More Sample Probability Questions
(Taken from Past AHSME/AMC/AIME competitions)

1) (2003 AMC 10 B Q 21) A bag contains two red beads and two green beads. You reach into the bag and pull out a bead, replacing it with a red bead regardless of the color you pulled out. What is the probability that all beads in the bag are red after three such replacements?

2) (1997 AHSME Q 10) Two six-sided dice are fair in the sense that each face is equally likely to turn up. However, one of the dice has the 4 replaced by 3 and the other die has the 3 replaced by 4. When these dice are rolled, what is the probability that the sum is an odd number?

3) (1984 AHSME Q 19) A box contains 11 balls numbered 1, 2, 3, …, 11. If 6 balls are drawn simultaneously at random, what is the probability that the sum of the numbers on the balls is odd?

4) (1986 AHSME Q 22) Six distinct integers are picked at random from {1, 2, 3, …, 10}. What is the probability that, among those selected, the second smallest is 3?

5) (1990 AHSME Q 18) First a is chosen at random from the set {1, 2, 3, …, 100}, and then b is chosen at random from the same set. What is the probability that the units digit of $3^a + 7^b$ is 8?

6) (1992 AHSME Q 29) An "unfair" coin has a 2/3 probability of turning up heads. If this coin is tossed 50 times, what is the probability that the total number of heads is even?

7) (1993 AHSME Q 24) A box contains 3 shiny pennies and 4 dull pennies. One by one, pennies are drawn at random from the box and not replaced. What is the probability that it will take more than four draws to remove all of the shiny pennies?

8) (1995 AHSME Q 20) If a, b and c are three (not necessarily distinct) numbers chosen randomly and with replacement from the set {1, 2, 3, 4, 5}, what is the probability that ab + c is even?

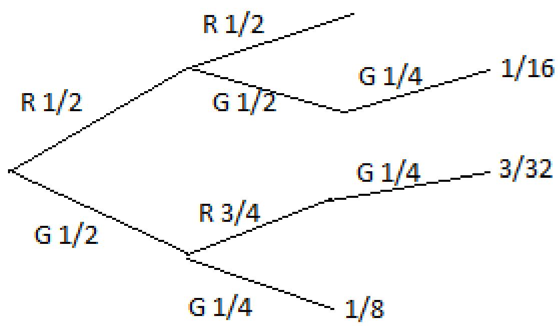
9) (1988 AIME Q 5) What is the probability that a randomly chosen positive divisor of 10^{99} is an integer multiple of 10^{88}?

10) (1989 AIME Q 5) When a certain biased coin is flipped 5 times, the probability of getting heads exactly once is not equal to 0 and is the same as that of getting heads exactly twice. What is the probability that the coin comes up heads exactly three times out of five?

11) (1990 AIME Q 9) A fair coin is to be tossed ten times. What is the probability that heads never comes up on consecutive tosses?

1) (2003 AMC 10 B Q 21) A bag contains two red beads and two green beads. You reach into the bag and pull out a bead, replacing it with a red bead regardless of the color you pulled out. What is the probability that all beads in the bag are red after three such replacements?

Let's draw a tree diagram. R means you pulled a red, G means you pulled a green. You have to pull 2 G's out of 3 to "win". Your chance of pulling a green bead after pulling one is just ¼, because the other three beads are red.



To get the final probability, just add the three separate branches labeled that correspond to ending up with all red beads in 3 pulls. (Note, for the branch labeled 1/8, the third pull is guaranteed to pull a red bead and replace it with a read bead…)

Answer = 1/16 + 3/32 + 1/8 = 2/32 + 3/32 + 4/32 = **9/32**

2) (1997 AHSME Q 10) Two six-sided dice are fair in the sense that each face is equally likely to turn up. However, one of the dice has the 4 replaced by 3 and the other die has the 3 replaced by 4. When these dice are rolled, what is the probability that the sum is an odd number?

Die one has 4 odd numbers, 2 even numbers.
Die two has 2 odd numbers, 4 even numbers.

$$p(SumOdd) = p(D1 = odd) \times p(D2 = even) + p(D1 = even) \times p(D2 = odd)$$

$$p(SumEven) = \frac{2}{3} \times \frac{2}{3} + \frac{1}{3} \times \frac{1}{3} = \mathbf{\frac{5}{9}}$$

3) (1984 AHSME Q 19) A box contains 11 balls numbered 1, 2, 3, …, 11. If 6 balls are drawn simultaneously at random, what is the probability that the sum of the numbers on the balls is odd?

The sample space is $\binom{11}{6} = \frac{11\times10\times9\times8\times7}{5\times4\times3\times2\times1} = 11 \times 6 \times 7 = 462$. To get a sum that is odd, we need an odd number of odd balls. Thus, we need either 1, 3 or 5 of the 6 balls to be odd. Separate our counting into these groups.

# of ways to get 1 odd ball (and 5 even ones) = $\binom{5}{1}\binom{6}{5} = 30$

# of ways to get 3 odd balls (and 3 even ones) = $\binom{5}{3}\binom{6}{3} = 200$

# of ways to get 5 odd balls (and 1 even one) = $\binom{5}{5}\binom{6}{1} = 6$

This gives us a probability of $\frac{30+200+6}{462} = \frac{236}{462} = \mathbf{\frac{118}{231}}$

4) (1986 AHSME Q 22) Six distinct integers are picked at random from $\{1, 2, 3, \ldots, 10\}$. What is the probability that, among those selected, the second smallest is 3?

Our sample space is $\binom{10}{6} = \frac{10\times9\times8\times7}{4\times3\times2\times1} = 10 \times 3 \times 7 = 210$.

We must count how many subsets of size 6 have 3 as the second smallest number. We must choose exactly 1 value from the set $\{1, 2\}$. We must choose 3. Then we must choose 4 values from the set $\{4,5,6,7,8,9,10\}$. We can make our choices is $\binom{2}{1}\binom{7}{4} = 70$ ways.

The desired probability is $\frac{70}{210} = \mathbf{\frac{1}{3}}$

5) (1990 AHSME Q 18) First a is chosen at random from the set $\{1, 2, 3, \ldots, 100\}$, and then b is chosen at random from the same set. What is the probability that the units digit of $3^a + 7^b$ is 8?

First, let's work out the first few units digits of 3^a and 7^b:

Base\Exp	0	1	2	3	4
3	1	3	9	7	1
7	1	7	9	3	1

Notice that both repeat every four, so instead of doing our probability out of 100 x 100, because 100 is divisible by 4, it's good enough for us to work out the probability using the set {1,2,3,4}.

Here is a grid showing the 16 possible additions:

Base 3/base 7	7	9	3	1
3	0	2	6	4
9	6	**8**	2	0
7	4	6	0	**8**
1	**8**	0	4	2

It follows that the desired probability is $\mathbf{\frac{3}{16}}$.

6) (1992 AHSME Q 29) An "unfair" coin has a 2/3 probability of turning up heads. If this coin is tossed 50 times, what is the probability that the total number of heads is even?

This is a binomial distribution, but we aren't being asked to find a single term. Rather, we need to find the sum of the even terms in the binomial probability distribution.

Consider the following, via the Binomial Theorem:

$$\left(\frac{2}{3}+\frac{1}{3}\right)^{50} = \sum_{i=0}^{50} \binom{50}{i} \left(\frac{2}{3}\right)^{i} \left(\frac{1}{3}\right)^{50-i}$$

$$\left(\frac{2}{3}-\frac{1}{3}\right)^{50} = \sum_{i=0}^{50} \binom{50}{i} \left(\frac{2}{3}\right)^{i} \left(-\frac{1}{3}\right)^{50-i}$$

The key trick here is realizing that when we change the sign on the 1/3 to be negative, the odd terms become negative. Thus, if we were to add the two equations above, the sum contains each even term twice and none of the odd terms. (We can subtract the equations to get rid of the even terms…) So, adding these two equations we get:

$$1^{50} + \left(\frac{1}{3}\right)^{50} = 2\sum_{i=0}^{25} \binom{50}{2i} \left(\frac{2}{3}\right)^{2i} \left(\frac{1}{3}\right)^{50-2i}$$

Notice that the sum on the RHS (without the multiplicative factor of 2) is exactly our desired probability of flipping an even number of heads. It follows that our desired probability is:

$$1^{50} + \left(\frac{1}{3}\right)^{50} = \sum_{i=0}^{25} \binom{50}{2i} \left(\frac{2}{3}\right)^{2i} \left(\frac{1}{3}\right)^{50-2i} = \frac{1+\left(\frac{1}{3}\right)^{50}}{2}$$

7) (1993 AHSME Q 24) A box contains 3 shiny pennies and 4 dull pennies. One by one, pennies are drawn at random from the box and not replaced. What is the probability that it will take more than four draws to remove all of the shiny pennies?

The sample space is $\binom{7}{3} = 35$, the number of ways of ordering 3 shiny pennies and 4 dull pennies (the order is the order they in which they are drawn.)

Of these, there are 4 orderings, SSSDDDD, SSDSDDD, SDSSDDD, and DSSSDDD, where all the shiny pennies are drawn within the first four draws. This means that there are 35 – 4 = 31 orderings where it takes more than 4 draws to remove all the shiny pennies. It follows that the desired probability is $1 - \frac{4}{35} = \mathbf{\frac{31}{35}}$.

8) (1995 AHSME Q 20) If a, b and c are three (not necessarily distinct) numbers chosen randomly and with replacement from the set {1, 2, 3, 4, 5}, what is the probability that ab + c is even?

We only care if a, b and c are even or odd. Since each are pulled from the same set, we have a probability of $\frac{3}{5}$ that a is odd and a probability of $\frac{2}{5}$ that a is even. The same probabilities are true for both b and c.

First, let's examine the product ab. It can only be odd if both a and b are odd. Thus, the probability that ab is odd is $\frac{3}{5} \times \frac{3}{5} = \frac{9}{25}$. It follows that the probability that ab is even is $1 - \frac{9}{25} = \frac{16}{25}$.

In order for ab + c to be even, either both ab and c are even, or both ab and c are odd. Thus, we calculate our desired probability as follows:

$$p(ab + c = even) = p(ab = even) \times p(c = even) + p(ab = odd) \times p(c = odd)$$

$$p(ab + c = even) = \frac{16}{25} \times \frac{2}{5} + \frac{9}{25} \times \frac{3}{5} = \mathbf{\frac{59}{125}}$$

9) (1988 AIME Q 5) What is the probability that a randomly chosen positive divisor of 10^{99} is an integer multiple of 10^{88}?

$10^{99} = 2^{99} \times 5^{99}$, thus, this number has (99 + 1) x (99 + 1) = 100 x 100 divisors.

Let's count how many of these divisors are multiples of 10^{88}. In order to be a multiple of 10^{88} and be a divisor of 10^{99}, a number must have the form 2^a5^b, where $88 \le a, b \le 99$. There are 12 integer values of a which satisfy this constraint and 12 integer values of b which satisfy this constraint. In total there are 12 x 12 = 144 divisors of 10^{99} that are also multiples of 10^{88}. It follows that our desired probability is $\frac{12\times 12}{100\times 100} = \frac{3\times 3}{25\times 25} = \mathbf{\frac{9}{625}}$.

10) (1989 AIME Q 5) When a certain biased coin is flipped 5 times, the probability of getting heads exactly once is not equal to 0 and is the same as that of getting heads exactly twice. What is the probability that the coin comes up heads exactly three times out of five?

Let p be the probability of getting heads on a single flip. (Assuming that the probability of getting exactly 1 head is non-zero, then p < 1 as well.)

The probability of getting 1 head (and 4 tails) is $\binom{5}{1} p^1(1-p)^4$.

The probability of getting 2 heads (and 3 tails) is $\binom{5}{2} p^2(1-p)^3$.

Set these two equal to each other and divide through by $p(1 – p)^3$:

$$\binom{5}{1} p^1(1-p)^4 = \binom{5}{2} p^2(1-p)^3$$

$$5p^1(1-p)^4 = 10p^2(1-p)^3$$

$$5(1-p) = 10p$$

$$5 - 5p = 10p$$

$$15p = 5$$

$$p = \frac{1}{3}$$

The probability of getting exactly 3 heads is $\binom{5}{3}(\frac{1}{3})^3(\frac{2}{3})^2 = 10 \times \frac{1}{27} \times \frac{4}{9} = \mathbf{\frac{40}{243}}$

11) (1990 AIME Q 9) A fair coin is to be tossed ten times. What is the probability that heads never comes up on consecutive tosses?

Slow Solution Method
Our sample space is 2^{10}. If there are no consecutive heads, then there must be in between 0 and 5 heads, inclusive. Given that we have k heads, k ≤ 5, we can calculate the number of orderings of coin flips with k heads where no two heads are consecutive as follows:

Use the 10 – k tails as separators: __ T __ T __ T … T __

There are (10 – k) + 1 = 11 – k gaps in between those tails where we can place the k heads. Thus, there are $\binom{11-k}{k}$ sequences of k H's and 10 – k T's where no two H's are consecutive. To get the number of different orderings, sum this value over all possible k:

$$\sum_{k=0}^{5}\binom{11-k}{k} = \binom{11}{0} + \binom{10}{1} + \binom{9}{2} + \binom{8}{3} + \binom{7}{4} + \binom{6}{5}$$

$$= 1 + 10 + 36 + 56 + 35 + 6 = 144$$

It follows that the desired probability is $\frac{144}{2^{10}} = \frac{9\times 2^4}{2^{10}} = \frac{9}{2^6} = \mathbf{\frac{9}{64}}$

Faster Solution Method
Define f(k) = number of valid sequences (without consecutive heads) of length k. So,

f(1) = 2, (H or T)
f(2) = 3, (HT, TT, TH)
f(3) = 5, (HTT, TTT, THT, HTH, TTH)

Notice that either, the strings listed either end in T or in HT, and that these groups are exclusive. We create all the strings ending in T by looking at the previous list and adding a T to them because, we can extend any valid sequence by adding a T to it. Alternatively, for any valid sequence we can create another valid one by adding TH to it, and all of these must be distinct from all sequences that end in T. Thus, to find f(k), we can sum together all sequences of length k-1 and add a T to the end of those, and find all sequences of length k-2 and add a TH to the end of those. This gives us the recurrence:

f(k) = f(k-1) + f(k-2)

This is the Fibonacci recurrence, and the initial conditions are such that f(k) = Fibonacci(k+2). Thus, our desired number of valid arrangements of sequences of 10 Hs or Ts without consecutive heads equals the 12th Fibonacci number.

The Fibonacci numbers are 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89 and **144.**

Using the sample space from before, it follows that the final probability is $\frac{144}{1024} = \boldsymbol{\frac{9}{64}}$