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MATDIP401

Fourth Semester B.E. Degree Examination, Dec.2015/Jan.2016

Advanced Mathematics – II

Time: 3 hrs. Max. Marks:100

Note: ***Answer any FIVE full questions.***

1 a. Find the direction cosines of the line which is perpendicular to the lines with direction cosines (3, -1, 1) an (-3, 2, 4). **(06 Marks)**

b. If cos α, cos β, cos γ are the direction cosines of a line, then prove the following:

i) $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$

ii) $\cos 2\alpha + \cos 2\beta + \cos 2\gamma = -1$ **(07 Marks)**

c. Find the projection of the line AB on the line CD where A = (1, 2, 3), B = (1, 1, 1), C = (0, 0, 1), D = (2, 3, 0). **(07 Marks)**

2 a. Find the equation of the plane through (1, -2, 2), (-3, 1, -2) and perpendicular to the plane $2x - y - z + 6 = 0$. **(06 Marks)**

b. Find the image of the point (1, -2, 3) in the plane 2x + y – z = 5. **(07 Marks)**

c. Find the shortest distance between the lines $\frac{x-8}{3} = \frac{y+9}{-16} = \frac{z-10}{7}$ and $\frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5}$. **(07 Marks)**

3 a. Find the constant 'a' so that the vectors $2i - j + k$, $i + 2j - 3k$ and $3i + aj + 5k$ are coplanar. **(06 Marks)**

b. Prove that $\left[\vec{a}+\vec{b},\ \vec{b}+\vec{c},\ \vec{c}+\vec{a}\right] = 2\left[\vec{a}, \vec{b}, \vec{c}\right]$. **(07 Marks)**

c. Find the unit normal vector to both the vectors $4i - j + 3k$ and $-2i + j - 2k$. Find also the sine of the angle between them. **(07 Marks)**

4 a. A particle moves along the curve $x = t^3 + 1$, $y = t^2$, $z = 2t + 5$ where t is the time. Find the components of its velocity and acceleration at time t = 1 in the direction of $2i + 3j + 6k$. **(06 Marks)**

b. Find the angle between the surfaces $x^2 + y^2 + z^2 = 9$ and $x = z^2 + y^2 - 3$ at the point (2, -1, 2). **(07 Marks)**

c. Find the directional derivative of $\phi = xy^2 + yz^3$ at the point (1, -2, -1) in the direction of the normal to the surface $x \log z - y^2 = -4$ at (-1, 2, 1). **(07 Marks)**

5 a. Prove that $\text{div}(\text{curl}\,\vec{A}) = 0$. **(06 Marks)**

b. Find $\text{div}\,\vec{F}$ and $\text{curl}\,\vec{F}$ where $\vec{F} = \nabla(x^3 + y^3 + z^3 - 3xyz)$. **(07 Marks)**

c. Show that the vector $\vec{F} = (3x^2 - 2yz)i + (3y^2 - 2zx)j + (3z^2 - 2xy)k$ is irrotational and find ϕ such that $\vec{F} = \text{grad}\,\phi$. **(07 Marks)**

Important Note : 1. On completing your answers, compulsorily draw diagonal cross lines on the remaining blank pages.
2. Any revealing of identification, appeal to evaluator, will be treated as malpractice.

6 a. Find: $L\{\cos t \ \cos 2t \ \cos 3t\}$. **(06 Marks)**

b. Find: i) $L\{e^{-t}\cos^2 t\}$, ii) $L\{te^{-t}\sin 3t\}$. **(07 Marks)**

c. Find: $L\left\{\dfrac{\cos at - \cos bt}{t}\right\}$. **(07 Marks)**

7 a. Find: $L^{-1}\left\{\dfrac{4s+5}{(s-1)^2(s+2)}\right\}$. **(06 Marks)**

b. Find: i) $L^{-1}\left\{\dfrac{s+2}{s^2-4s+13}\right\}$, ii) $L^{-1}\left\{\log\left(\dfrac{s+a}{s+b}\right)\right\}$. **(07 Marks)**

c. Find: $L^{-1}\left\{\dfrac{1}{s^2(s+1)}\right\}$. **(07 Marks)**

8 a. Using Laplace transforms, solve $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + y = e^{2t}$ with $y(0) = 0$, $y'(0) = 1$. **(10 Marks)**

b. Using Laplace transformation method solve the differential equation $y'' + 2y' - 3y = \sin t$, $y(0) = y'(0) = 0$. **(10 Marks)**

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Q.1 (a) Let (l, m, n) be the D.C's of the line L_1.

$(3, -1, 1)$ be the DC's of the line L_2.

$(-3, 2, 4)$ be the DC's of the line L_3.

Given that L_1 is perpendicular to both L_2 and L_3,

$$3l - m + n = 0 \quad \text{and}$$

$$-3l + 2m + 4n = 0$$

Solve the above equations for l, m, n by the method of cross multiplication.

$$\frac{l}{\begin{vmatrix} -1 & 1 \\ 2 & 4 \end{vmatrix}} = \frac{m}{-\begin{vmatrix} 3 & 1 \\ -3 & 4 \end{vmatrix}} = \frac{n}{\begin{vmatrix} 3 & -1 \\ -3 & 2 \end{vmatrix}}$$

$$\Rightarrow \frac{l}{-6} = \frac{m}{-15} = \frac{n}{3} = K \text{ (say)}$$

$$\Rightarrow l = -6K, \quad m = -15K, \quad n = 3K$$

We have, $l^2 + m^2 + n^2 = 1$

$$36K^2 + 225K^2 + 9K^2 = 1$$

$$\Rightarrow 270K^2 = 1 \Rightarrow K^2 = 1/270 \Rightarrow K = \frac{1}{3\sqrt{30}}$$

$$\therefore l = \frac{-2}{\sqrt{30}}, \quad m = -\frac{5}{\sqrt{30}}, \quad n = \frac{1}{\sqrt{30}}$$

(b) We know that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$$\Rightarrow \cancel{1 - \sin^2\alpha + 1 - \cos^2\beta + 1 - \cos^2\gamma = 1}$$

$$\Rightarrow 1 - \sin^2\alpha + 1 - \sin^2\beta + 1 - \sin^2\gamma = 1$$

$$\Rightarrow 3 - (\sin^2\alpha + \sin^2\beta + \sin^2\gamma) = 1$$

$$\Rightarrow \sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2 \quad \text{— (i)}$$

ii By using $\sin^2 A = \frac{1-\cos 2A}{2}$, the equation (i) becomes

$$\frac{1-\cos 2\alpha}{2} + \frac{1-\cos 2\beta}{2} + \frac{1-\cos 2\gamma}{2} = 2$$

$$\Rightarrow \frac{3 - (\cos 2\alpha + \cos 2\beta + \cos 2\gamma)}{2} = 2$$

$$\Rightarrow 3 - 4 = \cos 2\alpha + \cos 2\beta + \cos 2\gamma$$

or, $$\underline{\underline{\cos 2\alpha + \cos 2\beta + \cos 2\gamma = -1}}$$

(c) DR's of CD = $2-0, 3-0, 0-1 = 2, 3, -1$

DC's of CD = $\frac{2}{\sqrt{4+9+1}}, \frac{3}{\sqrt{4+9+1}}, \frac{-1}{\sqrt{4+9+1}}$

$$= \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}, \frac{-1}{\sqrt{14}} = l, m, n$$

~~DR~~'s of A = $(1, 2, 3) = (x_1, y_1, z_1)$

B = $(1, 1, 1) = (x_2, y_2, z_2)$

∴ Projection of AB on CD is

$$l(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)$$

$$= \frac{2}{\sqrt{14}}(1-1) + \frac{3}{\sqrt{14}}(1-2) + \frac{-1}{\sqrt{14}}(1-3)$$

$$= 0 - \frac{3}{\sqrt{14}} + \frac{2}{\sqrt{14}} = \frac{-1}{\sqrt{14}}$$

Projection = $1/\sqrt{14}$

Q2 (a) Equation of a plane passing through $(1,-2,2)$ is

$$a(x-1)+b(y+2)+c(z-2)=0 \quad \ldots (1)$$

the plane (1) is passing through $(-3,1,-2)$ hence

$$a(-3-1)+b(1+2)+c(-2-2)=0$$

$$\Rightarrow -4a+3b-4c=0 \quad \ldots (2)$$

DR's of normal to the plane (1) are a, b, c and DR's of normal to the plane $2x-y-z+6=0$ are $2, -1, -1$, hence

$$2a-b-c=0 \quad \ldots (3)$$

Eliminating a, b, c from (1), (2) and (3), we get

$$\begin{vmatrix} x-1 & y+2 & z-2 \\ -4 & 3 & -4 \\ 2 & -1 & -1 \end{vmatrix} = 0$$

$$\Rightarrow (x-1)(-3-4)-(y+2)(4+8)+(z-2)(4-6)=0$$

$$\Rightarrow -7x+7-12y-24-2z+4=0$$

$$\Rightarrow 7x+12y+2z+13=0$$

(b) Let $(1,-2,3)$ be the given point and $B(\alpha, \beta, \gamma)$ be its reflection in the plane $2x+y-z=5$, then AB is perpendicular to the plane and C mid point of AB must lie on the plane.

$$C = \left(\frac{\alpha+1}{2}, \frac{\beta-2}{2}, \frac{\gamma+3}{2}\right)$$

$$2\left(\frac{\alpha+1}{2}\right)+\frac{\beta-2}{2}-\frac{\gamma+3}{2}=5$$

$2\alpha + 2 + \beta - 2 - \gamma + 3 = 10$

$\Rightarrow \quad 2\alpha + \beta - \gamma = 7 \quad \ldots \quad (1)$

Now, AB is $\perp^r$ to the plane, therefore it is parallel to the normal to the plane.

$$\therefore \quad \frac{\alpha - 1}{2} = \frac{\beta + 2}{1} = \frac{\gamma - 3}{-1} = k$$

$\Rightarrow \quad \alpha = 2k + 1, \quad \beta = k - 2, \quad \gamma = 3 - k$

$\therefore$ from (1) we get

$4k + 2 + k - 2 - 3 + k = 7$

$6k = 10 \quad \Rightarrow \quad k = 5/3$

$$\therefore \quad \alpha = \frac{13}{3}, \quad \beta = -\frac{1}{3}, \quad \gamma = \frac{4}{3}$$

Image point (α, β, γ) is $\left(\frac{13}{3}, \frac{-1}{3}, \frac{4}{3}\right)$

(c) the line of shortest distance is $\perp^r$ to both the lines, therefore

$3a - 16b + 7c = 0$

$3a + 8b - 5c = 0$

$$\frac{a}{\begin{vmatrix} -16 & 7 \\ 8 & -5 \end{vmatrix}} = \frac{b}{-\begin{vmatrix} 3 & 7 \\ 3 & -5 \end{vmatrix}} = \frac{c}{\begin{vmatrix} 3 & -16 \\ 3 & 8 \end{vmatrix}}$$

$$\frac{a}{24} = \frac{b}{36} = \frac{c}{72} \quad \Rightarrow \quad \frac{a}{2} = \frac{b}{3} = \frac{c}{6}$$

DR's of the line is $2, 3, 6$

D.C's of the SD line is $\frac{2}{\sqrt{4+9+36}}, \frac{3}{\sqrt{4+9+36}}, \frac{6}{\sqrt{4+9+36}} =$

$2/7, \; 3/7, \; 6/7$

$$S.D = \frac{2}{7}(15-8) + \frac{3}{7}(29+9) + \frac{6}{7}(5-10) = 98/7.$$

3 (a) Let $\bar{A} = 2i - j + k$, $\bar{B} = i + 2j - 3k$, $C = 3i + aj + 5k$

$$\vec{A} \cdot (\bar{B} \times \bar{C}) = \begin{vmatrix} 2 & -1 & 1 \\ 1 & 2 & -3 \\ 3 & a & 5 \end{vmatrix} = 0$$

$\Rightarrow 2(10 + 3a) + 1(5 + 9) + 1(a - 6) = 0$

$\Rightarrow 6a + 20 + 14 + a - 6 = 0$

$\Rightarrow 7a + 28 = 0$

$a = -\frac{28}{7} = -4$

(b) $[\bar{a} + \bar{b}, \bar{b} + \bar{c}, \bar{c} + \bar{a}] = (\bar{a} + \bar{b}) \cdot \{(\bar{b} + \bar{c}) \times (\bar{c} + \bar{a})\}$

$= (\bar{a} + \bar{b}) \cdot \{\bar{b} \times \bar{c} + \bar{c} \times \bar{c} + \bar{b} \times \bar{a} + \bar{c} \times \bar{a}\}$

$= (\bar{a} + \bar{b}) \cdot \{\bar{b} \times \bar{c} + 0 + \bar{b} \times \bar{a} + \bar{c} \times \bar{a}\}$

$= \bar{a} \cdot (\bar{b} \times \bar{c}) + \bar{a} \cdot (\bar{b} \times \bar{a}) + \bar{a} \cdot (\bar{c} \times \bar{a}) +$
$\bar{b} \cdot (\bar{b} \times \bar{c}) + \bar{b} \cdot (\bar{b} \times \bar{a}) + \bar{b} \cdot (\bar{c} \times \bar{a})$

$= [\bar{a}, \bar{b}, \bar{c}] + [\bar{a}, \bar{b}, \bar{a}] + [\bar{a}, \bar{c}, \bar{a}]$
$+ [\bar{b}, \bar{b}, \bar{c}] + [\bar{b}, \bar{b}, \bar{a}] + [\bar{b}, \bar{c}, \bar{a}]$

$= [\bar{a}, \bar{b}, \bar{c}] + 0 + 0 + 0 + 0 + [\bar{b}, \bar{c}, \bar{a}]$

$= [\bar{a}, \bar{b}, \bar{c}] + [\bar{a}, \bar{b}, \bar{c}]$

$= 2[\bar{a}, \bar{b}, \bar{c}]$

(c) Let $\bar{a} = 4i - j + 3k$, $\bar{b} = -2i + j - 2k$

$$\bar{a} \times \bar{b} = \begin{vmatrix} i & j & k \\ 4 & -1 & 3 \\ -2 & 1 & -2 \end{vmatrix} = i(2-3) - j(-8+6) + k(4-2)$$

$$= -i - 2j + 2k$$

$$|\bar{a} \times \bar{b}| = \sqrt{1+4+4} = 3$$

Unit normal vector to given vectors is

$$\hat{n} = \frac{\bar{a} \times \bar{b}}{|\bar{a} \times \bar{b}|} = \frac{-i - 2j + 2k}{3} = -\frac{1}{3} i - \frac{2}{3} j + \frac{2}{3} k$$

$$\sin\theta = \frac{|\bar{a} \times \bar{b}|}{|\bar{a}|\,|\bar{b}|} = \frac{3}{\sqrt{16+1+9}\ \sqrt{4+1+4}} = \frac{3}{\sqrt{26}\ \sqrt{9}} = \frac{1}{\sqrt{26}}$$

$$\therefore \sin\theta = \frac{1}{\sqrt{26}}$$

4 (a) $\bar{r} = xi + yj + zk = (t^3+1)i + t^2 j + (2t+5)k$

$$\bar{v} = \frac{d\bar{r}}{dt} = 3t^2 i + 2t j + 2k$$

$$\bar{v}\Big|_{t=1} = 3i + 2j + 2k$$

$$\bar{a} = \frac{d\bar{v}}{dt} = 6t i + 2j$$

$$\bar{a}\Big|_{t=1} = 6i + 2j$$

Let $\bar{b} = 2i + 3j + 6k$, $|\bar{b}| = \sqrt{4+9+36} = 7$

$$\hat{b} = \frac{\bar{b}}{|b|} = \frac{2i + 3j + 6k}{7}$$

Component of $\bar{v}$ along vector $\bar{b}$ = $\bar{v} \cdot \hat{b} = \frac{1}{7}(6+6+12) = 24/7$

Component of $\bar{a}$ along vector $\bar{b}$ = $\bar{a} \cdot \hat{b} = \frac{1}{7}(12+6) = 18/7$

(b) Let $\phi = x^2+y^2+z^2-9$

$\nabla\phi = 2xi + 2yj + 2zk$

$\therefore \vec{n}_1 = \nabla\phi|_{(2,-1,2)} = 4i - 2j + 4k$

$|\bar{n}_1| = \sqrt{16+4+16} = 6$

$\psi = z^2+y^2-x-3$

$\nabla\psi = -i + 2yj + 2zk$

$\therefore \vec{n}_2 = \nabla\psi|_{(2,-1,2)} = -i - 2j + 4k$

$|\bar{n}_2| = \sqrt{1+4+16} = \sqrt{21}$

$\therefore \cos\theta = \dfrac{\bar{n}_1 \cdot \bar{n}_2}{|\bar{n}_1|\,|\bar{n}_2|} = \dfrac{\cancel{-4} + \cancel{4} + 16}{6\sqrt{21}} = \dfrac{16}{6\sqrt{21}}$

Angle between given two surfaces is $\cos^{-1}\left(\dfrac{8}{3\sqrt{21}}\right)$

(c) $\phi = xy^2 + yz^3$

$\nabla\phi = y^2 i + (2xy+z^3)j + 3yz^2 k$

$\nabla\phi|_{(1,-2,-1)} = 4i - 5j - 6k$

Let $\psi = x\log z - y^2 + 4$

$\nabla\psi = \log z\, i - 2yj + \frac{x}{z}k$

$\vec{n} = \nabla\psi|_{(-1,2,1)} = -4j - k$

$|\vec{n}| = \sqrt{16+1} = \sqrt{17}$

$\hat{n} = \frac{1}{\sqrt{17}}(-4j - k)$

Directional derivative of ϕ in the direction of normal to the surface $\psi = 0$ is

$$\nabla\phi \cdot \hat{n} = (4i - 5j - 6k) \cdot \frac{(-4j - k)}{\sqrt{17}}$$

$$= \frac{20+6}{\sqrt{17}} = \frac{26}{\sqrt{17}}$$

5 (a) Let $\bar{A} = A_1 i + A_2 j + A_3 k$

$$\text{curl } \bar{A} = \nabla \times A = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}$$

$$= i\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right) - j\left(\frac{\partial A_3}{\partial x} - \frac{\partial A_1}{\partial z}\right) + k\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)$$

$$\text{div}(\text{curl } \bar{A}) = \nabla \cdot \text{curl } \bar{A}$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right) - \frac{\partial}{\partial y}\left(\frac{\partial A_3}{\partial x} - \frac{\partial A_1}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)$$

$$= \cancel{\frac{\partial^2 A_3}{\partial x \partial y}} - \cancel{\frac{\partial^2 A_2}{\partial x \partial z}} - \cancel{\frac{\partial^2 A_3}{\partial y \partial x}} + \cancel{\frac{\partial^2 A_1}{\partial y \partial z}} + \cancel{\frac{\partial^2 A_2}{\partial x \partial z}} - \cancel{\frac{\partial^2 A_1}{\partial z \partial y}}$$

$$= 0$$

(b) $$\bar{F} = \nabla(x^3 + y^3 + z^3 - 3xyz)$$

$$= (3x^2 - 3yz)\,i + (3y^2 - 3xz)\,j + (3z^2 - 3xy)\,k$$

$$= 3(x^2 - yz)\,i + 3(y^2 - xz)\,j + 3(z^2 - xy)\,k$$

$$\text{div } \bar{F} = \cancel{6xi + 6yj + 6zk}\; 6(x+y+z)$$

$$\text{Curl } \bar{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3(x^2 - yz) & 3(y^2 - xz) & 3(z^2 - xy) \end{vmatrix}$$

$$= i(-3x + 3x) - j(-3y + 3y) + k(-3z + 3z)$$

$$= \bar{0}$$

(c) $\bar{F} = (3x^2 - 2yz)i + (3y^2 - 2zx)j + (3z^2 - 2xy)k$

$$\text{Curl } \bar{F} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 3x^2 - 2yz & 3y^2 - 2xz & 3z^2 - 2xy \end{vmatrix}$$

$$= i(-2x + 2x) - j(-2y + 2y) + k(-2z + 2z)$$

$$= \bar{0}$$

Since, curl $\bar{F} = \bar{0}$, therefore $\bar{F}$ is irrotational.

Let $\bar{F} = \nabla\phi = \frac{\partial \phi}{\partial x} i + \frac{\partial \phi}{\partial y} j + \frac{\partial \phi}{\partial z} k$

$\Rightarrow \frac{\partial \phi}{\partial x} = 3x^2 - 2yz \quad \Rightarrow \phi = x^3 - 2xyz + f_1(y, z)$

$\frac{\partial \phi}{\partial y} = 3y^2 - 2xz \quad \Rightarrow \phi = y^3 - 2xyz + f_2(x, z)$

$\frac{\partial \phi}{\partial z} = 3z^2 - 2xy \quad \Rightarrow \phi = z^3 - 2xyz + f_3(x, y)$

Comparing three relations, $f_1(y, z) = y^3 + z^3$,
$f_2(x, z) = x^3 + z^3$, $f_3(x, y) = x^3 + y^3$

$\phi(x, y, z) = x^3 + y^3 + z^3 - 2xyz.$

6 (a) Let $f(t) = (\cos t \ \cos 2t) \cos 3t$

$= \frac{1}{2}(\cos 3t + \cos t) \cos 3t$

$= \frac{1}{2}\{\cos^2 3t + \cos t \cos 3t\}$

$\cos(A+B) + \cos(A-B) = 2\cos A \cos B$

$1 + \cos 2A = 2\cos^2 A$

$= \frac{1}{2}\left\{\frac{1}{2}(1+\cos 6t) + \frac{1}{2}(\cos 4t + \cos 2t)\right\}$

$= \frac{1}{4}\{1 + \cos 6t + \cos 2t + \cos 4t\}$

$\therefore L\{f(t)\} = \frac{1}{4}\left[L\{1\} + L\{\cos 2t\} + L\{\cos 4t\} + L\{\cos 6t\}\right]$

$= \frac{1}{4}\left\{\frac{1}{s} + \frac{s}{s^2+4} + \frac{s}{s^2+16} + \frac{s}{s^2+36}\right\}$

(b) i) Let $f(t) = \cos^2 t = \frac{1+\cos 2t}{2}$

$L\{f(t)\} = \frac{1}{2}\left[L\{1\} + L\{\cos 2t\}\right]$

$= \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+4}\right] = \frac{s^2+4+s^2}{2s(s^2+4)}$

$= \frac{s^2+2}{s(s^2+4)} = F(s)$, say

$L\{e^{-t} f(t)\} = F(s+1)$ by first shifting property

$= \frac{(s+1)^2+2}{(s+1)\left((s+1)^2+4\right)} = \frac{s^2+2s+3}{(s+1)(s^2+2s+5)}$

(ii) $L\{\sin 3t\} = \frac{3}{s^2+9}$

$L\{t \sin 3t\} = -\frac{d}{ds}\left(\frac{3}{s^2+9}\right)$

$L\{t f(t)\} = -\frac{d}{ds} F(s)$ where $L\{f(t)\} = F(s)$

$= -3 \times \frac{-1}{(s^2+9)^2} \times 2s = \frac{6s}{(s^2+9)^2}$

$$L\{e^{-t}\, t \sin 3t\} = \left.\frac{6s}{(s^2+9)^2}\right|_{s \to s+1}$$

$$= \frac{6(s+1)}{((s+1)^2+9)^2} = \frac{6(s+1)}{(s^2+2s+10)^2}$$

ⓒ $$L\left\{\frac{\cos at - \cos bt}{t}\right\} = \int_s^\infty \left(\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right) ds$$

$$= \left.\frac{1}{2}\log(s^2+a^2) - \frac{1}{2}\log(s^2+b^2)\right|_s^\infty$$

$$= \left.\frac{1}{2}\log\frac{s^2+a^2}{s^2+b^2}\right|_s^\infty$$

$$= \frac{1}{2}\lim_{s\to\infty}\log\frac{s^2+a^2}{s^2+b^2} - \frac{1}{2}\log\frac{s^2+a^2}{s^2+b^2}$$

$$= \frac{1}{2}\lim_{s\to\infty}\log\frac{1+a^2/s^2}{1+b^2/s^2} - \frac{1}{2}\log\frac{s^2+a^2}{s^2+b^2}$$

$$= \frac{1}{2}\log 1 - \frac{1}{2}\log\frac{s^2+a^2}{s^2+b^2}$$

$$= \frac{1}{2}\log\frac{s^2+b^2}{s^2+a^2}.$$

7 ⓐ $$\frac{4s+5}{(s-1)^2(s+2)} = \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{s+2}$$

$$= \frac{A(s-1)(s+2) + B(s+2) + C(s-1)^2}{(s-1)^2(s+2)}$$

$\therefore\ 4s+5 = A(s-1)(s+2) + B(s+2) + C(s-1)^2$

$s = 1 \implies 9 = 0 + 3B + 0 \implies B = 3$

$s = -2 \implies -3 = 0 + 0 + 9C \implies C = -1/3$

$s = 0 \implies 5 = -2A + 2B + C \implies A = -2/3$

$$\therefore \frac{4s+5}{(s-1)^2(s+2)} = -\frac{2}{3}\frac{1}{s-1} + \frac{3}{(s-1)^2} + -\frac{1}{3}\frac{1}{s+2}$$

$$L^{-1}\left\{\frac{4s+5}{(s-1)^2(s+2)}\right\} = -\frac{2}{3}L^{-1}\left\{\frac{1}{s-1}\right\} + 3L^{-1}\left\{\frac{1}{(s-1)^2}\right\} - \frac{1}{3}L^{-1}\left\{\frac{1}{s+2}\right\}$$

$$= -\frac{2}{3}e^{t} + 3e^{t}L^{-1}\left\{\frac{1}{s^2}\right\} - \frac{1}{3}e^{-2t}$$

$$= -\frac{2}{3}e^{t} + 3te^{t} - \frac{1}{3}e^{-2t}$$

(b) i

$$L^{-1}\left\{\frac{s+2}{s^2-4s+13}\right\} = L^{-1}\left\{\frac{s+2}{(s-2)^2+3^2}\right\}$$

$$= L^{-1}\left\{\frac{s-2+4}{(s-2)^2+3^2}\right\}$$

$$= e^{2t}L^{-1}\left\{\frac{s+4}{s^2+3^2}\right\}$$

$$= e^{2t}L^{-1}\left\{\frac{s}{s^2+3^2}\right\} + \frac{4e^{2t}}{3}L^{-1}\left\{\frac{3}{s^2+3^2}\right\}$$

$$= e^{2t}\cos 3t + \frac{4}{3}e^{2t}\sin 3t$$

$$= e^{2t}\left(\cos 3t + \frac{4}{3}\sin 3t\right).$$

ii Let $F(s) = \log\frac{s+a}{s+b} = \log s+a - \log s+b$

$$F'(s) = \frac{1}{s+a} - \frac{1}{s+b}$$

We know $L\{t f(t)\} = -F'(s)$ where $L\{f(t)\} = F(s)$

$$\therefore L\{t f(t)\} = -\frac{1}{s+a} + \frac{1}{s+b}$$

$$\therefore t f(t) = L^{-1}\left\{\frac{1}{s+b}\right\} - L^{-1}\left\{\frac{1}{s+a}\right\}$$

$$= e^{-bt} - e^{-at}$$

$$\therefore f(t) = \frac{e^{-bt} - e^{-at}}{t}$$

So, $$L^{-1}\left\{\log\left(\frac{s+a}{s+b}\right)\right\} = \frac{e^{-bt} - e^{-at}}{t}$$

c Let $F(s) = \frac{1}{s^2}$; $G(s) = \frac{1}{s+1}$

$$\therefore f(t) = L^{-1}\{F(s)\} = t \quad ; \quad g(t) = L^{-1}\{G(s)\} = e^{-t}$$

By convolution theorem

$$L^{-1}\{F(s) \cdot G(s)\} = \int_0^t f(u)\, g(t-u)\, du$$

$$\therefore L^{-1}\left\{\frac{1}{s^2(s+1)}\right\} = \int_0^t u\, e^{-(t-u)}\, du$$

$$= e^{-t} \int_0^t u e^u\, du$$

$$= e^{-t}\left(u e^u \Big|_0^t - \int_0^t e^u\, du \right)$$

$$= e^{-t}\left(t e^t - 0 - e^t + 1\right)$$

$$= e^{-t}\left(1 + t e^t - e^t\right)$$

$$= e^{-t} + t - 1$$

8 a $$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = e^{2t}$$

Take laplace transform ~~on~~ in both sides

$$s^2 L\{y(t)\} - s\,y(0) - y'(0) - 2\{s L\{y(t)\} - y(0)\} + L\{y(t)\} = L\{e^{2t}\}$$

Let $L\{y(t)\} = Y(s)$

$$\Rightarrow s^2 Y(s) - 0 - 1 - 2sY(s) + 0 + Y(s) = \frac{1}{s-2}$$

$$\Rightarrow (s^2 - 2s + 1)\,Y(s) = \frac{1}{s-2} + 1 = \frac{s-1}{s-2}$$

$$(s-1)^2\, Y(s) = \frac{s-1}{s-2}$$

$$\therefore Y(s) = \frac{1}{(s-1)(s-2)} = \frac{1}{s-2} - \frac{1}{s-1}$$

Taking inverse we get

$$y(t) = L^{-1}\left\{\frac{1}{s-2}\right\} - L^{-1}\left\{\frac{1}{s-1}\right\}$$

$$\underline{y(t) = e^{2t} - e^{t}}$$

b/ $y'' + 2y' - 3y = \sin t$

Taking laplace transform

$$s^2 Y(s) - s\,y(0) - y'(0) + 2sY(s) - 2y(0) - 3Y(s) = \frac{1}{1+s^2}$$

$$(s^2 + 2s - 3)\,Y(s) = \frac{1}{1+s^2}$$

$$\therefore Y(s) = \frac{1}{(s^2+1)(s^2+2s-3)}$$

$$Y(s) = \frac{1}{(s-1)(s+3)(s^2+1)} \quad \text{---} \quad (1)$$

Consider

$$\frac{1}{(s^2+1)(s+3)(s-1)} = \frac{A}{s-1} + \frac{B}{s+3} + \frac{Cs+D}{s^2+1}$$

$$\frac{1}{(s^2+1)(s+3)(s-1)} = \frac{A(s+3)(s^2+1) + B(s-1)(s^2+1) + (Cs+D)(s-1)(s+3)}{(s^2+1)\ (s+3)\ (s-1)}$$

$$1 = A(s+3)(s^2+1) + B(s-1)(s^2+1) + (Cs+D)(s-1)(s+3)$$

put $s = -3$

$$1 = B(-4)(9+1) \Rightarrow B = -1/40$$

put $s = 1$

$$1 = A(4)(2) \Rightarrow A\ \cancel{B} = 1/8$$

put $s = 0$

$$1 = 3A - B + D(-3)$$

$$\Rightarrow\ 1 = \frac{3}{8} + \frac{1}{40} - 3D \Rightarrow D = -1/5$$

put $s = 2$

$$1 = A(5)(5) + B(5) + (Cs + D)(1)(5)$$

$$\frac{1}{5} = 5A + B + D + 2C$$

$$\Rightarrow\ 2C = \frac{1}{5} - 5A - B - D = \frac{1}{5} - \frac{5}{8} + \frac{1}{40}$$

$$2C = -2/5 \Rightarrow C = -1/5$$

$$\therefore\ Y(s) = \frac{1}{8}\frac{1}{s-1} - \frac{1}{40}\frac{1}{s+3} - \frac{1}{5}\frac{(s+1)}{s^2+1}$$

$$y(t) = \frac{1}{8}e^{t} - \frac{1}{40}e^{-3t} - \frac{1}{5}(\sin t + \cos t)$$