USN 1CR15CV416

MATDIP401

Fourth Semester B.E. Degree Examination, June/July 2016
Advanced Mathematics – II

Time: 3 hrs. Max. Marks:100

Note: ***Answer any FIVE full questions.***

1 a. Find the angle between any two diagonals of a cube. **(07 Marks)**

b. Prove that the general equation of first degree in x, y, z represents a plane. **(07 Marks)**

c. Find the angle between the lines,

$\frac{x-1}{1}=\frac{y-5}{0}=\frac{z+1}{5}$ and $\frac{x+3}{3}=\frac{y}{5}=\frac{z-5}{2}$. **(06 Marks)**

2 a. Prove that the lines,

$\frac{x-5}{3}=\frac{y-1}{1}=\frac{z-5}{-2}$ and $\frac{x+3}{1}=\frac{y-5}{3}=\frac{z}{5}$ are perpendicular. **(07 Marks)**

b. Find the shortest distance between the lines.

$\frac{x-8}{3}=\frac{y+9}{-16}=\frac{z-10}{7}$ and $\frac{x-15}{3}=\frac{y-29}{8}=\frac{z-5}{-5}$. **(07 Marks)**

c. Find the equation of the plane containing the point (2, 1, 1) and the line, $\frac{x+1}{2}=\frac{y-2}{3}=\frac{z+1}{-1}$ **(06 Marks)**

3 a. Find the constant 'a' so that the vectors $2\hat{i}-\hat{j}+\hat{k}$, $\hat{i}+2\hat{j}-3\hat{k}$ and $3\hat{i}+a\hat{j}+5\hat{k}$ are co-planar. **(07 Marks)**

b. If $\vec{a}=2\hat{i}+3\hat{j}-4\hat{k}$ and $\vec{b}=8\hat{i}-4\hat{j}+\hat{k}$ then prove that $\vec{a}$ is perpendicular to $\vec{b}$ and also find $\left|\vec{a}\times\vec{b}\right|$. **(07 Marks)**

c. Find the volume of the parallelopiped whose co-terminal edges are represented by the vectors,

$\vec{a}=\hat{i}+\hat{j}+\hat{k}$, $\vec{b}=2\hat{i}+3\hat{j}-\hat{k}$ and $\vec{c}=\hat{i}-\hat{j}-\hat{k}$ **(06 Marks)**

4 a. Find the velocity and acceleration of a particle moves along the curve $\vec{r}=e^{-2t}\hat{i}+2\cos 5t\hat{j}+5\sin 2t\hat{k}$ at any time 't'. **(07 Marks)**

b. Find the directional derivative of x^2yz^3 at (1, 1, 1) in the direction of $\hat{i}+\hat{j}+2\hat{k}$. **(07 Marks)**

c. Find the divergence of the vector $\vec{F}=(xyz+y^2z)\hat{i}+(3x^2+y^2z)\hat{j}+(xz^2-y^2z)\hat{k}$. **(06 Marks)**

5 a. $\vec{F}=(x+y+1)\hat{i}+\hat{j}-(x+y)\hat{k}$, show that $\vec{F}\cdot \text{curl}\,\vec{F}=0$. **(07 Marks)**

b. Show that the vector field, $\vec{F}=(3x+3y+4z)\hat{i}+(x-2y+3z)\hat{j}+(3x+2y-z)\hat{k}$ is solenoidal. **(07 Marks)**

c. Find the constants a, b, c such that the vector field,

$\vec{F}=(x+y+az)\hat{i}+(x+cy+2z)\hat{j}+(bx+2y-z)\hat{k}$ is irrotational. **(06 Marks)**

6 a. Prove that $L(\sin at) = \frac{a}{s^2 + a^2}$. **(07 Marks)**

b. Find $L[\sin t \quad \sin 2t \quad \sin 3t]$. **(07 Marks)**

c. Find $L[\cos^3 t]$. **(06 Marks)**

7 a. Find the inverse Laplace transform of $\frac{1}{(s+1)(s+2)(s+3)}$. **(07 Marks)**

b. Find $L^{-1}\left[\log\left(1+\frac{a^2}{s^2}\right)\right]$. **(07 Marks)**

c. Find $L^{-1}\left[\frac{s+2}{s^2-4s+13}\right]$. **(06 Marks)**

8 a. Solve the differential equation, $y'' + 2y' + y = 6te^{-t}$ under the conditions $y(0) = 0 = y'(0)$ by Laplace transform techniques. **(10 Marks)**

b. Solve the differential equation, $y'' - 3y' + 2y = 0$ $y(0) = 0$, $y'(0) = 1$ by Laplace transform techniques. **(10 Marks)**

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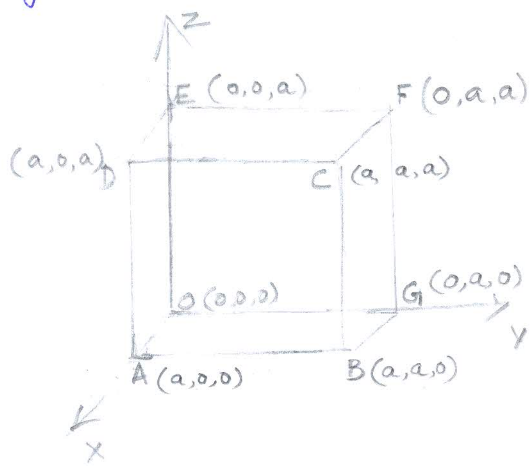
Advanced Mathematics - II (June / July 2016)

1 a. Find the angle between two diagonals of a cube.

Soln. Let 'a' be the side of the cube. Consider the coordinate system as in figure. The coordinates of the vertices



are O : (0,0,0) D : (a,0,a)

A : (a,0,0) E : (0,0,a)

B : (a,a,0) F : (0,a,a)

C : (a,a,a) G : (0,a,0).

The four diagonals of the cube are OC, EB, DG and AF.

D.rs of OC are $a-0, a-0, a-0 = a, a, a$.

Drs of EB are $a-0, a-0, 0-a = a, a, -a$.

$\therefore$ Angle b/w OC and EB: $\cos\theta = \dfrac{a \cdot a + a \cdot a + a \cdot (-a)}{\sqrt{a^2+a^2+a^2}\sqrt{a^2+a^2+(-a)^2}} = \dfrac{a^2}{\sqrt{3a^2}\sqrt{3a^2}} = \dfrac{1}{3}$

$\underline{\theta = \cos^{-1}\left(\frac{1}{3}\right)}$

b. Prove that the general equation of first degree in x, y, z represents a plane.

Ans: Let the first degree equation in x, y + z be

$ax + by + cz + d = 0$ —— (1)

Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be any two points on this surface.

then, $ax_1 + by_1 + cz_1 + d = 0$ —— (2)

$ax_2 + by_2 + cz_2 + d = 0$ —— (3)

Let R be a point on PQ, dividing it in the ratio $k:1$.

Then $R = \left(\dfrac{kx_2 + x_1}{k+1}, \dfrac{ky_2 + y_1}{k+1}, \dfrac{kz_2 + z_1}{k+1}\right)$

③ × k + ② gives

$$kax_2 + kby_2 + kcz_2 + kd + ax_1 + by_1 + cz_1 + d = 0$$

$$\Rightarrow a(kx_2 + x_1) + b(ky_2 + y_1) + c(kz_2 + z_1) + d(k+1) = 0$$

$$\Rightarrow a\left(\frac{kx_2 + x_1}{k+1}\right) + b\left(\frac{ky_2 + y_1}{k+1}\right) + c\left(\frac{kz_2 + z_1}{k+1}\right) + d = 0$$

$\Rightarrow R\left(\frac{kx_2 + x_1}{k+1}, \frac{ky_2 + y_1}{k+1}, \frac{kz_2 + z_1}{k+1}\right)$ lies on ①

$\Rightarrow$ Every point on PQ lies on ①.

$\Rightarrow$ The surface represented by ① is a plane.

c. Find the angle b/w the lines

$$\frac{x-1}{1} = \frac{y-5}{0} = \frac{z+1}{5} \text{ and } \frac{x+3}{3} = \frac{y}{5} = \frac{z-5}{2}.$$

Ans: the drs of the first line are 1, 0, 5

and d.rs of second line are 3, 5, 2.

If θ is the angle b/w the lines, then

$$\cos\theta = \frac{1(3) + 0(5) + 5(2)}{\sqrt{1^2+0^2+5^2}\sqrt{3^2+5^2+2^2}} = \frac{13}{\sqrt{26}\sqrt{38}} = \frac{13}{\sqrt{190}}$$

$$\theta = \cos^{-1}\left(\frac{13}{\sqrt{190}}\right)$$

2\. a. Prove that the lines $\frac{x-5}{3} = \frac{y-1}{1} = \frac{z-5}{-2}$ and $\frac{x+3}{1} = \frac{y-5}{3} = \frac{z}{5}$ are perpendicular.

Ans: D.rs of first line $a_1, b_1, c_1 = 3, 1, -2$

D.rs of second line $a_2, b_2, c_2 = 1, 3, 5$

$$a_1a_2 + b_1b_2 + c_1c_2 = 3(1) + 1(3) + (-2)(5)$$
$$= 3 + 3 - 10$$
$$= -4.$$

$\therefore$ the given lines are not perpendicular.

b. Find the shortest distance b/w the lines

$\frac{x-8}{3} = \frac{y+9}{-16} = \frac{z-10}{7}$ and $\frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5}$

Soln: the d.rs of the given lines are

$3, -16, 7$ and $3, 8, -5$.

Since shortest distance is perpendicular to both the lines,

$$3a - 16b + 7c = 0$$
$$3a + 8b - 5c = 0$$

$$\frac{a}{\begin{vmatrix} -16 & 7 \\ 8 & -5 \end{vmatrix}} = \frac{-b}{\begin{vmatrix} 3 & 7 \\ 3 & -5 \end{vmatrix}} = \frac{c}{\begin{vmatrix} 3 & -16 \\ 3 & 8 \end{vmatrix}}$$

$$\frac{a}{24} = \frac{-b}{-36} = \frac{c}{72} \Rightarrow \frac{a}{2} = \frac{b}{3} = \frac{c}{6}$$

$a, b, c = 2, 3, 6$

$$l, m, n = \frac{2}{\sqrt{4+9+36}}, \frac{3}{\sqrt{4+9+36}}, \frac{6}{\sqrt{4+9+36}} = \frac{2}{7}, \frac{3}{7}, \frac{6}{7}$$

$$S.D = l(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)$$

$$= \frac{2}{7}(15-8) + \frac{3}{7}(29+9) + \frac{6}{7}(5-10)$$

$$= \frac{1}{7}(14 + 114 - 30) = \frac{98}{7} \text{ units}$$

c. Find the equation of the plane containing the point $(2, 1, 1)$ and the line $\frac{x+1}{2} = \frac{y-2}{3} = \frac{z+1}{-1}$

Soln: Equation of plane containing the given line is

$a(x+1) + b(y-2) + c(z+1) = 0$ — ①, with

$2a + 3b - c = 0$ — ②

Since the plane ① contains the point $(2, 1, 1)$,

$a(2+1) + b(1-2) + c(1+1) = 0$.

ie $3a - b + 2c = 0$ — ③

Solving ①, ② and ③,

$$\begin{vmatrix} x+1 & y-2 & z+1 \\ 2 & 3 & -1 \\ 3 & -1 & 2 \end{vmatrix} = 0$$

$(x+1)5 - (y-2)7 + (z+1)(-11) = 0$

$5x - 7y - 11z + 8 = 0$

3 a. Find the constant 'a' so that the vectors $2\hat{i}-\hat{j}+\hat{k}$, $\hat{i}+2\hat{j}-3\hat{k}$ and $3\hat{i}+a\hat{j}+5\hat{k}$ are coplanar.

Soln: $\vec{A} = 2\hat{i}-\hat{j}+\hat{k}$

$\vec{B} = \hat{i}+2\hat{j}-3\hat{k}$

$\vec{C} = 3\hat{i}+a\hat{j}+5\hat{k}$

$$\vec{A}\cdot\vec{B}\times\vec{C} = \begin{vmatrix} 2 & -1 & 1 \\ 1 & 2 & -3 \\ 3 & a & 5 \end{vmatrix} = 0$$

$$2(10+3a)+1(5+9)+1(a-6)=0$$

$$7a+28=0$$

$$a = \frac{-28}{7} = -4$$

b. If $\vec{a} = 2\hat{i}+3\hat{j}-4\hat{k}$ and $\vec{b} = 8\hat{i}-4\hat{j}+\hat{k}$ then prove that $\vec{a}$ is perpendicular to $\vec{b}$ and also find $|\vec{a}\times\vec{b}|$

Soln: W.k.t if $\vec{a}\perp\vec{b}$ then $\vec{a}\cdot\vec{b}=0$

$$\vec{a}\cdot\vec{b} = 2(8)+3(-4)+(-4)(1) = 16-12-4=0$$

$\therefore \vec{a}\perp\vec{b}$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & -4 \\ 8 & -4 & 1 \end{vmatrix} = (3-16)\hat{i} - (2+32)\hat{j} + (-8-24)\hat{k}$$

$$= -13\hat{i} - 34\hat{j} - 32\hat{k}$$

$$|\vec{a}\times\vec{b}| = \sqrt{(-13)^2+(-34)^2+(-32)^2} = \sqrt{2349}$$

c. Find the volume of the parallelopiped whose coterminal edges are represented by the vectors $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{b} = 2\hat{i} + 3\hat{j} - \hat{k}$ and $\vec{c} = \hat{i} - \hat{j} - \hat{k}$.

Soln:

Volume of parallelopiped $= \vec{a} \cdot (\vec{b} \times \vec{c})$

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & -1 \\ 1 & -1 & -1 \end{vmatrix}$$

$$= 1(-3-1) - 1(-2+1) + 1(-2-3)$$

$$= -4 + 1 - 5$$

$$= -8$$

Volume $= \underline{\underline{8 \text{ units}}}$.

4 a. Find the velocity and acceleration of a particle moving along the curve $\vec{r} = e^{-2t}\hat{i} + 2\cos 5t\,\hat{j} + 5\sin 2t\,\hat{k}$ at any time 't'.

Soln: $\vec{r} = e^{-2t}\hat{i} + 2\cos 5t\,\hat{j} + 5\sin 2t\,\hat{k}$

$$\vec{v} = \frac{d\vec{r}}{dt} = -2e^{-2t}\hat{i} - 10\sin 5t\,\hat{j} + 10\cos 2t\,\hat{k}$$

$$\vec{a} = \frac{d^2\vec{r}}{dt^2} = 4e^{-2t}\hat{i} - 50\cos 5t\,\hat{j} - 20\sin 2t\,\hat{k}$$

b. Find the directional derivative of x^2yz^3 at $(1,1,1)$ in the direction of $\hat{i}+\hat{j}+2\hat{k}$

Soln: $\phi = x^2yz^3$

$$\nabla\varphi = \frac{\partial\varphi}{\partial x}\hat{i} + \frac{\partial\varphi}{\partial y}\hat{j} + \frac{\partial\varphi}{\partial z}k$$

$$= 2xyz^3\hat{i} + x^2z^3\hat{j} + 3x^2yz^2\hat{k}$$

$$\nabla\phi \text{ at } (1,1,1) = 2\hat{i} + \hat{j} + 3\hat{k}$$

$$\vec{a} = \hat{i} + \hat{j} + 2\hat{k}, \quad |\vec{a}| = \sqrt{6}$$

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{1}{\sqrt{6}}(\hat{i} + \hat{j} + 2\hat{k})$$

$$\nabla\phi\cdot\hat{a} = \frac{1}{\sqrt{6}}(2\times1 + 1\times1 + 3\times2) = \frac{9}{\sqrt{6}}$$

c. Find the divergence of the vector $\vec{F} = (xyz + y^2z)\hat{i} + (3x^2 + y^2z)\hat{j} + (xz^2 - y^2z)\hat{k}$.

Soln: $\text{div}\,\vec{F} = \frac{\partial}{\partial x}(xyz + y^2z) + \frac{\partial}{\partial y}(3x^2 + y^2z) + \frac{\partial}{\partial z}(xz^2 - y^2z)$

$$= yz + 2yz + 2xz - y^2$$

$$= 3yz + 2xz - y^2$$

5 a. $\vec{F} = (x+y+1)\hat{i} + \hat{j} - (x+y)\hat{k}$. Show that $\vec{F}\cdot \text{curl}\,\vec{F} = 0$

<u>Soln</u>: $\text{curl}\,\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y+1 & 1 & -x-y \end{vmatrix}$

$= \hat{i}(-1) - \hat{j}(-1) + \hat{k}(-1)$

$= -\hat{i} + \hat{j} - \hat{k}$

$\vec{F}\cdot \text{curl}\,\vec{F} = (x+y+1)(-1) + 1\times 1 - (x+y)(-1)$

$= -x-y-1+1+x+y = \underline{0}$

b. Show that the vector field

$\vec{F} = (3x+3y+4z)\hat{i} + (x-2y+3z)\hat{j} + (3x+2y-z)\hat{k}$ is solenoidal.

<u>Soln</u>: $\nabla\cdot\vec{F} = \frac{\partial}{\partial x}(3x+3y+4z) + \frac{\partial}{\partial y}(x-2y+3z) + \frac{\partial}{\partial z}(3x+2y-z)$

$= 3-2-1 = 0$

Since $\nabla\cdot\vec{F} = 0$, $\vec{F}$ is solenoidal.

c. Find the constants a, b, c such that the vector field $\vec{F} = (x+y+az)\hat{i} + (bx+2y-z)\hat{j} + (x+cy+2z)\hat{k}$ is irrotational.

<u>Soln</u>: $\vec{F}$ is irrotational $\Rightarrow \nabla\times\vec{F} = 0$

$\nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y+az & bx+2y-z & x+cy+2z \end{vmatrix}$

$= \hat{i}(c+1) - \hat{j}(1-a) + \hat{k}(b-1) = 0$

$c+1 = 0 \qquad 1-a = 0 \qquad b-1 = 0$

$c = -1 \qquad a = 1, \; b = 1$

5 a. Prove that $L(\sin at) = \dfrac{a}{s^2+a^2}$

Soln: $L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$

$$L(\sin at) = \int_0^\infty e^{-st} \sin at\,dt$$

$$\int e^{ax} \sin bx\,dx = \frac{e^{ax}}{a^2+b^2}\left[a \sin bx - b\cos bx\right]$$

$$\therefore L(\sin at) = \left.\frac{e^{-st}}{(-s)^2+a^2}\left[-s \sin at - a\cos at\right]\right)_0^\infty$$

$$= \frac{0 - 1(0-a)}{s^2+a^2} = \underline{\underline{\frac{a}{s^2+a^2}}}$$

b. Find $L[\sin t \sin 2t \sin 3t]$

Soln: $\sin 3t \sin 2t \sin t = \frac{1}{2}[\cos t - \cos 5t] \sin t$

$\left(\because \sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]\right)$

$$= \frac{1}{2}\cos t \sin t - \frac{1}{2}\cos 5t \sin t$$

$$= \frac{1}{4}\sin 2t - \frac{1}{4}(\sin 6t - \sin 4t) \quad \left(\because \cos A \sin B = \frac{1}{2}(\sin(A+B) - \sin(A-B))\right)$$

$$= \frac{1}{4}(\sin 2t - \sin 6t + \sin 4t)$$

$$\therefore L[\sin t \sin 2t \sin 3t] = \underline{\underline{\frac{1}{4}\left[\frac{2}{s^2+4} - \frac{6}{s^2+36} + \frac{4}{s^2+16}\right]}}$$

c. Find $L(\cos^3 t)$

Soln: $\cos^3 t = \frac{1}{4}[\cos 3t + 3\cos t]$; $L(\cos at) = \dfrac{s}{s^2+a^2}$

$$\therefore L(\cos^3 t) = \underline{\underline{\frac{1}{4}\left[\frac{s}{s^2+9} + 3 \cdot \frac{s}{s^2+1}\right]}}$$

7 a. Find the inverse Laplace transform of $\frac{1}{(s+1)(s+2)(s+3)}$

<u>Soln</u> : Let $\frac{1}{(s+1)(s+2)(s+3)} = \frac{A}{(s+1)} + \frac{B}{(s+2)} + \frac{C}{(s+3)}$

$A = \frac{1}{2} \quad B = -1 \quad C = \frac{1}{2}$

$\therefore L^{-1}\left[\frac{1}{(s+1)(s+2)(s+3)}\right] = \frac{1}{2} L^{-1}\left(\frac{1}{s+1}\right) - L^{-1}\left(\frac{1}{s+2}\right) + \frac{1}{2} L^{-1}\left(\frac{1}{s+3}\right)$

$= \frac{1}{2}e^{-t} - e^{-2t} + \frac{1}{2}e^{-3t}$

b. Find $L^{-1}\left(\log\left(1+\frac{a^2}{s^2}\right)\right)$

<u>Soln</u> $F(s) = \log\left(1+\frac{a^2}{s^2}\right) = \log\left(\frac{s^2+a^2}{s^2}\right)$

$= \log(s^2+a^2) - \log s^2$

$\frac{d}{ds}F(s) = \frac{2s}{s^2+a^2} - \frac{2\cancel{s}}{s^{\cancel{2}}}$

$L^{-1}\left[\frac{d}{ds}F(s)\right] = 2L^{-1}\left[\frac{s}{s^2+a^2}\right] - 2L^{-1}\left(\frac{1}{s}\right)$

$= 2\cos at - 2$

ie $(-1)^1 t f(t) = 2\cos at - 2$

$\therefore f(t) = \frac{2\cos at - 2}{-t} = \frac{2(1-\cos at)}{t}$

c. Find $L^{-1}\left[\frac{s+2}{s^2-4s+13}\right]$

<u>Soln</u> : $\frac{s+2}{s^2-4s+13} = \frac{s+2}{(s-2)^2+9} = \frac{(s-2)+4}{(s-2)^2+9}$

$\therefore L^{-1}\left[\frac{s+2}{s^2-4s+13}\right] = L^{-1}\left[\frac{s-2}{(s-2)^2+9}\right] + 4L^{-1}\left[\frac{1}{(s-2)^2+9}\right]$

$= e^{2t}\cos 3t + \frac{4}{3}e^{2t}\sin 3t$

8 a. Solve the D.E: $y'' + 2y' + y = 6te^{-t}$ under the conditions $y(0) = 0 = y'(0)$ by Laplace transform techniques.

Soln: $y'' + 2y' + y = 6te^{-t}$

$$L[y''] + 2L[y'] + L[y] = 6L[te^{-t}]$$

$$s^2 L[y] - s\,y(0) - y'(0) + 2\left(s L[y] - y(0)\right) + L[y] = 6\frac{1}{(s+1)^2}$$

$$s^2 L[y] + 2s L[y] + L[y] = \frac{6}{(s+1)^2}$$

$$(s^2 + 2s + 1) L[y] = \frac{6}{(s+1)^2}$$

$$L[y] = \frac{6}{(s^2+2s+1)(s+1)^2} = \frac{6}{(s+1)^2(s+1)^2} = \frac{6}{(s+1)^4}$$

$$y = L^{-1}\left[\frac{6}{(s+1)^4}\right] = \frac{\cancel{6}\, e^{-t} t^3}{\cancel{3!}} = e^{-t} t^3$$

b. Solve the D.E $y'' - 3y' + 2y = 0$, $y(0) = 0$, $y'(0) = 1$ by Laplace transform techniques.

Soln: $L[y''] - 3L[y'] + 2L[y] = L[0]$

$$s^2 L[y] - s\,y(0) - y'(0) - 3\left(s L[y] - y(0)\right) + 2L[y] = 0$$

$$s^2 L[y] - 1 - 3s L[y] + 2L[y] = 0$$

$$(s^2 - 3s + 2) L[y] = 1$$

$$L[y] = \frac{1}{s^2 - 3s + 2} = \frac{1}{(s-1)(s-2)}$$

Let $\frac{1}{(s-1)(s-2)} = \frac{A}{(s-1)} + \frac{B}{(s-2)}$

Then $A = -1$, $B = 1$

$$L[y] = \frac{-1}{s-1} + \frac{1}{s-2}$$

$$y = L^{-1}\left[\frac{-1}{s-1}\right] + L^{-1}\left[\frac{1}{s-2}\right] = -e^{t} + e^{2t}$$