Week 2

27 January 2025 23:42

rightpdf_W... 2 Lecture

Linear transformations

$$f : \mathbb{R}^n \longrightarrow \mathbb{R}$$

special structure

Vector space

Principle of Superposition

for any $\alpha, \beta \in \mathbb{R}$

& $u, v \in \mathbb{R}^n$

a f^n: f satisfies (SP) iff

$$\boxed{f(\alpha u + \beta v) = \alpha f(u) + \beta f(v)}$$

[satisfy this to get superposition]

1) $f(x)$ $\quad x \in \mathbb{R}^3$

$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$f\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = f(\boxed{x}) = \boxed{x_1^2 - 2x_2 + x_3}$$

$\mathbb{R}^3$

$f : \mathbb{R}^n \longrightarrow \mathbb{R}$

2) Ex: $f : \mathbb{R}^n \longrightarrow \mathbb{R}$

Fix $a \in \mathbb{R}^n$ $\quad a = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$

$\forall x \in \mathbb{R}^n$ ~~f~~ $x \to$ belongs to $\mathbb{R}^n$ and apply f

$$f(x) = a^T x = (a_1 \cdots a_n)\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

Result in $\to f(x) = \underbrace{\boxed{a_1x_1 + \cdots + a_nx_n}}_{}$ $= \sum_{i=1}^{n} a_i x_i$

$\in \mathbb{R}$

Linear map / transformation / function

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function which satisfies superposition principle. Then f is a linear function from $\mathbb{R} \to \mathbb{R}$.

$$f(\alpha u + \beta v) = \alpha f(u) + \beta f(v) \quad \forall\, \alpha, \beta \in \mathbb{R},\ u, v \in \mathbb{R}^n$$

Ex: $f : \mathbb{R}^n \to \mathbb{R}$. Fix $\underbrace{a \in \mathbb{R}^n}$,

$$f_a(x) = a^T x = \sum_{i=1}^{n} a_i x_i$$

every fixed vector in $\mathbb{R}^n$ will give rise to a linear transformation from $\mathbb{R}^n \to \mathbb{R}$. [important]

$$f(\alpha x + \beta y) = a^T(\alpha x + \beta y) = \alpha a^T x + \beta a^T y$$

$$= a^T(\alpha x) + a^T(\beta y)$$

$$= \alpha f(x) + \beta f(y)$$

Ex: $f : \mathbb{R}^n \to \mathbb{R}$

$$f(v) = f\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = v_1 + v_2 + \cdots + v_n$$

$$v = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$

$$f(\alpha u + \beta v) = f\begin{pmatrix} \alpha u_1 + \beta v_1 \\ \vdots \\ \alpha u_n + \beta v_n \end{pmatrix} = \alpha f(u) + \beta f(v)$$

$$f(v) = a^T \cdot v = [1\ 1\ 1\ 1 \cdots] \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$

$$a^T = 1_n^T$$

$$f(v) = v_1 + v_2 + \cdots + v_n = \underbrace{[1\ \ 1\ \cdots\ 1]}_{n \text{ times}} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

Ex: $f : \mathbb{R}^n \to \mathbb{R}$

This fn is NOT a linear transformation.

$n = 2$

$f\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \max\{v_1, \dots, v_n\}$ $u = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$, $v = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$f(u+v) = 0 \neq f(u) + f(v)$ $u+v = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $f(u+v) = 0 \neq f(u) + f(v)$

Ex: $f: \mathbb{R}^n \to \mathbb{R}$ $u = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ $v = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

$u+v = \begin{pmatrix} 5 \\ 5 \end{pmatrix}$

$f\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \text{avg}(v_1, \dots, v_n)$

$= \frac{1}{n}(v_1 + v_2 + \dots + v_n)$

$f(u+v) = f(u) + f(v)$

$= \frac{5}{2} + \frac{5}{2}$

$a = \frac{1}{n}\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \in \mathbb{R}^n$

$= \frac{1}{n} 1_n \in \mathbb{R}^n$

This f is also a linear transformation.

$f(v) = a^T v$ (inner product)

Ex: $f: \mathbb{R}^n \to \mathbb{R}$ NOT a linear map.

$f\begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = |v_1| + |v_2|$

Inner product representation of a linear map. (from $\mathbb{R}^n \to \mathbb{R}$)

let $v \in \mathbb{R}^n$, $f: \mathbb{R}^n \to \mathbb{R}$ be a linear map.

Let $\{e_1, e_2, \dots, e_n\}$ be the std. basis of $\mathbb{R}^n$.

$v = v_1 e_1 + v_2 e_2 + \dots + v_n e_n$ where $v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$

$f(v) = f(v_1 e_1 + v_2 e_2 + \dots + v_n e_n)$

$= v_1 f(e_1) + v_2 f(e_2) + \dots + v_n f(e_n)$ .... because f is linear

$$= \underbrace{\begin{bmatrix} f(e_1) & f(e_2) & \cdots & f(e_n) \end{bmatrix}}_{a^T} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

$$\boxed{f(v) = a^T v} \quad \text{where} \quad a = \begin{bmatrix} f(e_1) \\ f(e_2) \\ \vdots \\ f(e_n) \end{bmatrix}$$

Linear map as inner product representation is unique.

$f(x) = a^T x = b^T x \Rightarrow (a^T - b^T) x = 0 \quad \forall x \in \mathbb{R}^n$

take $x = e_i$ $(a_i - b_i) = 0 \Rightarrow a_i = b_i \quad \forall i = 1, \ldots, n$

Ex: $f : \mathbb{R}^3 \to \mathbb{R}$

$$f\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = 2v_1 + v_2 - v_3 + 4$$

$$= \underbrace{\begin{bmatrix} 2 & 1 & -1 \end{bmatrix}}_{= a^T} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} + 4 \rightarrow \text{affine}$$

, they move from the initial original position.

$f(v) = a^T v + \underset{\mathbb{R}}{4}$

Affine functions.

A function $f : \mathbb{R}^n \to \mathbb{R}$ is called as an affine function if & only if it can be represented as

$f(v) = a^T v + b$

Every linear map is an affine map but every affine map is not a linear map. (***)

for some fixed $a \in \mathbb{R}$ & $b \in \mathbb{R}$.

If $f : \mathbb{R}^n \to \mathbb{R}$ is affine, then

$$f(\boxed{\alpha u + \beta v}) = \alpha f(u) + \beta f(v)$$

$\boxed{\text{where } \alpha + \beta = 1}$

$$f(\alpha u + \beta v) = a^T(\alpha u + \beta v) + b$$

$$= \alpha a^T u + \beta a^T v + \boxed{(\alpha + \beta)} b$$

$$= \alpha (a^T u + b) + \beta (a^T v + b)$$

$$= \alpha f(u) + \beta f(v)$$

Linear functions $\mathbb{R}^n \to \mathbb{R}$

representation of linear fns as inner products

Affine fns: $\mathbb{R}^n \to \mathbb{R}$

representation

Affine maps

$f: \mathbb{R}^n \longrightarrow \mathbb{R}$ is called as affine function

if $\exists\ a \in \mathbb{R}^n$ & $b \in \mathbb{R}$

s.t. $f(v) = a^T v + b \quad \forall v \in \mathbb{R}^n$

$f(\alpha u + \beta v) = \alpha f(u) + \beta f(v)$ for $\alpha, \beta \in \mathbb{R}$ s.t. $\alpha + \beta = 1$.

Ex: Recall several calculus.

$f: \mathbb{R}^n \longrightarrow \mathbb{R}$ differentiable.

fix $z \in \mathbb{R}^n$

$$f(x) \approx \hat{f}(x) = f(z) + \left.\frac{\partial f}{\partial x_1}\right|_{x=z} (x_1 - z_1) + \cdots + \left.\frac{\partial f}{\partial x_n}\right|_{x=z} (x_n - z_n)$$

$$\boxed{\hat{f}(x) = f(z) + (\nabla f)^T (x - z)}$$

$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$, $z = \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}$

$$\nabla f \Big|_{x=z} = \begin{bmatrix} \frac{\partial f}{\partial x_1}\Big|_{x=z} \\ \vdots \\ \frac{\partial f}{\partial x_n}\Big|_{x=z} \end{bmatrix}$$

[evaluated at x.]

$\hat{f}$ is an affine function of x.

$$\hat{f}(x) = \underbrace{\boxed{(\nabla f)^T}}_{a^T} x + \underbrace{\boxed{(f(z) - (\nabla f)^T z)}}_{b}$$

When x is "close" to z, then $\hat{f}(x)$, an affine function, approximates the function f very well.

(***)

Ex: # $f : \mathbb{R}^2 \to \mathbb{R}$

$f\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = f(v) = a^T v + b$

$z = f(v) = [a_1 \quad a_2]\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} + b \Rightarrow$

$z = a_1 v_1 + a_2 v_2 + b$

$\Rightarrow \boxed{a_1 v_1 + a_2 v_2 - z + b = 0}$

Ex: Regression model

$$\boxed{y = x^T \beta + \alpha}$$

x : feature vector

y : dependent variables

y = house price

$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$,

x_1 = area of the house

x_2 = no. of bedrooms

x_3 = no. of bathrooms

Let V, W be [...]

•) $f : V \to W$ is (

$f(\alpha v_1 + \beta v_2)$

•) $g : V \to W$ is a

$b \in V$ such t

So for easiness

Linear : 5 —

10

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Affine: 5 —

$x^{(1)} = \begin{pmatrix} \vdots \end{pmatrix} = y_1$, $x^{(2)} = \begin{pmatrix} \vdots \end{pmatrix} = y_2$, $\dots$, $x = \begin{pmatrix} \vdots \end{pmatrix} = y_N$

Norm of vector.

Norm :- basically

A function $\underbrace{\| \overset{x}{\cdot} \|}_{g(x)} : \mathbb{R}^n \to \mathbb{R}$ is defined as a norm

if it satisfies following properties:

i) $\|x\| \geq 0 \quad \forall x \in \mathbb{R}^n$ non-negative property

$\|x\| = 0$ if & only if $x = 0$ positive definiteness

ii) $\|\alpha x\| = |\alpha| \|x\|$ absolute homogenity

iii) $\|x + y\| \leq \|x\| + \|y\|$ triangle inequality

$\sqrt{T} \quad \bullet$ Euclidean norm, 2-norm, spectral norm (important)

Ex: $\|x\|_2 = \sqrt{x^T x}$ • Euclidean norm, ℓ_2 , spectral norm (important)

i) $\|x\|_2 \geq 0 \quad \forall x \in \mathbb{R}^n$

$\|x\|_2 = 0 \iff x = 0$

$(\Leftarrow)$: $x = 0$, then $x_1^2 + \cdots + x_n^2 = 0 \Rightarrow \|x\|_2 = 0$

$(\Rightarrow)$: $\|x\|_2 = 0 \Rightarrow \|x\|_2^2 = 0 \Rightarrow x_1^2 + x_2^2 + \cdots + x_n^2 = 0 \Rightarrow x_i = 0 \quad \forall i = 1, 2, \ldots, n$

ii) $\|\alpha x\|_2 = |\alpha| \, \|x\|_2$

iii) triangle inequality $\|x+y\|_2 \leq \|x\|_2 + \|y\|_2$

To prove.

Cauchy-Schwarz inequality

$x, y \in \mathbb{R}^n$ $\quad x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}; \quad y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$

$$\boxed{|x^T y| \leq \|x\|_2 \|y\|_2}$$

$$\Rightarrow \left| \sum_{i=1}^n x_i y_i \right| \leq \sqrt{\sum_{i=1}^n x_i^2} \cdot \sqrt{\sum_{i=1}^n y_i^2}$$

C-S inequality

$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \quad y = ($

$|x^T y| \leq \|x\|_2 \|$

$\left[\left| \sum_{i=1}^n x_i y_i \right| \leq \sqrt{\sum} \right.$

Proof: set $\alpha = \|x\|_2$, $\beta = \|y\|_2$

$$0 \le \|\beta x - \alpha y\|_2^2 = (\beta x - \alpha y)^T (\beta x - \alpha y)$$
$$= \beta^2 x^T x - \alpha\beta \boxed{x^T y} - \alpha\beta \boxed{y^T x} + \alpha^2 y^T y$$
$$= \alpha^2\beta^2 - 2\alpha\beta\, x^T y + \alpha^2 \beta$$

$\Rightarrow \quad x^T y \le \alpha\beta \quad \Rightarrow \quad x^T y \le \|x\|_2 \|y\|_2$; Apply on $-x$ & y

$x^T y \le \|x\|_2 \|y\|_2$ $\qquad x$ & y

$-x^T y \le \|x\|_2 \|y\|_2$ $\qquad -x$ & y

$\Rightarrow \quad |x^T y| \le \|x\|_2 \|y\|_2$

Triangle inequality

$$\|x+y\|_2^2 = (x+y)^T (x+y) = x^T x + x^T y + y^T x + y^T y$$
$$= \|x\|_2^2 + 2 x^T y + \|y\|_2^2$$
$$\le \|x\|_2^2 + 2\|x\|_2 \|y\|_2 + \|y\|_2^2$$
$$= \left(\|x\|_2 + \|y\|_2\right)^2$$

$\Rightarrow \quad \|x+y\|_2 \le \|x\|_2 + \|y\|_2$

Examples of affine fns:

Example

Norm

$\|\cdot\|_2$

Ex: $x \in \mathbb{R}^n$

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \qquad \|x\|_2 = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} = \sqrt{x^T x}$$

(i), (ii), $\boxed{\text{(iii)}}$

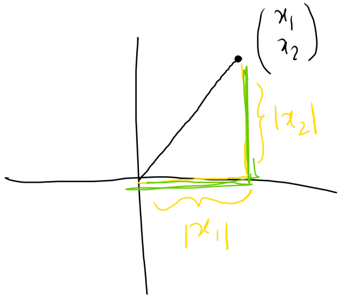
Ex: $x \in \mathbb{R}^n$

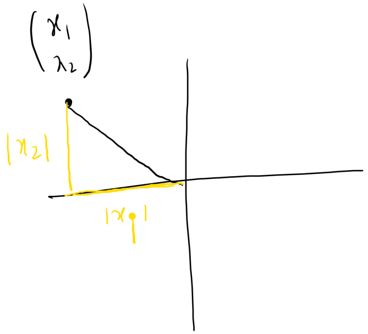
p - norm $\qquad \|x\|_p = \left(|x_1|^p + \cdots + |x_n|^p \right)^{1/p} = \sqrt[p]{\sum_{i=1}^{n} |x_i|^p} \qquad p \geq 1$

$p = 1 \qquad \|x\|_1 = |x_1| + |x_2| + \cdots + |x_n|$

$\mathbb{R}^2 \qquad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$\|x\|_2 = \sqrt{x_1^2 + x_2^2}$



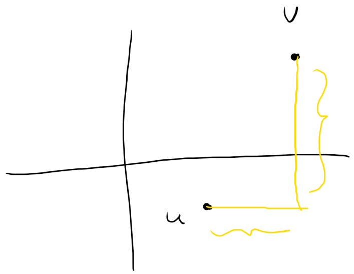


$\|x\|_1 = |x_1| + |x_2|$

Distance between the vectors $u, v \in \mathbb{R}^n$

$\|u - v\|_2$ = 2-norm distance betn u & v

$= \sqrt{\sum_{i=1}^{n} (u_i - v_i)^2}$



$\|u - v\|_1 = |u_1 - v_1| + |u_2 - v_2| + \cdots + |u_n - v_n|$ ← taxi-cab distance

Ex: Let $u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_d \end{pmatrix} \qquad u_1 \in \mathbb{R}^{n_1}, \; u_2 \in \mathbb{R}^{n_2}, \; \ldots, \; u_d \in \mathbb{R}^{n_d}$

$u \in \mathbb{R}^n \Rightarrow n_1 + n_2 + \cdots + n_d = n$

2 norm:

$\|u\|_2 = \sqrt{\|u_1\|_2^2 + \ldots}$

$\|u\|_2^2 = u^T u = (u_1 \; u_2 \; \cdots \; u_d) \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_d \end{pmatrix} = u_1 u_1 + u_2 u_2 + \cdots$

$= \|u_1\|_2^2 + \|u_2\|_2^2 + \cdots + \|u_d\|_2^2$

$\Rightarrow \quad \|u\|_2 = \sqrt{\|u_1\|_2^2 + \cdots + \|u_d\|_2^2}$

Angle between the vectors

let $u, v \in \mathbb{R}^n$

$\theta = \arccos\left(\frac{u^T v}{\|u\|_2 \|v\|_2}\right)$

$u^T v = \|u\|_2 \|v\|_2 \cos\theta$

$\theta \in [0, \pi]$

arccos : inverse cosine with values $[0, \pi]$

$\theta = \arccos\left(\frac{u^T v}{\|u\|_2 \|v\|_2}\right)$

i) $u^T v = 0 \Rightarrow \theta = \frac{\pi}{2} \Rightarrow$ vectors u & v are orthogonal / perpendicular

ii) $u^T v = \|u\|_2 \|v\|_2 \Rightarrow u$ & v are on the same line, $\theta = 0$

iii) $u^T v = -\|u\|_2 \|v\|_2 \Rightarrow u$ & v are on the same line, $\theta = \pi$

C-S inequality $\frac{|u^T v|}{\|u\|_2 \|v\|_2} \leq 1 \quad \Rightarrow \quad -1 \leq \frac{u^T v}{\|u\|_2 \|v\|_2} \leq 1$

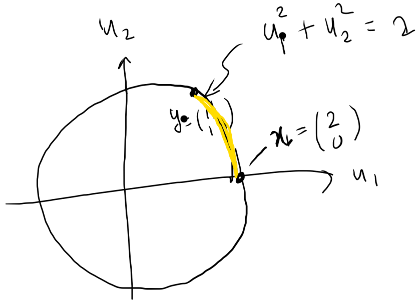
case dim 2 vec...

$\|u - v\|_2 = \sqrt{\ldots}$

1 norm:

$\|x\|_1 = |x_1| \ldots$

u



$$x = \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \quad y = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\theta = \arccos\left(\frac{x^T y}{\|x\|_2 \, \|y\|_2}\right) = \arccos\left(\frac{2}{2 \cdot \sqrt{2}}\right)$$

$$= \arccos\left(\frac{1}{\sqrt{2}}\right)$$

$$\boxed{\theta = \pi/4}$$

Computational Complexity.

- number of floating point operations (flops) additions / multiplications

$x, y \in \mathbb{R}^n$ $\quad x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ $\quad y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$

$$\boxed{x+y} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{pmatrix}$$

'n' additions

$$\boxed{\alpha x} = \alpha \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \alpha x_1 \\ \alpha x_2 \\ \vdots \\ \alpha x_n \end{pmatrix}$$

'n' multiplications

$$a^T x = [a_1 \ a_2 \ \cdots \ a_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

(n-1) additions

$$= \sum_{i=1}^{n} a_i x_i = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n$$

'n' multiplications

Computational complexity

$$n + (n-1) = 2n - 1$$

$$\|x\|_2 = \sqrt{x^T x}$$

require $(2n-1) + 1 = 2n$ flops

2n-1 flops

Norm

- 2, 1
- geometrical interpretion

Angle

Computational complexity

flops

Matrix - Vector multiplication

$A \in \mathbb{R}^{m \times n}$, $x \in \mathbb{R}^n$

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} ; \quad x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$c = Ax \in \mathbb{R}^m$

$$c = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{bmatrix}$$

$$c_j = \sum_{k=1}^{n} a_{jk} x_k \quad \text{for } j = 1, 2, \dots, m$$

$$c = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Row interpretation

Computational complexity

m - number of inner products of length n vectors.

$m(2n-1) = 2mn - m$

$\approx 2mn$

i^{th} entry c is the inner product of i^{th} row of A and x.

Column intrepretation

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & \vdots & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$\Rightarrow \quad c = Ax = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} x_1 + \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} x_2 + \cdots + \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} x_n$$

Linear function from $\mathbb{R}^n \longrightarrow \mathbb{R}^m$

A function $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is called as a linear function if $\forall$ $u, v \in \mathbb{R}^n$ and scalars $\alpha, \beta \in \mathbb{R}$

$$f(\alpha u + \beta v) = \alpha f(u) + \beta f(v)$$

— superposition principle.

$\mathbb{R}^n \qquad \mathbb{R}^m$

Ex: $f: \mathbb{R}^n \longrightarrow \mathbb{R}^{m}$, $m = n$

$$f: \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \longmapsto \begin{pmatrix} x_1 + x_2 \\ x_2 + x_3 \\ \vdots \\ x_n + x_1 \end{pmatrix}$$

Easy to verify that f is linear.

Ex: $f: \mathbb{R}^n \longrightarrow \mathbb{R}^{2n-1}$

$$f: \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \longmapsto \begin{pmatrix} x_1 \\ \frac{x_1 + x_2}{2} \\ x_2 \\ \vdots \\ x_{n-1} \\ \frac{x_{n-1} + x_n}{2} \\ x_n \end{pmatrix}$$

$$f(\alpha x + \beta y) = f\begin{pmatrix} \alpha x_1 + \beta y_1 \\ \vdots \\ \alpha x_n + \beta y_n \end{pmatrix} = \begin{pmatrix} \alpha x_1 + \beta y_1 \\ \frac{\alpha x_1 + \beta y_1 + \alpha x_2 + \beta y_2}{2} \\ \vdots \end{pmatrix}$$

$$= \alpha \begin{pmatrix} x_1 \\ \frac{x_1 + x_2}{2} \\ \vdots \end{pmatrix} + \beta \begin{pmatrix} y_1 \\ \frac{y_1 + y_2}{2} \\ \vdots \end{pmatrix} = \alpha f(x) + \beta f(y)$$

$\Rightarrow f$ is linear

Ex: $f: \mathbb{R}^n \longrightarrow \mathbb{R}^n$

Ex: f: ℝ ...

$$f: \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \longmapsto \begin{pmatrix} |x_1| \\ \vdots \\ |x_n| \end{pmatrix}$$

NOT a linear transformation.

Ex: Let $A \in \mathbb{R}^{m \times n}$

Define $f_A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$

$$f(x) = \underbrace{Ax}_{\in \mathbb{R}^m} \quad \forall\, x \in \mathbb{R}^n$$

$$f(\alpha x + \beta y) = A(\alpha x + \beta y) = \alpha A x + \beta A y = \alpha f(x) + \beta f(y)$$

$\Rightarrow$ f is a linear transform.

Can every linear transformation be represented as matrix-vector multiplication??

Let $e_1, e_2, \dots, e_n \in \mathbb{R}^n$ be std. basis vectors.

$$x \in \mathbb{R}^n \Rightarrow x_1 e_1 + x_2 e_2 + \dots + x_n e_n$$

Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be any linear function.

$$f(x) = f(x_1 e_1 + x_2 e_2 + \dots + x_n e_n)$$

$$= x_1 f(e_1) + x_2 f(e_2) + \dots + x_n f(e_n)$$

linear comb. of 'n' vectors in

$= x_1 f(e_1) + \dots$

$\in \mathbb{R}^m$ $\in \mathbb{R}^m$ $\in \mathbb{R}^m$ $\mathbb{R}^m$

$$\boxed{f(x) = A_f x} \quad \text{where } A_f = \begin{bmatrix} f(e_1) & f(e_2) & \cdots & f(e_n) \end{bmatrix}_{m \times n}$$

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$$

Two important subspaces:

$f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is linear.

$N_f = \{ x \in \mathbb{R}^n \mid f(x) = 0 \} \subseteq \mathbb{R}^n$ = Nullspace of f

$R_f = \{ y \in \mathbb{R}^m \mid \exists\, x \in \mathbb{R}^n \text{ s.t. } f(x) = y \} \subseteq \mathbb{R}^m$

= Image or range space of f.

Claim: $N_f \subseteq \mathbb{R}^n$ is a subspace of $\mathbb{R}^n$

Given: $\alpha, \beta \in \mathbb{R}$ & $x, y \in N_f$

To show: $\alpha x + \beta y \in N_f$,

$\alpha x + \beta y \in N_f$

$\Uparrow$

$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y) = \alpha \cdot 0 + \beta \cdot 0 = 0$

Claim: $Im(f) = R_f \subseteq \mathbb{R}^m$ is a subspace of $\mathbb{R}^m$.

Take: $y_1, y_2 \in R_f$, $\alpha, \beta \in \mathbb{R}$

To show: $\alpha y_1 + \beta y_2 \in R_f$

$y_1 \in R_f \Rightarrow \exists\ x_1$ s.t. $f(x_1) = y_1$

$y_2 \in R_f \Rightarrow \exists\ x_2$ s.t. $f(x_2) = y_2$

$$\alpha y_1 + \beta y_2 = \alpha f(x_1) + \beta f(x_2)$$

$$= f(\alpha x_1 + \beta x_2)$$

$\Rightarrow \alpha y_1 + \beta y_2$ has a pre-image $\alpha x_1 + \beta x_2 \in \mathbb{R}^n$

$\Rightarrow \alpha y_1 + \beta y_2 \in R_f$

Given an $f : \mathbb{R}^n \to \mathbb{R}^m$, a linear fn:

$\exists\ A_f \in \mathbb{R}^{m \times n}$ s.t.

$$f(x) = A_f x$$

$$N(A_f) = \{x \in \mathbb{R}^n \mid A_f x = 0\} \subseteq \mathbb{R}^n$$

$$R(A_f) = \{y \in \mathbb{R}^m \mid \exists\ x \in \mathbb{R}^n \text{ s.t. } Ax = y\} \subseteq \mathbb{R}^m$$

null space of A_f

Null space

Range space of A_f

Matrix - vector multiplication

→ row - interpretation

→ column - interpretation.

Linear functions from $\mathbb{R}^n \to \mathbb{R}^m$

Null space

Image space

Linear function $\mathbb{R}^n \to \mathbb{R}^m$ $\longleftrightarrow$ $A_f \in \mathbb{R}^{m \times n}$

$f : \mathbb{R}^n \to \mathbb{R}^m$

Let $A_f \in \mathbb{R}^{m \times n}$

$$f(x) = A_f\, x = \begin{bmatrix} A_1^T \\ A_2^T \\ \vdots \\ A_m^T \end{bmatrix}_{m \times n} \begin{bmatrix} \\ x \\ \\ \end{bmatrix} = \begin{bmatrix} \cdot \\ \vdots \\ \cdot \end{bmatrix}$$

$$f(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_m(x) \end{pmatrix} = \begin{pmatrix} A_1^T x \\ A_2^T x \\ \vdots \\ A_m^T x \end{pmatrix}$$

$f_i \in \mathbb{R}^n \to \mathbb{R}$, $i = 1, 2, \dots, m$

$$\boxed{f_i(x) = A_i^T x}$$

Affine function from $\mathbb{R}^n$ to $\mathbb{R}^m$

A function $g : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is called as an affine function if there exists a matrix $A_g \in \mathbb{R}^{m \times n}$ and $b_g \in \mathbb{R}^m$ s.t.

$$g(x) = \boxed{A_g\, x} + \boxed{b_g} \in \mathbb{R}^m \quad \text{fr } \forall x \in \mathbb{R}^n$$

$$g(x) = \begin{pmatrix} g_1(x) \\ \vdots \\ g_m(x) \end{pmatrix} = \begin{pmatrix} f_1(x) \\ \vdots \\ f_m(x) \end{pmatrix} + \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} = \begin{pmatrix} \boxed{f_1(x) + b_1} \\ \vdots \\ \boxed{f_m(x) + b_m} \end{pmatrix}$$

$g_1(x) =$, $g_m(x) =$ affine

$f_i : \mathbb{R}^n \to \mathbb{R}$

$g_i : \mathbb{R}^n \longrightarrow \mathbb{R}$

$i = 1, 2, \ldots, m$

$f_i : \mathbb{R} \to \ldots$

linear

Ex:

g is an affine function from $\mathbb{R}^n$ to $\mathbb{R}^m$

if & only if

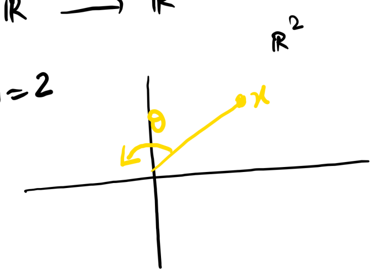
$$g(\alpha x + \beta y) = \alpha g(x) + \beta g(y)$$

whenever $\alpha, \beta \in \mathbb{R}$, $\alpha + \beta = 1$ & $x, y \in \mathbb{R}^n$

Examples:

$f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$

Ex: $m = n = 2$



$f :=$ Rotation by an angle θ in anti-clockwise direction.

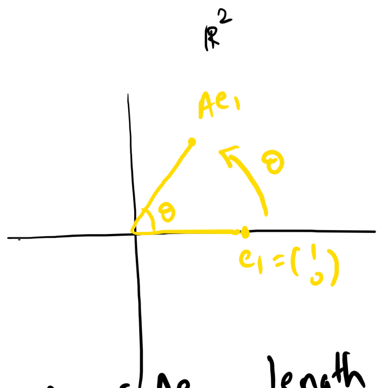
Find the matrix of this linear transformation.

$x \in \mathbb{R}^2 \Rightarrow x_1 e_1 + x_2 e_2$

& $f =$ Rotation:

$$f(x) = f(x_1 e_1 + x_2 e_2) = f(e_1) x_1 + f(e_2) x_2$$

$$= \begin{bmatrix} f(e_1) & f(e_2) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$



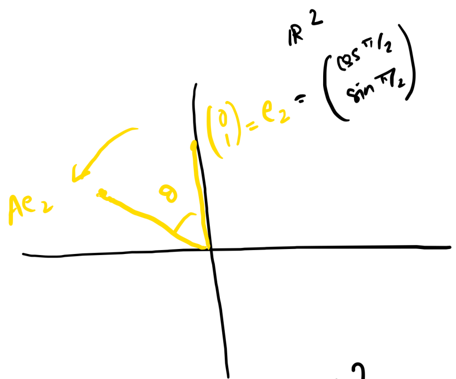
$\in \mathbb{R}^2$

$= A_f x$

$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = r \begin{pmatrix} \cos 0 \\ \sin 0 \end{pmatrix}$$

$$Ae_1 = 1 \cdot \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} = f(e_1)$$

length of Ae_1 = length of e_1



$$Ae_2 = r \begin{pmatrix} \cos\left(\frac{\pi}{2}+\theta\right) \\ \sin\left(\frac{\pi}{2}+\theta\right) \end{pmatrix} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

$$= f(e_2)$$

$$A_f = \begin{bmatrix} f(e_1) & f(e_2) \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \in \mathbb{R}^{2\times 2}$$

Rotator matrix

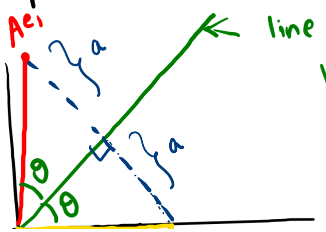
Rotation action does <u>NOT</u> change the length (in 2-norm) of the vector.

$\|x\|_2 = \|A_f x\|_2$

Rotation will <u>not</u> change angle betn the vectors.

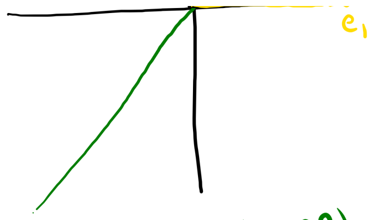
$\langle x, y \rangle = \langle A_f x, A_f y \rangle$

Ex: Reflection through a line making an angle θ with positive x-axis. $(m = n = 2)$



line through which we want to reflect

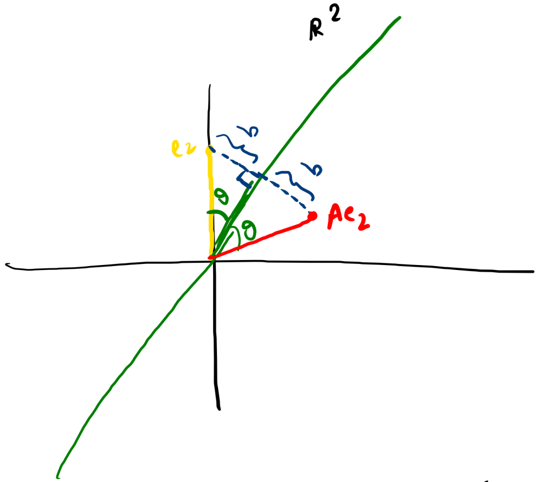
Notice that length of Ae_1 is same



as length of $e_1 = 1$

$\|Ae_1\|_2 = \|e_1\|_2 = 1$

$$Ae_1 = \begin{pmatrix} \cos 2\theta \\ \sin 2\theta \end{pmatrix}$$



$\|Ae_2\|_2 = \|e_2\|_2 = 1$

$$Ae_2 = \begin{pmatrix} \sin(\pi + 2\theta) \\ \cos(\pi + 2\theta) \end{pmatrix}$$

$$Ae_2 = \begin{pmatrix} \sin(2\theta) \\ -\cos(2\theta) \end{pmatrix}$$

Rotating 2θ in clockwise direction

Matrix of reflection action thru' line passing thru' origin & angle θ with positive X-axis

$$= \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} \rightarrow \text{Reflector matrix}$$

Affine functions

- Affine combinition

Examples of linear maps

- Rotators $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$
- Reflectors $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$