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MATDIP401

Fourth Semester B.E. Degree Examination, Dec.2014/Jan.2015
Advanced Mathematics – II

Time: 3 hrs. Max. Marks: 100

Note: ***Answer any FIVE full questions.***

1 a. If l, m, n are the direction cosines of a line then prove that $l^2 + m^2 + n^2 = 1$ **(06 Marks)**

b. Find angle between any two diagonals of a cube. **(07 Marks)**

c. Find angle between two lines whose direction cosines satisfy the equations, $l + m + n = 0$ and $2l + 2m - mn = 0$. **(07 Marks)**

2 a. With the usual notations derive the equation of the plane in the form $lx + my + nz = 0$ **(06 Marks)**

b. Find the equation of the plane through (1, 2, -1) and perpendicular to the planes $x + y - 2z = 5$ and $3x - y + 4z = 12$. **(07 Marks)**

c. Find the shortest distance between the lines,

$$\frac{x-6}{3} = \frac{y-7}{-1} = \frac{z-4}{1} \text{ and }$$

$$\frac{x}{-3} = \frac{y+9}{2} = \frac{z-2}{4}$$ **(07 Marks)**

3 a. Prove that $\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}\left(\vec{c}.\vec{a}\right) - \vec{c}\left(\vec{a}.\vec{b}\right)$. **(06 Marks)**

b. Find the sine of angle between the vectors $\vec{a} = 2\hat{i} - 2\hat{j} + \hat{k}$ and $\vec{b} = \hat{i} - 2\hat{j} + 2\hat{k}$. **(07 Marks)**

c. Show that the vectors $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$, $\vec{b} = 2\hat{i} + \hat{j} + \hat{k}$ and $\vec{c} = 3\hat{i} + 4\hat{j} - \hat{k}$ are coplanar. **(07 Marks)**

4 a. Find the unit normal vector to the space curve $\vec{r} = 4\sin t\,\hat{i} + 4\cos t\,\hat{j} + 3t\,\hat{k}$. **(06 Marks)**

b. A particle moves along the curve $\vec{r} = \cos 2t\,\hat{i} + \sin 2t\,\hat{j} + t\,\hat{k}$. Find the velocity and acceleration at $t = \frac{\pi}{8}$ along $\sqrt{2}\,\hat{i} + \sqrt{2}\,\hat{j} + \hat{k}$. **(07 Marks)**

c. Find angle between the surfaces $x^2 + y^2 + z^2 = 9$ and $x = z^2 + y^2 - 3$ at (2, -1, 2) **(07 Marks)**

5 a. Find the directional derivative of x^2yz^3 at (1, 1, 1) in the direction of $\hat{i} + \hat{j} + 2\hat{k}$. **(06 Marks)**

b. If $\vec{F} = (x + y + 1)\hat{i} + \hat{j} - (x + y)\hat{k}$ then show that $\vec{F}.\text{curl}\,\vec{F} = 0$ **(07 Marks)**

c. Show that the vector $\vec{F} = (3x^2 - 2yz)\hat{i} + (3y^2 - 2zx)\hat{j} + (3z^2 - 2xy)\hat{k}$ is irrotational. **(07 Marks)**

6 a. Prove that $L[\sin at] = \dfrac{a}{s^2 + a^2}$. **(05 Marks)**

b. Find $L[\sin t \sin 2t \sin 3t]$. **(05 Marks)**

c. Find $L[te^{-t} \sin 2t]$. **(05 Marks)**

d. Find $L\left[\dfrac{e^{at} - e^{bt}}{t}\right]$. **(05 Marks)**

7 a. If $L[f(t)] = \int_0^\infty e^{-st} f(t)dt$ then prove that $L[f''(t)] = s^2 L[f(t)] - sf(0) - f'(0)$. **(05 Marks)**

b. Find $L^{-1}\left[\dfrac{s+2}{s^2 - 4s + 13}\right]$. **(05 Marks)**

c. Find $L^{-1}\left[\dfrac{s+1}{(s-2)^3}\right]$. **(05 Marks)**

d. Find $L^{-1}\left[\log\left(\dfrac{s-a}{s-b}\right)\right]$. **(05 Marks)**

8 a. Using Laplace transform solve $y'' - 2y' + y = e^{2t}$ with $y(0) = 0$, $y'(0) = 1$. **(10 Marks)**

b. Using Laplace transform solve the simultaneous equation,

$$\frac{dx}{dt} + y = \sin t$$

$$\frac{dy}{dt} + x = \cos t$$

given that x(0)=1, y(0) = 0 **(10 Marks)**

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SOLUTIONS OF ADVANCED MATHEMATICS-II. Dec14 / Jan15

1 (a) If l, m and n are the direction Cosines of a line then $l = \cos\alpha = \frac{x}{r}$

$$m = \cos\beta = \frac{y}{r}$$

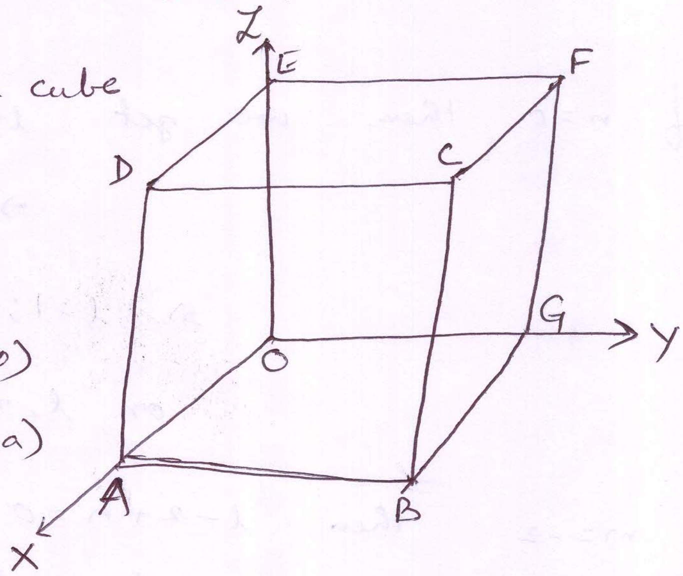
$$n = \cos\gamma = \frac{z}{r}$$

$$\Rightarrow l^2 + m^2 + n^2 = \cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{x^2}{r^2} + \frac{y^2}{r^2} + \frac{z^2}{r^2}$$

$$= \frac{x^2 + y^2 + z^2}{r^2} = \frac{r^2}{r^2} = 1.$$

$$\therefore \quad l^2 + m^2 + n^2 = 1.$$

(b) Let us consider a cube as shown



Let 'a' be the side of a cube

The co-ordinate of the vertice can be given as

$O(0,0,0)$, $A(a,0,0)$ $B(a,a,0)$

$C(a,a,a)$ $D(a,0,a)$, $E(0,0,a)$

$F(0,a,a)$ $G(0,a,0)$.

The diagonals of this cube are OC, EB, AF and DG.

Let us consider the diagonals OC & EB.

The Direction ratios of OC are $a-0,\ a-0,\ a-0 = a, a, a$

The Direction ratios of EB are $a-0,\ a-0,\ 0-a = a, a, -a$

So the angle between these two diagonals is given by

$$\cos\theta = \frac{a(a) + a(a) + a(-a)}{\sqrt{a^2 + a^2 + a^2} \cdot \sqrt{a^2 + a^2 + (-a)^2}} = \frac{a^2 + a^2 - a^2}{\sqrt{3a^2} \cdot \sqrt{3a^2}} = \frac{a^2}{3a^2} = \frac{1}{3}$$

$$\Rightarrow \theta = \cos^{-1}\frac{1}{3} = \text{arc}\cos\left(\tfrac{1}{3}\right)$$

Similarly we can show angle between DG & AF also as $\cos^{-1}\left(\frac{1}{3}\right)$.

(c) Given $l + m + n = 0 \longrightarrow (1)$

& $2l + 2m - nm = 0 \longrightarrow (2)$.

from eqn (1) we can write $l = -(m+n)$.

Using this in (2) we get $2(-(m+n)) + 2m - nm = 0$

$$-2m - 2n + 2m - nm = 0 \Rightarrow -2n - nm = 0.$$

$$\Rightarrow -n(2+m) = 0$$

$$\Rightarrow n = 0 \text{ or } m = -2$$

if $n = 0$ then we get $l + m = 0 \Rightarrow l = -m$

$\Rightarrow$ if $l = 1$ then $m = -1$

$\Rightarrow l = 1;\ m = -1;\ n = 0;$

or l, m & n are proportional to $1, -1$ & 0

if $m = -2$ then $l - 2 + n = 0 \Rightarrow l = 2 - n$

if $n = 1$ then $l = 1$

So if $m = 2$ then $l = 1$ for $n = 1$

$\therefore$ l, m & n are proportional to $1, -2, 1$.

$$\therefore \cos\theta = \frac{1(1) + (-1)(-2) + 0(1)}{\sqrt{1^2 + (-1)^2 + 0^2} \cdot \sqrt{1^2 + (-2)^2 + 1^2}} = \frac{1+2}{\sqrt{2} \cdot \sqrt{6}} = \frac{3}{\sqrt{2} \cdot \sqrt{6}} = \frac{\sqrt{3}}{2}$$

$$\Rightarrow \boxed{\theta = \pi/6}$$

2 (a) Here l, m, n are direction Cosines ~~ad~~ of the normal to the plane and p is the length of the perpendicular to the plane from the origin.

Spse the plane meets the Co-ordinate axes at A, B, C with intercepts a, b and c respectively.

$\therefore$ The equation of the plane is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

Now $l = \cos\alpha = \frac{p}{a}$

$m = \cos\beta = \frac{p}{b}$ & $n = \frac{p}{c}$

$\Rightarrow \frac{lx+my+nz}{p}=1 \Rightarrow lx+my+nz=p.$

(b) The equation of the plane passing thro' the point $(1,2,-1)$ is given by $a(x-1)+b(y-2)+c(z-(-1))=0$

$\Rightarrow a(x-1)+b(y-2)+c(z+1)=0$

The Direction ratios normal to this plane are a, b, c.

The Direction ratios of the ~~planes~~ normal to the given planes are $1, 1, -2$ & $3, -1, 4$ respectively.

$\therefore$ we can write $a+b-2c=0$

$3a-b+4c=0.$

Eliminating a, b & c from the above three eqns we get

$$\begin{vmatrix} x-1 & y-2 & z+1 \\ 1 & 1 & -2 \\ 3 & -1 & 4 \end{vmatrix}=0$$

$\Rightarrow (x-1)(4-2) - (y-2)(4+6) + (z+1)(-1-3) = 0$

$\Rightarrow 2(x-1) - 10(y-2) - 4(z+1) = 0$

$\Rightarrow 2x - 2 - 10y + 20 - 4z - 4 = 0$

$\Rightarrow 2x - 10y - 4z = 2 - 20 + 4 = -14$

$\Rightarrow x - 5y - 2z = -7 \Rightarrow x - 5y - 2z + 7 = 0$

is the requisite plane.

(c) The shortest distance between the two given lines is given by $l(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)$

where $x_1 = 6$ $\quad x_2 = 0$

$y_1 = 7$ $\quad y_2 = -9$

$z_1 = 4$ $\quad z_2 = 2$

l, m & n are given by $\frac{a}{\sqrt{a^2+b^2+c^2}}$, $\frac{b}{\sqrt{a^2+b^2+c^2}}$ & $\frac{c}{\sqrt{a^2+b^2+c^2}}$

where a, b & c are obtained from a_1 b_1 c_1 & a_2 b_2 & c_2

which are $a_1 = 3$; $b_1 = -1$, $c_1 = 1$ $\Rightarrow 3a - b + c = 0$

$a_2 = -3$; $b_2 = 2$; $c_2 = 4$. $\Rightarrow -3a + 2b + 4c = 0$

($\because$ The shortest distance is $\perp$r to both lines.

$\Rightarrow \frac{a}{\begin{vmatrix} -1 & 1 \\ 2 & 4 \end{vmatrix}} = \frac{b}{-\begin{vmatrix} 3 & 1 \\ -3 & 4 \end{vmatrix}} = \frac{c}{\begin{vmatrix} 3 & -1 \\ -3 & 2 \end{vmatrix}}$ $\Rightarrow \frac{a}{-6} = \frac{b}{-15} = \frac{c}{3}$ $\Rightarrow \frac{a}{-2} = \frac{b}{-5} = \frac{c}{-1}$

$\Rightarrow \quad a = -2; \quad b = -5; \quad c = 1.$

So $\quad l = \dfrac{-2}{\sqrt{(-2)^2+(-5)^2+1^2}} = \dfrac{-2}{\sqrt{30}}; \quad m = \dfrac{-5}{\sqrt{(-2)^2+(-5)^2+1}} = \dfrac{-5}{\sqrt{30}}$ & $n = \dfrac{1}{\sqrt{30}}.$

$\therefore$ The shortest distance $= \dfrac{-2}{\sqrt{30}}(0-6) + \left(\dfrac{-5}{\sqrt{30}}\right)(-9-7) + \dfrac{1}{\sqrt{30}}(2-4)$

$= \dfrac{12}{\sqrt{30}} + \dfrac{80}{\sqrt{30}} - \dfrac{2}{\sqrt{30}} = \dfrac{90}{\sqrt{30}} = 3\sqrt{30}$

3 (a) To prove $\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}\,(\vec{c}\cdot\vec{a}) - \vec{c}\,(\vec{a}\cdot\vec{b})$

Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$; $\vec{b} = b_1\hat{i} + b_2\hat{j}$ & $\vec{c} = (c_1\hat{i} + c_2\hat{j} + c_3\hat{k})$

Now $\vec{a} \times (\vec{b} \times \vec{c}) = \vec{a} \times \vec{d} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ d_1 & d_2 & d_3 \end{vmatrix}$

$\vec{b} \times \vec{c} = \begin{vmatrix} i & j & k \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \vec{d}$

$= (b_2c_3 - c_2b_3)\hat{i} + (b_1c_3 - b_3c_1)\hat{j} + (b_1c_2 - b_2c_1)\hat{k} = d_1\hat{i} + d_2\hat{j} + d_3\hat{k}$

So $\vec{a} \times (\vec{b} \times \vec{c}) = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ d_1 & d_2 & d_3 \end{vmatrix}$

$= \hat{i}\,(a_2d_3 - d_2a_3) - \hat{j}\,(a_1d_3 - a_3d_1) + \hat{k}\,(a_1d_2 - a_2d_1)$

$= \hat{i}\left[a_2(b_1c_2 - b_2c_1) - a_3(b_3c_1 - c_3b_1)\right] + \hat{j}\left[a_1(b_1c_2 - b_2c_1) - a_3(b_2c_3 - c_2b_3)\right]$

$$+\hat{k}\Big(a_1(b_3c_1-b_1c_3)-a_2(b_2c_3-c_2b_3)\Big)$$

$$=\left[(a_2c_2+a_3c_3)b_1-(a_2b_2+a_3b_3)c_1\right]\hat{i}$$

$$+\left[(a_1c_1+a_3c_3)b_2-(a_1b_1+a_3b_3)c_2\right]\hat{j}$$

$$+\left[(a_1c_1+a_2c_2)b_3-(a_1b_1+a_2b_2)c_3\right]\hat{k}$$

$$=\left[(a_1c_1+a_2c_2+a_3c_3)b_1-(a_1b_1+a_2b_2+a_3b_3)c_1\right]\hat{i}$$

$$+\left[(a_1c_1+a_2c_2+a_3c_3)b_2-(a_1b_1+a_2b_2+a_3b_3)c_2\right]\hat{j}$$

$$+\left[(a_1c_1+a_2c_2+a_3c_3)b_3-(a_1b_1+a_2b_2+a_3b_3)c_3\right]\hat{k}$$

$$=\left[(\vec{a}\cdot\vec{c})b_1-(\vec{a}\cdot\vec{b})c_1\right]\hat{i}$$

$$+\left[(\vec{a}\cdot\vec{c})b_2-(\vec{a}\cdot\vec{b})c_2\right]\hat{j}$$

$$+\left[(\vec{a}\cdot\vec{c})b_3-(\vec{a}\cdot\vec{b})c_3\right]\hat{k}$$

$$=(\vec{a}\cdot\vec{c})(b_1\hat{i}+b_2\hat{j}+b_3\hat{k})-(\vec{a}\cdot\vec{b})(c_1\hat{i}+c_2\hat{j}+c_3\hat{k}).$$

$$=(\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c}$$

$$\therefore\quad \vec{a}\times(\vec{b}\times\vec{c})=(\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c}.$$

3 (b)

$\sin\theta$ between a & b is given by

$$\sin\theta = \frac{|\vec{a}\times\vec{b}|}{|\vec{a}|\,|\vec{b}|}$$

So $\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -2 & 1 \\ 1 & -2 & 2 \end{vmatrix} = \hat{i}(-4+2) - \hat{j}(4-1) + \hat{k}(-4+2)$

$= -2\hat{i} - 3\hat{j} - 2\hat{k}$

$\therefore |\vec{a}\times\vec{b}| = \sqrt{(-2)^2+(-3)^2+(-2)^2} = \sqrt{17}$.

$|\vec{a}| = \sqrt{(2)^2+(-2)^2+1^2} = \sqrt{9} = 3$; $|\vec{b}| = \sqrt{1^2+(-2)^2+2^2} = \sqrt{9} = 3$

$\therefore \sin\theta = \frac{\sqrt{17}}{3\times 3} = \frac{\sqrt{17}}{9}$

3 (c) To show that the three vectors are coplanar we need to show $\vec{a}\cdot(\vec{b}\times\vec{c}) = 0$.

So $\vec{a}\cdot(\vec{b}\times\vec{c}) = \begin{vmatrix} 1 & -2 & 3 \\ 2 & 1 & 1 \\ 3 & 4 & -1 \end{vmatrix} = 1(-1-4) - (-2)(-2-3) + 3(8-3)$

$= 1(-5) + 2(-5) + 3(5) = 0$.

$\therefore \vec{a}, \vec{b}$ ad $\vec{c}$ are Coplanar.

4 (a) Given $\vec{r} = 4\sin t\,\hat{i} + 4\cos t\,\hat{j} + 3t\,\hat{k}$

Then $\vec{T} = \dfrac{d\vec{r}}{dt} = 4\cos t\,\hat{i} - 4\sin t\,\hat{j} + 3\,\hat{k}$

$\Rightarrow |\vec{T}| = \sqrt{(4\cos t)^2 + (-4\sin t)^2 + 3^2} = \sqrt{16\cos^2 t + 16\sin^2 t + 9}$

$= \sqrt{16(1) + 9} = \sqrt{25} = 5$

So $\hat{T} = \dfrac{\vec{T}}{|\vec{T}|} = \dfrac{4\cos t\,\hat{i} - 4\sin t\,\hat{j} + 3\hat{k}}{5}$

Further $\dfrac{d\hat{T}}{dt} = -\dfrac{4}{5}\sin t\,\hat{i} - \dfrac{4}{5}\cos t\,\hat{j} + 0$

$\Rightarrow \dfrac{d\hat{T}}{ds} = \dfrac{d\hat{T}/dt}{ds/dt} = \dfrac{d\hat{T}/dt}{|d\vec{r}/dt|} = \dfrac{-\frac{4}{5}(\sin t\,\hat{i} + \cos t\,\hat{j})}{5}$

$\Rightarrow \dfrac{d\hat{T}}{ds} = -\dfrac{4}{25}(\sin t\,\hat{i} + \cos t\,\hat{j})$

$\Rightarrow \left|\dfrac{d\hat{T}}{ds}\right| = \sqrt{\left(\dfrac{-4}{25}\sin t\right)^2 + \left(\dfrac{-4}{25}\cos t\right)^2} = \sqrt{\dfrac{16}{625}(\cos^2 t + \sin^2 t)}$

$= \dfrac{4}{25}$

$\therefore$ The unit normal vector to the given (space) curve is $\hat{n} = \dfrac{d\hat{T}/ds}{\left|\frac{d\hat{T}}{ds}\right|} = \dfrac{-\frac{4}{25}(\sin t\,\hat{i} + \cos t\,\hat{j})}{4/25}$

$\Rightarrow \hat{n} = -(\sin t\,\hat{i} + \cos t\,\hat{j})$

4 (b) given $\vec{r} = \cos 2t\,\hat{i} + \sin 2t\,\hat{j} + t\,\hat{k}$

$$\vec{v} = \frac{d\vec{r}}{dt} = -2\sin 2t\,\hat{i} + 2\cos 2t\,\hat{j} + \hat{k}$$

$$\Rightarrow \vec{v}\Big|_{t=\frac{\pi}{8}} = -2\sin\frac{\pi}{4}\,\hat{i} + 2\cos\frac{\pi}{4}\,\hat{j} + \hat{k} = -\sqrt{2}\,\hat{i} + \sqrt{2}\,\hat{j} + \hat{k}$$

$\therefore\ \vec{v} = \dfrac{d\vec{r}}{dt} = -\sqrt{2}\,\hat{i} + \sqrt{2}\,\hat{j} + \hat{k}$ is the velocity at $t = \pi/8$

$|||^{ly}$ $\vec{a} = \dfrac{d\vec{v}}{dt} = -4\cos 2t\,\hat{i} - 4\sin 2t\,\hat{j}$

$$\Rightarrow \vec{a}\Big|_{t=\frac{\pi}{8}} = -4\cos\frac{\pi}{4}\,\hat{i} - 4\sin\frac{\pi}{4}\,\hat{j} = -4/\sqrt{2}\,\hat{i} - 4/\sqrt{2}\,\hat{j}$$

$\Rightarrow \vec{a} = -2\sqrt{2}\,\hat{i} - 2\sqrt{2}\,\hat{j}$ at $t = \pi/8$

Now velocity along the vector $\vec{b}$ is $\vec{v}\cdot\hat{b} = \vec{v}\Big|_{t=\pi/8} \cdot \dfrac{\vec{b}}{|\vec{b}|}$

we know $\vec{v}\Big|_{t=\pi/8} = -\sqrt{2}\,\hat{i} + \sqrt{2}\,\hat{j} + \hat{k}$ & $\dfrac{\vec{b}}{|\vec{b}|} = \dfrac{\sqrt{2}\,\hat{i} + \sqrt{2}\,\hat{j} + \hat{k}}{\sqrt{(\sqrt{2})^2 + (\sqrt{2})^2 + (1)^2}}$

So $\vec{v}\cdot\hat{b} = \left(-\sqrt{2}\,\hat{i} + \sqrt{2}\,\hat{j} + \hat{k}\right)\cdot\left(\dfrac{\sqrt{2}\,\hat{i} + \sqrt{2}\,\hat{j} + \hat{k}}{\sqrt{5}}\right) = \dfrac{-2+2+1}{\sqrt{5}} = \dfrac{1}{\sqrt{5}}$

$|||^{ly}$ acceleration $\vec{a}$ along $\vec{b}$ $= \vec{a}\Big|_{t=\pi/8} \cdot \dfrac{\vec{b}}{|\vec{b}|}$.

$$= \left(-2\sqrt{2}\,\hat{i} - 2\sqrt{2}\,\hat{j}\right)\cdot\left(\frac{\sqrt{2}\,\hat{i} + \sqrt{2}\,\hat{j} + \hat{k}}{\sqrt{5}}\right) = \frac{-4-4}{\sqrt{5}} = \frac{-8}{\sqrt{5}}$$

4 (c) Let the two given surfaces be $\phi = x^2+y^2+z^2-9$

and $\psi = x^2+y^2-x-3$.

The angle between these two surfaces be the angle between their normals. Say $\hat{n}_1$ and $\hat{n}_2$ respectively.

We know $\hat{n}_1 = \dfrac{\vec{n_1}}{|\vec{n_1}|}$ and $\vec{n_1} = [\nabla\phi]_{at\ (2,-1,2)}$.

Ⅲly $\hat{n}_2 = \dfrac{\vec{n_2}}{|\vec{n_2}|}$ and $\vec{n_2} = [\nabla\psi]_{at\ (2,-1,2)}$.

So $\vec{n_1} = (\nabla\phi)_{(2,-1,2)} = \left(2x\,\hat{i} + 2y\,\hat{j} + 2z\,\hat{k}\right)_{at\ (2,-1,2)} = 4\hat{i}-2\hat{j}+4\hat{k}$

$\Rightarrow |\vec{n_1}| = \sqrt{4^2+(-2)^2+4^2} = \sqrt{36} = 6.$

$\therefore \hat{n}_1 = \dfrac{4\hat{i}-2\hat{j}+4\hat{k}}{6}$

Now $\vec{n_2} = (\nabla\psi)_{(2,-1,2)} = \left(-\hat{i} + 2y\hat{j} + 2z\hat{k}\right]_{(2,-1,2)} = -\hat{i}-2\hat{j}+4\hat{k}$

$\Rightarrow |\vec{n_2}| = \sqrt{(-1)^2+(-2)^2+4^2} = \sqrt{21}$. $\therefore \hat{n}_2 = \dfrac{\vec{n_2}}{|\vec{n_2}|} = \dfrac{-\hat{i}-2\hat{j}+4\hat{k}}{\sqrt{21}}$

Now $\cos\theta = \dfrac{\vec{n_1}\cdot\vec{n_2}}{|\vec{n_1}|\cdot|\vec{n_2}|} = \dfrac{4\hat{i}-2\hat{j}+4\hat{k}}{6}\cdot\dfrac{-\hat{i}-2\hat{j}+4\hat{k}}{\sqrt{21}}$

$= \dfrac{4(-1)+(-2)(-2)+4(4)}{6(\sqrt{21})} = \dfrac{-4+4+16}{6(\sqrt{21})} = \dfrac{8}{3\sqrt{21}}$

$\Rightarrow \theta = \cos^{-1}\left(\dfrac{8}{3\sqrt{21}}\right)$ is the angle b/w the two surfaces.

5 (a) We know the directional derivative of $\phi = x^2yz^3$ in the direction of $\vec{a}$ is $\nabla\phi \cdot \hat{a}$

$$\nabla\phi = 2xyz^3\,\hat{i} + x^2z^3\,\hat{j} + 3x^2yz^2\,\hat{k} \Rightarrow (\nabla\phi)_{at\,(1,1,1)} = 2\hat{i}+\hat{j}+3\hat{k}$$

$$\hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{\hat{i}+\hat{j}+2\hat{k}}{\sqrt{1^2+1^2+2^2}} = \frac{\hat{i}+\hat{j}+2\hat{k}}{\sqrt{6}}$$

$\therefore$ The directional derivative of ϕ along a is $(2\hat{i}+\hat{j}+3\hat{k}) \cdot \frac{(\hat{i}+\hat{j}+2\hat{k})}{\sqrt{6}}$

$$\nabla\phi \cdot \hat{a}\Big|_{at\,(1,1,1)} = \frac{2+1+6}{\sqrt{6}} = \frac{9}{\sqrt{6}}$$

5 (b) To show that $\vec{F} \cdot \text{Curl}\,\vec{F} = 0$ we need to show that $\vec{F} \cdot (\nabla \times F) = 0$; first $\text{Curl}\,\vec{F} = \nabla \times f = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y+1 & 1 & -(x+y) \end{vmatrix}$

$$= \hat{i}(-1-0) - \hat{j}(-1-0) + \hat{k}(0-1) = -\hat{i} + \hat{j} - \hat{k}.$$

now $\vec{F} \cdot \nabla \times F = \left((x+y+1)\hat{i} + \hat{j} - (x+y)\hat{k}\right) \cdot (-\hat{i} + \hat{j} - \hat{k})$

$$= (x+y+1)(-1) + 1 + (x+y) = 0$$

5 (c) To show that the Vector $\vec{F}$ is irrotational, we show $\text{Curl}\,\vec{F} = 0$

So $\text{Curl}\,\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2-2yz & 3y^2-2zx & 3z^2-2xy \end{vmatrix} = \hat{i}(-2x+2x) - \hat{j}(-2y+2y) + \hat{k}(-2z+2z)$

$$= 0$$

$\Rightarrow \text{curl}\,\vec{F} = \nabla \times \vec{F} = 0 \Rightarrow \vec{f}$ is irrotational.

6 (a) $L(\sin at) = \int_0^\infty e^{-st} \sin at \, dt = \left[\frac{e^{-st}}{(-s)^2 + a^2} (-s \sin at - a \cos at) \right]_0^\infty$

$= \frac{e^{-\infty}}{s^2+a^2} (-s \sin at - a \cos at)_{at\ t=\infty} - \frac{e^{-0}}{s^2+a^2} (-s \sin 0 - a \cos 0) = 0 + \frac{a}{a^2+s^2}$

So $L(\sin at) = \frac{a}{s^2+a^2}$;

(b) $L(\sin t \sin 2t \sin 3t) = L\left(\sin t \cdot \left\{ \frac{1}{2} \cos(2t-3t) - \cos(2t+3t) \right\}\right)$

$= \frac{1}{2} L\left(\sin t \left(\cos t - \cos 5t\right)\right) = \frac{1}{2} L(\sin t \cos t) - \frac{1}{2} L(\sin t \cos 5t)$

$= \frac{1}{2} L\left(\frac{1}{2} \sin 2t\right) - \frac{1}{2} L\left(\frac{1}{2} \sin(5+1)t - \sin (5-1)t\right)$

$= \frac{1}{4} L(\sin 2t) - \frac{1}{4} L(\sin 6t) + \frac{1}{4} \sin 4t$

$= \frac{1}{4}\left(\frac{2}{s^2+2^2}\right) - \frac{1}{4} \frac{6}{s^2+6^2} + \frac{1}{4} \frac{4}{s^2+4^2} = \frac{2}{4}\left(\frac{1}{s^2+2^2} - \frac{3}{s^2+6^2} + \frac{2}{s^2+4^2}\right)$

$= \frac{1}{2}\left(\frac{1}{s^2+2^2} - \frac{3}{s^2+6^2} + \frac{2}{s^2+4^2}\right)$

(c) $L(t e^{-t} \sin 2t) = L(e^{-t} \cdot t \sin 3t)$.

we know $L(\sin 3t) = \frac{3}{s^2+3^2} \Rightarrow L(t \sin 3t) = (-1)^1 \frac{d}{ds}\left(\frac{3}{s^2+9}\right)$

$= - \frac{-3}{(s^2+9)^2} \cdot \frac{d}{ds}(s^2+9) = \frac{3}{(s^2+9)^2} \cdot 2s = \frac{6s}{(s^2+9)^2}$

$\Rightarrow L(e^{-t} t \sin 3t) = \frac{6(s+1)}{((s+1)^2+9)^2} = \frac{6(s+1)}{(s+1)^4 + 18(s+1)^2 + 81}$

(d) $L\left(\frac{e^{at} - e^{bt}}{t}\right) = L\left(\frac{e^{at}}{t} - \frac{e^{bt}}{t}\right) = L\left(\frac{e^{at}}{t}\right) - L\left(\frac{e^{bt}}{t}\right)$

$$L(e^{at}-e^{bt}) = \frac{1}{s-a} - \frac{1}{s-b}$$

$$\Rightarrow L\left(\frac{e^{at}-e^{bt}}{t}\right) = \int_s^\infty \left(\frac{1}{s-a} - \frac{1}{s-b}\right)ds = \log(s-a) - \log(s-b)\Big|_s^\infty$$

$$= \log\left(\frac{s-a}{s-b}\right)\Big|_s^\infty = \log\left(\frac{1-\frac{a}{s}}{1-\frac{b}{s}}\right)\Big|_s^\infty = \log 1 - \log\left(\frac{(s-a)}{s-b}\right)$$

$$= 0 - \log\frac{s-a}{s-b} = \log\frac{(s-b)}{s-a}.$$

7 (a). Given $L(f(t)) = \int_0^\infty e^{-st} f(t)\, dt$

$$\Rightarrow L(f'(t)) = \int_0^\infty e^{-st} f'(t)\, dt = e^{-st} f(t)\Big|_0^\infty - \int_0^\infty e^{-st}(-s) f(t)\,dt$$

$$= 0 - f(0) + s\int_0^\infty e^{-st} f(t)\, dt$$

$$\Rightarrow L(f'(t)) = s\,L(f(t)) - f(0).$$

now $L(f''(t)) = \int_0^\infty e^{-st} f''(t)\, dt = e^{-st} f'(t)\Big|_0^\infty - \int_0^\infty e^{-st}(-s) f'(t)\,dt$

$$= 0 - f'(0) + s\int_0^\infty e^{-st} f'(t)\, dt$$

$$= -f'(0) + s\,L(f'(t))$$

$$= -f'(0) + s\left[s\,L(f(t)) - f(0)\right]$$

$$\Rightarrow L(f''(t)) = s^2 L(f(t)) - s f(0) - f'(0).$$

7 (b) Let $F(s) = \frac{s+2}{s^2-4s+13} = \frac{s+2}{(s-2)^2+9} = \frac{(s-2)+4}{(s-2)^2+9}$

$$\Rightarrow L^{-1}(F(s)) = L^{-1}\left(\frac{(s-2)}{(s-2)^2+3^2}\right) + L^{-1}\left[\frac{4}{(s-2)^2+3^2}\right] = e^{2t}\left[L^{-1}\left(\frac{s}{s^2+3^2}\right) + L^{-1}\left(\frac{4}{s^2+3^2}\right)\right]$$

$$= e^{2t}\left[\cos 3t + \frac{4}{3}\sin 3t\right].$$

$$\therefore L^{-1}\left[\frac{s+2}{(s-2)^2+3^2}\right] = e^{2t}\left[\cos 3t + \frac{4}{3}\sin 3t\right].$$

7 (c) Let $F(s) = \frac{s+1}{(s-2)^3} = \frac{s-2+3}{(s-2)^3} = \frac{(s-2)}{(s-2)^3} + \frac{3}{(s-2)^3}$

$$= \frac{1}{(s-2)^2} + \frac{3}{(s-2)^3}$$

$$\Rightarrow L^{-1}(F(s)) = L^{-1}\left(\frac{1}{(s-2)^2} + \frac{3}{(s-2)^3}\right) = L^{-1}\left(\frac{1}{(s-2)^2}\right) + 3L^{-1}\left(\frac{1}{(s-2)^3}\right).$$

$$= e^{2t}L^{-1}\left(\frac{1}{s^2}\right) + 3e^{2t}L^{-1}\left(\frac{1}{s^3}\right).$$

$$= e^{2t}\frac{t^{2-1}}{(2-1)!} + 3e^{2t}\cdot\frac{t^{3-1}}{(3-1)!} = e^{2t}\cdot\frac{t}{1} + 3e^{2t}\frac{t^2}{2!}$$

$$\Rightarrow L^{-1}\left(\frac{s+1}{(s-2)^3}\right) = te^{2t} + \frac{3}{2}e^{2t}\cdot t^2 = te^{2t}\left[1+\frac{3t}{2}\right].$$

7 (d) Let $F(s) = \log\frac{(s-a)}{(s-b)}$ So $L^{-1}(F(s)) = L^{-1}\left(\log\frac{s-a}{s-b}\right)$

Let $f(t) = L^{-1}(F(s)) = L^{-1}\left(\log\left(\frac{s-a}{s-b}\right)\right).$

$$\Rightarrow t f(t) = L^{-1}\left(-\frac{d}{ds}\log\left(\frac{s-a}{s-b}\right)\right) = -L^{-1}\left[\frac{d}{ds}\{\log(s-a) - \log(s-b)\}\right]$$

$$= -L^{-1}\left[\frac{1}{s-a} - \frac{1}{s-b}\right] = L^{-1}\left(\frac{1}{s-b}\right) - L^{-1}\left(\frac{1}{s-a}\right)$$

$$tf(t) = e^{bt} - e^{at}$$

$$\Rightarrow f(t) = \frac{e^{bt} - e^{at}}{t}$$

$$\therefore L^{-1}\left(\log\left(\frac{s-a}{s-b}\right)\right) = \frac{e^{bt} - e^{at}}{t}$$

8 (a) Taking Laplace Transform on both sides of the given diff eqn.

we get $L(y''-2y'+y) = L(e^{2t}) \Rightarrow L(y'') - 2L(y') + L(y) = \frac{1}{s-2}$

$s^2 L(y) - s y(0) - y'(0) - 2(sL(y) - y(0)) + L(y) = \frac{1}{s-2}$

$\Rightarrow (s^2 - 2s + 1) L(y) - 1 = \frac{1}{s-2} \Rightarrow (s-1)^2 L(y) = \frac{1}{s-2} + 1 = \frac{s-1}{(s-2)}$

$\Rightarrow L(y) = \frac{s-1}{s-2} \times \frac{1}{(s-1)^2} = \frac{1}{(s-2)(s-1)} = \frac{1}{s-2} - \frac{1}{s-1}$

$\Rightarrow y = L^{-1}\left(\frac{1}{s-2} - \frac{1}{s-1}\right) = e^{2t} - e^{t}$

So the soln of the given differential eqn is $\boxed{y = e^{2t} - e^{t}}$

8 (b) Taking Laplace Transforms on both sides of the two equations

$L(x'+y) = L(\sin t)$ and $L(y'+x) = L(\cos t)$

$\Rightarrow L(x') + L(y) = \frac{1}{s^2+1}$ $\quad\quad$ $\Rightarrow L(y') + L(x) = \frac{s}{s^2+1}$

$\Rightarrow sL(x) - x(0) + L(y) = \frac{1}{s^2+1} \longrightarrow (1)$ $\quad\quad$ $\Rightarrow sL(y) - y(0) + L(x) = \frac{s}{s^2+1} \longrightarrow (2)$

$\Rightarrow sL(x) + L(y) = \frac{1}{s^2+1} + 1 \quad \times s \Rightarrow s^2 L(x) + \cancel{sL(y)} = s\left(1 + \frac{1}{1+s^2}\right)$

$L(x) + sL(y) = \frac{s}{s^2+1} \quad \times 1 \Rightarrow L(x) + \cancel{sL(y)} = \frac{s}{s^2+1}$

$(s^2-1)L(x) = s + \frac{s}{1+s^2} - \frac{s}{1+s^2} = s$

$\Rightarrow L(x) = \frac{s}{s^2-1} \Rightarrow x = L^{-1}\left(\frac{s}{s^2-1^2}\right) = \cosh t$

also we have $y = \sin t - x' = \sin t - \sinh t \quad (\because x = \cosh t)$

$\therefore$ The soln of the given Simultaneous equations are

$\boxed{y = \sin t - \sinh t}$ & $\boxed{x = \cosh t}$.