CS 473ug: Algorithms

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Part I

Problems

Why Algorithms?

World is full of algorithmic problems.

- decision problems (example: given n, *is* n prime?)
- search problems (example: given n, *find a* factor of n if it exists)
- optimization problems (example: find the *fewest* class rooms to schedule all classes)

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Given a new or unfamiliar problem how do we know whether it is easy or not?
We don't! In fact one can formally show that this meta-problem is difficult.

What do we do?

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- take an algorithms class to learn
  - some standard methods: greedy, divide and conquer, dynamic programming, optimization, reductions, ...
  - some standard problems to use as templates and for use in reductions
  - some standard methods to prove intractability or difficulty of problems (lower bounds) so that we do not waste time looking for an algorithm when there is none

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- creativity in devising new ideas/algorithms
- pay someone else to do it

Problem

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Language Decision Problem

- fix some alphabet Σ (say binary).
- problem $\Pi \subseteq \Sigma^*$ (essentially a language)
- Goal: given $x \in \Sigma^*$, is $x \in \Pi$?

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Example: given an ascii string, is it a valid Java program?

Search and Optimization Problems

- problem Π is a subset of Σ^* (essentially a language)
- given a string $I \in \Sigma^*$, I is an *instance* of Π if $I \in \Pi$
- assumption: given $I \in \Sigma^*$ there is an efficient algorithm to tell if I is an instance or not
- each instance I has a set $sol(I)$ - set of all *feasible solutions* for I
- implicit assumption: given instance I and $y \in \Sigma^*$, some reasonable efficient way to check if $y \in sol(I)$

Decision problem: given I, is $sol(I)$ empty?
Search problem: given I find *some* $y \in sol(I)$.

Optimization Problem

more information!

- a *valuation function* v that for each $y \in sol(I)$ assigns a number $v(y)$
- minimization problem: given I, find $\min_{y \in sol(I)} v(y)$
- maximization problem: given I, find $\max_{y \in sol(I)} v(y)$

Continuous vs Discrete Problems

Computers: discrete input only

Nevertheless, $sol(I)$ can be an infinite continuous set (example: linear programming)

Discrete/Combinatorial problems: $sol(I)$ is a discrete set (potentially infinite)

Combinatorial Optimization Problems

A typical problem:

- instance I consists of a set of objects N
- each object i may have a weight/value w_i
- $sol(I) \subseteq P(N)$ (powerset) — some subsets of N are solutions
- goal: find a set $S \in sol(I)$ to minimize/maximize $w(S) = \sum_{i \in S} w_i$

There is always an exponential time algorithm for such problems. Why?

Part II

Greedy Algorithms via a Strong Exchange Property

A Really Simple Problem

Informal

- Given n items each with a non-negative weight w_i
- Pick at most k items to maximize their total weight

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- instance I: $(n, w_1, w_2, \ldots, w_n)$ properly encoded as a string
- $sol(I)$: $S \in sol(I)$ iff $S \subseteq \{1, 2, \ldots, n\}$ and $|S| \leq k$
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Why does it work? Assume that weights are distinct

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Algorithm? Sort and pick the k items of largest weight

Why does it work? Assume that weights are distinct
Exchange argument: if one of the heaviest k items is not in an optimal solution, put it in and remove some lighter element

Exchange argument in more detail

$sol(I)$ has the property: for every $S \in sol(I)$ and item i

- either $S + i \in sol(I)$ or
- there is some $j \in S$ such that $S + i - j$ is in $sol(I)$.

The weights do not play a role in this exchange property and hence algorithm works for any given weights on the items.

This is true only for a class of problems related to matroids (out of scope for this class)

Note the difference with the interval selection problem

A More Interesting Problem

- Given n items N each with a non-negative weight w_i
- N partitioned into sets $N_1, N_2, \ldots, N_\ell$
- Goal: pick at most k items overall but at most k_j items from N_j for $1 \leq j \leq \ell$.

Example: $k = 4$, $k_1 = 1$, $k_2 = 1$, $k_3 = 3$.

	N_1		N_2			N_3		
Item	1	2	3	4	5	6	7	8
w_i	5	3	10	2	9	1	3	2

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Optimal solution: $S = \{1, 3, 7, 8\}$ and $w(S) = 20$

A More Interesting Problem: Algorithm

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S = ∅
for i = 1 to n do
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Proof: exchange argument

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Theorem

If weights are distinct then there is a unique optimum solution and the Greedy algorithm finds it.

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- Lemma: can add i_j to O and remove a lighter item from O

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- Lemma: can add i_j to O and remove a lighter item from O
- Contradicts optimality of O

□

Proof of Lemma

Lemma

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Proof.

- Let $A = O - \{i_1, i_2, \ldots, i_{j-1}\}$. $A \neq \emptyset$.

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- Suppose $i_j \in N_r$.

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- Let $A = O - \{i_1, i_2, \ldots, i_{j-1}\}$. $A \neq \emptyset$.
- Suppose $i_j \in N_r$.
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- Claim: $O' = O - t + i_j$ is feasible.

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- Can it be that $|O' \cap N_r| > k_r$?

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- $|O'| = |O| \leq k$.
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- No. Exercise.

Part III

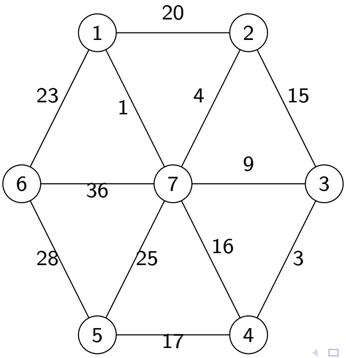
Greedy Algorithms: Minimum Spanning Tree

Minimum Spanning Tree

Input Connected graph $G = (V, E)$ with edge costs

Goal Find $T \subseteq E$ such that (V, T) is connected and total cost of all edges in T is smallest

- T is the minimum spanning tree (MST) of G



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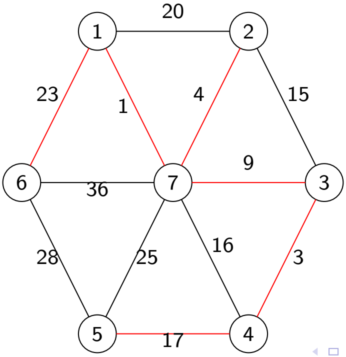


Applications

- Network Design
  - Designing networks with minimum cost but that satisfy connectivity requirements
- Approximation algorithms
  - Can be used to bound the optimality of algorithms to approximate Travelling Salesman Problem, Steiner Trees, etc.
- Cluster Analysis

Greedy Template

```
Initially E is the set of all edges in G
T is empty (* T will store edges of a MST *)
while E is not empty
    choose i ∈ E
    if (T+i is feasible)
        add i to T
end while
return the set T
```

Main Task: In what order should edges be processed?

KA PA

Kruskal's Algorithm

Process edges in the order of their costs (starting from the least) and add edges to T as long as they don't form a cycle.

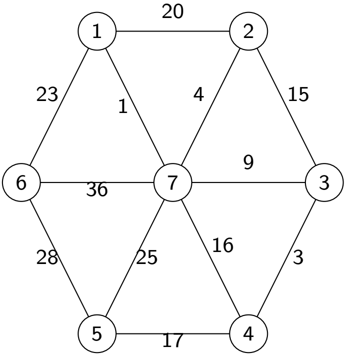
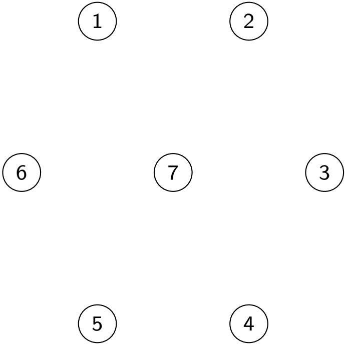


Figure: Graph G



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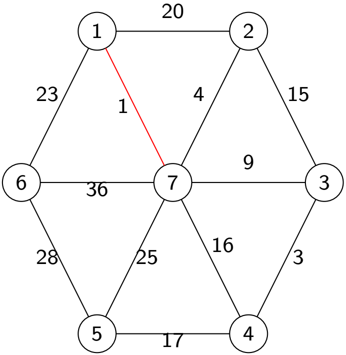

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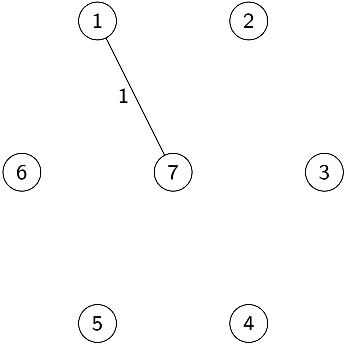

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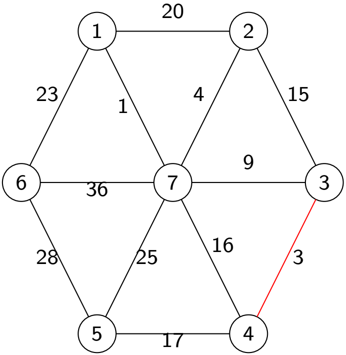

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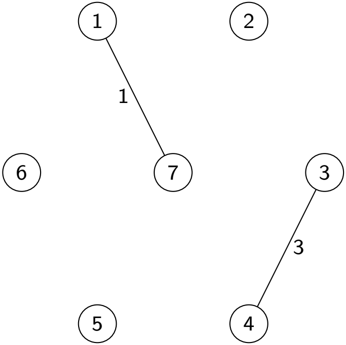

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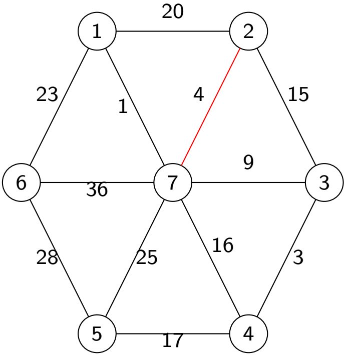

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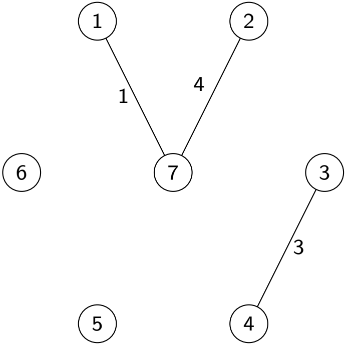

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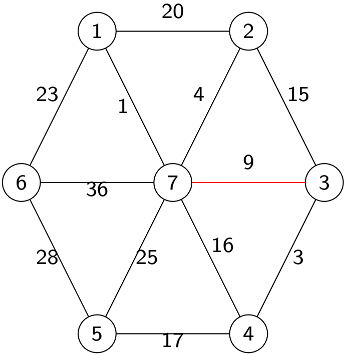

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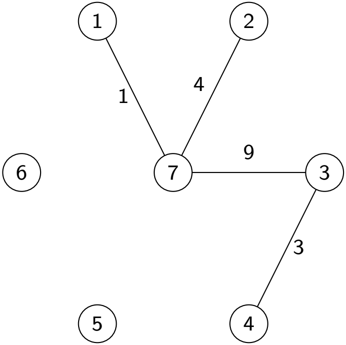

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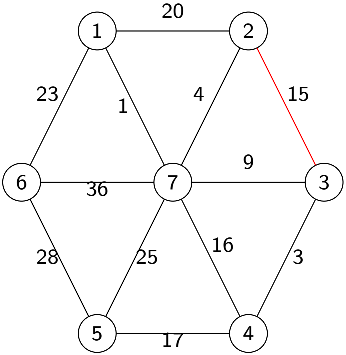

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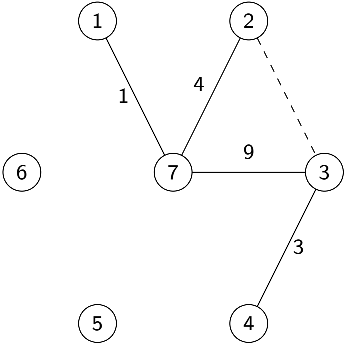

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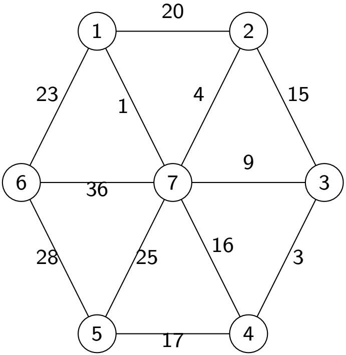

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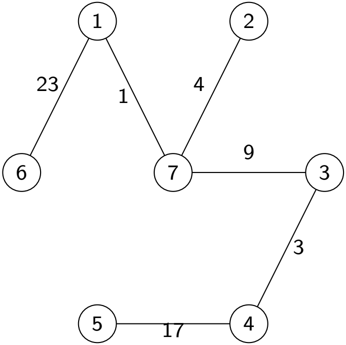

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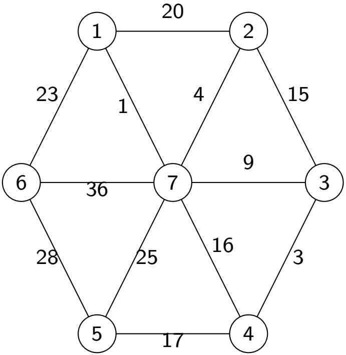
Prim's Algorithm

T maintained by algorithm will be a tree. In each iteration, pick edge with least attachment cost to T.



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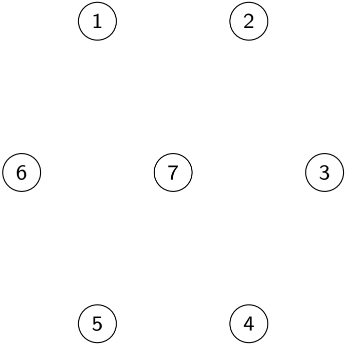

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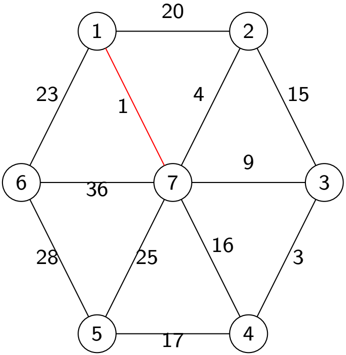

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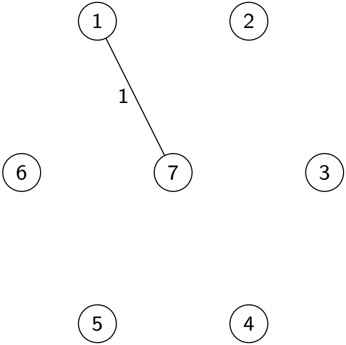

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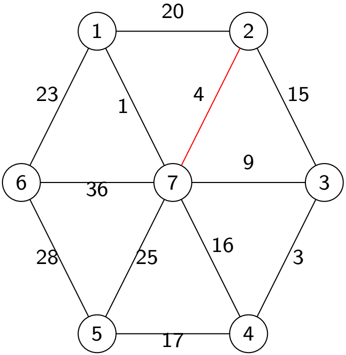

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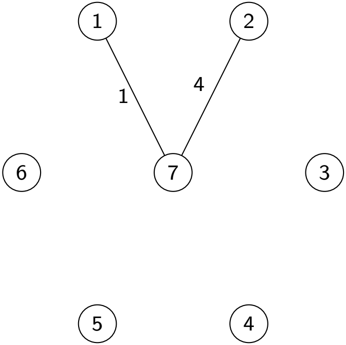

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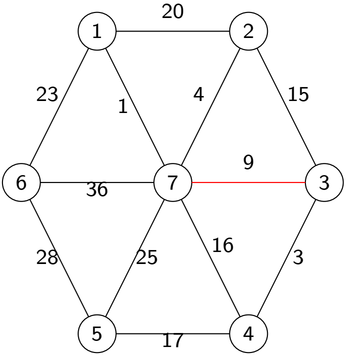

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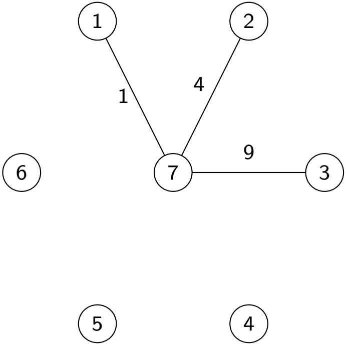
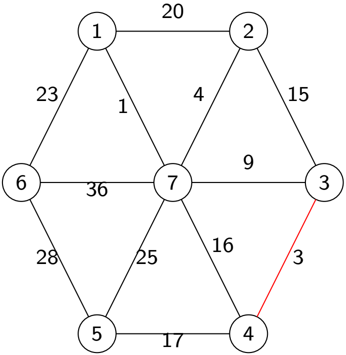


Figure: MST of G

Prim's Algorithm

T maintained by algorithm will be a tree. In each iteration, pick edge with least attachment cost to T.



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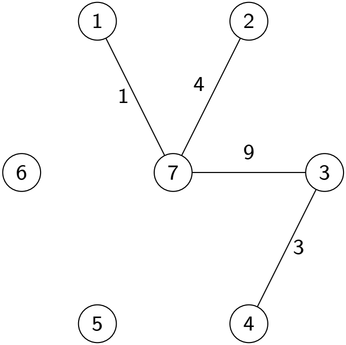

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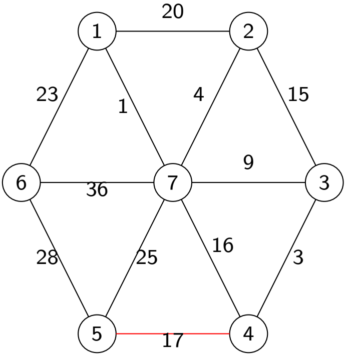

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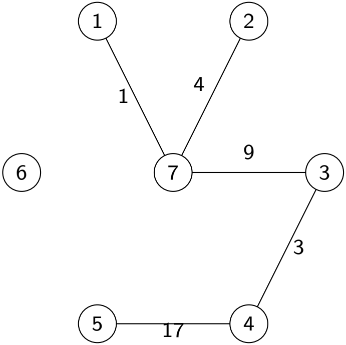

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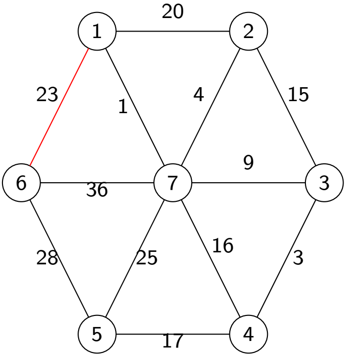

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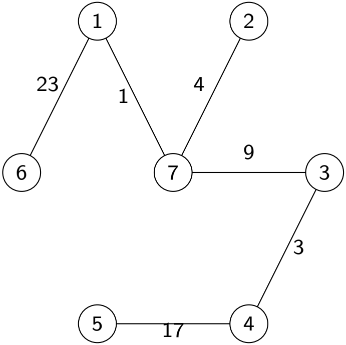

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Kruskal's Algorithm

```
Sort edges by weight and assume w_1 ≤ w_2 ≤ ... ≤ w_m
T is empty (* T will store edges of a MST *)
for i = 1 to m do
    if (T+i is feasible (does not contain a cycle))
        add i to T
end while
return the set T
```

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Suppose not. More than one connected component. $V_1, V_2, \ldots, V_r$ vertices in connected components. $r > 1$ by assumption.

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Consider V_1. Since G is connected, there is an edge $e = (u, v)$ in G with $u \in V_1$, $v \in V \setminus V_1$.

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Why did algorithm not add e? □

And for now . . .

Assumption

No two edge costs are equal, that is $w_1 < w_2 < \ldots < w_m$.

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Theorem

Kruskal's algorithm outputs the unique optimum solution when edge weights are distinct.

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- Let $i_1, i_2, \ldots, i_{n-1}$ be the edges added by algorithm in order.

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- Lemma: there is an edge $e \in T' \setminus \{i_1, i_2, \ldots, i_{j-1}\}$ such that $T' - e + i_j$ is a spanning tree.
- Since i_j is lighter than e, contradicts optimality of T'.

$\square$

Proof of Lemma

Lemma

There is an edge $e \in T' \setminus \{i_1, i_2, \ldots, i_{j-1}\}$ such that $T' - e + i_j$ is a spanning tree.

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- If P contains only edges in $\{i_1, i_2, \ldots, i_{j-1}\}$ then $P + i_j$ contains a cycle and algorithm will not pick i_j.

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- There must be $e \in P$ such that $e \notin \{i_1, i_2, \ldots, i_{j-1}\}$. Why?
- If P contains only edges in $\{i_1, i_2, \ldots, i_{j-1}\}$ then $P + i_j$ contains a cycle and algorithm will not pick i_j.
- Easy claim: $T - e + i_j$ is a spanning tree. Why?

Illustration for Proof

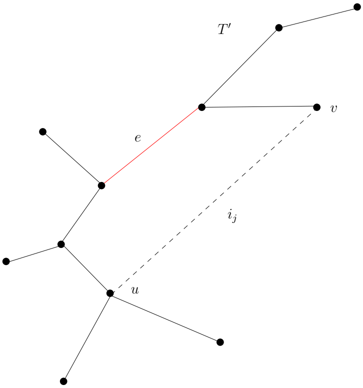

Figure: $e \not\in \{i_1, i_2, \ldots, i_j\}$. $T' + i_j - e$ is cheaper than T'

When edge costs are not distinct

Order edges lexicographically to break ties

- $i \prec j$ if either $w_i < w_j$ or $(w_i = w_j$ and $i < j)$

Assume edge weights distinct. Saw proof that MST is unique (this is not obvious btw).

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Questions:

- When is an edge e in all possible MSTs?
- When is an edge e in no MST?

Cut Property

Lemma

An edge $e = (u, v)$ is in every MST iff the following is true. There is some set $S \subset V$ with $u \in S, v \in V \setminus S$ such that $w(e)$ is the unique smallest weight edge between *S and $V \setminus S$.*

Proof Sketch.

Exchange property!

- If T does not contain e, add e to T to form cycle C.
- C must contain edge e' that crosses cut $(S, V \setminus S)$.
- $T - e' + e$ has strictly less weight than T (since e is strictly lighter than e' by assumption).
- Contradicts optimality of T.

$\square$

Cycle Property

Lemma

An edge $e = (u, v)$ is no MST iff there is a cycle C containing e such that e is the unique maximum weight edges on C.

Proof Sketch.

- Suppose e is in some MST T.
- Removing e from T generates two components S and $V \setminus S$.
- Pick $e' \in C$ *not* in T that has one end point in S and the other in $V \setminus S$.
- Why should such an edge e' exist? Exercise.
- $T - e + e'$ is strictly smaller weight than T, T cannot be an MST.

$\square$

Other MST Algorithms

Many variants of MST algorithms.

All of them rely essentially on the exchange property via the Cut or Cycle Properties.

To be done: implementation issue for Kruskal/Prim's algorithm.