MATH 446/OR 481
Project 2

September 13, 2022

Project Conclusion - Using different $g(x)$ for fixed point interation, we compute the three real roots of the given function. An analsyis of convergence rates done using hand calculations and MATLAB are in agreement with the condition for convergence of fixed point iteration, i.e. $\left|g'(r)\right| < 1$.

Q1. For the first root, we rewrite the given equation as

$$x = \frac{\exp(x) + x^3 + 2 - \sin(x)}{5}$$

Using an initial guess of 0 for the root and a tolerance value of 10^{-9} to check for convergence, we get the approximate value of the root as $\boxed{0.81346772}$. The number of iterations required are 53.

For the second root, we rewrite the given equation as

$$x = (5x + \sin(x) - 2 - \exp(x))^{1/3}$$

Using an initial guess of 0 for the root and a tolerance value of 10^{-9} to check for convergence, we get the approximate value of the root as $\boxed{1.14019958}$. The number of iterations required are 38.

For the third root, we rewrite the given equation as

$$x = -\sqrt{\left(\frac{5x + \sin(x) - 2 - \exp(x)}{x}\right)}$$

Using an initial guess of -2 for the root and a tolerance value of 10^{-9} to check for convergence, we get the approximate value of the root as $\boxed{-2.46912418}$. The number of iterations required are 13.

A summary of results is presented in the table below.

$g(x)$	Initial Guess	Iteration Count	Obtained Root
$\frac{\exp(x)+x^3+2-\sin(x)}{5}$	0	53	0.81346772
$(5x + \sin(x) - 2 - \exp(x))^{1/3}$	0	38	1.14019958
$-\sqrt{\left(\frac{5x+\sin(x)-2-\exp(x)}{x}\right)}$	-2	13	-2.46912418

The MATLAB code used to implement the fixed point iteration method and the Command Window output can be found in the appendices.

Q2. For the first root, we have

$$g(x) = \frac{\exp(x) + x^3 + 2 - \sin(x)}{5}$$

Differentiating w.r.t x, we have

$$g'(x) = \frac{\exp(x) + 3x^2 - \cos(x)}{5}$$

Subsituting the value obtained for the root, we have the convergence rate.

$$\boxed{S = \left|g'(r)\right| = \left|g'(0.81346772)\right| = 0.71078}$$

For the second root, we have

$$g(x) = (5x + \sin(x) - 2 - \exp(x))^{1/3}$$

Differentiating w.r.t x, we have

$$g'(x) = \frac{1}{3}(5 + \cos(x) - \exp(x))^{-2/3}$$

Subsituting the value obtained for the root, we have

$$\boxed{S = \left|g'(r)\right| = \left|g'(1.14019958)\right| = 0.58716}$$

For the third root, we have

$$g(x) = -\sqrt{\left(\frac{5x + \sin(x) - 2 - \exp(x)}{x}\right)}$$

Differentiating w.r.t x, we have

$$g'(x) = -\frac{x\cos(x) - \sin(x) + 2 + \exp(x)(1 - x)}{2x^2}\sqrt{\left(\frac{x}{5x + \sin(x) - 2 - \exp(x)}\right)}$$

Subsituting the value obtained for the root, we have

$$\boxed{S = \left|g'(r)\right| = \left|g'(-2.46912418)\right| = 0.16103}$$

Q3. Using MATLAB, we obtain the following convergence rates for each of the roots.

obtained root	convergence rate
0.81346772	0.71074
1.14019958	0.58712
-2.46912418	0.16103

The MATLAB code used to compute the convergence rate and the Command Window output can be found in the appendices.

Appendix - Command Window Output

```
Command Window
  Root #1 is 0.81346772 obtained in 53 fixed point iterations.
  Root #1 is 1.14019958 obtained in 38 fixed point iterations.
  Root #1 is -2.46912418 obtained in 13 fixed point iterations.

  Approx. rate of convergence of fixed iteration for root #1 is 0.71074
  Approx. rate of convergence of fixed iteration for root #2 is 0.58712
  Approx. rate of convergence of fixed iteration for root #3 is 0.16103
fx >>
```

Appendix - MATLAB Code

```
clear, clc

%% Q1 - Find all real roots, correct to 8 decimal places

% root #1
g = @(x) (exp(x) + x^3 + 2 - sin(x))/5;

tol = 1e-9;     % accuracy
iter = 0;       % iteration counter
MAXITER = 100;  % maximum number of iterations

initialGuess = 0;
oldGuess = initialGuess;

while (iter < MAXITER)
% update iteration counter
iter = iter + 1;
% fixed point iteration
newGuess = g(oldGuess);
%fprintf('%.9f\n', newGuess)
% check for accuracy
if (abs(newGuess - oldGuess) < tol)
break
else
oldGuess = newGuess;
end
end

root_1 = newGuess;
fprintf('Root #1 is %.8f obtained in %d fixed point iterations.\n', ...
root_1, iter)

% root #2
g = @(x) (5*x + sin(x) - 2 - exp(x))^(1/3);

tol = 1e-9;
iter = 0;
MAXITER = 100;
```

```
initialGuess = 0;
oldGuess = initialGuess;

while (iter < MAXITER)
iter = iter + 1;
newGuess = g(oldGuess);
%fprintf('%.9f\n', newGuess)
if (abs(newGuess - oldGuess) < tol)
break
else
oldGuess = newGuess;
end
end

root_2 = newGuess;
fprintf('Root #1 is %.8f obtained in %d fixed point iterations.\n', ...
root_2, iter)

% root #3
g = @(x) -sqrt((5*x + sin(x) - 2 - exp(x))/x);

tol = 1e-9;
iter = 0;
MAXITER = 100;

initialGuess = -2;
oldGuess = initialGuess;

while (iter < MAXITER)
iter = iter + 1;
newGuess = g(oldGuess);
%fprintf('%.9f\n', newGuess)
if (abs(newGuess - oldGuess) < tol)
break
else
oldGuess = newGuess;
end
end

root_3 = newGuess;
fprintf('Root #1 is %.8f obtained in %d fixed point iterations.\n', ...
```

```
root_3, iter)

fprintf('\n')
%% Q3 - Compute rate of convergence

% assume true root = computed above

% root #1
g = @(x) (exp(x) + x^3 + 2 - sin(x))/5;

initialGuess = 0;
oldGuess = initialGuess;

iter = 0;      % iteration counter
MAXITER = 100; % max number of iterations

oldErrorRatio = 1e4;
newErrorRatio = 1;

tol = 1e-5;

while (iter < MAXITER)
% update iteration counter
iter = iter + 1;
% fixed point iteration
newGuess = g(oldGuess);
% compute ratio of errors
newErrorRatio = abs(newGuess - root_1)/abs(oldGuess - root_1);
%fprintf('%.4f\n', newErrorRatio)
% check if errorRatio has converged
if (abs(newErrorRatio - oldErrorRatio) < tol)
break
else
oldErrorRatio = newErrorRatio;
end
% update oldGuess for next iteration
oldGuess = newGuess;
end

fprintf(['Approx. rate of convergence of fixed iteration for root #1 ' ...
'is %.5f \n'], newErrorRatio)
```

```
% root #2
g = @(x) (5*x + sin(x) - 2 - exp(x))^(1/3);

initialGuess = 0;
oldGuess = initialGuess;

iter = 0;
MAXITER = 100;

oldErrorRatio = 1e4;
newErrorRatio = 1;

tol = 1e-5;

while (iter < MAXITER)
iter = iter + 1;
newGuess = g(oldGuess);
newErrorRatio = abs(newGuess - root_2)/abs(oldGuess - root_2);
%fprintf('%.4f\n', errorRatio)
if (abs(newErrorRatio - oldErrorRatio) < tol)
break
else
oldErrorRatio = newErrorRatio;
end
oldGuess = newGuess;
end

fprintf(['Approx. rate of convergence of fixed iteration for root #2 ' ...
'is %.5f \n'], newErrorRatio)

% root #3
g = @(x) -sqrt((5*x + sin(x) - 2 - exp(x))/x);

initialGuess = -2;
oldGuess = initialGuess;

iter = 0;
MAXITER = 100;

oldErrorRatio = 1e4;
```

```
newErrorRatio = 1;

tol = 1e-5;

while (iter < MAXITER)
iter = iter + 1;
newGuess = g(oldGuess);
newErrorRatio = abs(newGuess - root_3)/abs(oldGuess - root_3);
%fprintf('%.4f\n', errorRatio)
if (abs(newErrorRatio - oldErrorRatio) < tol)
break
else
oldErrorRatio = newErrorRatio;
end
oldGuess = newGuess;
end

fprintf(['Approx. rate of convergence of fixed iteration for root #3 ' ...
'is %.5f \n'], newErrorRatio)
```