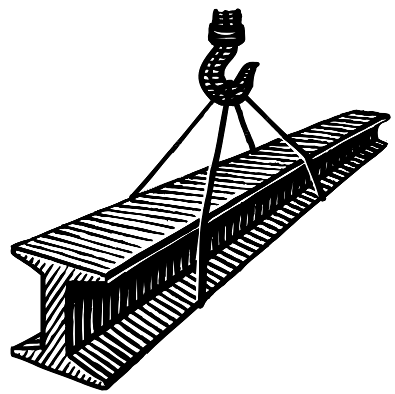
CEE 213—Deformable Solids

CP 3

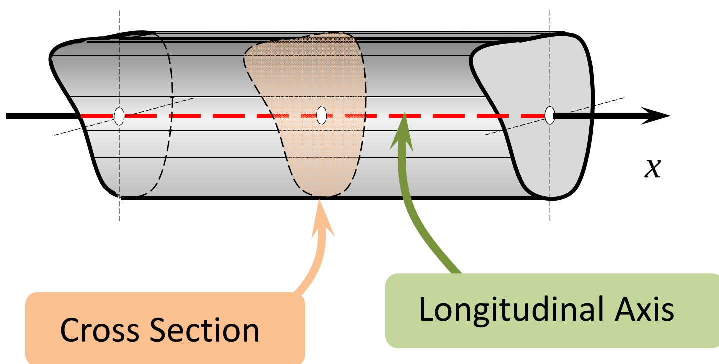
Properties of polygonal plane areas

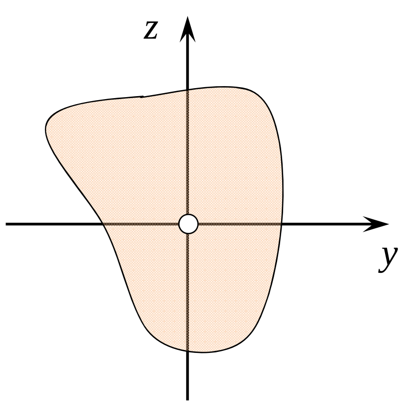


Keith D. Hjelmstad
School for Sustainable Engineering and the Built Environment
Ira A. Fulton Schools of Engineering
Arizona State University

CP 3—Properties of Areas

The properties of cross sections





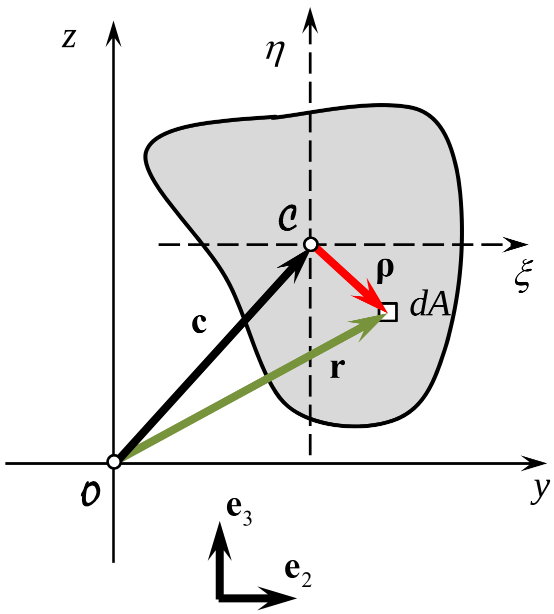
Cross Section

The properties of cross sections. In the study of axially loaded bars, bars subjected to torques, and beams subjected to transverse loads, it becomes evident that much of the behavior is dictated by the properties of the *cross sectional geometry*. In fact, the resistance to deformation is always a function of the distribution of material in a cross section. For the axial bar the *cross sectional area* is key; for torsion the *polar moment of inertia* shows up; for flexure the *moment of inertia* about the axis of bending is an important property. These properties are essential to determining the deformation under load.

Most of the classical "tricks of the trade" that can be used in service of the computation of geometric properties amount to making use of the properties of simple geometric pieces (often rectangles) to build up the overall cross sectional properties. What we show in this set of notes is that it is possible to derive the integrals needed to compute the cross sectional properties for a general triangle and then use that basic foundation to create a method to compute the cross sectional properties of *any* closed polygonal region. This approach is based on two key observations: (1) all integrals can be divided into pieces and (2) it is possible to add in an integral that is not in the physical region as long as you subtract it right back out.

CP 3—Properties of Areas

The key cross sectional properties



The *tensor* (or *outer*) product of vectors

$$\mathbf{v} \otimes \mathbf{v} = \mathbf{v}\,\mathbf{v}^T$$

$$= \begin{Bmatrix} v_1 \\ v_2 \end{Bmatrix} \begin{Bmatrix} v_1 & v_2 \end{Bmatrix} = \begin{bmatrix} v_1^2 & v_1 v_2 \\ v_1 v_2 & v_2^2 \end{bmatrix}$$

The key cross sectional properties. Consider the general cross section defined at left. We establish a (y, z) coordinate system with origin *O*, and imagine another coordinate system (ξ, η) passing through the *centroid* *C*. For beam theory we need to compute area and moment of inertia (about the centroid). To compute the moment of inertia we need the location of the centroid.

The *area* is simply defined as the sum of the infinitesimal areas that make up the cross section:

$$A = \int_A dA$$

The *centroid* of an area is defined as follows:

$$\mathbf{c} = \frac{1}{A}\int_A \mathbf{r}\, dA$$

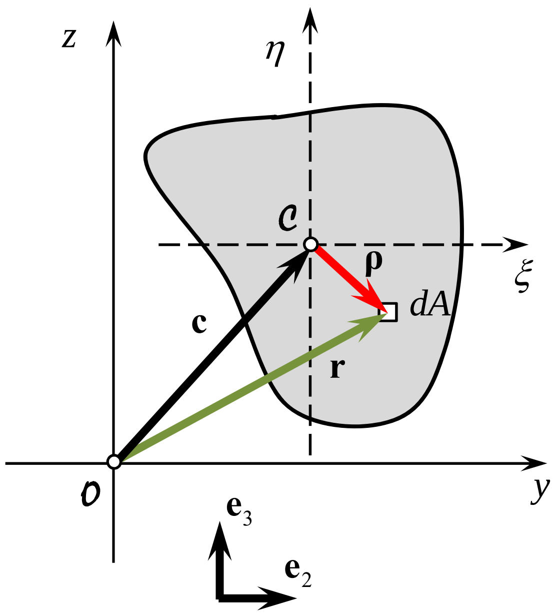
In other words, it is the constant vector **c** that is the average over the area of the position **r** of the infinitesimal areas. Finally, the *moment of inertia* (tensor) is defined as

$$\mathbf{J} = \int_A \boldsymbol{\rho} \otimes \boldsymbol{\rho}\; dA$$

Where the vector ρ is the distance from the centroid *C* to the infinitesimal area of integration.

CP 3—Properties of Areas

The moment of inertia tensor



$$\mathbf{c}\otimes\mathbf{c}=\mathbf{c}\,\mathbf{c}^T=\begin{Bmatrix}c_1\\c_2\end{Bmatrix}\{c_1\quad c_2\}=\begin{bmatrix}c_1^2 & c_1c_2\\ c_1c_2 & c_2^2\end{bmatrix}$$

$$\mathbf{r}\otimes\mathbf{r}=\mathbf{r}\,\mathbf{r}^T=\begin{Bmatrix}y\\z\end{Bmatrix}\{y\quad z\}=\begin{bmatrix}y^2 & yz\\ yz & z^2\end{bmatrix}$$

The moment of inertia tensor. The problem with computing $\mathbf{J}$ is that it is defined in terms of the centroidal coordinate system. We can compute it in the regular coordinates noting that (by vector addition)

$$\boldsymbol{\rho}=\mathbf{r}-\mathbf{c}$$

Thus, we can compute

$$\begin{aligned}\mathbf{J}&=\int_A\boldsymbol{\rho}\otimes\boldsymbol{\rho}\,dA\\&=\int_A(\mathbf{r}-\mathbf{c})\otimes(\mathbf{r}-\mathbf{c})\,dA\\&=\int_A(\mathbf{r}\otimes\mathbf{r}-\mathbf{c}\otimes\mathbf{r}-\mathbf{r}\otimes\mathbf{c}+\mathbf{c}\otimes\mathbf{c})\,dA\\&=\int_A(\mathbf{r}\otimes\mathbf{r})\,dA-\mathbf{c}\otimes\Big(\int_A\mathbf{r}\,dA\Big)-\Big(\int_A\mathbf{r}\,dA\Big)\otimes\mathbf{c}+\mathbf{c}\otimes\mathbf{c}\Big(\int_A dA\Big)\\&=\int_A(\mathbf{r}\otimes\mathbf{r})\,dA-\mathbf{c}\otimes(A\mathbf{c})-(A\mathbf{c})\otimes\mathbf{c}+\mathbf{c}\otimes\mathbf{c}(A)\\&=\int_A[\mathbf{r}\otimes\mathbf{r}]\,dA-A[\mathbf{c}\otimes\mathbf{c}]\end{aligned}$$

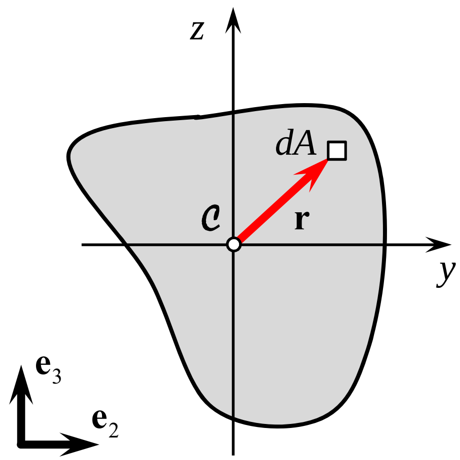
Therefore, we can compute the *moment of inertia* tensor in the original coordinate system as

$$\mathbf{J}=\int_A[\mathbf{r}\otimes\mathbf{r}]\,dA-A[\mathbf{c}\otimes\mathbf{c}]$$

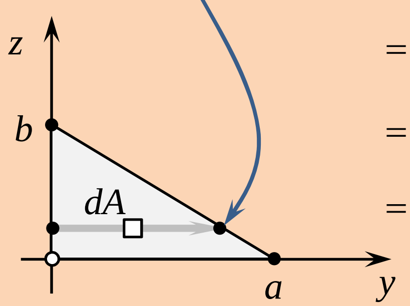
We need the centroidal distance **c**, and we need to figure out how to compute the integral of **r**⊗**r**.

CP 3—Properties of Areas

The moment of inertia tensor



$$y = a\left(1 - \frac{z}{b}\right)$$



$$\begin{aligned} I_{yy} &= \int_0^b \int_0^{a(1-z/b)} z^2 dy\, dz \\ &= \int_0^b z^2 \big[y\big]_0^{a(1-z/b)} dz \\ &= \int_0^b z^2 a\big(1 - z/b\big) dz \\ &= a\left(\tfrac{1}{3}z^3 - \tfrac{1}{4}z^4/b\right)_0^b \\ &= \tfrac{1}{12}ab^3 \end{aligned}$$

The moment of inertia tensor. Let's look at the components of **J** to see what we are computing. The components of the vector from the *centroid* to the typical particle and the elemental area of the particle are

$$\mathbf{r} = \begin{Bmatrix} y \\ z \end{Bmatrix}, \qquad dA = dy\, dz$$

Thus, we can compute the following integral

$$\int_A \mathbf{r} \otimes \mathbf{r}\, dA = \int_A \begin{bmatrix} y^2 & yz \\ yz & z^2 \end{bmatrix} dy\, dz$$

The integrals themselves are not difficult at all, but the limits of integration can be very difficult (see box at left). We will note a few things. The integrals

$$I_{yy} = \int_A z^2\, dA, \qquad I_{zz} = \int_A y^2\, dA$$

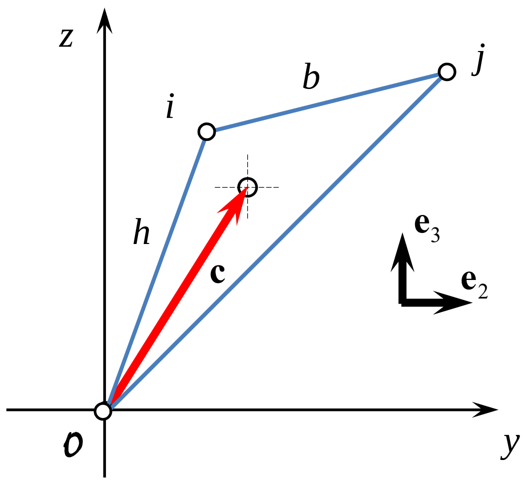
Are the moments of the area about the y and z axes. These are important for beam bending. The sum of these two gives

$$I_{yy} + I_{zz} = \int_A \left(y^2 + z^2\right) dA = \int_A r^2\, dA = J$$

which is the *polar moment of the area*, the property most important in the torsion problem.

CP 3—Properties of Areas

The general triangular region



The basic triangle

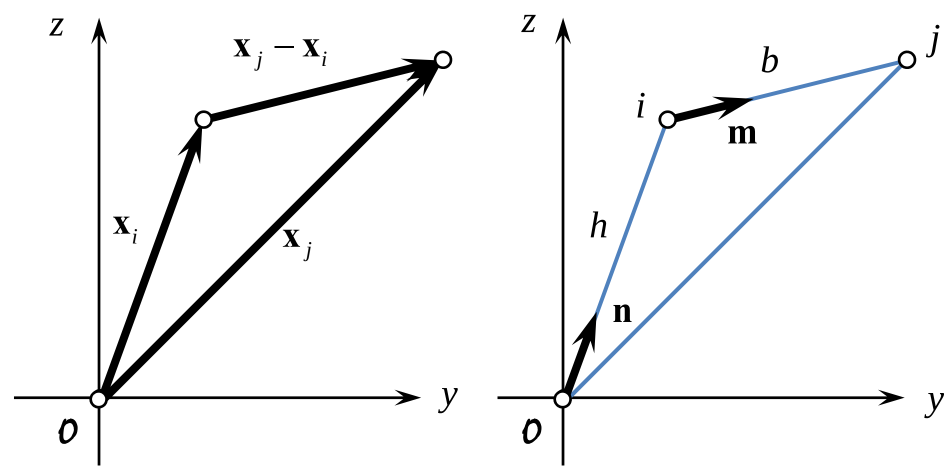
All of these key quantities can be computed from the vectors $\mathbf{x}_i$ and $\mathbf{x}_j$

$$h = \left\| \mathbf{x}_i \right\| \qquad \mathbf{n} = \frac{\mathbf{x}_i}{h}$$

$$b = \left\| \mathbf{x}_j - \mathbf{x}_i \right\| \qquad \mathbf{m} = \frac{\mathbf{x}_j - \mathbf{x}_i}{b}$$

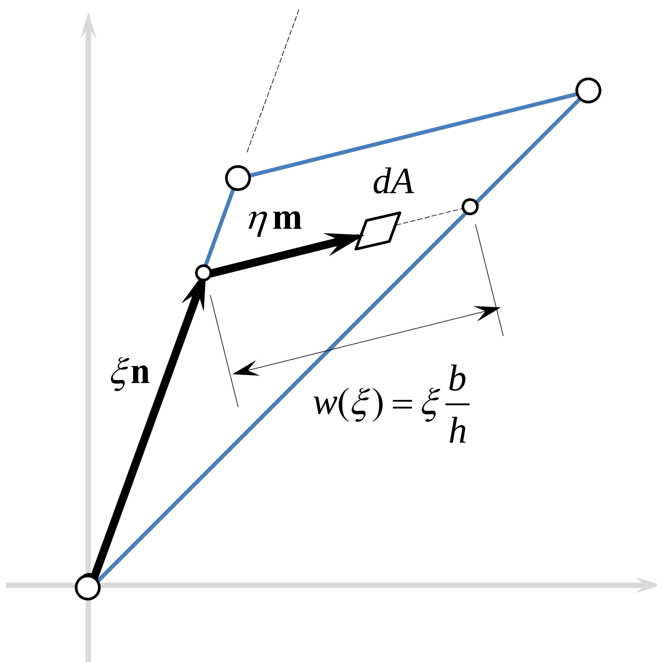
The general triangular region. We can specialize the notion of our cross sectional properties to a general triangular region (shown at left). We will need to get specific about how to do the integrals over this region. Once we have found the results, we can use them as a basic building block in an algorithm to compute the three basic properties of any polygonal cross section (and, by taking points close together we can approximate cross sections with curved edges, too).

We will construct our concept of the triangle from the position vectors of the two vertices i and j (which will lie on an edge of the polygon later). These vectors are $\mathbf{x}_i$ and $\mathbf{x}_j$. From these vectors we can compute h, b, and the unit vectors $\mathbf{n}$ and $\mathbf{m}$ (as shown in the box at left).



CP 3—Properties of Areas

Integration over the triangle

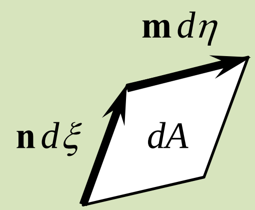


The variables of integration are ξ (measures distance in **n** direction) and η (measures distance in **m** direction), which are in the range

$$0 \le \xi \le h \qquad 0 \le \eta \le b$$

Note how the limits of integration are done in the box at right.

Integration over the triangular region. It will be convenient to set up the area integration in the coordinate system **n** and **m** (which happens not to be rectangular).



Thus, the element of integration is

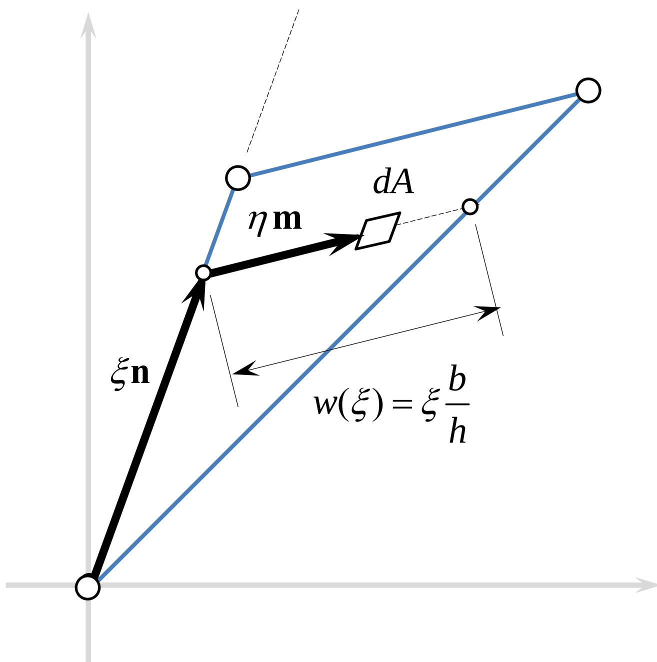
$$
\begin{aligned}
dA\mathbf{e}_1 &= (\mathbf{n}\,d\xi)\times(\mathbf{m}\,d\eta)\\
dA &= (\mathbf{n}\times\mathbf{m})\cdot\mathbf{e}_1\,d\eta\,d\xi\\
&= \sin\varphi\,d\eta\,d\xi\\
\sin\varphi &= (\mathbf{n}\times\mathbf{m})\cdot\mathbf{e}_1 = n_2 m_3 - n_3 m_2
\end{aligned}
$$

And the integration over the area is done as

$$
\begin{aligned}
\int_A (\bullet)\,dA &= \sin\varphi \int_0^h \int_0^{w(\xi)} (\bullet)\,d\eta\,d\xi\\
&= \sin\varphi \int_0^h \int_0^{\xi\frac{b}{h}} (\bullet)\,d\eta\,d\xi
\end{aligned}
$$

CP 3—Properties of Areas

Area of the general triangle



For beam theory we need to compute area, centroidal location, and moment of inertia (about the centroid):

Area of the general triangle. We can compute *area* of the triangle as follows:

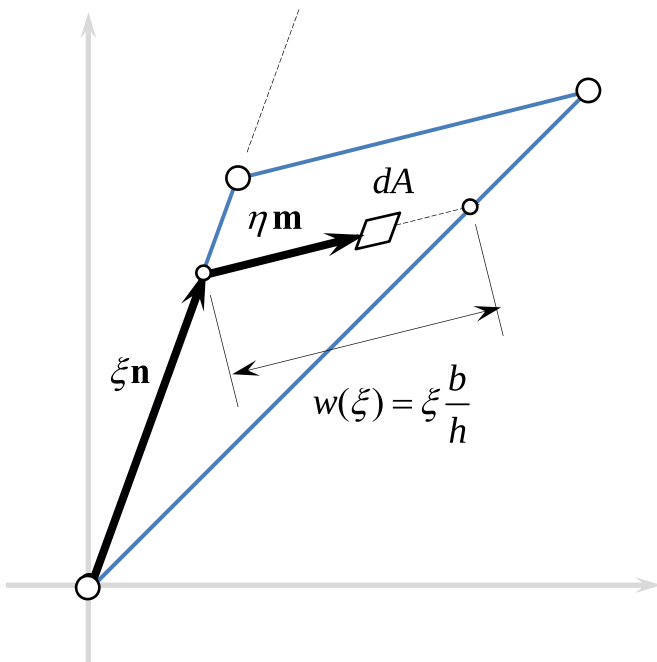
$$
\begin{aligned}
A &= \int_A dA \\
&= \sin\varphi \int_0^h \int_0^{\xi\frac{b}{h}} d\eta\, d\xi \\
&= \sin\varphi \int_0^h \big[\eta\big]_0^{\xi\frac{b}{h}} d\xi \\
&= \sin\varphi \int_0^h \xi\frac{b}{h} d\xi \\
&= \sin\varphi \frac{b}{h}\Big[\tfrac{1}{2}\xi^2\Big]_0^h \\
&= \tfrac{1}{2}\sin\varphi\, b\, h
\end{aligned}
$$

Note how "one half base times height" still seems to be part of the picture. The factor $\sin\varphi$ takes care of two things: (1) that the triangle is not a right triangle and (2) if it turns out to be negative then the area is "negative." The *area* of the general triangle is

$$A = \tfrac{1}{2} b\, h \sin\varphi$$

CP 3—Properties of Areas

The centroid of the triangle



We will sum the contributions for each triangle so we will not compute **c** (the location of the centroid) for the triangle until we add up all of the cross section until we add up all of the **p** contributions.

The centroid of the triangle. We can compute the integral of **r** over the triangle as follows:

$$
\begin{aligned}
\mathbf{p} &= \int_A \mathbf{r}\,dA \\
&= \sin\varphi \int_0^h \int_0^{\xi\frac{b}{h}} (\xi\mathbf{n}+\eta\mathbf{m})\,d\eta\,d\xi \\
&= \sin\varphi \int_0^h \left[\xi\eta\mathbf{n}+\tfrac{1}{2}\eta^2\mathbf{m}\right]_0^{\xi\frac{b}{h}} d\xi \\
&= \sin\varphi \int_0^h \left[\xi^2\left(\frac{b}{h}\right)\mathbf{n}+\tfrac{1}{2}\xi^2\left(\frac{b}{h}\right)^2\mathbf{m}\right] d\xi \\
&= \sin\varphi \left[\tfrac{1}{3}\xi^3\left(\frac{b}{h}\right)\mathbf{n}+\tfrac{1}{6}\xi^3\left(\frac{b}{h}\right)^2\mathbf{m}\right]_0^h \\
&= \sin\varphi\left(\tfrac{1}{3}bh^2\mathbf{n}+\tfrac{1}{6}b^2h\mathbf{m}\right) \\
&= A\left(\tfrac{2}{3}h\mathbf{n}+\tfrac{1}{3}b\mathbf{m}\right)
\end{aligned}
$$

Note that **p** is one third of the distance from the edges, but adjusted by the general area A and measured in the directions **n** and **m**. The *location of the centroid* of the general triangle is

$$\mathbf{p}=\int_A \mathbf{r}\,dA = A\left(\tfrac{2}{3}h\mathbf{n}+\tfrac{1}{3}b\mathbf{m}\right)$$

CP 3—Properties of Areas

The integral of $\mathbf{r} \otimes \mathbf{r}$

The vector $\mathbf{r}$ to the elemental point of integration is

$$\mathbf{r} = \xi\,\mathbf{n} + \eta\,\mathbf{m}$$

It is convenient to define tensors

$$\mathbf{A} = \mathbf{n} \otimes \mathbf{n}$$
$$\mathbf{B} = \mathbf{n} \otimes \mathbf{m} + \mathbf{m} \otimes \mathbf{n}$$
$$\mathbf{C} = \mathbf{m} \otimes \mathbf{m}$$

These tensors are constant over the triangle and therefore can be pulled out of the integral.

Note that we will not compute the actual moment of inertia about the centroid at this point because we plan to add together multiple pieces to construct the moment of inertia for polygonal regions. So we will simply compute (and accumulate) the integrals of the outer product.

The integral of $\mathbf{r} \otimes \mathbf{r}$. We can compute the integral of the *outer product* of $\mathbf{r}$ with itself for the triangular region as follows:

$$\begin{aligned}
\int_A \mathbf{r} \otimes \mathbf{r}\, dA &= \sin\varphi \int_0^h \int_0^{\xi\frac{b}{h}} (\xi\mathbf{n} + \eta\mathbf{m}) \otimes (\xi\mathbf{n} + \eta\mathbf{m})\, d\eta\, d\xi \\
&= \sin\varphi \int_0^h \int_0^{\xi\frac{b}{h}} \left[\xi^2\mathbf{A} + \xi\eta\,\mathbf{B} + \eta^2\mathbf{C}\right] d\eta\, d\xi \\
&= \sin\varphi \int_0^h \left[\xi^2\eta\mathbf{A} + \tfrac{1}{2}\xi\eta^2\,\mathbf{B} + \tfrac{1}{3}\eta^3\mathbf{C}\right]_0^{\xi\frac{b}{h}} d\xi \\
&= \sin\varphi \int_0^h \left[\xi^3\left(\frac{b}{h}\right)\mathbf{A} + \tfrac{1}{2}\xi^3\left(\frac{b}{h}\right)^2\mathbf{B} + \tfrac{1}{3}\xi^3\left(\frac{b}{h}\right)^3\mathbf{C}\right] d\xi \\
&= \sin\varphi \left[\tfrac{1}{4}\xi^4\left(\frac{b}{h}\right)\mathbf{A} + \tfrac{1}{8}\xi^4\left(\frac{b}{h}\right)^2\mathbf{B} + \tfrac{1}{12}\xi^4\left(\frac{b}{h}\right)^3\mathbf{C}\right]_0^h \\
&= \sin\varphi\left(\tfrac{1}{4}bh^3\mathbf{A} + \tfrac{1}{8}b^2h^2\,\mathbf{B} + \tfrac{1}{12}b^3h\mathbf{C}\right) \\
&= A\left(\tfrac{1}{2}h^2\mathbf{A} + \tfrac{1}{4}bh\mathbf{B} + \tfrac{1}{6}b^2\mathbf{C}\right)
\end{aligned}$$

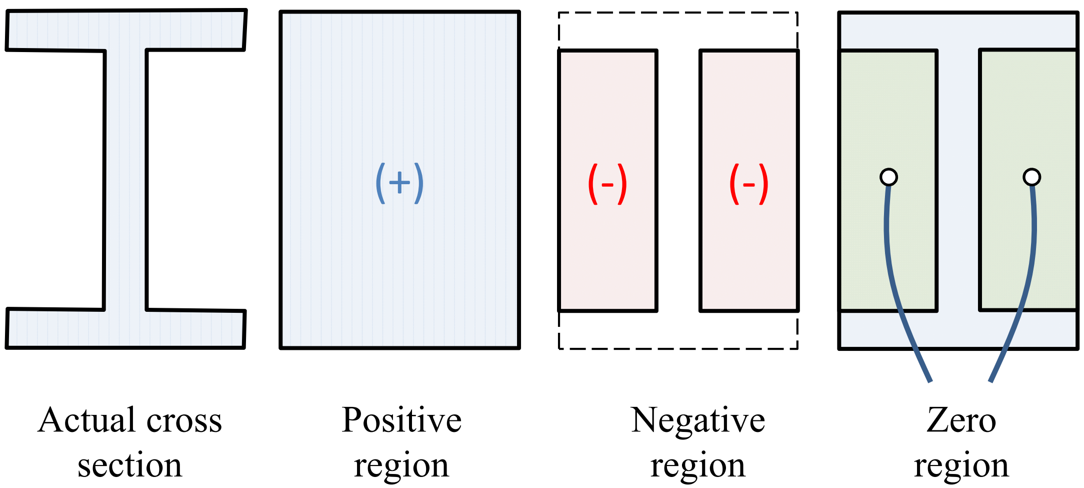
Hence, we have the following result (which we will eventually use to compute the moment of inertia)

$$\mathbf{R} = \int_A \mathbf{r} \otimes \mathbf{r}\, dA = A\left(\tfrac{1}{2}h^2\mathbf{A} + \tfrac{1}{4}bh\mathbf{B} + \tfrac{1}{6}b^2\mathbf{C}\right)$$

CP 3—Properties of Areas

Positive and negative regions

We can compute integrals over a cross section by considering positive and negative areas. In regions where there are both, they cancel each other out. The I-beam, for example, can be constructed from the large (positive) rectangle and the two smaller (negative) rectangles.



$$
\begin{aligned}
\int_A w\,dA &= \int_{A^+} w\,dA + \int_{A^-} (-w)\,dA \\
&= \int_{A^+} w\,dA - \int_{A^-} w\,dA
\end{aligned}
$$

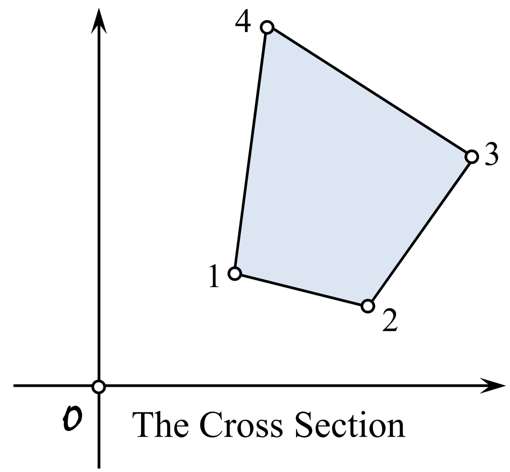
$$\boxed{(+)} + \boxed{(-)} = \boxed{(0)}$$

We will use this general idea to accumulate the contributions of the triangles that define the sides of the polygon. If $\sin\varphi$ is greater than zero the area will be positive and if it is less than zero then the area will be negative. For any region that has both a positive and negative contribution the net effect (in all of our integrals) will be zero. Hence, this is a good way to lay out the calculation.

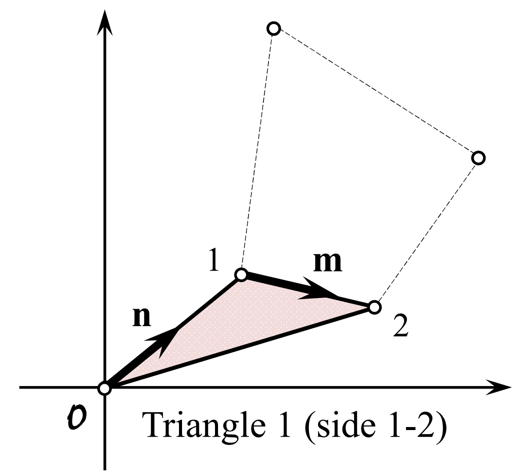
CP 3—Properties of Areas

Accumulation of properties

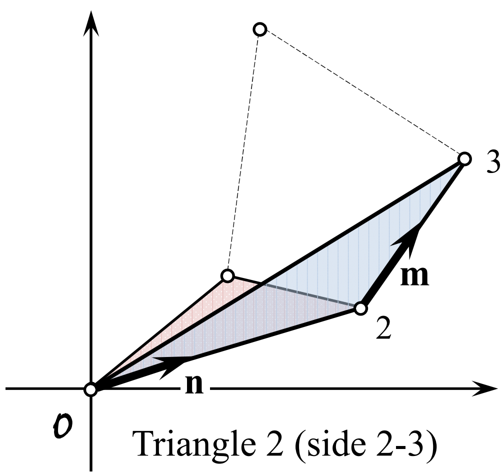
Consider the cross section shown at right. Taking the points sequentially we see the contributions of four triangles. If **n**×**m** is into the paper then the area is negative (red), otherwise it is positive (blue). Notice how the areas outside the cross section get exactly one positive and one negative contribution. In the end, only the blue region remains.



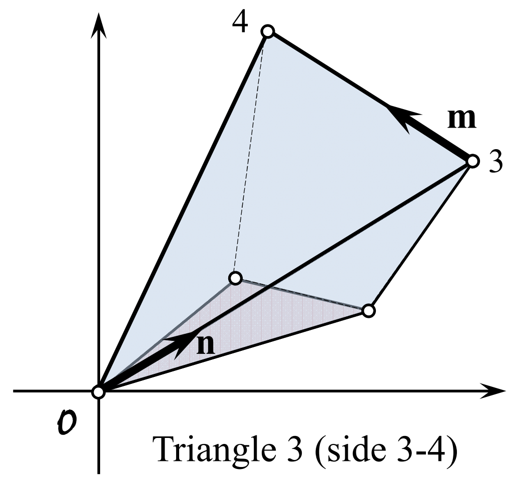
The Cross Section



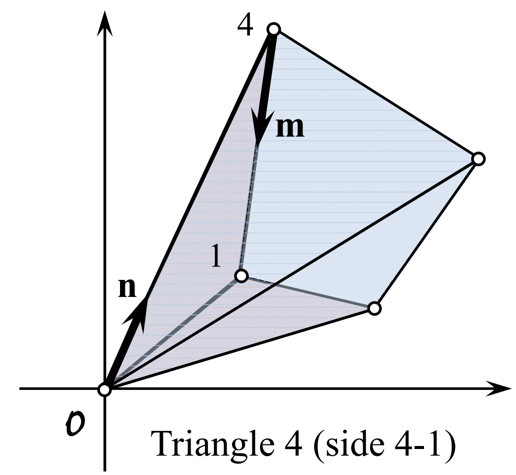
Triangle 1 (side 1-2)



Triangle 2 (side 2-3)



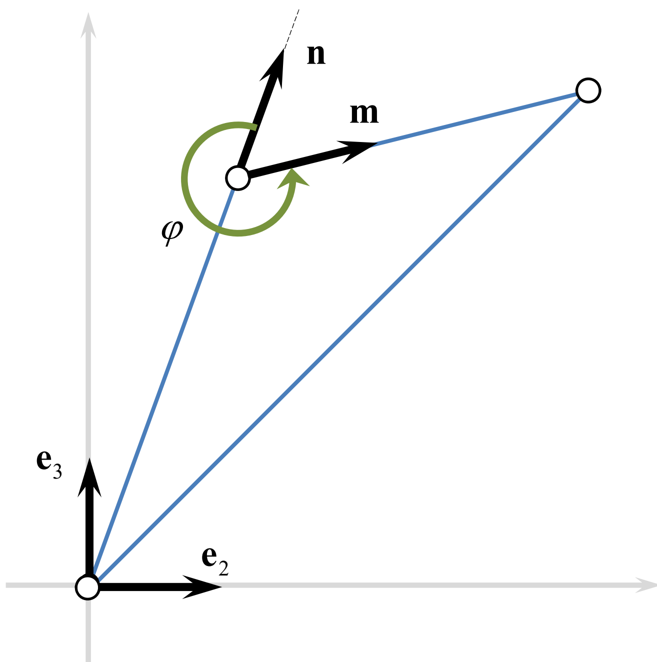
Triangle 3 (side 3-4)



Triangle 4 (side 4-1)

CP 3—Properties of Areas

Positive regions and negative regions

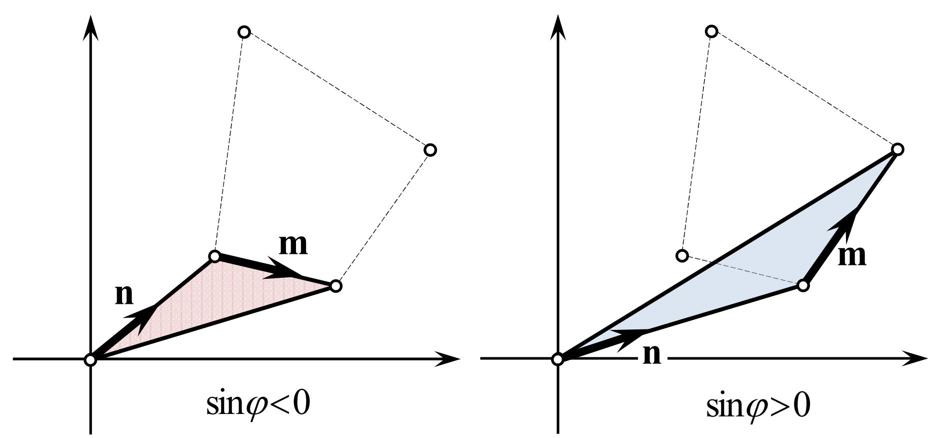


The sine function is positive in the first and second quadrant and negative in the third and fourth, and this gives the integrals their algebraic sign. Also, as the vectors tend to point along the same line the magnitude gets smaller (to reflect that the region of the triangle gets smaller).

Positive regions and negative regions. The power of the approach we are taking is the ability to distinguish between *positive regions* and *negative regions*. When we use the general triangle as the building block for computing integrals over a polygonal shape we must be able to assign a positive or negative value to each piece in accord with the need to either add that portion to the sum or subtract it from the sum. The term $\sin\varphi$ takes care of this issue. Recall the definition

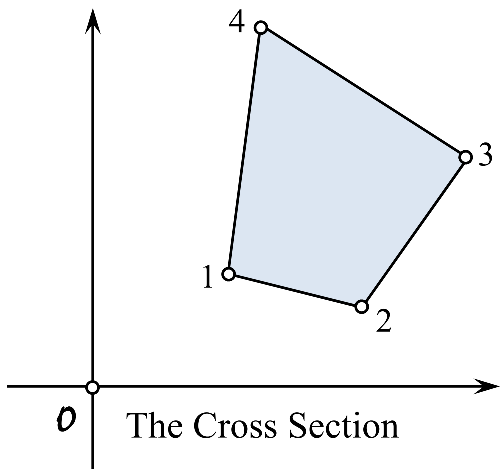
$$\sin\varphi = (\mathbf{n}\times\mathbf{m})\cdot\mathbf{e}_1 = n_2 m_3 - n_3 m_2$$

According to the definition of cross product, the angle is clockwise from the vector **n** to the vector **m**. You can see how this plays out in the two examples below (the triangles for sides 1-2 and 2-3 for our example).



CP 3—Properties of Areas

Algorithm



The Cross Section

The process can be put into a program with the steps outlined at right. In the loop we carry out the computations associated with the triangle whose edge is defined by the current vertices. We then accumulate those into sums for each of the properties (i.e., we are adding up all of the contributions of all triangles). Once the loop is complete we can finish the computations of $\mathbf{c}$ and $\mathbf{J}$, find the principal values, and print and/or plot the results.

Algorithm. We can organize the computation as follows:

1. Store coordinates of vertices in array $\mathbf{x}$

$$\mathbf{x} = \begin{Bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_N & y_N \\ x_1 & y_1 \end{Bmatrix}$$

2. Initialize properties $A = 0$, $\mathbf{p} = \mathbf{0}$, $\mathbf{R} = \mathbf{0}$
3. Loop over the sides, $i = 1:N$
    a. Compute the unit vectors $\mathbf{n}_i$ and $\mathbf{m}_i$ from $\mathbf{x}$

    b. Compute $\sin\varphi_i = (\mathbf{n}_i \times \mathbf{m}_i)\cdot \mathbf{e}_1$

    c. Compute the triangle contributions

    $$A_i = \int_{A_i} dA,\ \mathbf{p}_i = \int_{A_i} \mathbf{r}\, dA,\ \text{and } \mathbf{R}_i = \int_{A_i} \mathbf{r}\otimes\mathbf{r}\ dA$$

    d. Add the contributions to the whole

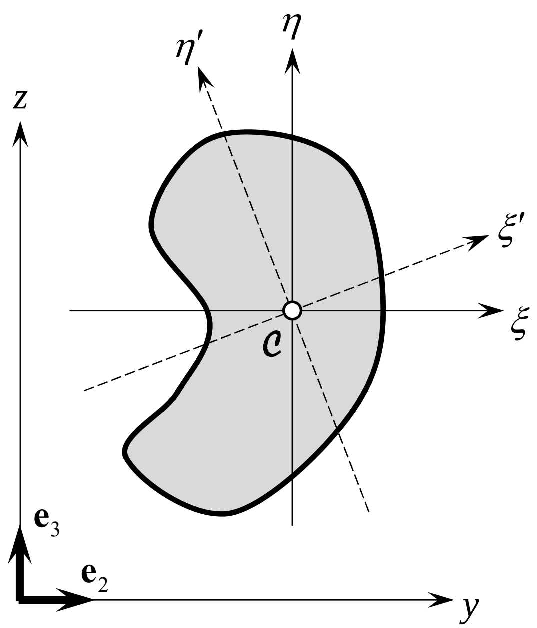
    $$A \leftarrow A + A_i,\ \mathbf{p} \leftarrow \mathbf{p} + \mathbf{p}_i,\ \mathbf{R} \leftarrow \mathbf{R} + \mathbf{R}_i$$
4. Finish the computation of $\mathbf{c}$ and $\mathbf{J}$

$$\mathbf{c} = \mathbf{p}/A,\quad \mathbf{J} = \mathbf{R} - A[\mathbf{c}\otimes\mathbf{c}]$$

5. Compute principal values of $\mathbf{J}$
6. Print and plot results

CP 3—Properties of Areas

*Principal values of **J***



The maximum and minimum values of the inertia tensor **J** are its eigenvalues. These are the values of **J** that we would compute in the (ξ',η') coordinate system (if we knew what it was in advance). For a symmetric cross section **J** is diagonal and the the max/min values are equal to the diagonal elements of **J**.

Principal values of J. When the off-diagonal elements of the tensor **J** are not zero that implies that the J_{yy} and J_{zz} values are not the maximum or minimum moments of inertia of the cross section.

The way we computed **J** had the first step of laying down a coordinate system (y, z). We found the location of the centroid $\mathcal{C}$ and that allowed us to nail down the coordinate system (ξ,η), which is simply a translation of the original coordinate system to pass through the centroid. These two coordinate systems were probably chosen because of some other aspect of the problem we are trying to solve (e.g., bending about the *y*-axis). But, as far as the properties of the cross section are concerned, the choice is arbitrary.

An interesting question to ask is this: Could we pick a coordinate system (ξ',η') in such a way that the values of J_{yy} and J_{zz} are the biggest or smallest possible? The answer is "yes" and the coordinate system we need points in the direction of the *eigenvectors* of **J**. Similarly, the actual max/min values are the *eigenvalues* of **J**.

MATLAB gives us a very easy way to compute the eigenvalues of a matrix:

```
Jmax = eig(J)
```