DFS in Directed Graphs, Strong Connected Components, and DAGs

Lecture 2

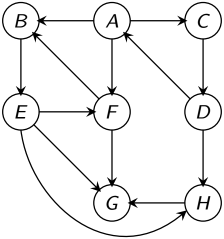
January 19, ~~2011~~ 2012

Strong Connected Components (SCCs)

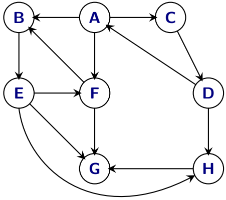
Algorithmic Problem

Find all SCCs of a given directed graph.

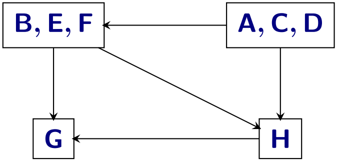
Previous lecture:
Saw an $\mathbf{O(n \cdot (n + m))}$ time algorithm.
This lecture: $\mathbf{O(n + m)}$ time algorithm.



Graph of SCCs



Graph G



Graph of SCCs G^{SCC}

Meta-graph of SCCs

Let $S_1, S_2, \ldots S_k$ be the strong connected components (i.e., SCCs) of G. The graph of SCCs is G^{SCC}

- Vertices are $S_1, S_2, \ldots S_k$
- There is an edge (S_i, S_j) if there is some $u \in S_i$ and $v \in S_j$ such that (u, v) is an edge in G.

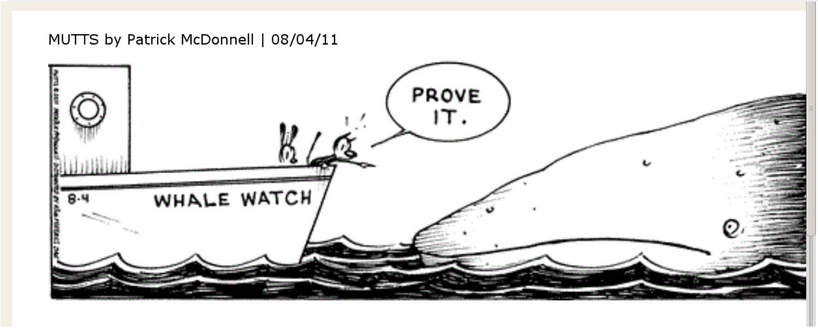
Reversal and SCCs

Proposition

For any graph G, the graph of SCCs of $\mathbf{G}^{\mathrm{rev}}$ is the same as the reversal of G^{SCC}.

Proof.

Exercise. $\square$



SCCs and DAGs

Proposition

For any graph G, the graph G^{SCC} has no directed cycle.

Proof.

If G^{SCC} has a cycle $S_1, S_2, \ldots, S_k$ then $S_1 \cup S_2 \cup \cdots \cup S_k$ should be in the same SCC in G. Formal details: exercise. $\square$

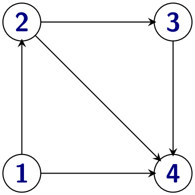
Part I

Directed Acyclic Graphs

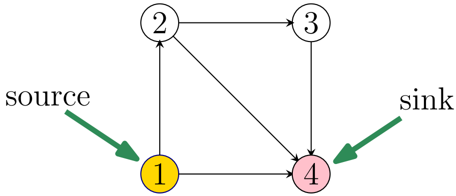
Directed Acyclic Graphs

Definition

A directed graph G is a **directed acyclic graph** (DAG) if there is no directed cycle in G.



Sources and Sinks



Definition

- A vertex **u** is a **source** if it has no in-coming edges.
- A vertex **u** is a **sink** if it has no out-going edges.

Simple DAG Properties

- Every DAG G has at least one source and at least one sink.
- If G is a DAG if and only if G^{rev} is a DAG.
- G is a DAG if and only each node is in its own strong connected component.

Formal proofs: exercise.

Simple DAG Properties

- Every DAG G has at least one source and at least one sink.
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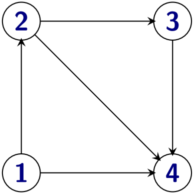
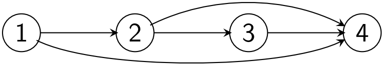
Formal proofs: exercise.

Topological Ordering/Sorting



Graph G



Topological Ordering of G

Definition

A **topological ordering**/**topological sorting** of $G = (V, E)$ is an ordering $\prec$ on V such that if $(u, v) \in E$ then $u \prec v$.

Informal equivalent definition:

One can order the vertices of the graph along a line (say the x-axis) such that all edges are from left to right.

DAGs and Topological Sort

Lemma

A directed graph G can be topologically ordered iff it is a DAG.

Proof.

$\Longrightarrow$: Suppose G is not a DAG and has a topological ordering $\prec$. G has a cycle $C = u_1, u_2, \ldots, u_k, u_1$.

Then $u_1 \prec u_2 \prec \ldots \prec u_k \prec u_1$!

That is... $u_1 \prec u_1$.

A contradiction (to $\prec$ being an order).

Not possible to topologically order the vertices. $\square$

DAGs and Topological Sort

Lemma

A directed graph G can be topologically ordered iff it is a DAG.

Continued.

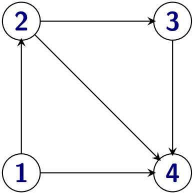
$\Leftarrow$: Consider the following algorithm:

- Pick a source $\mathbf{u}$, output it.
- Remove $\mathbf{u}$ and all edges out of $\mathbf{u}$.
- Repeat until graph is empty.
- Exercise: prove this gives an ordering.

$\square$

Exercise: show above algorithm can be implemented in $\mathbf{O(m+n)}$ time.

Topological Sort: An Example



Output: 1 2 3 4

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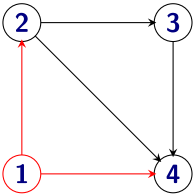

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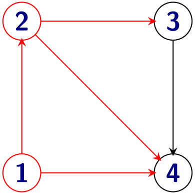

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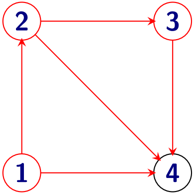

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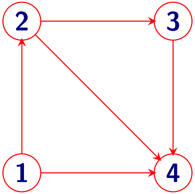

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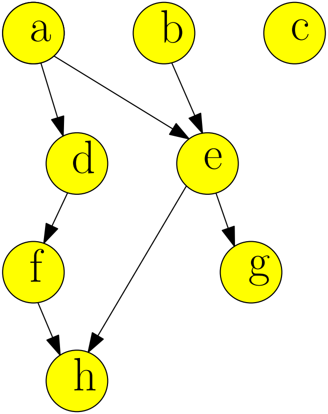
Topological Sort: Another Example



a c d f b e h g

DAGs and Topological Sort

Note: A DAG G may have many different topological sorts.

Question: What is a DAG with the most number of distinct topological sorts for a given number n of vertices?

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Using DFS...

... to check for Acylicity and compute Topological Ordering

Question

Given G, is it a DAG? If it is, generate a topological sort.

DFS based algorithm:

- Compute **DFS**$(\mathbf{G})$
- If there is a back edge then G is not a DAG.
- Otherwise output nodes in decreasing post-visit order.

Correctness relies on the following:

Proposition

G is a DAG iff there is no back-edge in **DFS**$(\mathbf{G})$.

Proposition

If G is a DAG and $\mathrm{post}(\mathbf{v}) > \mathrm{post}(\mathbf{u})$, then $(\mathbf{u}, \mathbf{v})$ is not in G.

Using DFS...

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Given G, is it a DAG? If it is, generate a topological sort.

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Proof

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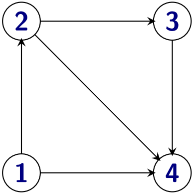
Proof.

Assume $\mathrm{post}(\mathbf{v}) > \mathrm{post}(\mathbf{u})$ *and* $(\mathbf{u}, \mathbf{v})$ is an edge in $\mathbf{G}$. We derive a contradiction. One of two cases holds from DFS property.

- Case 1: $[\mathbf{pre}(\mathbf{u}), \mathbf{post}(\mathbf{u})]$ is contained in $[\mathbf{pre}(\mathbf{v}), \mathbf{post}(\mathbf{v})]$. Implies that $\mathbf{u}$ is explored during **DFS**$(\mathbf{v})$ and hence is a descendent of $\mathbf{v}$. Edge $(\mathbf{u}, \mathbf{v})$ implies a cycle in G but G is assumed to be DAG!
- Case 2: $[\mathbf{pre}(\mathbf{u}), \mathbf{post}(\mathbf{u})]$ is disjoint from $[\mathbf{pre}(\mathbf{v}), \mathbf{post}(\mathbf{v})]$. This cannot happen since $\mathbf{v}$ would be explored from $\mathbf{u}$.

$\square$

Example



Back edge and Cycles

Proposition

G has a cycle iff there is a back-edge in **DFS(G)**.

Proof.

If: $(\mathbf{u}, \mathbf{v})$ is a back edge implies there is a cycle $\mathbf{C}$ consisting of the path from $\mathbf{v}$ to $\mathbf{u}$ in **DFS** search tree and the edge $(\mathbf{u}, \mathbf{v})$.

Only if: Suppose there is a cycle $\mathbf{C} = \mathbf{v_1} \rightarrow \mathbf{v_2} \rightarrow \ldots \rightarrow \mathbf{v_k} \rightarrow \mathbf{v_1}$.
Let $\mathbf{v_i}$ be first node in $\mathbf{C}$ visited in **DFS**.
All other nodes in $\mathbf{C}$ are descendants of $\mathbf{v_i}$ since they are reachable from $\mathbf{v_i}$.
Therefore, $(\mathbf{v_{i-1}}, \mathbf{v_i})$ (or $(\mathbf{v_k}, \mathbf{v_1})$ if $\mathbf{i = 1}$) is a back edge. □

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DAGs and Partial Orders

Definition

A **partially ordered set** is a set $\mathbf{S}$ along with a binary relation $\preceq$ such that $\preceq$ is

1. **reflexive** ($\mathbf{a} \preceq \mathbf{a}$ for all $\mathbf{a} \in \mathbf{V}$),
2. **anti-symmetric** ($\mathbf{a} \preceq \mathbf{b}$ and $\mathbf{a} \neq \mathbf{b}$ implies $\mathbf{b} \not\preceq \mathbf{a}$), and
3. **transitive** ($\mathbf{a} \preceq \mathbf{b}$ and $\mathbf{b} \preceq \mathbf{c}$ implies $\mathbf{a} \preceq \mathbf{c}$).

Example: For numbers in the plane define $(\mathbf{x}, \mathbf{y}) \preceq (\mathbf{x}', \mathbf{y}')$ iff $\mathbf{x} \leq \mathbf{x}'$ and $\mathbf{y} \leq \mathbf{y}'$.

Observation: A *finite* partially ordered set is equivalent to a DAG. (No equal elements.)

Observation: A topological sort of a DAG corresponds to a complete (or total) ordering of the underlying partial order.

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Example

- **V**: set of **n** products (say, **n** different types of tablets).
- Want to buy one of them, so you do market research...
- Online reviews compare only pairs of them.
  ...Not everything compared to everything.
- Given this partial information:
  - Decide what is the best product.
  - Decide what is the ordering of products from best to worst.
  - ...

What DAGs got to do with it?

Or why we should care about DAGs

- DAGs enable us to represent partial ordering information we have about some set (very common situation in the real world).
- Questions about DAGs:
  - Is a graph G a DAG?

    $\Longleftrightarrow$

    Is the partial ordering information we have so far is consistent?
  - Compute a topological ordering of a DAG.

    $\Longleftrightarrow$

    Find an a consistent ordering that agrees with our partial information.
  - Find comparisons to do so DAG has a unique topological sort.

    $\Longleftrightarrow$

    Which elements to compare so that we have a consistent ordering of the items.

Part II

Linear time algorithm for finding all strong connected components of a directed graph

Finding all SCCs of a Directed Graph

Problem

Given a directed graph $G = (V, E)$, output *all* its strong connected components.

Straightforward algorithm:

```
Mark all vertices in V as not visited.
for each vertex u ∈ V not visited yet do
    find SCC(G, u) the strong component of u:
        Compute rch(G, u) using DFS(G, u)
        Compute rch(G^rev, u) using DFS(G^rev, u)
        SCC(G, u) ⇐ rch(G, u) ∩ rch(G^rev, u)
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```

Running time: $O(n(n+m))$

Is there an $O(n+m)$ time algorithm?

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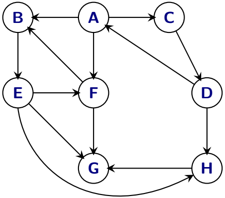
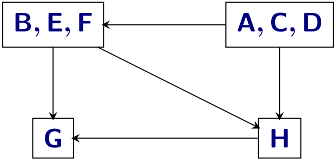
Structure of a Directed Graph



Graph G



Graph of SCCs G^{SCC}

Reminder

G^{SCC} is created by collapsing every strong connected component to a single vertex.

Proposition

For a directed graph G, its meta-graph G^{SCC} is a DAG.

Linear-time Algorithm for SCCs: Ideas

Exploit structure of meta-graph...

Wishful Thinking Algorithm

- Let $\mathbf{u}$ be a vertex in a *sink* SCC of G^{SCC}
- Do **DFS**$(\mathbf{u})$ to compute $\mathrm{SCC}(\mathbf{u})$
- Remove $\mathrm{SCC}(\mathbf{u})$ and repeat

Justification

- **DFS**$(\mathbf{u})$ only visits vertices (and edges) in $\mathrm{SCC}(\mathbf{u})$
- **DFS**$(\mathbf{u})$ takes time proportional to size of $\mathrm{SCC}(\mathbf{u})$
- Therefore, total time $\mathbf{O(n+m)}$!

Big Challenge(s)

How do we find a vertex in a sink SCC of G^{SCC}?

Can we obtain an *implicit* topological sort of G^{SCC} without computing G^{SCC}?

Answer: **DFS(G)** gives some information!

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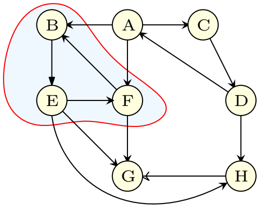
Post-visit times of SCCs

Definition

Given G and a SCC **S** of G, define $\mathrm{post}(\mathbf{S}) = \max_{\mathbf{u} \in \mathbf{S}} \mathrm{post}(\mathbf{u})$ where post numbers are with respect to some **DFS**(**G**).

An Example



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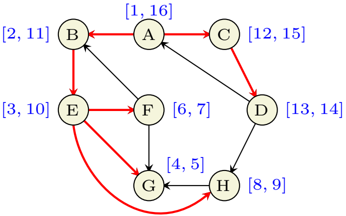

Graph with pre-post times for **DFS**(**A**); black edges in tree

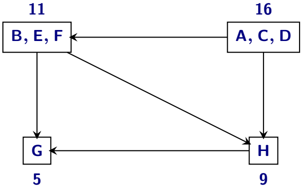

Figure: G^{SCC} with post times

Graph of strong connected components

... and post-visit times

Proposition

If $\mathbf{S}$ *and* $\mathbf{S'}$ *are* SCC*s in* G *and* $(\mathbf{S}, \mathbf{S'})$ *is an edge in* G^{SCC} *then* $\mathrm{post}(\mathbf{S}) > \mathrm{post}(\mathbf{S'})$.

Proof.

Let $\mathbf{u}$ be first vertex in $\mathbf{S} \cup \mathbf{S'}$ that is visited.

- If $\mathbf{u} \in \mathbf{S}$ then all of $\mathbf{S'}$ will be explored before $\mathbf{DFS}(\mathbf{u})$ completes.
- If $\mathbf{u} \in \mathbf{S'}$ then all of $\mathbf{S'}$ will be explored before any of $\mathbf{S}$.

$\square$

A False Statement: If $\mathbf{S}$ and $\mathbf{S'}$ are SCCs in G and $(\mathbf{S}, \mathbf{S'})$ is an edge in G^{SCC} then for *every* $\mathbf{u} \in \mathbf{S}$ and $\mathbf{u'} \in \mathbf{S'}$, $\mathrm{post}(\mathbf{u}) > \mathrm{post}(\mathbf{u'})$.

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Topological ordering of the strong components

Corollary

Ordering SCC*s in decreasing order of* $\mathbf{post(S)}$ *gives a topological ordering of* G^{SCC}

Recall: for a DAG, ordering nodes in decreasing post-visit order gives a topological sort.

So...

DFS(G) gives some information on topological ordering of $\mathbf{G}^{\mathrm{SCC}}$!

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Finding Sources

Proposition

The vertex $\mathbf{u}$ *with the highest post visit time belongs to a source* SCC *in* G^{SCC}

Proof.

- $\mathbf{post(SCC(u)) = post(u)}$
- Thus, $\mathbf{post(SCC(u))}$ is highest and will be output first in topological ordering of $\mathbf{G^{SCC}}$.

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$\square$

Finding Sinks

Proposition

The vertex $\mathbf{u}$ *with highest post visit time in* **DFS**$(\mathbf{G^{rev}})$ *belongs to a sink SCC of* G*.*

Proof.

- $\mathbf{u}$ belongs to source SCC of $\mathbf{G^{rev}}$
- Since graph of SCCs of $\mathbf{G^{rev}}$ is the reverse of G^{SCC}, $\mathrm{SCC}(\mathbf{u})$ is sink SCC of G. □

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Linear Time Algorithm

...for computing the strong connected components in G

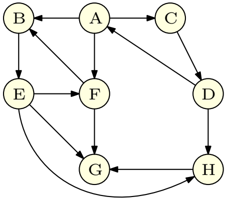
do DFS(G^{rev}) and sort vertices in decreasing post order.
Mark all nodes as unvisited
for each u in the computed order **do**
    if u is not visited **then**
        DFS(u)
        Let S_u be the nodes reached by u
        Output S_u as a strong connected component
        Remove S_u from G

Analysis

Running time is $O(n + m)$. (Exercise)

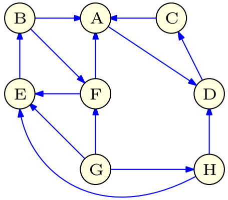
Linear Time Algorithm: An Example - Initial steps

Graph G:



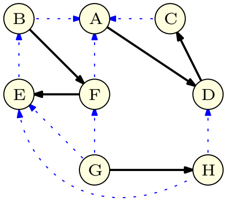
$\Longrightarrow$

Reverse graph G^{rev}:



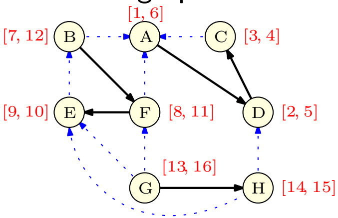
DFS of reverse graph:

$\Longrightarrow$



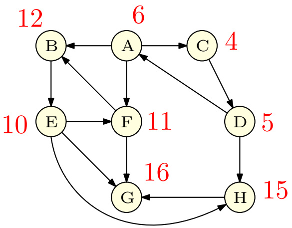
$\Longrightarrow$

Pre/Post **DFS** numbering of reverse graph:



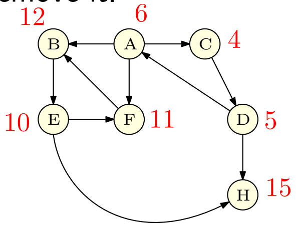
Removing connected components: 1

Original graph G with rev post numbers:



$\Longrightarrow$

Do **DFS** from vertex G remove it.

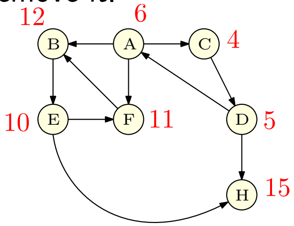


SCC computed:
{G}

Linear Time Algorithm: An Example

Removing connected components: 2

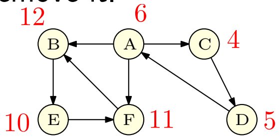
Do **DFS** from vertex G remove it.



SCC computed:
{G}

$\Longrightarrow$

Do **DFS** from vertex **H**, remove it.

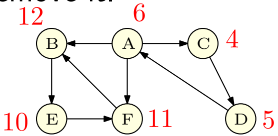


SCC computed:
{G}, {H}

Linear Time Algorithm: An Example

Removing connected components: 3

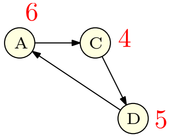
Do **DFS** from vertex **H**, remove it.



$\Longrightarrow$

Do **DFS** from vertex ~~**F**~~ B

Remove visited vertices: **{F, B, E}**.



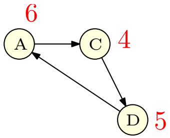
SCC computed: **{G}, {H}**

SCC computed: **{G}, {H}, {F, B, E}**

Linear Time Algorithm: An Example

Removing connected components: 4

Do **DFS** from vertex **F**
Remove visited vertices:
{F, B, E}.



SCC computed:
{G}, {H}, {F, B, E}

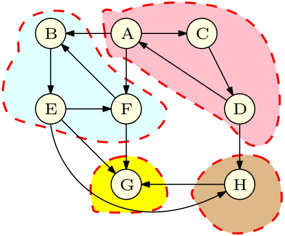
$\Longrightarrow$

Do **DFS** from vertex **A**
Remove visited vertices:
{A, C, D}.

SCC computed:
{G}, {H}, {F, B, E}, {A, C, D}

Linear Time Algorithm: An Example

Final result



SCC computed:

{G}, {H}, {F, B, E}, {A, C, D}

Which is the correct answer!

Obtaining the meta-graph...

Once the strong connected components are computed.

Exercise:

Given all the strong connected components of a directed graph $\mathbf{G} = (\mathbf{V}, \mathbf{E})$ show that the meta-graph $\mathrm{G}^{\mathrm{SCC}}$ can be obtained in $\mathbf{O}(\mathbf{m} + \mathbf{n})$ time.

Correctness: more details

- let $S_1, S_2, \ldots, S_k$ be strong components in G
- Strong components of G^{rev} and G are same and meta-graph of G is reverse of meta-graph of G^{rev}.
- consider $\mathbf{DFS}(G^{rev})$ and let $u_1, u_2, \ldots, u_k$ be such that $\mathrm{post}(u_i) = \mathrm{post}(S_i) = \max_{v \in S_i} \mathrm{post}(v)$.
- Assume without loss of generality that $\mathrm{post}(u_k) > \mathrm{post}(u_{k-1}) \geq \ldots \geq \mathrm{post}(u_1)$ (renumber otherwise). Then $S_k, S_{k-1}, \ldots, S_1$ is a topological sort of meta-graph of G^{rev} and hence $S_1, S_2, \ldots, S_k$ is a topological sort of the meta-graph of G.
- u_k has highest post number and $\mathbf{DFS}(u_k)$ will explore all of S_k which is a sink component in G.
- After S_k is removed u_{k-1} has highest post number and $\mathbf{DFS}(u_{k-1})$ will explore all of S_{k-1} which is a sink component in remaining graph $G - S_k$. Formal proof by induction.

Part III

An Application to `make`

`make` Utility [Feldman]

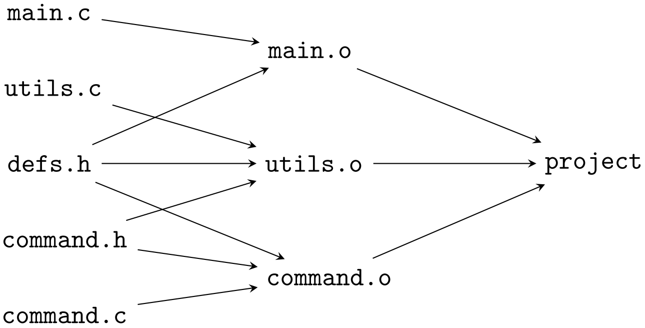
- Unix utility for automatically building large software applications
- A makefile specifies
  - Object files to be created,
  - Source/object files to be used in creation, and
  - How to create them

An Example `makefile`

```
project:  main.o utils.o command.o
    cc -o project main.o utils.o command.o

main.o:  main.c defs.h
    cc -c main.c
utils.o:  utils.c defs.h command.h
    cc -c utils.c
command.o:  command.c defs.h command.h
    cc -c command.c
```

makefile as a Digraph



Computational Problems for `make`

- Is the `makefile` reasonable?
- If it is reasonable, in what order should the object files be created?
- If it is not reasonable, provide helpful debugging information.
- If some file is modified, find the fewest compilations needed to make application consistent.

Algorithms for make

- Is the `makefile` reasonable? Is G a DAG?
- If it is reasonable, in what order should the object files be created? Find a topological sort of a DAG.
- If it is not reasonable, provide helpful debugging information. Output a cycle. More generally, output all strong connected components.
- If some file is modified, find the fewest compilations needed to make application consistent.
  - Find all vertices reachable (using **DFS**/**BFS**) from modified files in directed graph, and recompile them in proper order. Verify that one can find the files to recompile and the ordering in linear time.

Take away Points

- Given a directed graph G, its SCCs and the associated acyclic meta-graph G^{SCC} give a structural decomposition of G that should be kept in mind.
- There is a **DFS** based linear time algorithm to compute all the SCCs and the meta-graph. Properties of **DFS** crucial for the algorithm.
- DAGs arise in many application and topological sort is a key property in algorithm design. Linear time algorithms to compute a topological sort (there can be many possible orderings so not unique).

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