 \cdot \frac{1}{8}$$

$\Rightarrow$ $\sigma_x^2 = 3/4$

Now, standard deviation; $\sigma_x = \sqrt{3}/2$.

Q.2 $X \rightarrow$ ch. r.v. $f(x) = \begin{cases} e^{-x} & ; x \geq 0 \\ 0 & ; x < 0 \end{cases}$

(i) $Var(X)$ (ii) σ_x

$\Rightarrow$ $\sigma_x^2 = Var(X) = E[(X-\mu)^2] = \int_{-\infty}^{\infty} (x-\mu)^2 \cdot f(x) \cdot dx$

$= \int_0^{\infty} (x-1)^2 \cdot e^{-x} dx$

$= \left[-(x-1)^2 e^{-x}\right]_0^{\infty} - \int_0^{\infty} \left[2(x-1)(-e^{-x})\right] dx$

$= [0+1] + 2 \cdot \int_0^{\infty} (x-1) \cdot e^{-x} \cdot dx$

$= 1 + 2 \cdot \left\{\left[-(x-1) \cdot e^{-x}\right]_0^{\infty} - \left[e^{-x}\right]_0^{\infty}\right\}$

$= 1 + 2 \left\{[0 + (-1)] - [0-1]\right\}$

$= 1 + 2\{0\} = 1$

$\Rightarrow$ $\sigma_x^2 = 1$

(ii) Std. deviation $\sigma_x = \sqrt{Var(X)} = 1$

$\because E(1) = 1$

Theorem on Variance

Th^m 1) If x is random variable (discrete / cts) then $Var(x) = E(x^2) - [E(x)]^2$

Proof.

By def^n of variance;

$Var(x) = E[(x-\mu)^2]$

$\Rightarrow Var(x) = E[x^2 - 2x\mu + \mu^2]$

$\Rightarrow \quad = E(x^2) - 2\mu \cdot E(x) + \mu^2 \cdot E(1)$

$\Rightarrow \quad = E(x^2) - 2\mu \cdot \mu + \mu^2$

$\because E(x) = \mu$

$\Rightarrow \quad = E(x^2) - \mu^2$

$\Rightarrow \boxed{Var(x) = E(x^2) - [E(x)]^2}$

Th^m 2) If $c \in$ Constant, then $Var(cx) = c^2 \cdot Var(x)$.

Th^m 3) P.T. $Var(ax+b) = a^2\, Var(x)$, where a & b are constants.

In particular; if $b=0$; $Var(ax) = a^2\, Var(x)$

$a=0$; $Var(b) = 0$

$a=1$; $Var(x+b) = Var(x)$.

Th^m 4) If x & Y are independent r.v then $Var(x \pm Y) = Var(x) + Var(Y)$

OR

$$\sigma^2_{x \pm y} = \sigma_x^2 + \sigma_y^2$$

(discrete)

Q A random variable x is defined by

$$X = \begin{cases} -2 & ; \text{prob. } 1/3 \\ 3 & ; \text{prob. } 1/2 \\ 1 & ; \text{prob. } 1/6 \end{cases}$$

Find (i) $E(x)$ (ii) $E(2x+3)$ (iii) $E(x^2)$ (iv) $E(x^2+5x)$ (v) $Var(x)$.

Sol^n

(i) $E(x) = \sum x \cdot f(x) = \mu$

$\mu = (-2)\cdot\left(\frac{1}{3}\right) + 3\left(\frac{1}{2}\right) + 1\cdot\left(\frac{1}{6}\right) = 1$

$\boxed{\mu = 1}$

(ii) $E(2x+3) = 2 \cdot E(x) + 3 = 2\cdot(1) + 3 = 5$

(iii) $E(x^2) = \sum x^2 \cdot f(x) = (-2)^2 \cdot \frac{1}{3} + (3)^2 \cdot \frac{1}{2} + (1)^2 \cdot \frac{1}{6}$

$\boxed{E(x^2) = 6}$

(iv) $E(x^2 + 5x) = E(x^2) + 5\,E(x)$

$= 6 + 5\cdot(1) = 11$

(v) $Var(x) = E(x^2) - [E(x)]^2 = \sigma_x^2$

$\Rightarrow Var(x) = 6 - (1)^2 = 5 = \sigma_x^2$

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$(1+a)^{-1}$, $(1-a)^{-1}$ =

Ex HW Find mathematical expectation of discrete r.v. X whose probability f^n is $f(x) = \left(\frac{1}{2}\right)^x$, $x = 1, 2, 3, \ldots$ & find $P(|x-2|)$..

Sol^n To find : $E(x) = ?$

We have ; $E(x) = \sum x \cdot f(x)$

$\Rightarrow E(x) = 1\cdot\left(\frac{1}{2}\right)^1 + 2\cdot\left(\frac{1}{2}\right)^2 + 3\left(\frac{1}{2}\right)^3 + \ldots \rightarrow (1)$

Remember :- $(1+a)^{-n} = 1 - n\cdot a + \frac{(n+1)}{2}a^2 - \frac{(n+2)}{3}a^3 + \ldots$

$(1-a)^{-n} = 1 + na + \frac{(n+1)}{2}a^2 + \frac{(n+2)}{3}a^3 + \ldots$

$n=1$; $(1-a)^{-1} = 1 + a + a^2 + a^3 + \ldots$

$a = \frac{1}{2}$; $\left(1-\frac{1}{2}\right)^{-1} = 1 + \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \ldots$

$n=2$ $\left(1-\frac{1}{2}\right)^{-2} = 1 + 2\cdot\left(\frac{1}{2}\right) + \frac{3}{2}\left(\frac{1}{2}\right)^2 + \frac{4}{3}\cdot\left(\frac{1}{2}\right)^3 + \ldots$

Eq^n (1) $\Rightarrow$ $E(x) = \frac{1}{2}\left(1 + 2\cdot\left(\frac{1}{2}\right) + 3\cdot\left(\frac{1}{2}\right)^2 + \ldots\right)$

$\Rightarrow \quad = \frac{1}{2}\cdot\left\{\left(1-\frac{1}{2}\right)^{-2}\right\}$ .. by Binomial Th^m

$\Rightarrow E(x) = \frac{1}{2}\cdot\frac{1}{\left(1-\frac{1}{2}\right)^2} = \frac{1}{2}\times(2)^2 = \underline{\underline{2}}$

(ii) $P(|x-2| \le 2) = P(-2 \le (x-2) \le 2)$
$= P(0 \le x \le 4)$

$\Rightarrow = P(x=0) + P(x=1) + P(x=2) + P(x=3) + P(x=4)$

$= 0 + \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4$

$= 0 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{15}{16}$

Ex Let X be the random variable with density f^n $f(x) = \begin{cases} x^2/9 & ; 0 \le x \le 3 \\ 0 & ; \text{otherwise} \end{cases}$

Find $E(x)$ & $Var(x)$

$\Rightarrow$ $\mu = E(x) = \int_{-\infty}^{\infty} x\cdot f(x)\,dx = \int_0^3 x\cdot\frac{x^2}{9}\,dx$

$= \frac{1}{9}\cdot\left[\frac{x^4}{4}\right]_0^3 = \frac{9}{4}$

$\Rightarrow \boxed{\mu = E(x) = 9/4}$

$Var(x) = E[(X-\mu)^2] = E\left[\left(X - \frac{9}{4}\right)^2\right]$

$= \int_{-\infty}^{\infty}\left(x-\frac{9}{4}\right)^2\cdot f(x)\,dx = \int_0^3 \left(x-\frac{9}{4}\right)^2\left[\frac{x^2}{9}\right]dx$

$= \frac{1}{9}\int_0^3\left[x^4 - \frac{9}{2}x^3 + \frac{81}{16}x^2\right]dx$

$= \frac{1}{9}\cdot\left[\frac{x^5}{5} - \frac{9x^4}{8} + \frac{81x^3}{48}\right]_0^3$

$\boxed{Var(x) = 27/80}$

Or

$$Var(X) = E(x^2) - [E(x)]^2 \longrightarrow (1)$$

$$\sigma_x^2 = Var(x) = E(x^2) - \left(\frac{9}{4}\right)^2$$

$$\therefore E[x^2] = \int_{-\infty}^{\infty} x^2 \cdot f(x) \cdot dx = \int_0^3 x^2 \frac{x^2}{9} \cdot dx$$

$$= \frac{1}{9} \cdot \left[\frac{x^5}{5}\right]_0^3 = 27/5$$

Eqⁿ (1) ⇒

$$Var(x) = \frac{27}{5} - \left(\frac{9}{4}\right)^2 = \frac{27}{80}$$

H.W. If r.v. X is such that $E[(X-1)^2] = 10$, $E[(X-2)^2] = 6$

Find (i) $E[x]$ (ii) $Var(x)$ (iii) σ_x

Solⁿ:

$E[(X-1)^2] = 10 \Rightarrow E[x^2 - 2X + 1] = 10$

$\Rightarrow E[x^2] - 2E[x] + E(1) = 10$

$\Rightarrow E[x^2] - 2E[x] = 9 \longrightarrow (A)$ $E(1) = 1$

$E[(X-2)^2] = 6 \Rightarrow E[x^2 - 4X + 4] = 6$

(i) $\Rightarrow E[x^2] - 4E[x] = 6 - 4 = 2 \rightarrow (B) \because E[4] = 4$

from (A) & (B); (A) − (B) $\Rightarrow 2E[x] = 7 \Rightarrow \boxed{E[x] = 7/2}$

(ii) $Var(x) = E[x^2] - [E[x]]^2 = 16 - \left(\frac{7}{2}\right)^2$ $\quad \because E[x^2] = 16$

$\sigma_x^2 = 15/4$

(iii) $\sigma_x = \sqrt{\frac{15}{4}} = \frac{\sqrt{15}}{2}$

E[4] = 4.

Moments & Moment Generating functions.

"Moments about mean" (central moment) :- (Force of deviation of value from its mean).

Let X be the random variable with mean μ. Then rᵗʰ moment or rᵗʰ central moment of r.v. X about mean μ is denoted & defⁿ as

$$\mu_r = E[(X-\mu)^r] \; ; \; r = 0, 1, 2, \ldots$$

{ie Expected value of rᵗʰ power of deviation of X from its mean μ}.

In particular; if $r = 0 \Rightarrow \mu_0 = E[(x-\mu)^0] \Rightarrow E(1) = 1$

$r = 1$; $\mu_1 = E[(x-\mu)] = E[x] - \mu = \mu - \mu = 0$

$r = 2$; $\mu_2 = E[(x-\mu)^2] = Var\, X = \sigma_x^2$

ie 2ⁿᵈ moment of X about mean 'μ' is Variance of X & others are of fundamental importance.

Note. "Moment" is familiar mechanical term for measure of force with reference to its tendency to produce rotation.

* In statistics or Probability, it is use to describe to study nature of freq. distributⁿ (average / std. deviatⁿ का directⁿ फैलाव आकार)

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* If x is discrete r.v., then

r^{th} moment about mean; $\mu_r = E[(X-\mu)^r]$

$$\Rightarrow \boxed{\mu_r = \sum_x (x-\mu)^r \cdot f(x)}$$

*

If x is continuous r.v., then

$$\mu_r = E[(X-\mu)^r] = \int_{-\infty}^{\infty} (x-\mu)^r \cdot f(x) \cdot dx$$

* "MOMENT ABOUT ORIGIN" *

The r^{th} moment of r.v. 'X' about origin is denoted & defined as

$$\mu_r' = E[(X-0)^r] = E[(X)^r] \; ; r = 0, 1, 2, \ldots$$

* If X is discrete r.v. then

$$\mu_r' = E[X^r] = \sum_x x^r \cdot f(x)$$

* If X is continuous r.v., then

$$\mu_r' = E[x^r] = \int_{-\infty}^{\infty} x^r \cdot f(x)\, dx$$

$(a-b)^n = a_n - {}^nC_1 \cdot a_{n-1} \cdot b + {}^nC_2 \cdot a_{n-2} \cdot b^2 - {}^nC_3 \, a_{n-3} \cdot b^3 + \ldots$

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In particular;

i) $r = 0$, $\mu_0' = E[X^0] = E[1] = 1 = \mu_0$

$\Rightarrow \boxed{\mu_0' = \mu_0}$ i.e. $\mu_0' = 1 = \mu_0$.

$r = 1$, $\mu_1' = E[X] = \mu$ $\Rightarrow \boxed{\mu_1' = \mu}$

(μ = mean of X).

* Relatn b/w. central moment & moment about origin *

By defn of central moment (i.e. moment about mean) of r.v. 'X' we have;

$$\mu_r = E[(X-\mu)^r]$$

$$\mu_r = E\left[X^r - r \cdot X^{r-1} \cdot \mu + \frac{r(r-1)}{2!} X^{r-2} \mu^2 - \frac{r(r-1)(r-2)}{3!} X^{r-3} \cdot \mu^3 + \ldots + (-1)^r \cdot \mu^r\right]$$

$\longrightarrow$ by binomial thm

$$\Rightarrow \mu_r = E[X^r] - r \cdot \mu \cdot E[X^{r-1}] + \frac{r(r-1)}{2!} \mu^2 E[X^{r-2}] - \frac{r(r-1)(r-2)}{3!} \mu^3 \cdot E[X^{r-3}] + \ldots + (-1)^r \cdot \mu^r \cdot E(1)$$

$$\Rightarrow \mu_r = \mu_r' - r \cdot \mu \cdot \mu_{r-1}' + \frac{r(r-1)}{2!} \mu^2 \cdot \mu_{r-2}' - \ldots + (-1)^r \cdot \mu^r \cdot \mu_0$$

$\hookrightarrow$ (1)

$\left\{ \because E(1) = \mu_0 , \; E(x^r) = \mu_r' \right.$

$\boxed{\because \mu_0' = 1 \ \& \ \mu = \mu_1'}$

If $r = 1, 2, 3, 4, \ldots$

Eq^n ① $\Rightarrow$ $\mu_1 = \mu_1' - 1 \cdot \mu \cdot \mu_0' = \mu - \mu = 0$

$\Rightarrow \boxed{\mu_1 = 0}$

$r=2$; eq^n ① $\Rightarrow$ $\mu_2 = \mu_2' - 2\,\mu \cdot \mu_1' + 1 \cdot \mu^2 \mu_0'$

$\mu_2 = \mu_2' - 2 \cdot \mu_1'^2 + \mu^2 \cdot (1) \quad (\because \mu_1' = \mu)$

$\mu_2 = \mu_2' - \mu^2 \Rightarrow \boxed{\mu_2 = \mu_2' - \mu_1'^2}$

$r=3$; Eq^n ① $\Rightarrow$ $\mu_3 = \mu_3' - 3 \cdot \mu \mu_2' + 3 \cdot \mu^2 \mu_1' - \mu^3 \mu_0'$

$\mu_3 = \mu_3' - 3 \cdot \mu_1' \cdot \mu_2' + 3 \cdot \mu_1'^2 \cdot \mu_1' - \mu_1'^3$

$\Rightarrow \boxed{\mu_3 = \mu_3' - 3 \mu_1' \mu_2' + 2 \cdot \mu_1'^3}$

Ily; $\boxed{\mu_4 = \mu_4' - 4 \mu_1' \cdot \mu_3' + 6 \cdot \mu_1'^2 \cdot \mu_2' - 3 \mu_1'^4}$

& so on.

4th moment about mean

4th moment about origin.

Note: If mean, $\mu = 0$ then $\mu_1' = 0 = \mu$ & in this case

$\boxed{\mu_1 = \mu_1'}$, $\boxed{\mu_2 = \mu_2'}$, $\boxed{\mu_3 = \mu_3'}$

& $\boxed{\mu_4 = \mu_4'}$ & so on.

(m.g.f) * Moment Generating function *

The moment generating f^n of r.v. X about origin denoted by $M_X(t)$, where t is real parameter & is def^n as expected value of $e^{t(x-0)} = e^{tx}$

ie $M_X(t) = E\left[e^{tX}\right]$

* If X is discrete r.v then,

$$M_X(t) = E(e^{tx}) = \sum_x e^{tx} \cdot f(x)$$

* If X is cts r.v. then

$$M_X(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} \cdot f(x)\, dx$$

Now,

$$M_X(t) = E(e^{tx})$$

$$= E\left[1 + tX + \frac{t^2X^2}{2!} + \frac{t^3 \cdot X^3}{3!} + \ldots + \frac{t^r X^r}{r!} + \ldots\right]$$

$$= E(1) + t\,E(X) + \frac{t^2}{2!} E(X^2) + \frac{t^3}{3!} E(X^3) + \ldots + \frac{t^r}{r!} E(X^r) + \ldots$$

$E(X) = \mu_1'$ $\boxed{\mu_0' = 1}$

$$M_X(t) = 1 + t\cdot(\mu_1') + \frac{t^2}{2!}(\mu_2') + \frac{t^3}{3!}(\mu_3') + \dots + \frac{t^r}{r!}(\mu_r')$$

$$M_X(t) = \mu_0' + (\mu_1')\cdot t + (\mu_2')\cdot\frac{t^2}{2!} + (\mu_3')\cdot\frac{t^3}{3!} + \dots + (\mu_r')\cdot\frac{t^r}{r!} + \dots$$

Where $\mu_r' = E[X^r]$

This shows that

μ_0' is coefficient of t^0

μ_1' " " " t^1

μ_2' is " " $\frac{t^2}{2!}$

μ_3' is " " $\frac{t^3}{3!}$

$\vdots$

μ_r' is " " $\frac{t^r}{r!}$

Th^m. If X is a random variable (discrete/cts) with m.g.f $M_X(t)$ then at $t=0$

$$\mu_r' = \frac{d^r}{dt^r}[M_X(t)]$$

* Moment generating f^n (m.g.f) about the point $x=a$ of r.v. X is def^n as

$$[M_X(t)]_{t=a} = E\left[e^{t(x-a)}\right]$$

* If X is discrete r.v then

$$[M_X(t)]_{t=a} = E\left[e^{t(x-a)}\right] = \sum_x e^{t(x-a)}\cdot f(x)$$

* If X is cts r.v. then

$$M_X(t) = E\left[e^{t(x-a)}\right] = \int_{-\infty}^{\infty} e^{t(x-a)}\cdot f(x)\,dx$$

Now,

$$M_X(t)\Big|_{t=a} = E\left[e^{t(x-a)}\right]$$

$$= E\left[1 + t(x-a) + \frac{t^2(x-a)^2}{2!} + \dots + \frac{t^r}{r!}(x-a)^r\right]$$

$$\Rightarrow M_X(t)\Big|_{t=a} = E(1) + t\cdot E(x-a) + \frac{t^2}{2!}E[(x-a)^2] + \dots + \frac{t^r}{r!}E[(x-a)^r]$$

$$\Rightarrow M_X(t)\Big|_{t=a} = 1 + \mu_1\cdot t + \mu_2\cdot\frac{t^2}{2!} + \mu_3\cdot\frac{t^3}{3!} + \dots + \mu_r\cdot\frac{t^r}{r!} + \dots$$

where $\mu_r = E[(x-a)^r]$ is r^{th} moment of X about $x=a$.

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This shows that

μ_0 is coefficient of t^0

μ_1 " " " t^1

μ_2 " " " $\frac{t^2}{2!}$

μ_3 " " " $\frac{t^3}{3!}$

Ex Find the moment generating f^n for the r.v. x having density function

$$f(x) = \begin{cases} e^{-x} & ; x \geq 0 \\ 0 & ; x < 0 \end{cases}$$

& determine 1st four moments about Origin. Also find 1st four moments about mean.

$\Rightarrow$

M.g.f^n of cts. r.v. X is g^n by

$M_x(t) = E[e^{t(x-0)}] = E[e^{tX}]$ (about origin)

$$= \int_{-\infty}^{\infty} e^{tx} \cdot f(x) \cdot dx$$

$$M_x(t) = \int_0^{\infty} e^{tx} \cdot e^{-x} \cdot dx = \int_0^{\infty} e^{-(1-t) \cdot x} dx$$

$$= \left[\frac{e^{-(1-t)x}}{-(1-t)} \right]_0^{\infty} = \left[0 + \frac{e^0}{1-t} \right]$$

$M_x(t) = \frac{1}{1-t}$ for $t < 1$

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We have;

$M_x(t) = (1-t)^{-1}$; for $|t| < 1$

$M_x(t) = 1 + t + t^2 + t^3 + t^4 + \dots$

but m.g.f. about origin

$M_x(t) = 1 + \mu_1' t + \mu_2' \cdot \frac{t^2}{2!} + \mu_3' \cdot \frac{t^3}{3!} + \mu_4' \frac{t^4}{4!} + \dots$

$\mu_0' = 1$; $\mu_1' = 1$, $\mu_2' = 2! = 2$; $\mu_3' = 3! = 6$, $\mu_4' = 4! = 24$

are the 1st four moments about Origin

Moments about mean are

$\mu_1 = E(X - \mu) = \mu - \mu = 0$

$\mu_2 = \mu_2' - \mu^2 = 2 - 1^2 = 1$ $\quad \boxed{\because \mu = \mu_1'} = 1$

$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3 = 6 - 3 \cdot (2) \cdot (1) + 2(1)^3$
$= 6 - 6 + 2 = 2$

$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4$
$= 24 - 4(6)(1) + 6(2) \cdot (1)^2 - 3(1)^4 = 9$

$E(X - \mu)^3$.

Ex Find m.g.f^n of r.v. $X = \begin{cases} \frac{1}{2} ; & \text{prob.} \frac{1}{2} \\ -\frac{1}{2} ; & \text{prob.} \frac{1}{2} \end{cases}$

Also find 1^{st} four moments about Origin.

Sol^n $M_X(t) = E\left[e^{Xt}\right] = \sum e^{tx} \cdot f(x)$

$\therefore M_X(t) = e^{\frac{1}{2}t} \cdot \frac{1}{2} + e^{-\frac{1}{2}t} \cdot \left(\frac{1}{2}\right)$

$M_X(t) = \frac{e^{t/2} + e^{-t/2}}{2} = \cosh\left(\frac{t}{2}\right)$.

using Taylor series expansion;

$M_X(t) = 1 + \frac{\left(\frac{t}{2}\right)^2}{2!} + \frac{\left(\frac{t}{2}\right)^4}{4!} + \dots$

But $M_X(t) = 1 + \mu_1' t + \mu_2' \cdot \frac{t^2}{2!} + \mu_3' \cdot \frac{t^3}{3!} + \mu_4' \cdot \frac{t^4}{4!} + \dots$

$\therefore \mu_1' = 0 ; \mu_2' = \frac{1}{4} ; \mu_3' = 0 ; \mu_4' = \frac{1}{2^4}$.

are the first four moments about Origin

H.W. A random variable X has density f^n given by

$$f(x) = \begin{cases} 2 e^{-2x} & ; x \geq 0 \\ 0 & ; x < 0 \end{cases}$$

Find (i) $E[X]$ (ii) $Var(X)$
(iii) $E\left[(X-1)^2\right]$ (iv) moment generating f^n

$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots$

(v) 1^{st} four moments about Origin

Ans (i) $E(X) = \frac{1}{2}$ (ii) $Var(X) = \frac{1}{4}$ (iii) $\frac{1}{2}$
(iv) m.g.$f^n = \frac{2}{2-t}$; $|t| < 2$
(v) $\frac{1}{2}, \frac{1}{2}, \frac{3}{4}, \frac{3}{2}$.

H.W. Let $X = \begin{cases} 1 & ; \text{Prob } \frac{1}{6} \\ 2 & ; \text{Prob. } \frac{1}{3} \\ 3 & ; \text{prob. } \frac{1}{2} \end{cases}$ Find (i) mean (ii) Variance (iii) m.g.f^n

Ans (i) $\frac{7}{3}$ (ii) $\frac{5}{9}$ (iii) $\frac{e^t + 2e^{2t} + 3e^{3t}}{6}$

Ex H.W. Find moment generating f^n of r.v. X

$X = \begin{cases} \frac{1}{2} ; & \text{prob} \frac{1}{2} \\ -\frac{1}{2} ; & \text{Prob} \frac{1}{2} \end{cases}$. Also find 1^{st} four moments about Origin

Ex Find $E[X]$, $Var(X)$ & m.g.f. for exponential distribu^n

$$f(x) = \begin{cases} \alpha \cdot e^{-\alpha x} & ; x > 0 \\ 0 & ; x \leq 0 \end{cases}$$

$\rightarrow E[X] = \frac{1}{\alpha}$, $Var(X) = \frac{1}{\alpha^2}$; $M_X(t) = \frac{\alpha}{\alpha - t}$, $t < \alpha$

H.W. Ex Find 1^{st} 4-moments about (i) Origin $\rightarrow \{\frac{8}{5}, 3, \frac{216}{35}, \frac{27}{2}\}$
(ii) mean for prob. distribu^n $\rightarrow \{0, \frac{11}{25}, -\frac{32}{875}, \frac{3693}{8750}\}$

$$f(x) = \begin{cases} \frac{4x(9-x^2)}{81} & ; 0 \leq x \leq 3 \\ 0 & ; \text{otherwise}. \end{cases}$$

Th^m on Expectation

If X is a discrete r.v. & a & b any constant then

(a) $E(a) = a$ (b) $E(aX) = a \cdot E(X)$ (c) $E(aX+b) = a \cdot E(X) + b$.

(d) If X & Y are two r.v. then

$$E(X+Y) = E(X) + E(Y) \text{ i.e. } \mu_{X+Y} = \mu_X + \mu_Y.$$

(e) If X & Y are independent r.v, then

$$E(X \cdot Y) = E(X) \cdot E(Y).$$ (||ly for. Cts R.V)

Variance

The mean or expected value of r.v. X is a special imp. in prob. & stat, becoz it describe where prob. distributn is centred.

H.W

~~If r.v. X is such that $E[(X-1)^2]$~~

The density f^n of r.v. is $f(x) = \begin{cases} c \cdot e^{-2x} & ; x \geq 0 \\ 0 & ; \text{otherwise} \end{cases}$

Find ① Constant 'c'.

② $E[X]$, $Var[X]$ & $E[(X-1)^2]$.

Ans ① $c = 2$ ② $\mu = \frac{1}{2}$, $\sigma_x^2 = \frac{1}{4}$, $E[(X-1)^2] = \frac{1}{2}$

Remember: Expansion of std. f^n

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \dots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \dots$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \dots$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots$$

$$\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + \dots$$

$$\tanh x = x - \frac{x^3}{3} + \frac{2}{15}x^5 - \dots$$

(9) $(1+x)^n = 1^n + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots$

$(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - \dots$

$(1-x)^{-1} = 1 + x + x^2 + x^3 + x^4 + \dots$

Ex A random variable X is defined by

$$X = \begin{cases} -2 & ; \text{ prob. } 1/3 \\ 3 & ; \text{ prob. } 1/2 \\ 1 & ; \text{ prob. } 1/6 \end{cases}$$

Find: (i) $E(x)$, $E(2X+5)$, $E(X^2)$, $Var(x)$, $Var(2X+3)$

(ii) M.g.f. about mean & find 1st four moments about mean

(iii) Coefficient of skewness & kurtosis.

Ans

(i) $E(X) = \sum x \cdot f(x) = \mu = 1$

(ii) $E(2X+5) = 2 \cdot E(X) + 5 = 7$

(iii) $E(X^2) = \sum x^2 \cdot f(x) = 6$

(iv) $Var(x) = E(x^2) - [E(x)]^2 = 6 - 1 = 5$

(v) $Var(2X+3) = 2^2 \cdot Var(x) = 4 \times 5 = 20$

(ii) M.g.f. of X about mean $(E(x) = \mu) = 1$

we have

$M_X(t) = E\left[e^{t(X-\mu)}\right] = E\left[e^{t(X-

$$M_x(t) = \sum_x e^{t(x-1)} \cdot f(x)$$

$$= e^{-3t} \cdot \left(\frac{1}{3}\right) + e^{2t} \cdot \left(\frac{1}{2}\right) + e^{0t} \cdot \left(\frac{1}{6}\right)$$

$$= \frac{1}{3}\left\{1 - 3t + \frac{(3t)^2}{2!} - \frac{(3t)^3}{3!} + \frac{(3t)^4}{4!} - \dots\right\}$$

$$+ \frac{1}{2}\left\{1 + 2t + \frac{(2t)^2}{2!} + \frac{(2t)^3}{3!} + \frac{(2t)^4}{4!} + \dots\right\}$$

$$+ \frac{1}{6}$$

$$M_x(t) = \left(\frac{1}{3} + \frac{1}{2} + \frac{1}{6}\right) + (-1+1)t + (3+2)\frac{t^2}{2!} + (-9+4)\frac{t^3}{3!} + (27+8)\frac{t^4}{4!}$$

$$M_x(t) = 1 + 0t + \frac{5}{2!}t^2 - \frac{5}{3!}t^3 + \frac{35}{4!}t^4 - \dots$$

$\therefore$ 1st four moments of x about the mean are

$\mu_1 = \text{Coeff of } t = 0$ $\quad \mu_1 = 0$

$\mu_2 = \text{Coeff of } \frac{t^2}{2!} = 5 = 5$; $\quad \mu_2 = 5$

$\mu_3 = \text{" " } \frac{t^3}{3!} = -5$; $\quad \mu_3 = -5$

$\mu_4 = \text{" " } \frac{t^4}{4!} = 35$; $\quad \mu_4 = 35$

Ex The density f^n of r.v. is

$$f(x) = \begin{cases} 2\,e^{-2x} & ; x \geq 0 \\ 0 & ; \text{otherwise} \end{cases}$$

Find: (i) $E(x)$ (ii) $Var(x)$ (iii) $E[(x-1)^2]$
(IV) Moment generating f^n about origin
(V) 1st four moments about origin
(Vi) Coefficient of Skewness & kurtosis

Sol^n: Let X be cts. r.v. with density f^n

$$f(x) = \begin{cases} 2\,e^{-2x} & ; x \geq 0 \\ 0 & ; \text{otherwise} \end{cases}$$

(i) $E(X) = \int_{-\infty}^{\infty} x \cdot f(x) \cdot dx = \int_{0}^{\infty} 2\,e^{-2x} \cdot x \cdot dx$

$$= 2 \cdot \left\{ x\left(\frac{e^{-2x}}{-2}\right) - 1 \cdot \left(\frac{e^{-2x}}{4}\right) \right\}_0^{\infty}$$

$$= 2 \cdot \left\{ (0-0) - \left[0 - \frac{1}{4}\right] \right\}$$

$\mu = E(x) = 2/4 = 1/2$

(ii) $Var(x) = E(x^2) - [E(x)]^2 \longrightarrow (A)$

$$E(x^2) = 2\int_0^{\infty} x^2 \cdot e^{-2x} \cdot dx$$

$$= 2 \cdot \left\{ x^2 \left(\frac{e^{-2x}}{-2}\right)\Big|_0^{\infty} - \int_0^{\infty} 2x \cdot \left(\frac{e^{-2x}}{-2}\right) \cdot dx \right.$$

$$E(x^2) = 2\left\{ \int_0^{\infty} x \cdot e^{-2x} \cdot dx \right\} = 1/2$$

Eq^n (A) ⇒

$$Var(x) = \frac{1}{2} - \left(\frac{1}{2}\right)^2 = \frac{1}{2} - 1/4 = 1/4$$

(iii) $E[(x-1)^2] = E[x^2 - 2x + 1] = E(x^2) - 2E(x) + E(1)$

$$= 1/2 - 2 \cdot \left(\frac{1}{2}\right) + 1 = 1/2$$

(iv) M.g.f^n; $M_x(t) = E\left[e^{t(x-0)}\right]$

$$M_x(t) = E\left[e^{tx}\right] = \int_0^{\infty} e^{tx} \cdot f(x) \cdot dx$$

$$= \int_0^{\infty} e^{tx} \cdot 2 \cdot e^{-2x} \cdot dx = 2\int_0^{\infty} e^{(t-2)x} \cdot dx$$

$$= 2 \cdot \cancel{\left[\frac{e^{(t-2)x}}{(t-2)}\right]_0^{\infty}} =$$

$$= 2\int_{-0}^{\infty} e^{-(2-t)x}\, dx = 2 \cdot \left[\frac{e^{-(2-t)x}}{-(2-t)}\right]_0^{\infty}$$

$$= \frac{2}{\cancel{2}(t-2)}\left[0 - 1\right] = \frac{-2}{t-2}$$

$$\boxed{M_x(t) = \frac{2}{2-t}} \quad ; |t| < 2$$

$$\Rightarrow \left|\frac{t}{2}\right| < 1$$

(v) We have; $M_x(t) = \frac{2}{2\cdot(1-t/2)}$; $\left|\frac{t}{2}\right| < 1$

$\Rightarrow M_x(t) = \left[1 - \frac{t}{2}\right]^{-1}$

$= 1 + \left(\frac{t}{2}\right) + \left(\frac{t}{2}\right)^2 + \left(\frac{t}{2}\right)^3 + \left(\frac{t}{2}\right)^4 + \dots$

$= 1 + \frac{t}{2} + \frac{t^2}{4} + \frac{t^3}{8} + \frac{t^4}{16} + \dots$

$= 1 + \frac{1}{2}\cdot t + \left(\frac{1}{2}\right)\frac{t^2}{2!} + \left(\frac{3}{4}\right)\frac{t^3}{3!} + \left(\frac{4!}{16}\right)\frac{t^4}{4!}$

Now,

$\mu_1' = \text{Coeff. of } t = \frac{1}{2}$

$\mu_2' = \text{'' '' } \frac{t^2}{2!} = \frac{1}{2}$

$\mu_3' = \text{Coeff of } \frac{t^3}{3!} = \frac{3}{4}$ & $\mu_4' = \text{Coeff of } \frac{t^4}{4!}$

$= \frac{3}{2}$

(Vi) Coeff of Skewness

$\cancel{\alpha = \frac{\mu_3}{\sigma^3} = \frac{\mu_3}{(\mu_2')^{3/2}} = \frac{3/4}{\left(\frac{1}{2}\right)^{3/2}}}$

$\alpha = \frac{\mu_3}{\sigma^3} = \frac{\mu_3}{(\mu_2)^{3/2}}$

but $\mu_2 = \mu_2' - \mu_1'^2 = \frac{1}{2} - \left(\frac{1}{2}\right)^2 = \frac{1}{4}$.

$\Rightarrow \alpha = \frac{\mu_3}{(\mu_2)^{3/2}} = \frac{\mu_3}{\left(\frac{1}{4}\right)^{3/2}}$

$\cancel{\mu_3 = \mu_3' - 3\mu_1'\cdot\mu_3' + 6\cdot\mu_1'^2\cdot\mu_2' - 3\mu_1'^4}$

$\cancel{\mu_3 = \frac{3}{4} - 3\left(\frac{1}{2}\right)\left(\frac{3}{4}\right) + 6\cdot\left(\frac{1}{2}\right)^2\cdot\frac{1}{2} - 3\left(\frac{1}{2}\right)^4}$

$\mu_3 = \mu_3' - 3\cdot\mu_1'\cdot\mu_2' + 2\cdot\mu_1'^3$

$\mu_3 = \left(\frac{3}{4}\right) - 3\cdot\left(\frac{1}{2}\right)\cdot\left(\frac{1}{2}\right) + 2\cdot\left(\frac{1}{2}\right)^3 = \frac{1}{4}$.

$\Rightarrow \alpha = \frac{\mu_3}{(1/4)^{3/2}} = \frac{1/4}{(1/4)^{3/2}} = \left(\frac{1}{4}\right)^{-1/2} = (4)^{1/2}$

$\boxed{\alpha = 2}$

Now, Coefficient of kurtosis; $\beta = \frac{\mu_4}{\sigma^4} = \frac{\mu_4}{(\mu_2)^2}$

Now, $\mu_4 = \mu_4' - 4\cdot\mu_1'\cdot\mu_3' + 6\cdot\mu_1'^2\cdot\mu_2' - 3\cdot\mu_1'^4$

$= \frac{3}{2} - 4\cdot\left(\frac{1}{2}\cdot\frac{3}{4}\right) + 6\cdot\left(\frac{1}{4}\cdot\frac{1}{2}\right) - 3\cdot\left(\frac{1}{16}\right)$

$\mu_4 = 9/16$

$\Rightarrow \beta = \frac{\mu_4}{(\mu_2)^2} = \frac{9/16}{(1/4)^2} = 9$