CS 473ug: Algorithms

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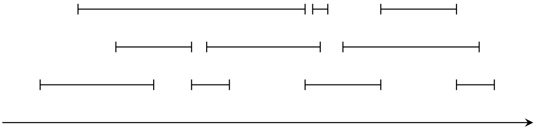
Part I

Greedy Algorithms: Tools and Techniques

Interval Scheduling

Input A set of jobs with start and finish times to be scheduled on a resource (example: classes and class rooms)

Goal Schedule as many jobs as possible



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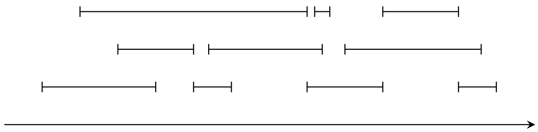

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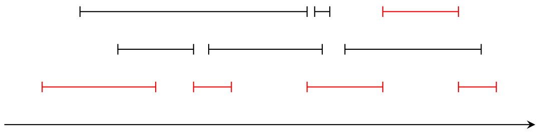

Greedy Template

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Initially R is the set of all requests
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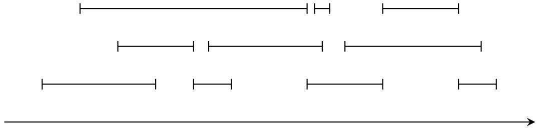
Main task: Decide the order in which to process requests in R

ES SP FC EF

Earliest Start Time

Process jobs in the order of their starting times, beginning with those that start earliest.



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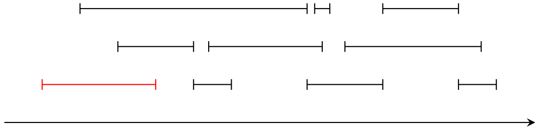

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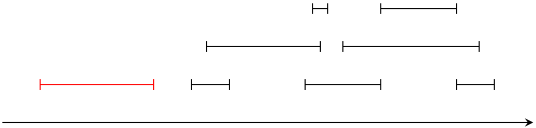

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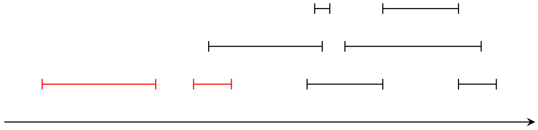

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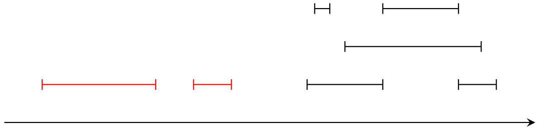

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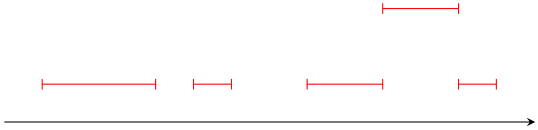

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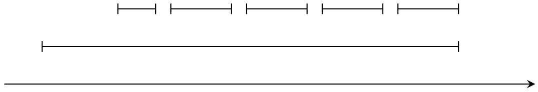

Figure: Counter example for earliest start time

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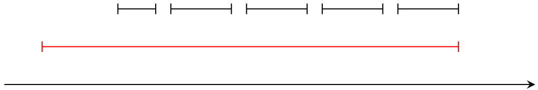

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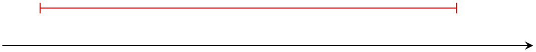

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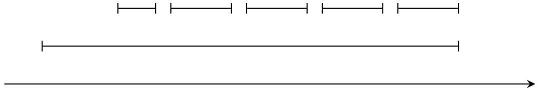

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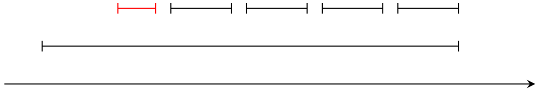

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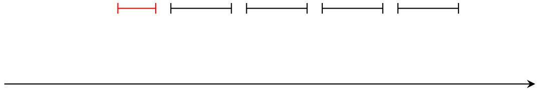

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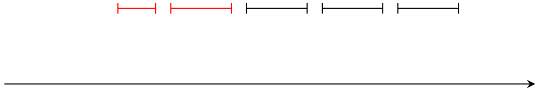

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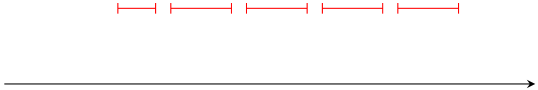

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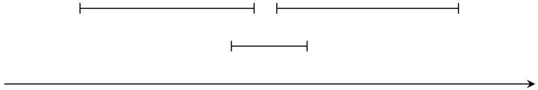

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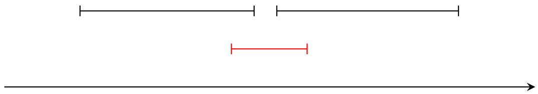

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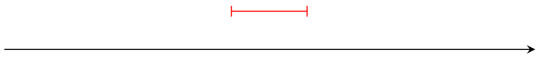

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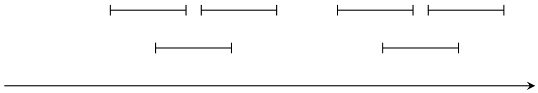

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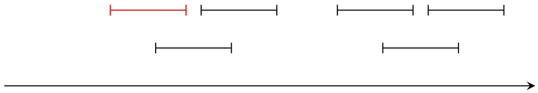

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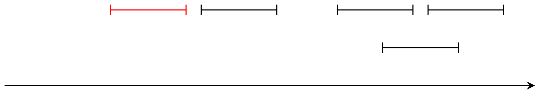

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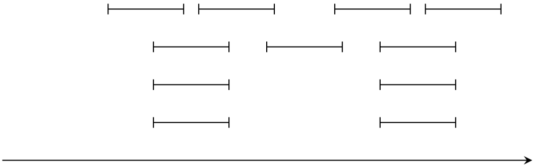

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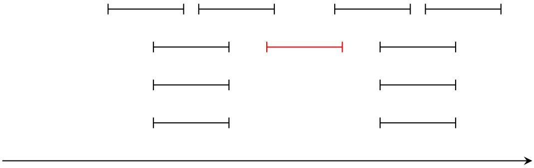

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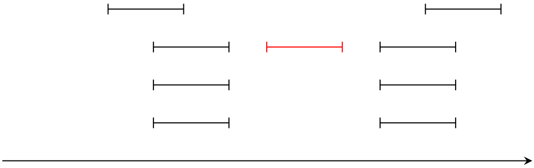

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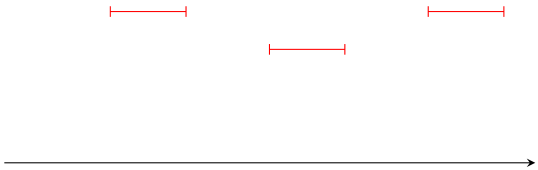

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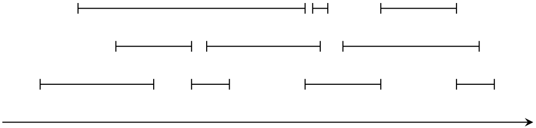

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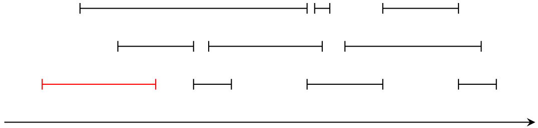

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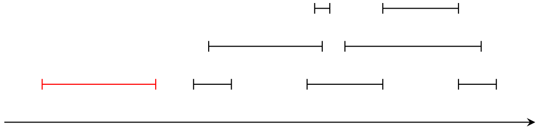

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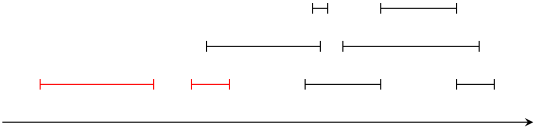

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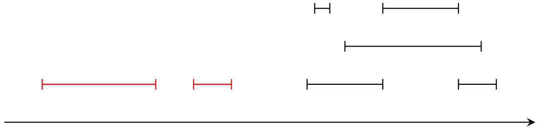

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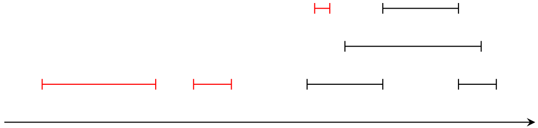

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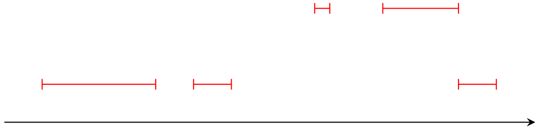

Optimal Greedy Algorithm

```
Initially R is the set of all requests
A is empty (* A will store all the jobs that will be scheduled *)
while R is not empty
    choose i ∈ R such that finishing time of i is least
    add i to A
    remove from R all requests that overlap with i
return the set A
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Theorem

The greedy algorithm that picks jobs in the order of their finishing times is optimal.

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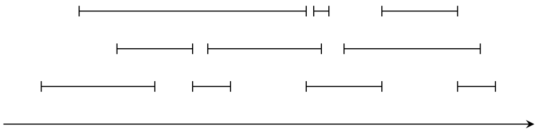

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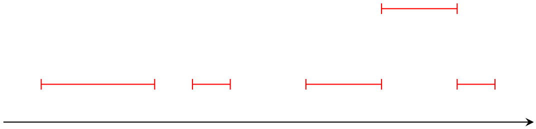

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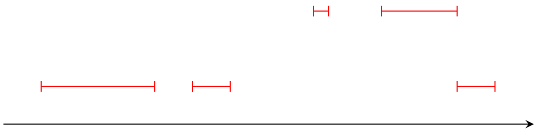

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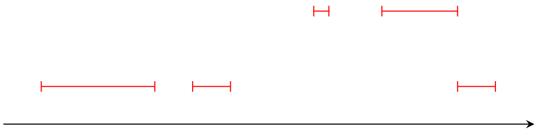

Instead we will show that $|O| = |A|$

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- Greedy algorithm will pick more requests! Contradiction. $\square$

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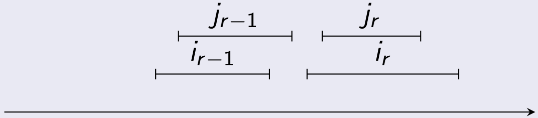
- Show Lemma for r. Is this a problem?



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- Since, $f(j_{r-1}) \leq s(j_r)$, we have that j_r is in R after removing the jobs conflicting with $i_1, \ldots i_{r-1}$
- Hence $f(i_r) \leq f(j_r)$

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Lemma

Let i_1 be first interval picked by Greedy. There exists an optimum solution that contains i_1.

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Let O be an arbitrary optimum solution and j_1 be the interval in O with the smallest finish time.

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Obtain a new solution O' by removing j_1 and adding i_1.
Claim: O' is also an optimum solution. $\square$

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Claim: $f(i_1) \le f(j_1)$ and i_1 and j_1 overlap.
If no overlap, can add i_1 to O and increase $|O|$!
Obtain a new solution O' by removing j_1 and adding i_1.
Claim: O' is also an optimum solution. □

Question: why does lemma imply Greedy is optimal? Exercise.

Implementation and Running Time

```
Initially R is the set of all requests
A is empty (* A will store all the jobs that will be scheduled *)
while R is not empty
    choose i ∈ R such that finishing time of i is least
    add i to A
    remove from R all requests that overlap with i
return the set A
```

Implementation and Running Time

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- Keep track of the finishing time of the last request added to A. Then check if starting time of i later than that
- Thus, checking non-overlapping is $O(1)$
- Total time $O(n \log n + n) = O(n \log n)$

Comments

- Interesting Exercise: smallest interval first picks at least half the optimum number of intervals.
- Instead of maximizing the total number of requests, associate *value/weight* with each job that is scheduled. Try to schedule jobs to maximize total value/weight. No greedy algorithm. Will be seen later in this course to illustrate dynamic programming.
- All requests need not be known at the beginning. Such *online* algorithms are a subject of research

Scheduling all Requests

Input A set of lectures, with start and end times

Goal Find the minimum number of classrooms, needed to schedule all the lectures such two lectures do not occur at the same time in the same room.

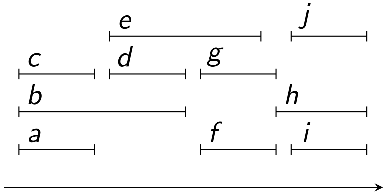

Figure: A schedule requiring 4 classrooms

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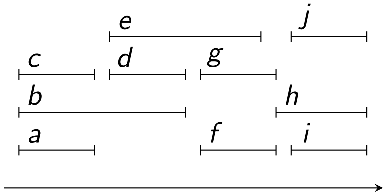

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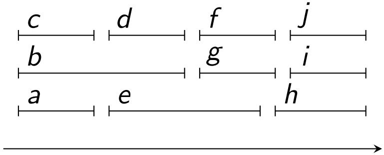

Figure: A schedule requiring 3 classrooms

Greedy Algorithm

```
Initially R is the set of all requests
d = 0 (* number of classrooms *)
while R is not empty
    choose i ∈ R
    if i can be scheduled in some class-room k ≤ d
        schedule lecture i in class-room k
    else
        allocate a new class-room d+1 and schedule lecture i in d+1
        d = d+1
```

What order should we process requests in?

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What order should we process requests in? According to start times (breaking ties arbitrarily)

Depth of Lectures

Definition

- For a set of lectures R, k are said to be in conflict if there is some time t such that there are k lectures going on at time t.

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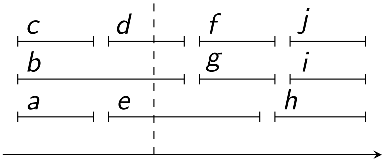

Depth and Number of Class-rooms

Lemma

For any set R of lectures, the number of class-rooms required is at least the depth of R.

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Proof.

All lectures that are in conflict must be scheduled in different rooms. ☐

Number of Class-rooms used by Greedy Algorithm

Lemma

Let d be the depth of the set of lectures R. The number of class-rooms used by the greedy algorithm is d.

Proof.

- Suppose the greedy algorithm uses more that d rooms. Let j be the first lecture that is scheduled in room $d + 1$.

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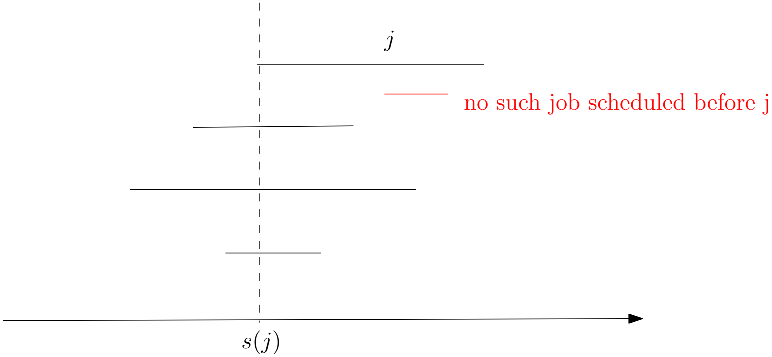
Proof.

- Suppose the greedy algorithm uses more that d rooms. Let j be the first lecture that is scheduled in room $d+1$.
- Since we process lectures according to start times, there are d lectures that start (at or) before j and which are in conflict with j.
- Thus, *at the start time of j*, there are at least $d+1$ lectures in conflict, which contradicts the fact that the depth is d. $\square$

Figure



Correctness

Observation

The greedy algorithm does not schedule two overlapping lectures in the same room.

Theorem

The greedy algorithm is correct and uses the optimal number of class-rooms.

Implementation and Running Time

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- Checking conflict takes $O(d)$ time.
- Total time $= O(n \log n + nd)$

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- Pre-sort according to start times. Picking lecture with earliest start time can be done in $O(1)$ time.
- Keep track of the finish time of last lecture in each room.
- With priority queues, checking conflict takes $O(\log d)$ time.
- Total time $= O(n \log n + n \log d) = O(n \log n)$

Scheduling to Minimize Lateness

- Given jobs with deadlines and processing times to be scheduled on a single resource.
- If a job i starts at time s_i then it will finish at time $f_i = s_i + t_i$, where t_i is its processing time.
- The lateness of a job is $l_i = \max(0, f_i - d_i)$.
- Schedule all jobs such that $L = \max l_i$ is minimized.

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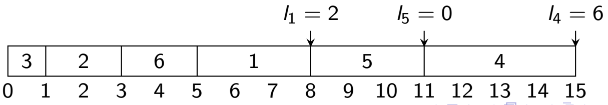
	1	2	3	4	5	6
t_i	3	2	1	4	3	2
d_i	6	8	9	9	14	15



A Simpler Feasibility Problem

- Given jobs with deadlines and processing times to be scheduled on a single resource.
- If a job i starts at time s_i then it will finish at time $f_i = s_i + t_i$, where t_i is its processing time.
- Schedule all jobs such that each of them completes before its deadline (in other words $L = \max_i l_i = 0$).

Definition

Feasible Schedule: a schedule in which all jobs finish before their deadline.

Greedy Template

```
Initially R is the set of all requests
curr-time = 0
while R is not empty
    choose i ∈ R
    curr-time = curr-time + t_i
    if (curr-time > d_i) then
        return ``no feasible schedule''
end while
return ``found feasible schedule''
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return ``found feasible schedule''
```

Main task: Decide the order in which to process jobs in R

SJ SS ED

Three Algorithms

- Shortest job first — sort according to t_i.
- Shortest slack first — sort according to $d_i - t_i$.
- Earliest deadline first — sort according to d_i.

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Counter examples for first two: exercise

Earliest Deadline First

Theorem

Greedy with EDF rule for picking requests correctly decides if there is a feasible schedule.

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Theorem

Greedy with EDF rule for picking requests correctly decides if there is a feasible schedule.

Proof via an exchange argument.

Idle time: time during which machine is not working.

Lemma

If there is a feasible schedule then there is one with no idle time before all jobs are finished.

Inversions

Definition

A schedule S is said to have an inversion if there are jobs i and j such that S schedules i before j, but $d_i > d_j$.

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A schedule S is said to have an inversion if there are jobs i and j such that S schedules i before j, but $d_i > d_j$.

Claim

If a schedule S has an inversion then there is an inversion between two adjacently *scheduled jobs.*

Proof: exercise.

Main Lemma

Lemma

If there is a feasible schedule, then there is one with no inversions.

Proof Sketch.

Let S be a schedule with minimum number of inversions.

- If S has 0 inversions, done.
- Suppose S has one or more inversions. By claim there are two adjacent jobs i and j that define an inversion.
- Swap positions of i and j.
- New schedule is still feasible. (Why?)
- New schedule has one fewer inversion — contradiction!

□

Back to Minimizing Lateness

Objective: schedule to minimize $L = \max_i l_i$.

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Given a lateness bound L, can we check if there is a schedule such that $\max_i l_i \leq L$?

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Given a lateness bound L, can we check if there is a schedule such that $\max_i l_i \leq L$?

Yes! Set $d'_i = d_i + L$ for each job i. Use feasibility algorithm with new deadlines.

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Yes! Set $d'_i = d_i + L$ for each job i. Use feasibility algorithm with new deadlines.

How can we find *minimum* L? Binary search!

Binary search for finding minimum lateness

$L = L_{\text{min}} = 0$
$L_{\text{max}} = \sum_i t_i$ // *why is this sufficient?*
While $L_{\text{min}} < L_{\text{max}}$ do
    $L = \lfloor (L_{\text{max}} + L_{\text{min}})/2 \rfloor$
    check if there is a feasible schedule with lateness L
    if ''yes'' then $L_{\text{max}} = L$
    else $L_{\text{min}} = L + 1$
endwhile
return L

Do we need binary search?

What happens in each call?
Greedy algorithm with deadlines $d'_i = d_i + L$.

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Greedy with EDF schedules the jobs in the same order for all L!!!

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Greedy algorithm with deadlines $d'_i = d_i + L$.

Greedy with EDF schedules the jobs in the same order for all L!!!

Maybe there is a direct greedy algorithm for minimizing maximum lateness?

Greedy Algorithm for Minimizing Lateness

```
Initially R is the set of all requests
curr-time = 0
curr-late = 0
while R is not empty
    choose i ∈ R with earliest deadline
    curr-time = curr-time + t_i
    late = curr-time - d_i
    curr-late = max (late, curr-late)
return curr-late
```

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Exercise: argue directly that above algorithm is correct (see book).

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Can be easily implemented in $O(n \log n)$ time after sorting jobs.

Greedy Analysis: Overview

- Greedy algorithm stays ahead. Show that after each step the solution of the greedy algorithm is at least as good as the solution of any other algorithm. Example, Interval scheduling.
- Structural property of solution. Observe some structural bound of every solution to the problem, and show that greedy algorithm achieves this bound. Example, Interval Partitioning.
- Exchange argument. Gradually transform any optimal solution to the one produced by the greedy algorithm, without hurting its optimality. Example, Minimizing lateness.