Math 3660 - Spring 2021
Mathematical Models in Economics

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Final project instructions

Instructions

Your final assignment is to complete an individual or group project, of a nature similar to our assignments throughout the semester, including modeling and computing aspects, that addresses an economic issue of interest to you. Complete a *Mathematica* notebook reporting on your individual project and submit by the scheduled day of the final exam Wed. May. 4 (but no final exam). If you like, you may coordinate on a more extensive project and presentation with one or two others in the class, but you should each submit a notebook (copies) reporting on your project. I have decided that this year presentation of projects will be optional for both individual and group projects. Your project will likely to be viewed favorably if there has been a chance for questions and answers about it, and I recommend the practice generally. Presentation will be short 5-10 minute progress reports to the class during the last day of class, Apr. 28. If you choose not to present, you need all of the details you wish to be graded on included report. And in any case I expect a written report for reference.

Your final project should describe a problem in economics and use our mathematical tools to model this problem, in order to better understand the nature of the problem and the information needed to find a definitive solution. You may wish to collect actual data, from a survey you conduct or from internet sources for example, and conduct the appropriate data analysis, fitting your model to the data. In lieu of collecting such data, you may choose to simulate data, explaining what real data you might collect and how you would analyze it, in the manner of a detailed research proposal. You should write up your problem, model, solution, analysis, computations, etc., preferably in the form of a *Mathematica* notebook as has been used for other assignments.

Ideas

A good way to start is to ask a specific economic question. Take a product or industry you're interested in. Why are prices the way they are? What competition is there in this market? What alternatives do consumers have? What are their preferences? How much would prices change if people had different alternatives, or if firms were organized differently? If you were running a business, how would you decide on products and set prices? Are the rules of the game in this market different than in other industries? What would the effect be of changing any of the existing conditions?

To answer your question(s), you should consider a model relating values pertinent to your question. This doesn't have to be a complex model; simpler models can provide insights into the workings of an economic system. You will need to identify what values are observable, at least in principle, what you want to predict or evaluate, and what parameters control the relationships involved. Basic information about your question can serve to constrain your model, determining at least reasonable ranges for parameters. Survey or historical data can be compared to the model to find the best fit for parameters if you can gather such information.

Projects may be theoretical, creating a model and simulating the system it describes for example, or they may be more practical, collecting data, characterizing consumer preferences, and predicting consumer behavior perhaps. It's important that you take something that interests you, something you want to know the answer to, but also a problem sufficiently accessible that you can demonstrate your analytical skills in solving it.

Here are some ideas about projects that I have previously posted.

1. I’ve imagined that each week the grocer gets a wholesale price for bananas, decides on a retail price he will charge and how much he will thus be able to sell, and then places a wholesale order for that quantity. But this assumes that retail demand for bananas is a known deterministic function of retail price. What happens if the consumer demand has a random component? The grocer might, because demand happened to be lower than an average week, get stuck with unsold bananas. Or if demand was higher than average, the grocer might run out of bananas. Simplest is to assume that the retail price will not be adjusted during the week to try to sell an excess of bananas or stretch a deficiency of bananas for the week, but bananas are a perishable good so there’s no carry over. What wholesale purchasing and retail pricing strategy is optimum for the grocer?

2. What about the same problem for a nonperishable good where the grocer can carry an inventory? Simplest is to assume the wholesale price is fixed. But goods that are purchased and put in inventory cost money, at least for the interest on the money used for the wholesale purchase pending the revenue from an eventual retail sale, but also possibly for warehouse space, security, etc. Again supposing weekly sales uncertainties but only weekly deliveries, how much inventory should the grocer carry to reduce the risk of lost sales due to running out while reducing the cost of carrying a suitable inventory? What about the case where wholesale prices also vary randomly week to week?

3. Determine a formula for the pass through rate (delta opt. price/delta marginal cost in the limit as delta marginal cost goes to zero) in terms of derivatives of demand function. Derive formulas for diversion ratios (find a definition) between products when there are two or more competing products. What other characteristics of demand or competition can be explained in detail using derivatives?

4. What happens if two products are complements instead of substitutes. That is, if lowering the price on one product increases the demand for both products, so the cross elasticity of demand is

negative, how will competing firms price their products? How will a merger of firms affect the optimum prices for the two products?

5. When a firm markets to subsets of consumers with identifiable characteristics, it may know that the distribution of values of consumers in the different subsets are different. If the firm can price discriminate, it would charge a different price to each subset. If it can't price discriminate, the knowledge about subsets of consumers would seem irrelevant. But suppose that the firm chooses which subsets to market to, reaching consumers in a group coming with some cost, and sales occurring only to those groups. Then marketing to low value groups may not be worth the cost, and limiting to high value groups may allow for a higher price. Can this model of consumer values be used to inform not just pricing but also marketing strategies?

6. To determine that a price maximizes profit we usually check the derivative of profit, computed in terms of the derivative of demand, is zero. We have estimated demand by fitting a model to sampled data. If we sample over a narrow range of prices, then will be giving up accuracy in the estimate of the derivative of demand at a particular price. If we sample over a wide range, then the accuracy of our estimate of demand at any point is likely less accurate due to model specification error, that is, a linear demand model fit for example will be less accurate if we sample a curved demand function over a wide range. If we are trying to determine a profit maximizing price, what demand sampling and fitting strategy is appropriate for determining this optimal price most accurately? What is the cost, in lost profit, for misestimating the optimal price?

7. Compare our specification and calculation of merger effects for a logit demand model to that computed by Luke Froeb's online merger simulation tool. Are the results identical or are there inconsistencies? For example, does it look like aggregate elasticity is defined the same there as we have defined it? If the online tool had an error in the coding, how would anyone find out? What kind of scientific peer review can we do on an online tool?

8. Develop an estimation procedure for the parameters of a logit demand model for two or three products. Illustrate the estimation procedure by generating random sample demand data at various prices for some defined logit demand parameters, in the manner of assignment 3, and then see how well you can recover the parameters from the sample data. How does the accuracy of your estimates depend on the combinations of prices you choose to sample demand at? In particular, illustrate that if you observe prices that are all at the same time relatively low or relatively high (collinearity) then it will be very difficult to estimate substitution between products.

9. What is the optimal wholesale pricing strategy for a supplier dealing with two retailers? Suppose the retailers sell the same good to consumers in overlapping markets, differentiated by retailer location, consumer awareness of offering, etc. That is, for some assumed retail demand function for the good through each of the retailers as a function of retail prices such that the retailers are partial but not perfect substitutes, determine the pass through rates from wholesale price to retail prices. Hence determine, implicitly, the wholesale demand as a function of the wholesale price and so solve for the supplier's optimal wholesale pricing. Does this retailer competition reduce the problem of double marginalization?

PROJECT : Pass-Through Rate and Diversion Ratios

1. Pass-Through Rate

The pass-through rate is defined as the change in optimal price relative to the change in marginal cost.

Consider a demand function q, which is a function of the price p, i.e.

$$q = q(p)$$

Then, in the limit as the change in marginal cost goes to zero, the pass-through rate can be written as a derivative

$$\text{pass-through rate } = \frac{\partial p^*}{\partial c}$$

where c is the marginal cost defined by the equation, $c = \dfrac{\partial C}{\partial q}$, C being the cost function, $C = C(q)$,

and, the price p^*, such that the profit is maximum.
The profit function is

$$\Pi(p) = p\, q(p) - C(q)$$

Since the profit is maximum, we must have

$$\left.\frac{\partial \Pi}{\partial p}\right|_{p=p^*} = 0$$

Differentiating the profit function and using the chain rule of derivatives

$$q(p^*) + p^* \left.\left(\frac{\partial q}{\partial p}\right)\right|_{p=p^*} - \left.\left(\frac{\partial C}{\partial q}\right)\left(\frac{\partial q}{\partial p}\right)\right|_{p=p^*} = 0$$

We recognise the marginal cost, $c = \dfrac{\partial C}{\partial q}$ and rearrange the above equation as

$$c = p^* + \frac{q(p^*)}{(\partial q / \partial p)\,|_{p=p^*}}$$

As the marginal cost varies, so does the optimal price for maximum profit. The optimal price can then be considered to be a function of the marginal cost, and inversely the marginal cost as the function of optimal price.

Differentiating the above equation, we have

$$\frac{\partial c}{\partial p^*} = 1 + \frac{\partial}{\partial p^*}\left(\frac{q(p^*)}{(\partial q / \partial p)\,|_{p=p^*}}\right)$$

Employing the quotient rule for derivatives, and simplifying, we have

$$\frac{\partial c}{\partial p^*} = 2 - q\,(p^*)\,\frac{\partial^2 q \big/ \partial p^2}{(\partial q / \partial p)^2}\,\Bigg|_{p=p^*}$$

The pass-through rate is then the inverse of $\partial c / \partial p^*$, and is given by

$$\text{pass-through rate } = \frac{\partial p^*}{\partial c} = \frac{1}{2 - q\,(p^*)\,\frac{\partial^2 q / \partial p^2}{(\partial q / \partial p)^2}\,\Big|_{p=p^*}}$$

In general, given a demand function $q = q\,(p)$ and a cost function $C = C\,(q)$, the pass-through rate is

$$\textbf{pass-through rate } = \frac{\mathbf{1}}{\mathbf{2} - q\,(p)\,\frac{\partial^2 q / \partial p^2}{(\partial q / \partial p)^2}}\,\Bigg|_{p=p^*} \quad \textbf{where } p^* \textbf{ is given by the equation } \frac{\partial \Pi}{\partial p}\,\Bigg|_{p=p^*} = \mathbf{0}$$

If we consider a linear demand function, then the second derivative is zero. The pass through rate, in that scenario, becomes an independent constant value of 0.5.

$$\text{If } q\,(p) = A\,p + B, \text{ then } \frac{\partial^2 q}{\partial p^2} = 0 \Longrightarrow \text{pass-through rate} = 0.5$$

In Assignment 1, we computed the average pass through rate for a linear demand function, which was indeed 0.5.

2. Diversion Ratios

When a firm increases the price of a product, the demand for it decreases. That demand is captured by competing firms offering substitute products. Diversion Ratios quantify this transfer/*diversion* of demand. Diversion Ratios are a measure of proportion of demand/sales captured by substitute products/competing firms, when the price of a product is increased.

Let us assume that there are $N \geq 2$ competing firms, and that firm #i increases the price of its product, p_i, leading to a decrease in demand q_i. The difference is demand is captured by other firms, leading to an increase in their demand. Then, mathematically, decrease in q_i equals the sum of increase in demand of all other firms. Hence,

$$\Delta q_i = \Sigma_{k=1,k\neq i}^{N} \Delta q_k$$

The same relationship can be expressed using differentials as follows.

$$-d\, q_i = \Sigma_{k=1,k\neq i}^{N} d\, q_k$$

The negative sign indicates that the demand q_i has decreased.

Rearranging, we have

$$d\, q_1 + \ldots + d\, q_i + \ldots + d\, q_N = 0$$

where $1 \leq i \leq N$

Dividing the entire equation by $d\, q_i$, we have

$$\frac{d\, q_1}{d\, q_i} + \ldots + 1 + \ldots + \frac{d\, q_N}{d\, q_i} = 0$$

The ratio of differentials are directly related to the diversion ratios. If the demand of product of firm #1 increased by an amount Δq_1, the the diversion ratio for firm #1 is $\Delta q_1 / \Delta q_i$. In differential form, this is the same as $-d\, q_1 / d\, q_i$. Thus, the diversion ratios may be written as

Diversion Ratio for firm #k, $r_k = -\dfrac{d\, q_k}{d\, q_i}$, $k \neq i$

Using this definition, we have

$$r_1 + \ldots + r_k + \ldots r_N = 1,\ 1 \leq k \leq N,\ k \neq i$$

The above equation says that the sum of the diversion ratios must be 1, which is obviously true - since we assume that the decrease in demand of one product is captured completely by firms producing substitute products.

The diversion ratios are directly related to the own and cross price elasticities of demand. We have

$$r_k = -\frac{d\, q_k}{d\, q_i} = -\frac{d\, q_k / d\, p_i}{d\, q_i / d\, p_i}$$

We have the following definitions of elasticities of demand.

$$\epsilon_{i\,i} = \frac{d\, q_i}{d\, p_i}\,\frac{p_i}{q_i} \text{ and } \epsilon_{k\,i} = \frac{d\, q_k}{d\, p_i}\,\frac{p_i}{q_k}$$

Using these definitions, we can rewrite the diversion ratio from k to i, r_k as

$$r_k = -\frac{\epsilon_{k\,i}\, q_k}{\epsilon_{i\,i}\, q_i}$$

Diversion ratios measure the degree of competition between products. They are also useful for finding closest substitutes of a product. A high diversion ratio implies a close substitute, and greater competition. They are also useful in analysis of effects of mergers. If firms with high diversion ratios merge, they would have a very large market share, which can be a concern for regulatory authorities. Diversion ratio evidence has been used to make decisions in several competition cases such as the Co-op/Somerfield supermarket merger, Ryanair/Aer Lingus merger case (which was blocked because of the firms being each other’s closest competitor as shown by high diversion ratios between the two airlines).