Week 1

Sunday, 26. January 2025 23:09

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1 Lecture

Applied Linear Algebra for AI & ML

- solving systems of linear equations
- optimization
- eigenvalue and eigenvector computations

Vector space:

- vectors & matrices

$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

$x + y = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \end{pmatrix}$ Component wise addition

for some $\alpha \in \mathbb{R}$ $\alpha x = \begin{pmatrix} \alpha x_1 \\ \alpha x_2 \end{pmatrix}$

Vector space

V : set of vectors.

field ($\mathbb{F}$) $\mathbb{R}$ - set of real numbers.

C - set of complex

numbers.

Two operation: vector addition + scalar multiplication.

closure properties: $\forall\, x, y \in V,\quad x+y \in V$
$\forall\, \alpha \in \mathbb{R},\ x \in V,\quad \alpha x \in V$

The set V is "closed" under vector addition & scalar multiplication

i) $\forall\, x, y \in V,\ x+y = y+x$
ii) $\forall\, x, y, z \in V,\ (x+y)+z = x+(y+z)$
iii) $\exists$ a zero vector in V, $0 \in V$ s.t. $x+0 = 0+x = x$
iv) $\exists\, y \in V$ for every x s.t. $x+y = y+x = 0$
v) $\forall\, x \in V,\ 1x = x$
vi) $\forall\, \alpha, \beta \in \mathbb{R},\ x \in V$ $(\alpha\beta)x = \alpha(\beta x)$
vii) $(\alpha+\beta)x = \alpha x + \beta x$
viii) $\alpha(x+y) = \alpha x + \alpha y$

Ex: $\mathbb{R}^2 = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \middle| x_1, x_2 \in \mathbb{R} \right\}$ $= V$; $\mathbb{R}^n = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} : x_1, \dots, x_n \in \mathbb{R} \right\}$ $= V$

vector addition: componentwise addition

$x, y \in \mathbb{R}^2$ $\quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},\ y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ $\quad \boxed{x+y} = \begin{pmatrix} x_1+y_1 \\ x_2+y_2 \end{pmatrix} \in \mathbb{R}^2$

$\alpha \in \mathbb{R}$ $\quad \boxed{\alpha x} = \alpha \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \alpha x_1 \\ \alpha x_2 \end{pmatrix} \in \mathbb{R}^2$ $\quad$ closure properties ✓

field of scalars ($\mathbb{F}$)

Because of commutativity & associativity of addition in $\mathbb{R}$, vector addition is also commutative & associative. (i), (ii)

Existence of zero vector $0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ scalars from $\mathbb{R}$ $\in \mathbb{R}^2$ (iii) vector zero

$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

(IV) Additive inverse: $y = \begin{pmatrix} -x_1 \\ -x_2 \end{pmatrix}$ $\in \mathbb{R}$ for $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$\Rightarrow x + y = 0 = y + x$

Ex: $\mathbb{R}^n$ is also a vector space over $\mathbb{R}$.

$\mathbb{R}^n$ is a real vector space

$0 = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, $\forall x \in \mathbb{R}^n$, $y = \begin{bmatrix} -x_1 \\ -x_2 \\ \vdots \\ -x_n \end{bmatrix}$

Ex: $\mathbb{R}^{m \times n}$ = set of all matrices of size $m \times n$

field of scalars $= \mathbb{R}$

$A \in \mathbb{R}^{m \times n}$

$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \in \mathbb{R}^{m \times n}$

$0 = \begin{pmatrix} 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ \vdots & & \\ 0 & \cdots & 0 \end{pmatrix} \in \mathbb{R}^{m \times n}$

$A + B$: componentwise addition

αA : componentwise scalar multiplication.

$-A = \begin{bmatrix} -a_{11} & \cdots & -a_{1n} \\ \vdots & & \\ -a_{m1} & \cdots & -a_{mn} \end{bmatrix}$

Ex: $P_n(\mathbb{R})$: collection / set of all polynomials in x with real coefficients having degree $\leq n$.

$s = s(x) = s_0 + s_1 x + s_2 x^2 + \cdots + s_n x^n \in P_n(\mathbb{R})$

where $s_0, s_1, \ldots, s_n \in \mathbb{R}$

$(c_n + d_n) x^n$

$s+t = s(x) + t(x) = (s_0+t_0) + (s_1+t_1)x + \dots + (s_n+t_n)x^n \in \mathbb{P}_n(\mathbb{R})$

$\forall \alpha \in \mathbb{R}, \quad \alpha s = \alpha s(x) = (\alpha s_0) + (\alpha s_1)x + \dots + (\alpha s_n)x^n \in \mathbb{P}_n(\mathbb{R})$

closure axioms

$0 = 0(x) = 0 + 0x + 0x^2 + \dots + 0x^n$: zero polynomial

vector space

- examples of vector spaces

$\mathbb{R}^n$, $\mathbb{R}^{m\times n}$, $\mathbb{P}_n(\mathbb{R})$

Vector space

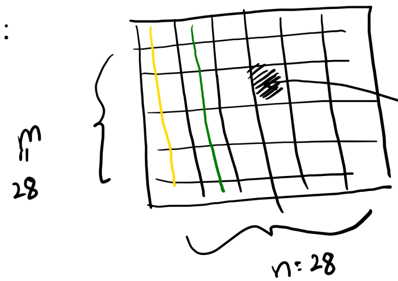
- definition
- examples

  $\mathbb{R}^n$, $\mathbb{R}^{m \times n}$, $\mathbb{P}_n(\mathbb{R})$

- Examples → vectors
- Vector subspace

(important)

Ex:



grey scale image

pixel

mn : pixels

intensity values are associated

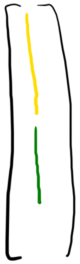
$m \times n$: matrix

with entries from $(0,1)$

the pixels

$[0,1]$

$A \in \mathbb{R}^{m \times n} \rightarrow v \in \mathbb{R}^{\overset{28 \times 28}{mn}} =$



MNIST

hand written

digit data

Ex: feature vectors:

Vector subspace:

A subset W of a vector space V (over $\mathbb{R}$) is called a subspace if W is a vector space $\mathbb{R}$ in its own right.

closure properties: $x, y \in W$, $x+y \in W$, $\alpha \in \mathbb{R}$, $x \in W$
$\alpha x \in W$

properties of addition & scalar multiplication] i, ii
v, vi, vii, viii

Existence of zero $\in W$

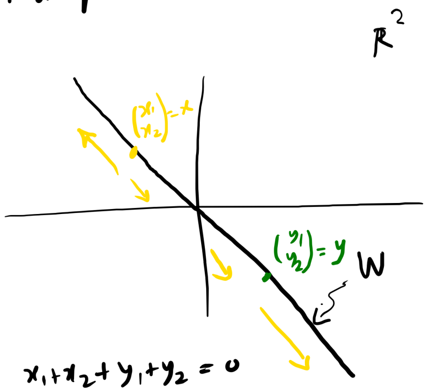
Existence of additive inverse in W

Result: Let $W \subseteq V$ where V is a real vector space.

W is a subspace of V

$\Updownarrow$ (if and only if)

W is closed under vector addition & scalar multiplication.



Examples: $V = \mathbb{R}^2$

$W = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \middle| \ x_1 + x_2 = 0 \right\}$

$x, y \in W \Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in W$

$x_1 + x_2 = 0 \ , \ y_1 + y_2 = 0 \Rightarrow x_1 + x_2 + y_1 + y_2 = 0$

$\Rightarrow \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \end{pmatrix} \in W$

$x_1 + x_2 = 0 \Rightarrow \alpha(x_1 + x_2) = 0 \quad \forall \alpha \in \mathbb{R}$

$\Rightarrow \begin{pmatrix} \alpha x_1 \\ \alpha x_2 \end{pmatrix} \in W$

$\Rightarrow W$ is a subspace of $\mathbb{R}^2$.

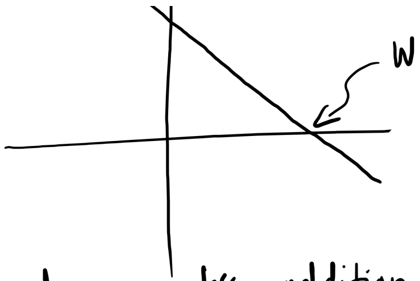
Ex: Any line passing through origin is a subspace of $\mathbb{R}^2$.

Ex: Why passing thru' origin is important ??

$\mathbb{R}^2$

$V = \mathbb{R}^2$

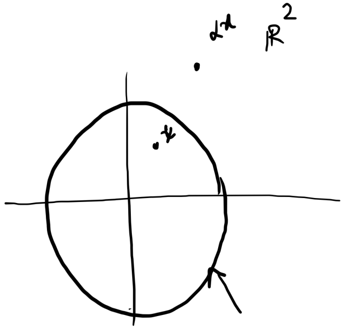
$W = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \middle| \; x_1 + x_2 = 1 \right\}$



clearly, W is NOT closed under vector addition & scalar multiplication. $\forall x \in W,\ 0 \cdot x = 0 \notin W$

Ex: $V = \mathbb{R}^2$

$W = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : x_1^2 + x_2^2 \leq 1 \right\}$



W : unit circle

$\alpha > 1 \qquad \alpha x \notin W$

W is NOT closed under scalar multiplication.

W is NOT a subspace.

Ex: $V = \mathbb{R}^2$, $W = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\} = \{0\}$ vector zero

Ex: $V = \mathbb{R}^2$, $W = V = \mathbb{R}^2$

trivial subspaces.

In $\mathbb{R}^2$

i) $\{0\}$, $\mathbb{R}^2$ are trivial subspaces.

ii) Any line passing thru' origin is a subspace of $\mathbb{R}^2$

Question: Are these the only subspaces of $\mathbb{R}^2$??

We defined

- vector subspace
- Result: W is subspace of V

  if & only if

  W is closed under

  vector addition & scalar multiplication

multiplication.

- Examples of subspaces of $\mathbb{R}^2$

Subspace of a vector space

Ex: $V = \mathbb{R}^2$,

$W = \{0\}$] – Trivial subspaces
$W = \mathbb{R}^2$

$W =$ any line passing through origin.

Ex: $V = \mathbb{R}^3$ plane passing thru' origin

$$W = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \middle| \alpha x_1 + \beta x_2 + \gamma x_3 = 0 \right\}$$

$\alpha, \beta, \gamma \in \mathbb{R}$
not all zero simultaneously.

(α, β, γ)

W is a subspace.

$x, y \in W$
$ax + by \in W$
$\forall a, b \in \mathbb{R}$
] closure axiom

Ex: $V = \mathbb{R}^3$

$W = \left\{ \begin{pmatrix} t \\ 2t \\ \end{pmatrix} : t \in \mathbb{R} \right\}$

W is a subspace.

$x, y \in W$

$x = \begin{pmatrix} t_1 \\ 2t_1 \\ -t_1 \end{pmatrix}$; $y = \begin{pmatrix} 2t_2 \\ -t_2 \end{pmatrix}$

for $t_1, t_2 \in \mathbb{R}$

$$\alpha x + \beta y = \begin{pmatrix} \alpha t_1 + \beta t_2 \\ 2\alpha t_1 + 2\beta t_2 \\ -\alpha t_1 - \beta t_2 \end{pmatrix} \in W$$

Ex: $V = \mathbb{R}^3$

$$W = \left\{ \begin{pmatrix} \alpha t \\ \beta t \\ \gamma t \end{pmatrix} : \begin{array}{l} t \in \mathbb{R} \\ \alpha, \beta, \gamma \\ \text{not all zero} \end{array} \right\}$$

: Line passing thru' origin in $\mathbb{R}^3$

$$\frac{x}{1} = \frac{y}{2} = \frac{z}{-1} = t$$

$V = \mathbb{R}^3$ Examples of subspaces.

- $\{0\}$, $\mathbb{R}^3$: trivial subspaces
- planes passing through origin
- lines passing through origin

$V = \mathbb{R}^2$

- $\{0\}$, $\mathbb{R}^2$
- lines passing through origin

$\boxed{\text{Are these the only subspaces.}}$

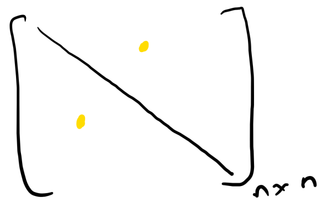
Generalize to $\mathbb{R}^n$

Ex: $V = \mathbb{R}^{m \times n}$: set of matrices of size $m \times n$.

Ex: $V = \mathbb{R}^{n \times n}$: square matrices.

S = set of all symmetric matrices.

$A \in S$, then $A_{ij} = A_{ji}$



Verify that S is a subspace of V.

$\alpha A + \beta B \in S$ if $A, B \in S$

$D \subseteq S$ where D : all diagonal matrices.

D also a subspace of V

D is a subspace of S.

Linear combination of vectors.

For any vectors $v_1, v_2, \ldots, v_k \in \mathbb{R}^n$

& scalars $\alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbb{R}$

the vector $w = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k = \sum_{i=1}^{k} \alpha_i v_i \in \mathbb{R}^n$

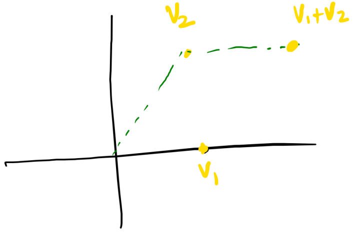
is called a linear combination of the vectors $v_1, v_2, \ldots, v_k$.

$v_1, v_2 \in \mathbb{R}^2$

$v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\alpha_1 = 1, \ \alpha_2 = 1$

$v_1 + v_2$



Consider a set $\boxed{S = \{v_1, \ldots, v_k\}} \subseteq \mathbb{R}^n$

Linear span of $S = \{\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k \mid \alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbb{R}\}$

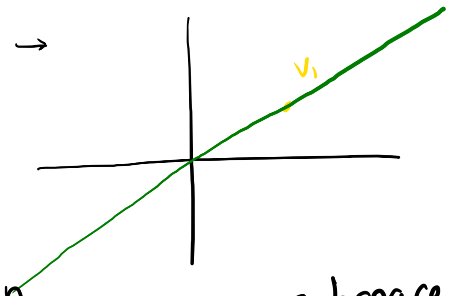
$\parallel$

$\boxed{\mathcal{L}(S)}$

• $\alpha_2 = \alpha_3 = \cdots = \alpha_k = 0$

$\alpha v_1 \rightarrow$

$\mathbb{R}^2$



$S = \{v_1, v_2\} \subseteq \mathbb{R}^2$

for $a, b \in \mathbb{R}$

$ax + by \in \mathcal{L}(S)$

$\boxed{ax + by}$

$= a(\alpha_1 v_1) + \cdots + a(\alpha_k v_k)$

$+ b(\beta_1 v_1) + \cdots + b(\beta_k v_k)$

$= \boxed{(a\alpha_1 + b\beta_1)} v_1 + \cdots$

Ex: $\mathcal{L}(S) \subseteq \mathbb{R}^n$ is a subspace.

$\ldots \mathcal{L}(S) \Rightarrow x = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k$

$x, y \in \mathcal{L}(S) \Rightarrow$

$y = \beta_1 v_1 + \beta_2 v_2 + \cdots + \beta_k v_k$ $\quad \cdots + \boxed{(a\alpha_k + b\beta_k)} v_k$

$\mathcal{L}(S)$ is a subspace of $\mathbb{R}^n$.

Is every subspace always a linear span ??

Linear dependence:

$A = \{v_1, \dots, v_k\} \subseteq \mathbb{R}^n$

We say $\{v_1, \dots, v_k\}$ is linearly dependent set when there exist scalars $\alpha_1, \alpha_2, \dots, \alpha_k$, not all zero, such that

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k = 0$$

Consider $\{v_1, \dots, v_k\} \subset \mathbb{R}^n$ linearly dependent.

$\exists$ scalars $\alpha_1, \alpha_2, \dots, \alpha_k$ (not all zero)

s.t. $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k = 0$

Assume that $\alpha_j \neq 0$

$\boxed{} \cdot \quad \boxed{\alpha_2} v \; - \cdots - \boxed{\alpha_{j-1}} v_{j-1}$

$$v_j = \boxed{-\frac{\alpha_1}{\alpha_j}} v_1 - \boxed{\frac{\quad}{\alpha_j}} v_2 \quad \boxed{\frac{\quad}{\alpha_j}}$$

$$- \boxed{\frac{\alpha_{j+1}}{\alpha_j}} v_{j+1} - \cdots - \boxed{\frac{\alpha_k}{\alpha_j}} v_k$$

important.

$\Rightarrow$ We can represent one vector v_j as a linear combination of the remaining vectors.

- Subspaces of $\mathbb{R}^n$
- Linear span
- Linear dependence

Linear dependence of vectors.

$S = \{v_1, \ldots, v_k\} \subseteq \mathbb{R}^n$

if $\exists$ scalars $\alpha_1, \ldots, \alpha_k$, not all zero,

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k = 0$$

$\Rightarrow$ One of the vectors in S can be written as a linear combination of the remaining vectors.

$\Rightarrow$ There is "<u>linear</u>" redundancy.

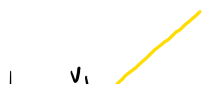
Ex: $S = \{v_1, \ldots, v_k, 0\} \subseteq \mathbb{R}^n$

zero vector — $0 = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{R}^n$

$\underset{0}{\alpha_1}, \underset{0}{\alpha_2}, \ldots, \underset{0}{\alpha_k}, \ \underline{\alpha_{k+1} = 1}$

$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k + 1 \cdot 0 = 0$ (only non zero vectors we need)

Ex: $S = \{v_1, v_2\} \subseteq \mathbb{R}^n$



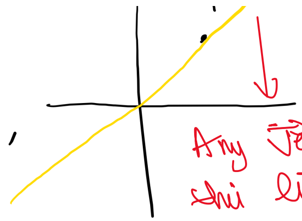
S is lin. dep.

$\exists\ \alpha_1, \alpha_2$, not zero simultaneously,

$$\alpha_1 v_1 + \alpha_2 v_2 = 0$$

let $\alpha_1 \neq 0 \Rightarrow v_1 = -\dfrac{\alpha_2}{\alpha_1} v_2$

$\Rightarrow v_1$ is a scalar multiple of v_2.



Any $\vec{v}$ on this line is dependent on v_1.

Linear independence of vectors.

$S = \{v_1, v_2, \ldots, v_k\} \subseteq \mathbb{R}^n$.

We say S is a linearly independent set,

if $\alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbb{R}$ s.t.

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k = 0$$

$$\Updownarrow \text{ if \& only if}$$

$$\alpha_1 = \alpha_2 = \cdots = \alpha_k = 0$$

Ex: $e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^n$, $e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^n, \ldots, e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \in \mathbb{R}^n$

$$e_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ 0 \end{bmatrix} \leftarrow i^{th} \text{ position}$$

$S = \{e_1, e_2, \cdots, e_n\} \subseteq \mathbb{R}$ is linearly independent.

$$\alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_n e_n = 0$$

$$\Rightarrow \alpha_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{pmatrix} + \cdots + \alpha_n \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Ex: $S = \left\{ \underset{e_1}{\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}}, \underset{e_2}{\begin{pmatrix} 1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}}, \cdots, \overset{e_1+e_2+\cdots+e_n}{\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}} \right\} \subseteq \mathbb{R}^n$

$$\alpha_1 \underbrace{e_1}_{v_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}} + \alpha_2 (\underbrace{e_1 + e_2}_{v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ \vdots \end{pmatrix}}) + \alpha_3 (\underbrace{e_1 + e_2 + e_3}_{v_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ \vdots \end{pmatrix}}) + \cdots + \alpha_n (\underbrace{e_1 + e_2 + \cdots + e_n}_{v_n = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}}) = 0$$

$$\Rightarrow \begin{bmatrix} \boxed{\alpha_1} + \alpha_2 + \cdots + \alpha_n \\ \boxed{\alpha_2} + \cdots + \alpha_n \\ \vdots \\ \boxed{\alpha_{n-1}} + \alpha_n \\ \boxed{\alpha_n} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \boxed{0} \end{bmatrix}$$

Ex: Let $S_1 \subseteq S_2 \subseteq \mathbb{R}^n$

If S_1 is linearly dependent, then S_2 is also linearly dependent.

Ex: Let $S_1 \subseteq S_2 \subseteq \mathbb{R}^n$.

If S_2 is linearly independent, then S_1 is also linearly independent.

How many Solutions ?

i) Row echelon Form

inconsistent pivot = ∞ consistent = !

free var = unique

→ Superset of any linearly dependent set is linearly dependent.

→ Subset of any linearly independent set is linearly independent.

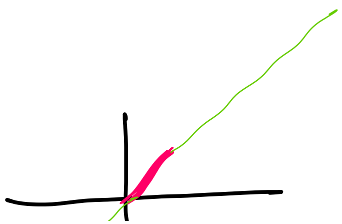
Ex: Let S be a linearly independent set in $\mathbb{R}^n$.

$S = \{v_1, \dots, v_k\}$ let $v \in \mathbb{R}^n$ s.t. $v \notin S$.

$S \cup \{v\}$ is linearly dependent if & only if $v \in \text{Span}\{S\}$

Pf: Assume that $v \in \text{span}(S)$

$\Rightarrow \exists\ \alpha_1, \dots, \alpha_k$ s.t.



$\Rightarrow \exists \ldots, \ldots_k$

$v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k$

$\Rightarrow \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k \boxed{-} v = 0$

$\Rightarrow \{v_1, \ldots, v_k\} \cup \{v\}$ is lin. dep.

Converse: $S \cup \{v\}$ is lin. dep.

To prove: $v \in \text{span}(S)$

$\exists$ scalars $\alpha_1, \ldots, \alpha_k$ and β, not all zero simultaneously,

s.t. $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k + \beta v = 0$

case (i) $\beta = 0$ ruled out

Some of $\alpha_1, \ldots, \alpha_k$ has to be not equal to zero.

$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k = 0$

case (ii) $\beta \neq 0$

$v = -\frac{\alpha_1}{\beta} v_1 - \frac{\alpha_2}{\beta} v_2 \cdots - \frac{\alpha_k}{\beta} v_k \Rightarrow v \in \text{Span}\{v_1, \ldots, v_k\}$

$v \in \text{span}(S)$ ☒

Independence - dimension inequality.

- Let $\{x_1, \ldots, x_k\} \subseteq \mathbb{R}^n$ be a linearly independent set.

  $\Rightarrow \boxed{k \leq n}$

- Any linearly independent collection of vectors in $\mathbb{R}^n$ can have at most n vectors.

- Any collection of $(n+1)$ vectors in $\mathbb{R}^n$ is always linearly dependent.

$\alpha_1 v_1 + \alpha_2 v_2 \cdots \alpha_n v_n = 0$

$\vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \alpha_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Span of $\vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is a line in

– Linear dependence /independence

– examples

Independence - dimension inequality.

- Let $\{x_1, \dots, x_k\} \subseteq \mathbb{R}^n$ be a linearly independent set.

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- Any linearly independent collection of vectors in $\mathbb{R}^n$ can have at most n vectors.
- Any collection of $(n+1)$ vectors in $\mathbb{R}^n$ is always linearly dependent.

— Linear dependence /independence

— examples

Independence - dimension inequality

Let $\{x_1, \dots, x_k\} \subseteq \mathbb{R}^n$ be linearly independent.

Then $k \leq n$

Proof: By induction on n.

$\boxed{\text{Base case: } k=1}$

Let $\{x_1, \dots, x_k\} \subseteq \mathbb{R}^1$

WLOG $x_1 \neq 0$

$x_2 = \boxed{\frac{x_2}{x_1}} \cdot x_1$, $x_3 = \boxed{\frac{x_3}{x_1}} \cdot x_1$ $\dots$, $x_k = \boxed{\frac{x_k}{x_1}} x_1$

$\Rightarrow$ x_i is a scalar multiple of x_1 for $i = 2, \dots, k$

which contradicts the assumption $\{x_1, \dots, x_k\}$ is linearly independent. $\Rightarrow k = 1$.

Assume for $n \geq 2$, independence - dimension inequality holds for $(n-1)$.

Let $\{x_1, \dots, x_k\} \subseteq \mathbb{R}^n$ be linearly independent.

To prove: $k \leq n$ $(n-1)\{[\,:\,]$

Partition $x_i = \begin{pmatrix} y_i \\ \alpha_i \end{pmatrix}$ $i = 1, 2, \dots, k$ components $\leftarrow x_i$

where $y_i \in \mathbb{R}^{n-1}$, $\alpha_i \in \mathbb{R}$

case (i): $\alpha_1 = \alpha_2 = \dots = \alpha_k = 0$

claim: $\{y_1, \dots, y_k\} \subseteq \mathbb{R}^{n-1}$ is lin. indep. $\leftarrow x_i$

$$\sum_{i=1}^{k} \beta_i y_i = 0 \Rightarrow \sum_{i=1}^{k} \beta_i x_i = 0$$

$$\Rightarrow \beta_1 = \beta_2 = \dots = \beta_k = 0$$

Induction principle $\Rightarrow k \le n-1 \Rightarrow \boxed{k \le n}$

case (ii): Suppose all of α_i's are not zero.

let $\alpha_j \neq 0$

Define $z_i = y_i - \frac{\alpha_i}{\alpha_j} y_j \in \mathbb{R}^{n-1}$ for $i = 1, 2, \dots, j-1$

$z_i = y_{i+1} - \frac{\alpha_{i+1}}{\alpha_j} y_j \in \mathbb{R}^{n-1}$ for $i = j, \dots, k-1$

claim: $\{z_i\}$ is linearly independent.

$$\sum_{i=1}^{k-1} \beta_i z_i = 0 \Rightarrow \boxed{\beta_i = 0 \ \forall i}$$

$$= \sum_{i=1}^{j-1} \boxed{\beta_i} \boxed{\begin{bmatrix} y_i \\ \alpha_i \end{bmatrix}} + \boxed{\gamma} \boxed{\begin{bmatrix} y_j \\ \alpha_j \end{bmatrix}} + \sum_{i=j+1}^{k} \boxed{\beta_{i-1}} \boxed{\begin{bmatrix} y_i \\ \alpha_i \end{bmatrix}} = 0$$

How to find linearly dependent or

$$\left\{ \begin{bmatrix} 8 \\ -4 \\ 1 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix} \begin{bmatrix} -13 \\ 6 \\ 3 \end{bmatrix} \right\} = V$$

Is V linearly independent?

$$\alpha_1 \begin{pmatrix} 8 \\ -4 \\ 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 3 \\ -2 \\ 5 \end{pmatrix} + \alpha_3 \begin{pmatrix} -13 \\ 6 \\ 3 \end{pmatrix} =$$

$\rightarrow$ Row echlon Form & Sol the solutions.

Subset: a collection of vectors

$$V = \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \text{ in } \mathbb{R}^4 \;\middle|\; x - 4z = 0 \right.$$

i) contain ϕ

ii) if you add 2 vectors in the you get a vector also in the

iii) If you multiply any vector in by a scalar, you get a vector with $\gamma = -\frac{1}{\alpha_j}\left(\sum_{i=1} \beta_i \alpha_i + \sum_{i=j+1} \ldots\right)$

$\therefore \{x_1, \ldots, x_k\}$ is linearly independent.

$\begin{pmatrix} y_1 \\ \alpha_1 \end{pmatrix} \cdots \begin{pmatrix} y_k \\ \alpha_k \end{pmatrix}$

$\Rightarrow \beta_1 = \cdots = \beta_{k-1} = \gamma = 0$

$\Rightarrow \beta_1 = \cdots = \beta_{k-1} = 0$

$\Rightarrow \{z_1, \ldots, z_{k-1}\} \subseteq \mathbb{R}^{n-1}$ is linearly independent.

$\Rightarrow k-1 \leq n-1$ (Assumption of independence dimension inequality in the induction step).

$\Rightarrow \boxed{k \leq n}$ ∎

set.

$\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix}$ $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}$ $\vec{a} + \vec{b} =$

$(a_1 + b_1) - 4(a_3 + b_3)$

In $\mathbb{R}^n$, any collection of $(n+1)$ vectors is linearly dependent.

Basis of a span means the vector are linearly independent in the v

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<u>Basis</u>:

A set $B = \{x_1, \ldots, x_n\} \subseteq \mathbb{R}^n$ of linearly independent vectors is called as a basis of $\mathbb{R}^n$. $x_1, \dots, x_n$ are called as basis vectors.

<u>Thm</u>: Any vector $v \in \mathbb{R}^n$ can be uniquely expressed as a linear combination of basis vectors.

<u>Proof</u>: $v \in \mathbb{R}^n$

$\{v, x_1, x_2, \dots, x_n\}$ is a linearly dependent set.

$\Rightarrow \exists$ scalars $\alpha_1, \alpha_2, \dots, \alpha_n$ & β, not all zero s.t.

$\alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_n x_n + \beta v = 0$

<u>case (i)</u> $\beta = 0$

$\Rightarrow \alpha_1 x_1 + \dots + \alpha_n x_n = 0$ & not all α_i's are zero.

$\Rightarrow \{x_1, \dots, x_n\}$ lin. dep.

which is a contradiction.

<u>case (ii)</u>: $\beta \neq 0$

$\Rightarrow v = -\frac{\alpha_1}{\beta} x_1 - \frac{\alpha_2}{\beta} x_2 \dots - \frac{\alpha_n}{\beta} x_n$

$\Rightarrow v$ is expressed as a linear combination of x_i's

$A = \{v_1, v_2 \dots v_n\}$ forms a basis in $\mathbb{R}^n$ i[f] the matrix A has a pivot in every row

Find the basis of subspace?

$V = \text{span}\left\{\begin{pmatrix}1\\-2\\2\end{pmatrix}, \begin{pmatrix}2\\-3\\4\end{pmatrix}, \begin{pmatrix}0\\4\\0\end{pmatrix},\right.$

Uniquess :

Assume $v = \sum_{i=1}^{n} \alpha_i x_i = \sum_{i=1}^{n} \beta_i x_i$

To prove $\alpha_i = \beta_i \quad \forall i = 1, 2, \ldots, n$

$$\sum_{i=1}^{n} \alpha_i x_i - \sum_{i=1}^{n} \beta_i x_i = 0$$

$$\Rightarrow \sum_{i=1}^{n} (\alpha_i - \beta_i) x_i = 0$$

$$\Rightarrow (\alpha_i - \beta_i) = 0 \quad \forall i = 1, 2, \ldots, n$$

$$\Rightarrow \alpha_i = \beta_i \quad i = 1, 2, \ldots, n$$

$\blacksquare$

Let $B = \{x_1, \ldots, x_n\} \subseteq \mathbb{R}^n$ be a basis.

$$\mathbb{R}^n = \mathcal{L}(B)$$

Ex: $\mathbb{R}^n$

$B = \{e_1, e_2, \ldots, e_n\}$ std. basis.

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$$

$$x = x_1 e_1 + x_2 e_2 + \cdots + x_n e_n$$

Ex: $\mathbb{R}^n$

Ex. " is a basis $\mathbb{R}$.

$B = \{e_1, e_1+e_2, \dots, e_1+e_2+\dots+e_n\}$

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \alpha_1 e_1 + \alpha_2(e_1+e_2) + \dots + \alpha_n(e_1+e_2+\dots+e_n)$$

- Independence - dimension inequality
- Basis
- $\mathbb{R}^n = \mathcal{L}(B)$
- dimension $(\mathbb{R}^n) = n$

  $\quad\quad\|$

  # of elements in B.

Null Space & Rank:

Rank: dim of column space A (Bas

Nullity: dim of null space A

Rank - nullity theorem: $\boxed{\text{rank}(A) + \text{null}(}$

Example:

1). let $\mathcal{F}(\mathbb{R})$ be a collection of all real-valued functions and W_n be a subset of

$\mathcal{F}(\mathbb{R})$ defined by, $\boxed{W = \{ f \in \mathcal{F}(\mathbb{R}) : f(0) = n \}}$.

For what values of n, W_n will be a subspace of the vector space $\mathcal{F}(\mathbb{R})$

Sol:- [zero]

2). How many subspace does $\mathbb{R}^2$ have? ∞ many

3). $W_1 = \left\{ x \in \mathbb{R}^2 : x = \begin{bmatrix} 2n \\ -\frac{3n}{2} + 2 \end{bmatrix}, n \in \mathbb{R} \right.$ b) $W_2 = \left\{ x \in \mathbb{R}^2 : x = \begin{bmatrix} -\frac{3n}{2} \\ n \end{bmatrix}, n \in \mathbb{R} \right\}$

c). $W_3 = \left\{ x \in \mathbb{R}^2 : x = \begin{bmatrix} m-n \\ \frac{3m}{2} + \frac{n}{2} \end{bmatrix}, m, n \in \mathbb{R} \right.$ d) $W_4 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ not a subspace of $\mathbb{R}^2$. (null & closure properties)

4). $S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix} \begin{bmatrix} 2 \\ 6 \\ -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \right\}$ dim of $V = ?$

$$\begin{bmatrix} 1 & 1 & 2 & 1 \\ 2 & 4 & 6 & 1 \\ 1 & -3 & -2 & 3 \end{bmatrix} \xRightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 1 & 3/2 \\ 0 & 1 & 1 & -1/2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad S = \left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 4 \\ -3 \end{pmatrix} \right\} \Rightarrow 2$$

5) $M_2(\mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{R} \right\}$ be the vector space over $\mathbb{R}$ of all 2×2 matrix

Let W be the subspace of $M_2(\mathbb{R})$

$W = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad W = \begin{bmatrix} a & b \\ b & d \end{bmatrix} = 3$ parameter

$$W = a \underline{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}} + b \underline{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}} + d \underline{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}$$