Math 3660 - Spring 2022
Mathematical Models in Economics

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1/26/22

Profit maximizing firms

■ Assignment 6

Instructions

Save a copy of this notebook, complete the exercises, save and submit on Brightspace your final version by start of class Tues. Mar. 1.

Nikhil Jayswal

Edit the line above that reads Enter your name here, inserting instead your name. You may also want to save to a file name that includes your name. By adding your name into the file and file name, you reduce the chance I will mix-up your submission with someone else's.

Exercises

For these problems, fix a consumer demand, say for example, a linear demand. We can take the quantity units so the amount demanded at zero price is some value, say 100 quantity units. We can take the monetary units so the maximum price consumers will support in the limit as quantity goes to is some value, say 100 monetary units. In this way, for illustrative purposes we may as well assume, for a linear demand, that $q(p) = 100 - p$ for $0 \leq p \leq 100$. Of course, for Cournot competition we need the inverse demand function which is simply $p(q) = 100 - q$, for $0 \leq q \leq 100$.

In[1]:=
```
inversedemand[q_] = 100. - q;
```

1. Suppose there are n firms, each so small that it assumes its quantity has no

impact on price. Suppose the firms are all similar, with increasing marginal costs, say each with the same cost function as given in terms of the individual firm's quantity sold qi below. Define the formula for a firm's marginal cost, say mc[qi_]=... (note why it was convenient to take a quadratic cost function). Determine the inverse supply function for the *n* firms (with *n* an unknown for now). I.e., for any given total quantity *q*, at what price will firms be willing to supply the total quantity *q* (note that by symmetry, each firm will supply the same amount, $q_i = q / n$)? Define this by taking, say, inversesupply[q_]=.... Now at what price and quantity will supply meet demand? How much profit will firms be making at this price? In a free market economy, if there is money to made supplying a product, new firms will enter the market to supply the product. How many firms will there be in this market (can there be if each must make a profit)?

```
In[2]:= Clear[qi, q, n]

In[3]:= cost[qi_] = 20. + 4. * qi + 1. * qi^2;

In[4]:= mc[qi_] = D[cost[qi], qi]
Out[4]= 4. + 2. qi

In[5]:= inversesupply[q_] = mc[q / n]
```

Out[5]= $4. + \frac{2.\,q}{n}$

Supply meets demand when

```
In[6]:= supplyEqualsDemand = inversesupply[q] == inversedemand[q]
```

Out[6]= $4. + \frac{2.\,q}{n} == 100. - q$

The total quantity when supply meets demand is

```
In[7]:= totalQuantity = Solve[supplyEqualsDemand, q][[1]]
```

Out[7]= $\left\{q \to \frac{96.}{1. + \frac{2.}{n}}\right\}$

and the corresponding price is

```
In[8]:= p = inversedemand[q] /. totalQuantity
```

Out[8]= $100. - \frac{96.}{1. + \frac{2.}{n}}$

In[9]:=
```
Simplify[p]
```

Out[9]=
$$\frac{200. + 4.\, n}{2. + 1.\, n}$$

The profit that the firms make is

In[10]:=
```
profit = {p * q / n - cost[q / n]} /. totalQuantity
```

Out[10]=
$$\left\{-20. - \frac{9216.}{\left(1. + \frac{2.}{n}\right)^2 n^2} - \frac{384.}{\left(1. + \frac{2.}{n}\right) n} + \frac{96. \left(100. - \frac{96.}{1.+\frac{2.}{n}}\right)}{\left(1. + \frac{2.}{n}\right) n}\right\}$$

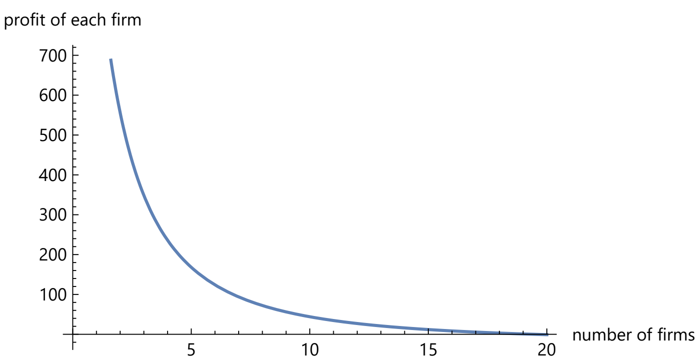
In[11]:=
```
Simplify[profit]
```

Out[11]=
$$\left\{\frac{9136. - 80.\, n - 20.\, n^2}{(2. + 1.\, n)^2}\right\}$$

If each firm has to make a profit, the profit must be positive.

In[12]:=
```
Plot[profit, {n, 0, 20}, AxesLabel → {"number of firms", "profit of each firm"}]
```

Out[12]=



In[13]:=
```
FindRoot[profit == 0, {n, 5}]
```

Out[13]=
$$\{n \to 19.4663\}$$

The profit that each firm makes steeply declines as more firms enter the market. To many any apprecia-ble profits though, there can be no more than 5-6 firms in the market. Beyond 19 firms, every firm will make a loss.

2. Imagine three firms supplying into the demand given above. Suppose each firm has a different constant marginal cost, as specified below, so cost can be taken as marginal cost times quantity supplied. Write down the profits of each firm as a function of the quantities supplied by the three firms, say profit1[q1_,q2_,q3_]=... and similarly profit2 and profit3. What are the Nash equilibrium conditions that must be satisfied if firms are profit maximizing?

Solve for the Nash equilibrium quantities each firm will supply to the market. What is the total quantity supplied and how much will consumers pay? How much will each firm make?

In[14]:=
```
mc1 = 10.;
mc2 = 12.;
mc3 = 14.;
```

The profit functions are

In[17]:=
```
profit1[q1_, q2_, q3_] = inversedemand[q1 + q2 + q3] * q1 - mc1 * q1
profit2[q1_, q2_, q3_] = inversedemand[q1 + q2 + q3] * q2 - mc2 * q2
profit3[q1_, q2_, q3_] = inversedemand[q1 + q2 + q3] * q3 - mc3 * q3
```

Out[17]=
```
-10. q1 + q1 (100. - q1 - q2 - q3)
```

Out[18]=
```
-12. q2 + q2 (100. - q1 - q2 - q3)
```

Out[19]=
```
-14. q3 + (100. - q1 - q2 - q3) q3
```

The conditions for Nash equilibrium are

In[20]:=
```
nashEquilibriumConditions = {D[profit1[q1, q2, q3], q1] == 0,
  D[profit2[q1, q2, q3], q2] == 0, D[profit3[q1, q2, q3], q3] == 0}
```

Out[20]=
```
{90. - 2 q1 - q2 - q3 == 0, 88. - q1 - 2 q2 - q3 == 0, 86. - q1 - q2 - 2 q3 == 0}
```

Solving these conditions gives us the quantities for which equilibrium is achieved.

In[21]:=
```
nashEquilibriumSolution = Solve[nashEquilibriumConditions, {q1, q2, q3}][[1]]
```

Out[21]=
```
{q1 → 24., q2 → 22., q3 → 20.}
```

The total quantity supplied is

In[22]:=
```
totalSupply = {q1 + q2 + q3} /. nashEquilibriumSolution
```

Out[22]=
```
{66.}
```

The price that consumers pay is

In[23]:=
```
p = inversedemand[q] /. {q → totalSupply}
```

Out[23]=
```
{34.}
```

The profits that the firms make are

In[24]:=
```
{profit1[q1, q2, q3], profit2[q1, q2, q3], profit3[q1, q2, q3]} /.
 nashEquilibriumSolution
```

Out[24]=
```
{576., 484., 400.}
```

3. Suppose that in the last problem, firms 2 and 3 merge. The combined firm

will prefer to supply at the lower marginal cost of firm 2, abandoning production from firm 3, that is, in the above calculation the merged firm will simply set q3=0. What are the quantities supplied by firms 1 and 2? What is the total quantity supplied and how much will consumers pay? How much will each firm make? Compare to the results from the last problem.

The profit of firm 1 is

```
In[25]:= profit1 = profit1[q1, q2, 0]
Out[25]=
-10. q1 + q1 (100. - q1 - q2)
```

The profit of the merged firm is

```
In[26]:= profit23 = profit2[q1, q2, 0]
Out[26]=
-12. q2 + (100. - q1 - q2) q2
```

The Nash Equilibrium equations are

```
In[27]:= nashEquilibriumConditions = {D[profit1, q1] == 0, D[profit23, q2] == 0}
Out[27]=
{90. - 2 q1 - q2 == 0, 88. - q1 - 2 q2 == 0}
```

The quantities supplied by firms 1 and 2 are

```
In[28]:= nashEquilibriumSolution = Solve[nashEquilibriumConditions, {q1, q2}][[1]]
Out[28]=
{q1 → 30.6667, q2 → 28.6667}
```

The total quantity supplied is

```
In[29]:= totalQuantity = {q1 + q2} /. nashEquilibriumSolution
Out[29]=
{59.3333}
```

This is lower than the supply in the case of 3 firms.

The price that consumers pay is

```
In[30]:= p = inversedemand[q] /. {q → totalQuantity}
Out[30]=
{40.6667}
```

This price is greater than the price consumers paid before the merger.

The profits that each firm make are

```
In[31]:= {profit1, profit23} /. nashEquilibriumSolution
Out[31]=
{940.444, 821.778}
```

The profit of firm 1 has increased significantly, whereas the combined profits of firm 2 and 3 have decreased.

4. Suppose two firms supplying into the demand above have cost functions cost1 and cost2 of their respective quantities produced as specified below. Note that marginal costs are decreasing so each firm would lower their marginal cost of production if they could capture more of the market, in particular, if they were the only firm supplying the total demand at a given price. Write the profit functions for the two firms in terms of the quantities each supplies to the market. Determine the Nash equilibrium quantities and resulting price. What net profits are the firms making in this case? Determine the optimum quantity, price, and profit for each firm if it were the only firm supplying the demand.

In[32]:=
```
Clear[cost1, cost2, q1, q2, profit1, profit2]
```

In[33]:=
```
cost1[q1_] = 1000. + 10. * q1 - 12 000. / (q1 + 30.);
cost2[q2_] = 800. + 15. * q2 - 7000. / (q2 + 40.);
```

The profit functions are

In[35]:=
```
profit1[q1_, q2_] = inversedemand[q1 + q2] * q1 - cost1[q1]
```

Out[35]=

$$-1000. - 10.\,q1 + \frac{12\,000.}{30. + q1} + q1\,(100. - q1 - q2)$$

In[36]:=
```
profit2[q1_, q2_] = inversedemand[q1 + q2] * q2 - cost2[q2]
```

Out[36]=

$$-800. - 15.\,q2 + (100. - q1 - q2)\,q2 + \frac{7000.}{40. + q2}$$

For Nash equilibrium

In[37]:=
```
nashEquilibriumConditions = {D[profit1[q1, q2], q1] == 0, D[profit2[q1, q2], q2] == 0}
```

Out[37]=

$$\left\{90. - 2\,q1 - \frac{12\,000.}{(30. + q1)^2} - q2 == 0,\ 85. - q1 - 2\,q2 - \frac{7000.}{(40. + q2)^2} == 0\right\}$$

The Nash equilibrium quantities are

In[38]:=
```
nashEquilibriumSolution = FindRoot[nashEquilibriumConditions, {q1, 10}, { q2, 10}]
```

Out[38]=

$\{q1 \to 29.9659,\ q2 \to 26.7311\}$

The resulting price is

In[39]:=
```
p = inversedemand[q1 + q2] /. nashEquilibriumSolution
```
Out[39]=

43.303

The profits made by each firm are

In[40]:=
```
{profit1[q1, q2], profit2[q1, q2]} /. nashEquilibriumSolution
```
Out[40]=

{198.069, 61.4688}

If firm 1 were the only firm supplying the demand, it’s profit function would be

In[41]:=
```
profit1[q1_] = inversedemand[q1] * q1 - cost1[q1]
```
Out[41]=

$$-1000. - 10.\,q1 + (100. - q1)\,q1 + \frac{12\,000.}{30. + q1}$$

Solving for maximum profit

In[42]:=
```
maxProfitEquation = D[profit1[q1], q1] == 0
```
Out[42]=

$$90. - 2\,q1 - \frac{12\,000.}{(30. + q1)^2} == 0$$

The maximum profit is obtained at a quantity

In[43]:=
```
maxProfitQuantity = FindRoot[maxProfitEquation, {q1, 10}]
```
Out[43]=

{q1 → 43.9014}

The resulting price would be

In[44]:=
```
p = inversedemand[q1] /. maxProfitQuantity
```
Out[44]=

56.0986

The profit made by firm 1 would be

In[45]:=
```
maxProfit = profit1[q1] /. maxProfitQuantity
```
Out[45]=

1186.17

If firm 2 were the only firm supplying the demand, it’s profit function would be

In[46]:=
```
profit2[q2_] = inversedemand[q2] * q2 - cost2[q2]
```
Out[46]=

$$-800. - 15.\,q2 + (100. - q2)\,q2 + \frac{7000.}{40. + q2}$$

Solving for maximum profit

In[47]:=
```
maxProfitEquation = D[profit2[q2], q2] == 0
```

Out[47]=

$$85. - 2\,q2 - \frac{7000.}{(40. + q2)^2} == 0$$

The maximum profit is obtained at a quantity

In[48]:=
```
maxProfitQuantity = FindRoot[maxProfitEquation, {q2, 10}]
```

Out[48]=

$\{q2 \to 41.9792\}$

The resulting price would be

In[49]:=
```
p = inversedemand[q2] /. maxProfitQuantity
```

Out[49]=

58.0208

The profit made by firm 2 would be

In[50]:=
```
maxProfit = profit2[q2] /. maxProfitQuantity
```

Out[50]=

1091.37