Introduction to Cryptography
Lecture 10

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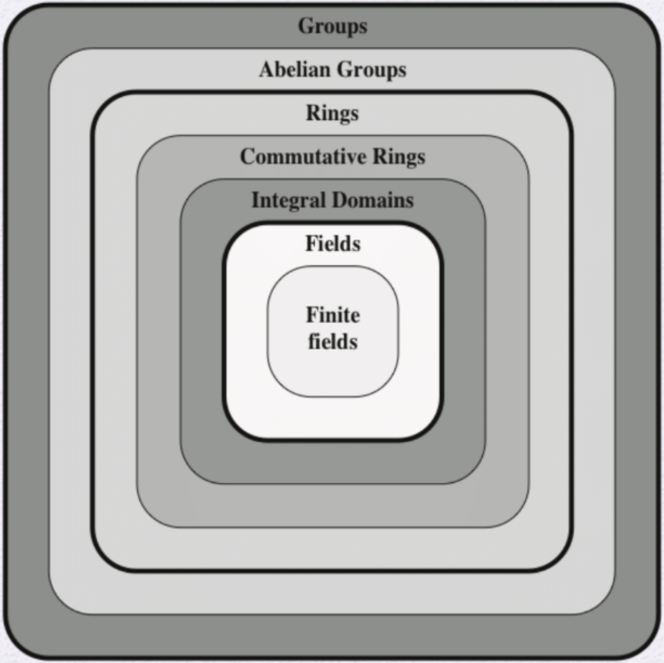

Groups
Abelian Groups
Rings
Commutative Rings
Integral Domains
Fields
Finite fields

Types of fields

Group

Abelian group

Ring

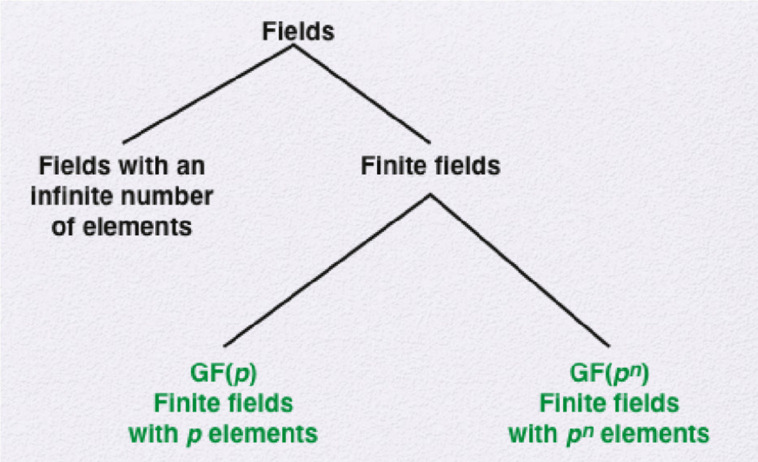
Commutative ring

Integral domain

Field

(A1) Closure under addition:	If a and b belong to S, then $a + b$ is also in S
(A2) Associativity of addition:	$a + (b + c) = (a + b) + c$ for all a, b, c in S
(A3) Additive identity:	There is an element 0 in R such that $a + 0 = 0 + a = a$ for all a in S
(A4) Additive inverse:	For each a in S there is an element $-a$ in S such that $a + (-a) = (-a) + a = 0$
(A5) Commutativity of addition:	$a + b = b + a$ for all a, b in S
(M1) Closure under multiplication:	If a and b belong to S, then ab is also in S
(M2) Associativity of multiplication:	$a(bc) = (ab)c$ for all a, b, c in S
(M3) Distributive laws:	$a(b + c) = ab + ac$ for all a, b, c in S $(a + b)c = ac + bc$ for all a, b, c in S
(M4) Commutativity of multiplication:	$ab = ba$ for all a, b in S
(M5) Multiplicative identity:	There is an element 1 in S such that $a1 = 1a = a$ for all a in S
(M6) No zero divisors:	If a, b in S and $ab = 0$, then either $a = 0$ or $b = 0$
(M7) Multiplicative inverse:	If a belongs to S and a 0, there is an element a^{-1} in S such that $aa^{-1} = a^{-1}a = 1$

Types of fields



Finite Fields of the Form $GF(p^m)$

Galois' Theorem

An order-n finite field exists if and only if $n = p^m$ for some prime p and some positive integer m.

- p is called the characteristic of this finite field
- The order of an finite field is its number of elements
- We usually use $GF(p^m)$ or $\mathbb{F}_{p^m}$ to represent the finite field of order p^m
- An order-n finite field is unique (up to isomorphism)
- Addition and multiplication modulo a prime number p form a finite field ($\mathbb{Z}_p = GF(p)$)

Finite Fields of the Form $GF(p^m)$

- If $m = 1$ then $\mathbb{Z}_p = GF(p)$
- One way to construct a finite field with $m > 1$ is using the polynomial basis. The field is constructed as a set of p^m polynomials along with two polynomial operations

Remark

Example. Consider $2^3 = 8$. We know $(\mathbb{Z}_8, +, \cdot)$ is not a field

Polynomial Arithmetic

- A polynomial $f(x)$ is a mathematical expression in the form $a_n x^n + a_{n-1} x^{n-1} + ... + a_0$
- The highest exponent of x is the degree of the polynomial
- $a_n, a_{n-1}, ..., a_0$ are called coefficients

Polynomial Arithmetic

We can:

- add polynomials
- subtract polynomials
- multiply polynomials
- divide polynomials

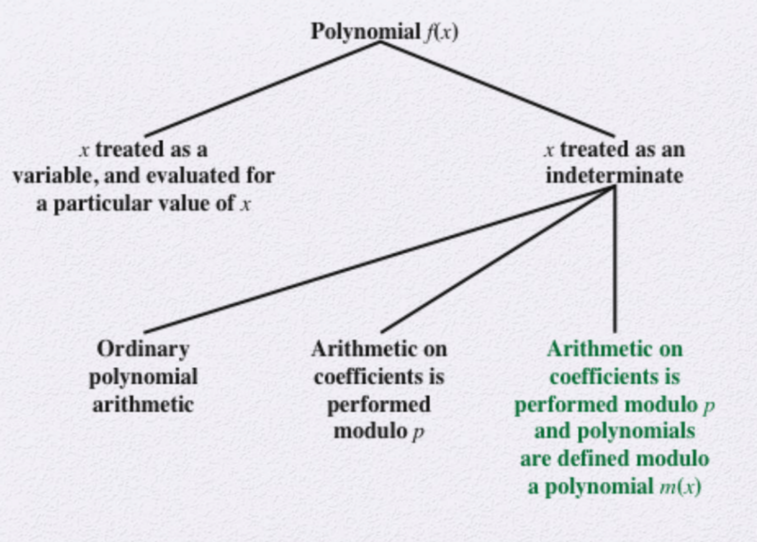
`Examples on whiteboard`

Polynomial Arithmetic over field

If the coefficients are taken from a field F, then we say it is a polynomial over F. With polynomials over field $GF(p)$, you can add and multiply polynomials just like you have always done but the coefficients need to be reduced modulo p.

`Examples on whiteboard`

Treatment of polynomials



Finite Fields of the form $GF(p^m)$

Irreducible polynomial

If a polynomial is divisible only by itself and constants, then we call this polynomial an irreducible polynomial

Polynomial GCD

$gcd[a(x), b(x)]$ is the polynomial of maximum degree that divides both $a(x)$ and $b(x)$

Similar to integers, you can do modular arithmetic with polynomials over a field. Now the operands and modulus are polynomials.
Always remember there are two moduli involved: a polynomial modulus and an integer modulus.

Finite Fields of the form $GF(p^m)$

Irreducible polynomial

If the modulus $g(x)$ is an irreducible polynomial of degree m over $GF(p)$, then the finite field $GF(p^m)$ can be constructed by the set of polynomials over $GF(p)$ whose degree is at most $m-1$, where addition and multiplication are done modulo $g(x)$.

`Examples on whiteboard`

Polynomial Arithmetic Modulo $(x^3 + x + 1)$

	+	000	001	010	011	100	101	110	111
		0	1	x	$x+1$	x^2	x^2+1	x^2+x	x^2+x+1
000	0	0	1	x	$x+1$	x^2	x^2+1	x^2+x	x^2+x+1
001	1	1	0	$x+1$	x	x^2+1	x^2	x^2+x+1	x^2+x
010	x	x	$x+1$	0	1	x^2+x	x^2+x+1	x^2	x^2+1
011	$x+1$	$x+1$	x	1	0	x^2+x+1	x^2+x	x^2+1	x^2
100	x^2	x^2	x^2+1	x^2+x	x^2+x+1	0	1	x	$x+1$
101	x^2+1	x^2+1	x^2	x^2+x+1	x^2+x	1	0	$x+1$	x
110	x^2+x	x^2+x	x^2+x+1	x^2	x^2+1	x	$x+1$	0	1
111	x^2+x+1	x^2+x+1	x^2+x	x^2+1	x^2	$x+1$	x	1	0

(a) Addition

	×	000	001	010	011	100	101	110	111
		0	1	x	$x+1$	x^2	x^2+1	x^2+x	x^2+x+1
000	0	0	0	0	0	0	0	0	0
001	1	0	1	x	$x+1$	x^2	x^2+1	x^2+x	x^2+x+1
010	x	0	x	x^2	x^2+x	$x+1$	1	x^2+x+1	x^2+1
011	$x+1$	0	$x+1$	x^2+x	x^2+1	x^2+x+1	x^2	1	x
100	x^2	0	x^2	$x+1$	x^2+x+1	x^2+x	x	x^2+1	1
101	x^2+1	0	x^2+1	1	x^2	x	x^2+x+1	$x+1$	x^2+x
110	x^2+x	0	x^2+x	x^2+x+1	1	x^2+1	$x+1$	x	x^2
111	x^2+x+1	0	x^2+x+1	x^2+1	x	1	x^2+x	x^2	$x+1$

(b) Multiplication

Thanks for Your attention.