Predictive Modelling of the Soil Water Profile for Effective Monitoring of Plant Growth Conditions

Choose-Focus-Analyse Exercise

BT5051: Transport Phenomena in Biological Systems

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Contents

1 Objective

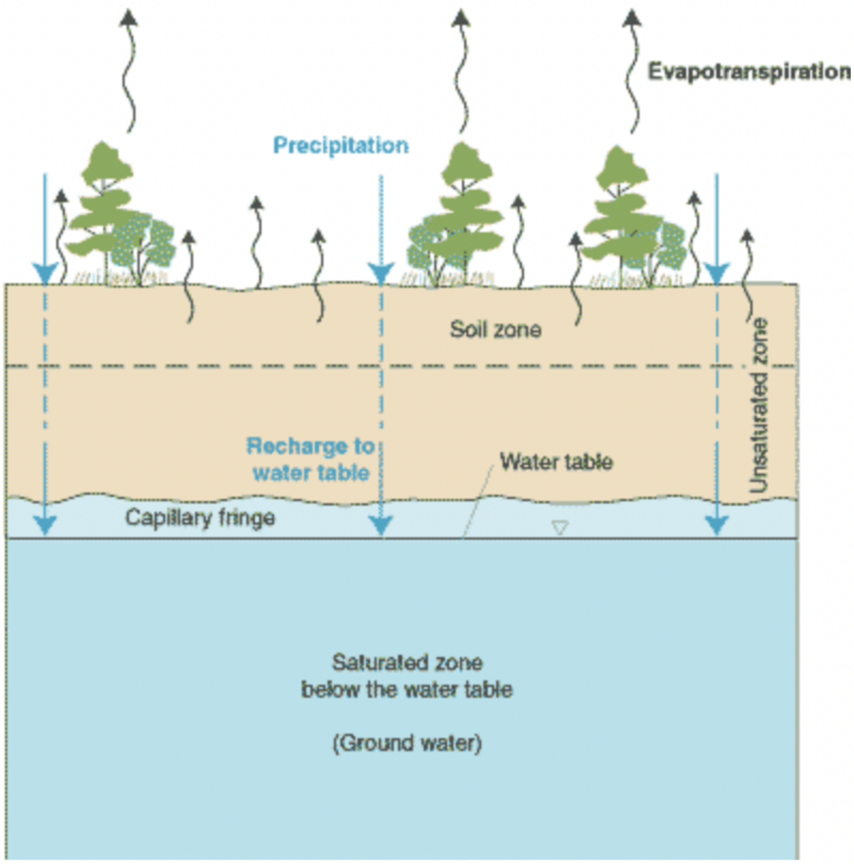
The aim of this study is to create an intuitive and powerful model that can help us computationally generate an accurate water profile depicting capillary action in the lower strata of the soil, depending on factors such as the soil particle size distribution, the heterogeneity of the soil, and the depth of the water bed. The water profile should tell us the volume of water at every height per unit volume of space in that volume.

2 Introduction

Coming from a place like Bangalore, I have seen various kinds of pollution in my surroundings, such as lake water pollution and air pollution. I've always considered it very important to fix this issue. Motivated by these problems, I decided to zero in on the concept of soil pollution, since I felt intuitively that there was something I could do about it.

Soil pollution tends to be caused because of soluble fertilisers and other chemicals being dissolved into the groundwater and therefore being absorbed into plants. Therefore, soil water is a very important component of any analysis related to soil pollution.

Before we formulate our model in order to proceed with our analysis, we first need to understand how the soil layers are structured and what might affect the percolation of water in the soil. Let's look at a basic diagram that entails what the dynamic soil structure might look like in normal conditions.



Source: https://pubs.usgs.gov/circ/circ1186/html/gen$_f$acts.html
Fig 1: A big picture view of the soil water profile

In order to analyse the soil water profile in any given situation, we need to consider two major components to the water profile – capillary action and rainwater infiltration. As a part of this study, we will only be analysing the capillary action. Rainwater infiltration has been touched upon in Section 5 on 'Further Avenues'.

Capillary action at the water bed: Every single soil profile will have a water bed somewhere deep down in the soil strata. The distance of this layer from the surface varies from place to place. For example, closer to streams, the groundwater layer may be closer to the surface, while it may be very far from the surface somewhere in the desert.

Groundwater isn't static. It moves very slowly, and its motion has been modelled according to Darcy's Law, to predict the rate of groundwater recharge by stream water and vice versa. The motion depends on the gradient of pressure as stated in Darcy's Law.[9]

Now, groundwater motion is definitely an important factor for the motion of nutrients in the soil and can influence quality of growth of plants as well. However, for the purpose of this model, we will only be modelling the vertical water profile, and will therefore ignore the horizontal motion of water. Furthermore, since our model mostly takes into account the motion of water during a particular day, we can safely assume even horizontal motion is negligible, considering how slow the motion tends to be in reality.

The level of groundwater varies from season to season, depending on various seasonal factors.[8] The fact is that even during a particular day, on account of heavy rainfall, groundwater level may rise and can be very dynamic. For example, within IIT Madras, there are small lakes near the Students' Activity Center which fill up during heavy rains and remain empty during the dry season. What this means is that during the dry season, the level of groundwater is below the land, while otherwise the level is above the land.

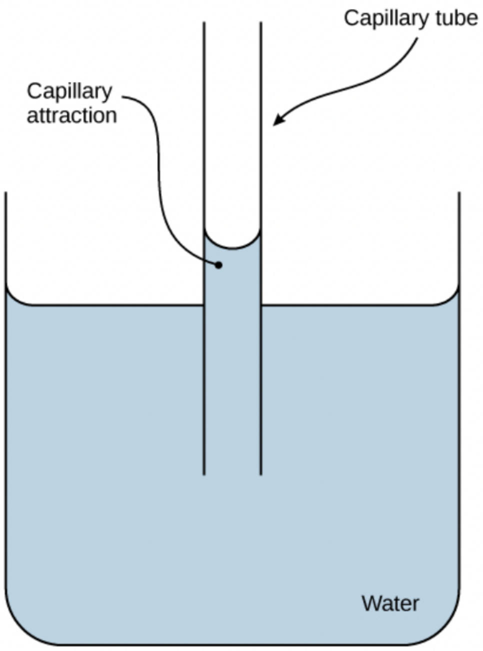
It's quite natural to imagine how the level of groundwater can vary with the level of rainfall. When it rains, percolation of rainwater to the bottom can increase the amount of groundwater and raise its level. However, we'll neglect the presence of rains for the purpose of the current model.

An additional aspect with regard to groundwater that can be important in certain cases is the capillary action of the soil near the water. Capillary action is a well-studied phenomenon related to surface tension, and the rise of the water in what are known as capillary fringes[8] near the water bed result from capillary action at the bed. Knowing that the soil is essentially a structure full of particles of different sizes, with small interstitial spaces, it becomes clear why the rise of water through capillary action might result in a way analogous to the rise of water in a small capillary tube. We will try to mathematically model this capillary rise and calculate the amount of water at different levels above the water bed, on account of capillary action.

These are not the only factors influencing the water profile. There are factors such as evaporation of water, which play a role in loss of water in various areas. Apart from this, it's natural to assume some quantity of water is being used up by the plant life in the area. However, both these factors have been neglected for the purpose of this study.

3 Background

3.1 Capillary Action



Source: https://bio.libretexts.org/Bookshelves
Fig 2: Capillary action

Capillary action is a well-studied phenomenon[10], and I will illustrate its fundamental principles here.

The principle behind this phenomenon is surface tension. Surface tension is a property of all fluids, which is a constant for a given liquid. We denote surface tension by γ.

If we consider the equilibrium height to which the water rises to be h, the radius of the capillary tube to be r, and the angle of contact to be θ. Thus, due to the excess pressure by the surface:

$$P_0 - P_A = \frac{2\gamma cos\theta}{r}$$

Now due to the gauge pressure offered by the water column,

$$P_A + \rho gh = P_B = P_0$$

Hence,

$$P_0 - P_A + P_A + \rho gh = \frac{2\gamma cos\theta}{r} + P_0$$

$$\Rightarrow \rho gh = \frac{2\gamma cos\theta}{r}$$

$$\Rightarrow \boxed{h = \frac{2\gamma cos\theta}{\rho gr}}$$

Thus,

$$\boxed{h \propto \frac{1}{r}}$$

We will use this result to model the capillary fringes.

3.2 Average soil particle size

The way we will tackle this is that every stratum of soil will have a specific average particle size. This may of course vary with depth. How do we define the average particle size?

Average particle size: At any specific location (x,y,z), we define the average particle size by considering an infinitesimal cuboid at the location (x,y,z). We then sum up the size of all particles in that cuboid and divide by the number of particles.

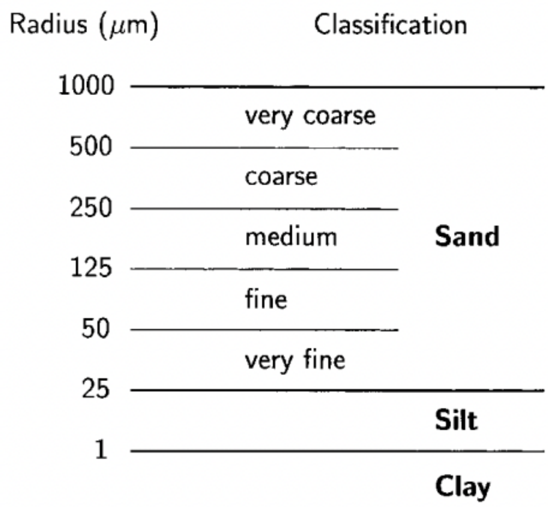
$$d_{avg} = \frac{\Sigma d_i}{N}$$

$$d_i = \text{diameter of the infinitesimal particles}$$
$$N = \text{number of particles}$$
$$d_{avg} = \text{average particle diameter}$$

This means that the average particle diameter can vary at different points in the soil, due to different compositions of the soil components at these different points.

In order to simplify things, we assume that the soil particle size remains approximately constant at the same horizontal level. Properties tend to be similar along the horizontal level, since soil forms in layers. The average size, however, may vary with depth.

Now, we can have a look at the general nomenclature of soils of different average particle sizes.



Source: https://www.ars.usda.gov/arsuserfiles/20361500/pdf$_p$ubs/P1788.pdf
Fig 3: Particle-size classification system

3.3 Depth of water bed

We will assume that the upper surface of the water bed has a depth of D below the surface.

Of course, the value of D varies with time in our model, as explained earlier.

3.4 Rainfall measurement

We define the rainfall R as volume of rainfall per unit area.

$$R = \frac{\text{Volume of rainfall}}{\text{Area of cross section of soil}}$$

We will consider the unit of rainfall to be in 'cm' or 'mm'. It is a unit of length, as expected from the dimensional analysis:

$$[R] = \frac{L^3}{L^2} = [L]$$

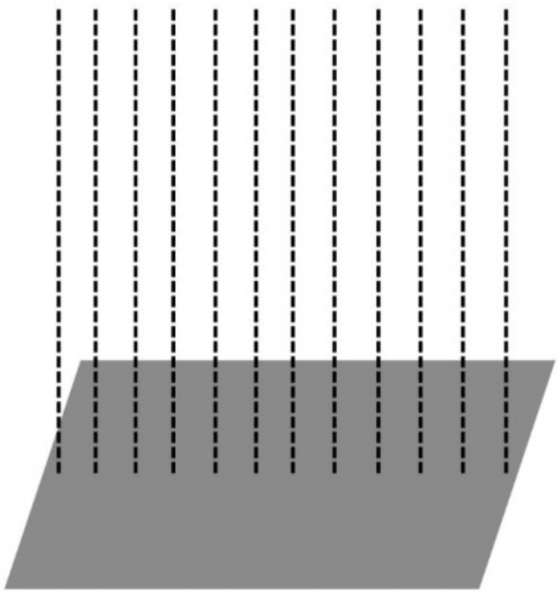


Fig 4: Rainfall per unit area

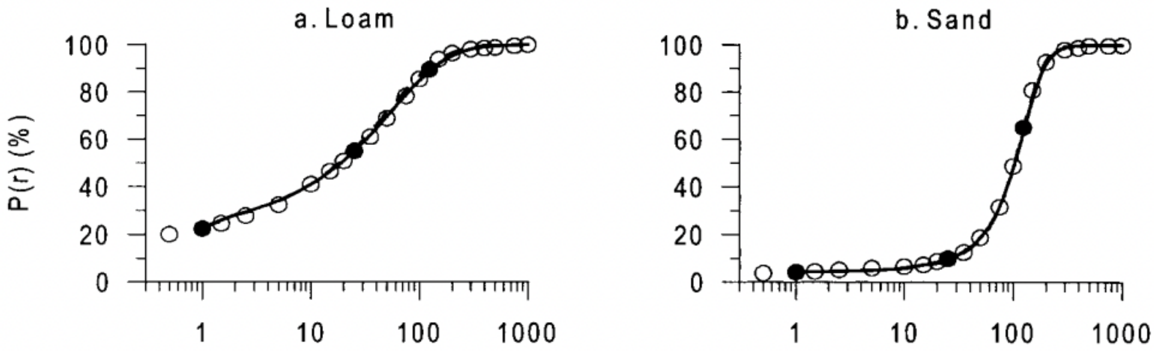
Although we shall not be using rainfall for the current study, a future study regarding rainfall percolation can make use of it, so this is a beneficial background to have.

4 Capillary Fringe Model

4.1 Soil Distribution Function

The average particle size discussed earlier forms an important measure that will help us model the distribution of capillary radius. Apart from this, we will also need an overall particle size distribution curve representing the composition of various particles at a given point in space.

The particle size distribution can be modelled from just a few data points as a logistic curve.[2]



Source: https://www.ars.usda.gov/arsuserfiles/20361500/pdf$_p$ubs/P1788.pdf

Fig 5: Logistic distribution of the Cumulative Probability Distribution functions for Loamy and Sandy soil particle size

$P(r)$ is the cumulative probability distribution function; i.e., it represents the fraction of those particles having a radius $\leq r$.

Thus, the fraction of particles having radius within the range $(r, r + \Delta r)$ is given by

$$P(r + \Delta r) - P(r)$$

$$\lim_{\Delta r \to 0} \frac{P(r + \Delta r) - P(r)}{\Delta r} = \frac{dP}{dr}$$

Now, $\frac{dP}{dr}$ represents the probability distribution function (PDF). We can verify this since the fraction of particles with radius from a to b is given by

$$\int_a^b \frac{dP}{dr} dr = P(b) - P(a)$$

We consider $P(r)$ to be estimated as a logistic function, based on the diagrams above[2]. Thus, it can be estimated as

$$P(r) = \frac{L}{1 + e^{-k(r - r_0)}}$$

$$\Rightarrow \frac{dP}{dr} = \frac{-L}{(1 + e^{-k(r-r_0)})^2} e^{-k(r-r_0)}(-k)$$

$$\Rightarrow PDF = \frac{dP}{dr} = \frac{Lke^{-k(r-r_0)}}{(1 + e^{-k(r-r_0)})^2}$$

The plot of the RHS in the above equation is visualised,

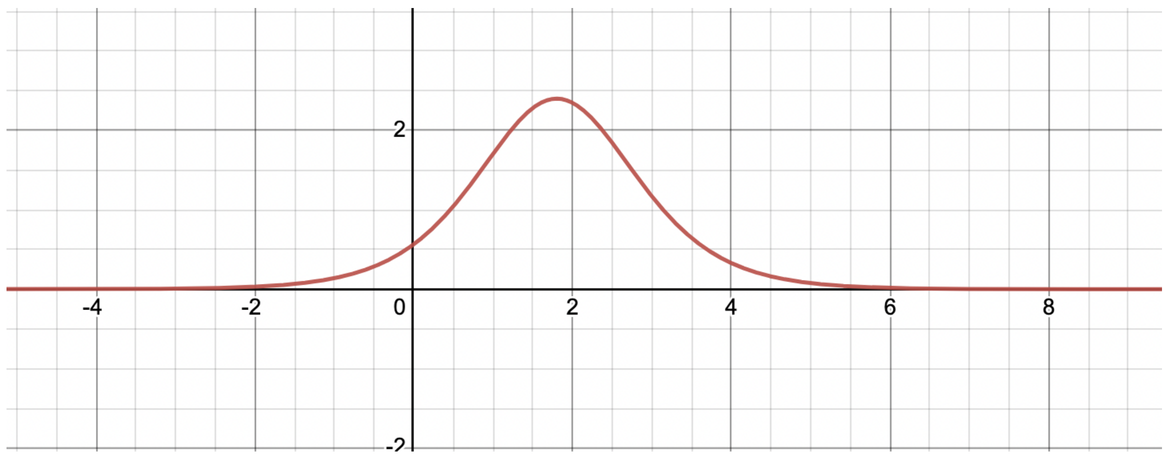


Fig 6: The probability distribution function approximates to a Gaussian function

Hence, we make the following claim:

The particle size distribution $\frac{dP}{dr}$ approximately follows a Gaussian distribution, with a certain 'mean' and 'standard deviation' value.

This claim should be true to a good approximation at every point. If we reduce the standard deviation, the number of particles having size closer to d_{avg} increases.

Hence,

$$PDF = \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(r-\mu^2)}{2\sigma^2}}$$

$$\mu = d_{avg}$$

$$\sigma = \text{standard deviation}$$

We now define a new term known as the heterogeneity factor η for the given soil sample. It represents how heterogeneous the soil is. Mathematically -

$$\eta = \sigma$$

i.e., η = standard deviation of the system

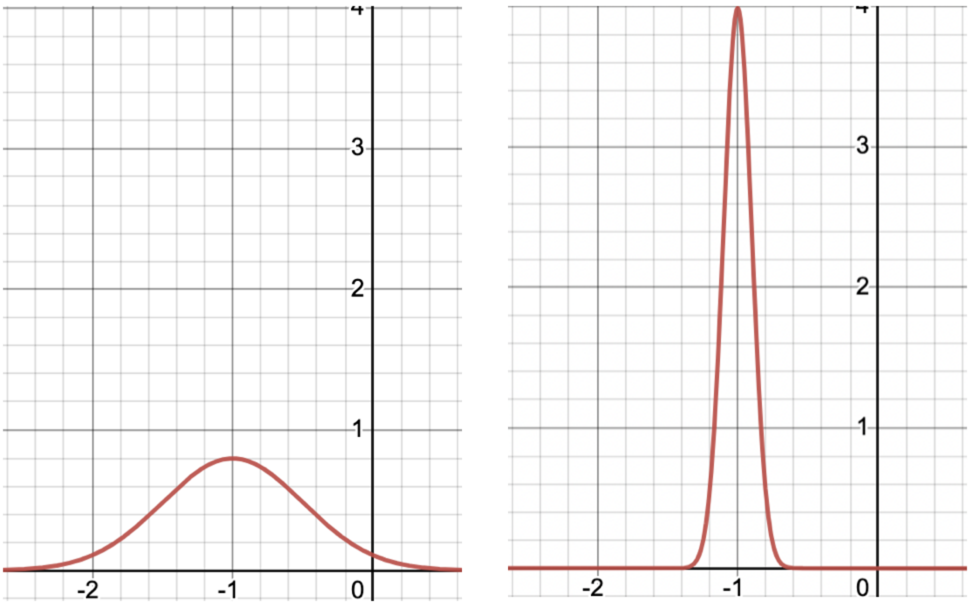


Fig 7: The nature of the Gaussian with 0.5 and 0.1 heterogeneity factors respectively. Clearly, decreasing the heterogeneity factors plays a role in concentrating more soil particles closer to the mean value.

Thus, if we define a soil distribution function (SDF), representing probabilities of finding soil particles of different radii r at a given location, is defined solely by the factors d_{avg} and η.

$$\boxed{SDF = \frac{1}{\sqrt{2\pi\eta^2}} e^{\frac{-(r-d_{avg})^2}{2\eta^2}}}$$

4.2 Variation of d_{avg} over depth

We have modelled the particle size distribution at a specific point in the soil. Now, we can model how the average particle size varies with depth in the soil.

Since soil forms in layers, the lower layers tend to have more pressure on them and are therefore more dense. We will therefore consider these lower layers to have particles of smaller radius than the higher layers. We also assume that the value of d_{avg} changes only slightly between the topmost and bottommost layers. So, we will model the variation to be linear.

Let $d_{avg,0}$ be the average diameter at the topmost layer, and m be some constant.

We model the average particle diameter as,

$$d_{avg}(z) = d_{avg,0} + mz$$

z = vertical coordinate in the soil, measured from the topmost layer

4.3 Analysis of Capillary Action

In order to analyse the capillary action, let us recall the relation for height of capillary rise in Section 3.1:

$$h = \frac{2\gamma \cos\theta}{\rho g r}$$

We assume that the water is more or less pure, and temperature is somewhere in the range of $20°C - 35°C$.

In these circumstances,

$$\gamma \approx 0.07275 J/m^2$$

$$\rho = 997 kg/m^3$$

$$g = 9.8 m/s^2$$

Thus,

$$h = \frac{0.00001489161 \cos\theta}{r}$$

$$\Rightarrow \boxed{h(\text{mm}) = \frac{29.78322 \cos\theta}{d(\text{mm})}} \text{ (Equation 3)}$$

Now we need to consider the particle size, since we claim that the particle size determines the capillary radius.

Since h is of the order of mm, we will assume that d_{avg} remains constant in this range. That is,

$$d_{avg} = d_{avg,0} + mD$$

We first consider the homogeneous case, where all particles are of constant size in the concerned region.

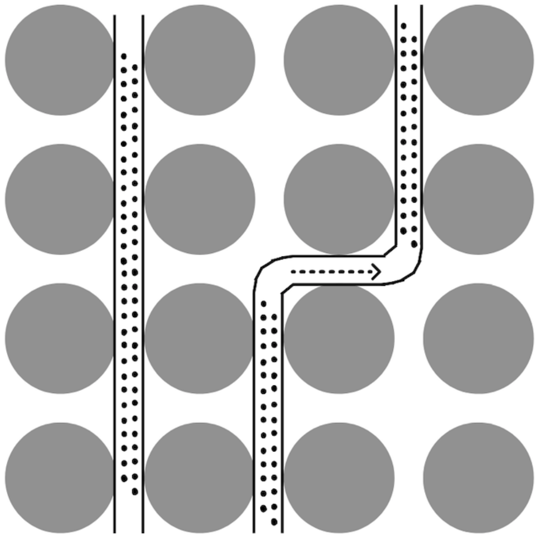


Fig 8: The water flows through interstitial spaces of soil particles, and these spaces are modelled as capillaries

Notice that the path may not necessarily be linear, it may in fact be bent. However, the relation for maximum height attained by the water remains the same even in the presence of bends (given our ideal assumption that surfaces are smooth and the contact angle and capillary radius remains constant along the path).

Now in this case, we model the diameter of interstitial gaps to be directly proportional to the particle radius, with proportionality constant K. That is,

$$d_{capillary} = K d_{particle}$$

However, the above diagram assumes soil particles to be homogeneous ($\eta = 0$). In reality, there will be some variation in the sizes of the particles. It may look something like this:

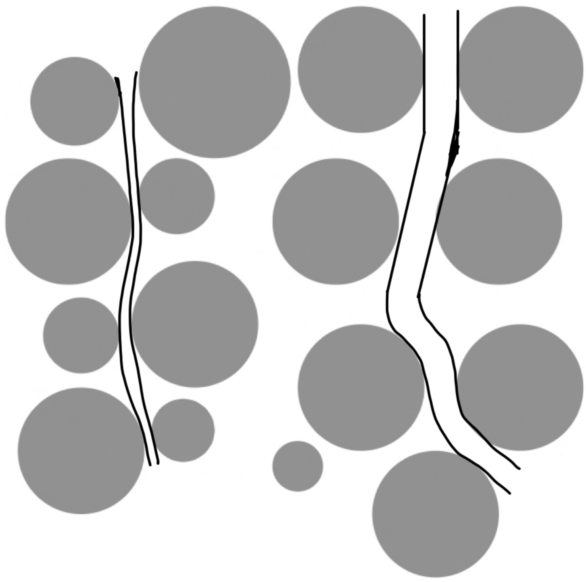


Fig 9: The particles are actually heterogeneous in their size distribution, and therefore there can be a heterogeneous distribution of capillary radii.

Thus, the capillary diameters still depend on the diameters of particles near them, and therefore there is a certain distribution of capillary diameters in a given region.

Now, $d_{capillary} = Kd_{particle}$, and we also saw that $d_{particle}$ has a Gaussian distribution. The X-axis for the distribution of $d_{particle}$ therefore has to be scaled by a factor of K in order to get the new distribution for $d_{capillary}$. This means η becomes $K\eta$ and d_{avg} becomes Kd_{avg}.

We can also show this because if X is a random variable following a Gaussian distribution $N(d_{avg}, \eta^2)$, then KX is a random variable following $N(Kd_{avg}, K^2\eta^2)$. [5]

So we define the soil capillary distribution function (SCDF) as follows:

$$SCDF(r) = \frac{1}{\sqrt{2\pi K^2\eta^2}} e^{\frac{-(r-Kd_{avg})^2}{2K^2\eta^2}}$$

$$where\ r = \text{radius of interstitial capillary}$$

Let's define the porosity of the soil,

$$\epsilon = \frac{\text{total area covered by capillaries}}{\text{area of soil}}$$

The expected value of capillary cross-sectional area is given by,

$$E[A_{\text{capillary}}] = \int_{-\infty}^{\infty} \pi r^2 SCDF(r)dr$$

Here, $SCDF(r)dr$ represents the fraction of capillaries having a radius in $(r, r+dr)$.

Total area of all capillaries is,

$$E[A_{capillary}] \times \text{No. of capillaries}$$

Hence, porosity is given as,

$$\epsilon = \frac{E[A_{capillary}] \times \text{No. of capillaries}}{\text{Unit area of soil}}$$

$$\Rightarrow \frac{\text{No. of capillaries}}{\text{Unit area of soil}} = \frac{\epsilon}{E[A_{capillary}]}$$

$$\Rightarrow \boxed{\text{No. of capillaries}/m^2 = \frac{\epsilon}{\int_{-\infty}^{\infty} \pi r^2 SCDF(r)dr}}$$

We want to find the water concentration as a function of height. At a given height, we define a new term called the Soil Water Content (SWC).

$$SWC(h) = \frac{\text{Area of capillaries with water in them}}{\text{Total soil area}}$$

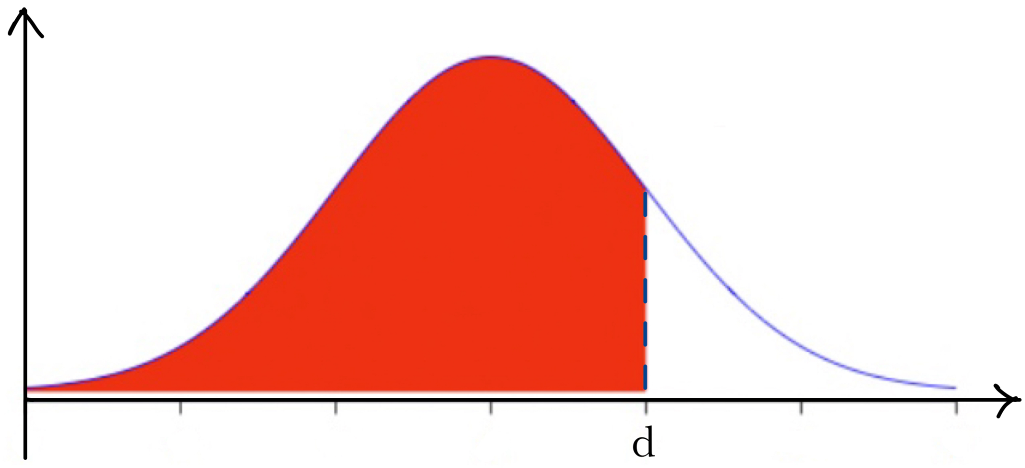
Now, note that if water has reached a particular height for a given diameter value, it implies that all smaller diameter capillaries would've reached at least that height.

Thus, if we are considering all the capillaries that reach a specific height h (at least), then we are considering all capillary diameters lesser than or equal to

$$d_{max} = \frac{29.78322\cos\theta}{h}$$

Hence,

$$d \leq \frac{29.78322\cos\theta}{h}$$



Source: https://freakonometrics.hypotheses.org/9404

Fig 10: The shaded region is essentially the cumulative distribution, and represents all the capillaries having diameter $\leq d$. Only these capillaries would thus contribute to the soil water content at that particular height.

Now, we try to utilise this information and calculate the Soil Water Content (SWC) at that height.

Since $SCDF(r)$ is effectively a representation of the fraction of all capillaries having specific diameters, hence $SCDF(r)\times$ No. of capillaries/m^2 would represent the number of capillaries having specific diameters. Let us denote the No. of capillaries/m^2 as A for now. Then, from equation (1),

$$A = \text{No. of capillaries}/m^2 = \frac{\epsilon}{\int_{-\infty}^{\infty} \pi r^2 SCDF(r)dr}$$

$$\Rightarrow A \cdot \int_{-\infty}^{\infty} \pi r^2 SCDF(r)dr = \epsilon$$

$$\Rightarrow \int_{-\infty}^{\infty} \pi r^2 SCDF(r)Adr = \epsilon \text{ (Equation 2)}$$

As explained earlier, $SCDF(r)Adr$ is essentially the total number of capillaries with radius $(r, r+dr)$. Thus, $\pi r^2 SCDF(r)Adr$ is the total area of all capillaries with radius $(r, r+dr)$. Finally, $\int_{-\infty}^{\infty} \pi r^2 SCDF(r)Adr$ is the total area of all capillaries (per unit area of soil). Not surprisingly, this is the definition of porosity (ϵ).

In a similar fashion, to find the total area of all capillaries which carry water, we have to calculate the same integral, except evaluate it from $-\infty$ to d.

$$\text{Area of capillaries carrying water}/m^2\text{soil} = \int_{-\infty}^{d} \pi r^2 SCDF(r)Adr$$

As we saw earlier, we have defined SWC(h) precisely as the LHS of the above equation. Hence, we can plug in appropriate values of d and A from equations (2) and (3) above to get:

$$SWC(h) = \frac{\epsilon \cdot \int_{-\infty}^{d} \pi r^2 SCDF(r)dr}{\int_{-\infty}^{\infty} \pi r^2 SCDF(r)dr}$$

$$\Rightarrow \boxed{SWC(h) = \frac{\epsilon \cdot \int_{-\infty}^{d} r^2 SCDF(r)dr}{\int_{-\infty}^{\infty} r^2 SCDF(r)dr}}$$

$$where\ d = \frac{29.78322 \cos\theta}{h}$$

Thus, we can now consider several values of h and compute the above expression for each of those values. As a result, we should be able to graph $SWC(h)$ as a function of h.

Before doing this, we will assume the angle-of-contact θ to be a constant at 0, and therefore, $\cos\theta = 1$.

Now we perform numerical integration using Python to plot the curve for $SWC(h)$. The code has been attached in Section (4) of this report.

4.3.1 Case Study: Silty Soils

Firstly, we assume the soil to be somewhat silty. As per Figure 3, the silty zone lies around where $d_{avg} = 0.01mm$. Also, we assume $K = 1/8 = 0.125$. Furthermore, we'll consider η to be a certain fraction of d_{avg}. According to the empirical rule in statistics (ref), the range $(\mu - 3\sigma, \mu + 3\sigma)$ is a 99.73% confidence interval. Now, we want to make sure that the entire major interval is in the positive zone, since it doesn't make sense for r to be negative (we initially assumed it to be a Gaussian distribution, but that was only an approximation). So we will ensure $\eta \leq \frac{d_{avg}}{3}$.

We also assume that its properties are not depth-dependent. Hence, d_{avg}, K, and η remain constant with depth.

Thus the values are,

$$d_{avg} = 0.01mm$$

$$K = 0.125$$

$$\eta = 0.2d_{avg} = 0.002$$

Based on these, we plot $SCDF(r)$ -

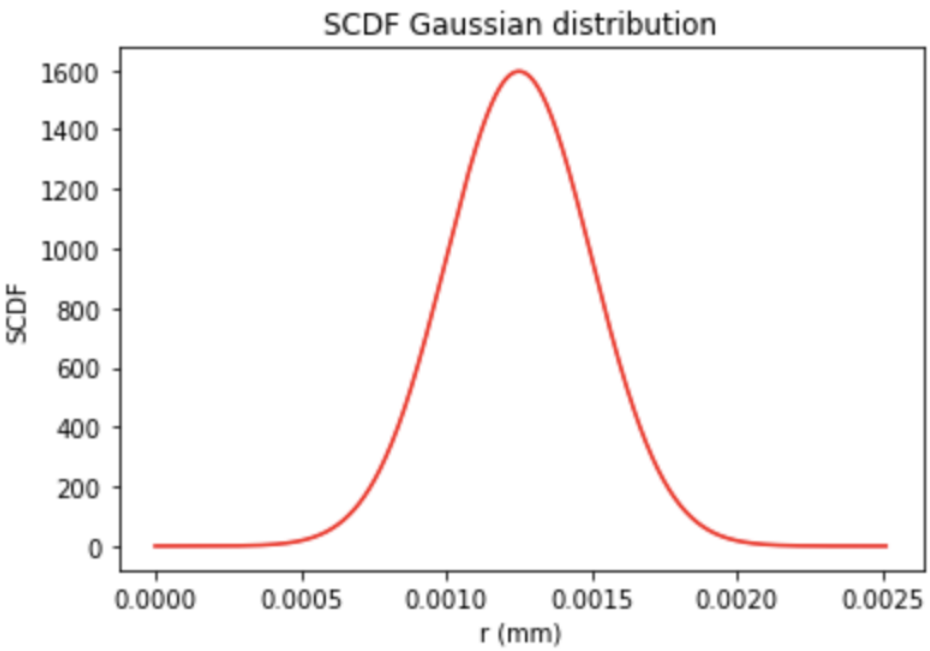


Fig 11: The Gaussian distribution SCDF(r) is plotted using matplotlib. It is centred at 0.00125 and has a standard deviation of 0.00025.

Further, we plot $r^2 SCDF(r)$ below -

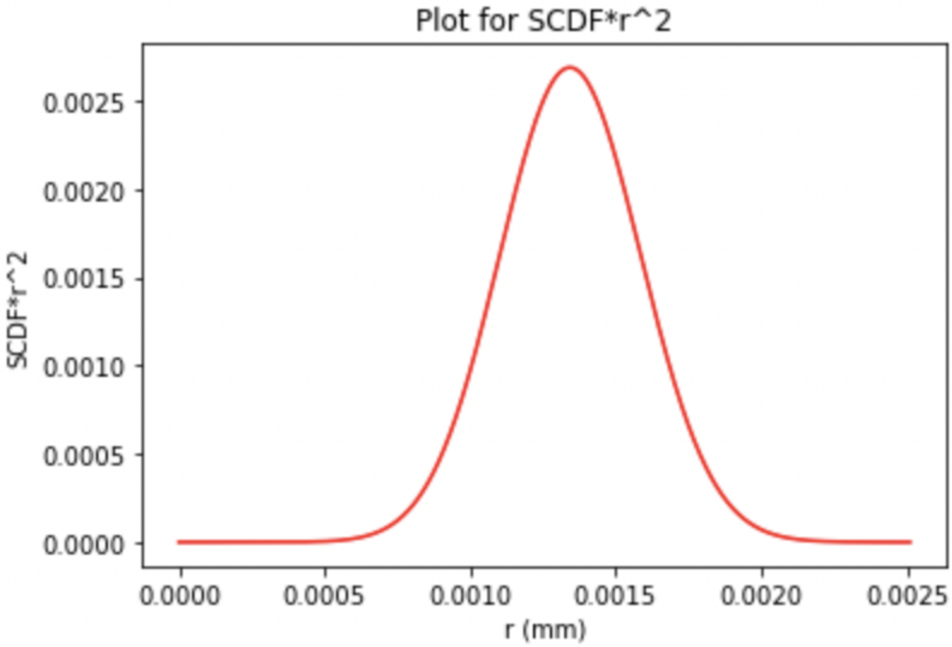


Fig 12: $r^2 SCDF(r)$ is plotted using matplotlib

Finally, the plot for $SWC(h)$ is made. We vary the value of h in the range $(d_{avg}, 2 \cdot \frac{29.78322}{Kd_{avg}})$. $SWC(h)$ is calculated for each h, and using the data points, we form the plot for $SWC(h)$.

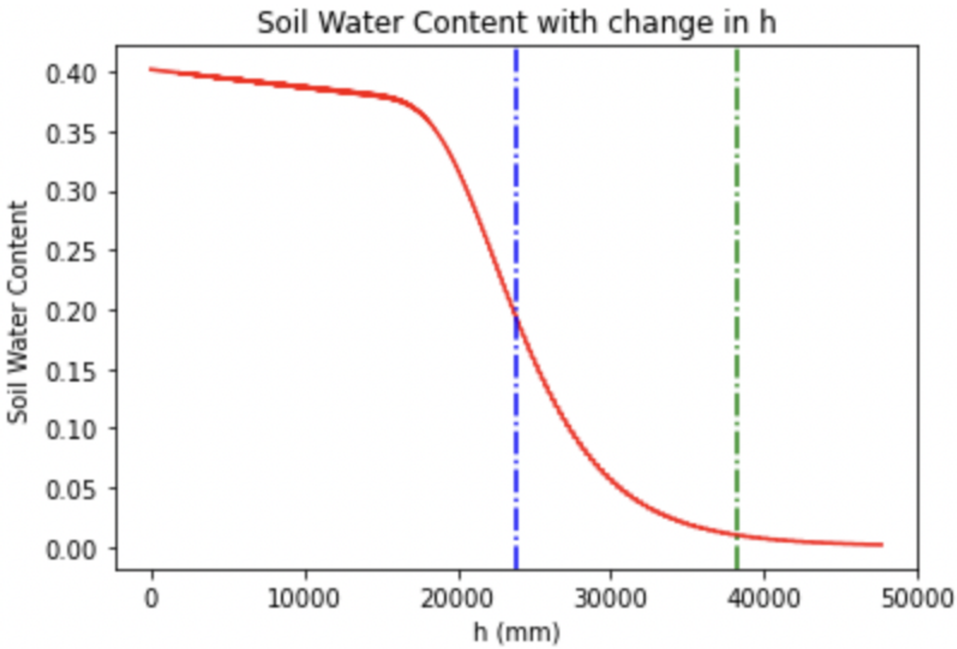


Fig 13: The Soil Water Content is found as a function of h. The blue line represents the height of rise corresponding to $d_{capillary} = Kd_{avg}$. The green line is the height beyond which $SWC \leq 0.01$.

$$h_{green-line} = 38188.29\text{mm}, \; h_{blue-line} = 23826.576\text{mm}$$

Now, we note some inferences:

- Silt has a very good water holding capacity, and manages to raise its ground water up to a height of 38 m above the groundwater level. This is quite good and proves why silt is an effective type of soil for plant growth.

- In silty regions which tend to have a decent amount of environmental humidity, our model will only be able to predict the contribution of the groundwater distribution effected by capillary action. To get the net groundwater distribution, we may also need to add up the contribution of the rainwater absorption in the upper layers. We'll talk more about this later.

Depth of the water bed tends to be very temporally dynamic, owing to the presence of extensive rainfall as well as evaporation in various areas. This causes it to rise and fall depending on the seasons.

It tends to be spatially dynamic as well – regions closer to lakes naturally have a higher groundwater level than those further from lakes.

Typically, the water bed depth in silty regions is in the range of 15-90m. Let's say the water bed depth is 20m.

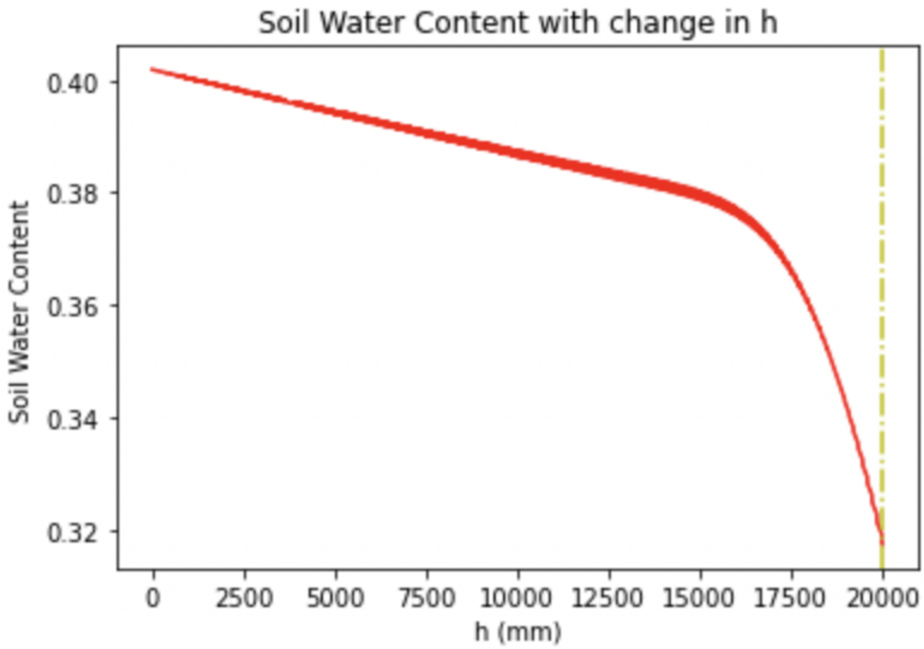


Fig 14: If water depth is 20m, we cut off the graph of SWC at 20m. The soil profile can be visualised in the plot above.

Similarly, if the soil depth is 90m, then we are not as much in luck. The contribution of the capillary flow will drop to zero by the time it reaches $38m$. Hence, the major source of water in the upper layers will be due to rainwater, which has been held back by the soil, as well as rainwater in the process of percolating down.

4.3.2 Case Study: Sandy Soils of the Sahara

We will now analyse a condition wherein the soil water profile could be more accurately predicted in terms of the capillary action contribution. In order to achieve this, we need to consider a type of soil where there is little to no rainfall, and whatever rain is present in the upper parts of the soil will get rapidly dried up by evaporation due to sunlight. Furthermore, we assume that the effect of evaporation in the lower strata of the soil is negligible, so that there is no loss of water there.

Thus, the perfect type of soil for our analysis would be the sandy soils of the desert.

Now, if we consider constant properties of the soil with depth in this case as well (i.e, constant porosity ϵ, constant η and constant K), then we can easily model this situation in terms of our model. We can approximate and consider this assumption to simplify our analysis.

Again, just like in the previous situation, the depth of groundwater is spatially and temporally irregular, and hence I will assume the groundwater bed depth to be D.

We'll assume that for a majority of the soil profile (at least beyond the first few layers), the sand is of medium coarseness. Hence, as per Figure 3, we take the radius to be $200\mu m$. Hence, diameter of the particle is $400\mu m = 0.4mm$.

We'll take K to be 0.125 again, and η to be $0.3d_{avg}$, which is in the upper ranges of heterogeneity.

So, the final values we take are:

$$d_{avg} = 0.4$$

$$K = 0.125$$

$$\eta = 0.3d_{avg} = 0.12$$

We plot our SCDF Gaussian distribution -

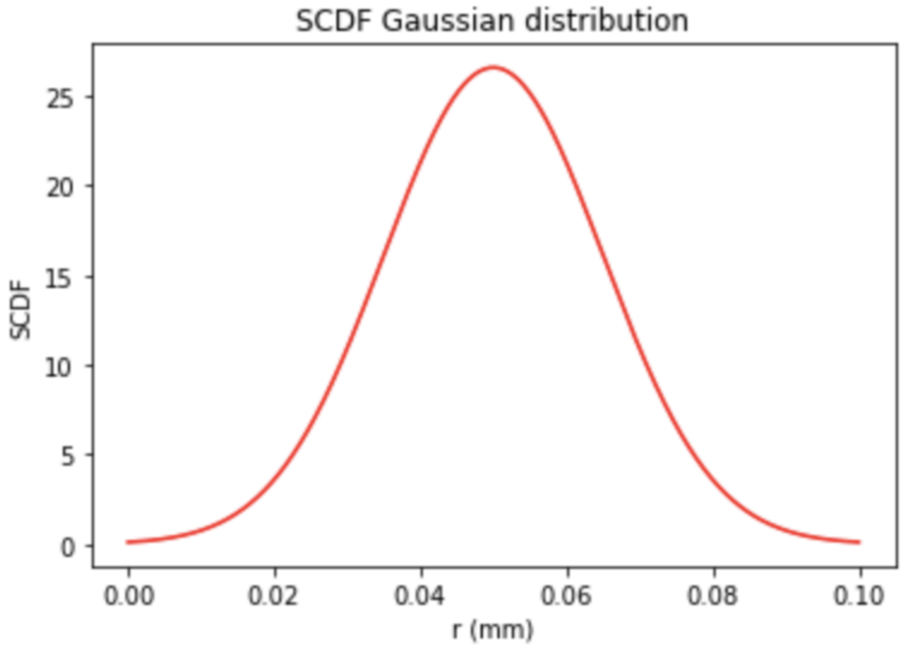


Fig 15: The Gaussian distribution SCDF(r) is plotted using matplotlib. It is centred at 0.05 and has a standard deviation of 0.015.

Further, we plot $r^2SCDF(r)$ -

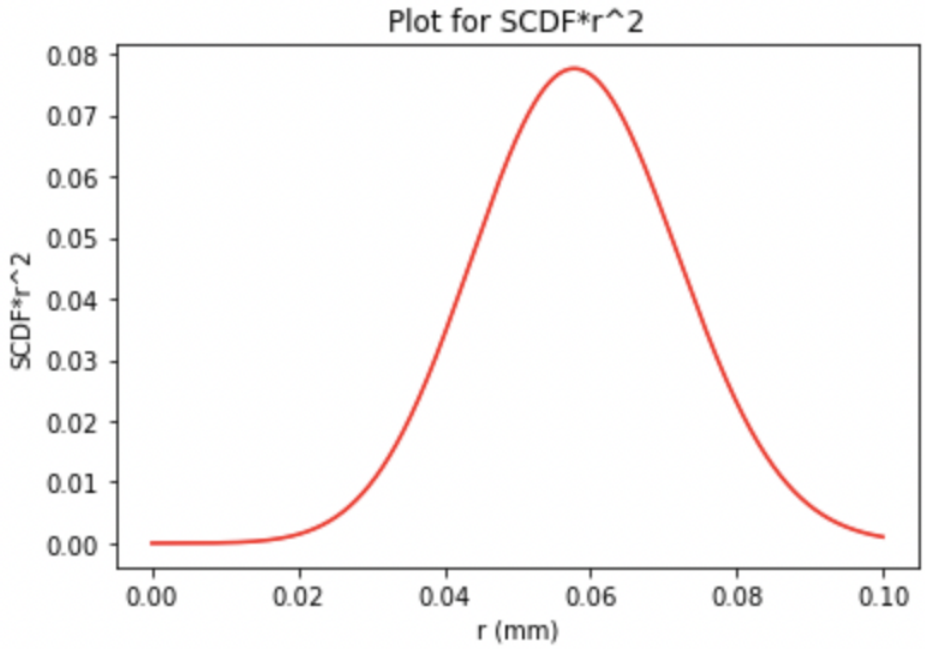


Fig 16: $r^2 SCDF(r)$ is plotted using matplotlib

Finally, we plot $SWC(h)$. Again, we use the bounds $(d_{avg}, 2 \cdot \frac{29.78322}{Kd_{avg}})$.

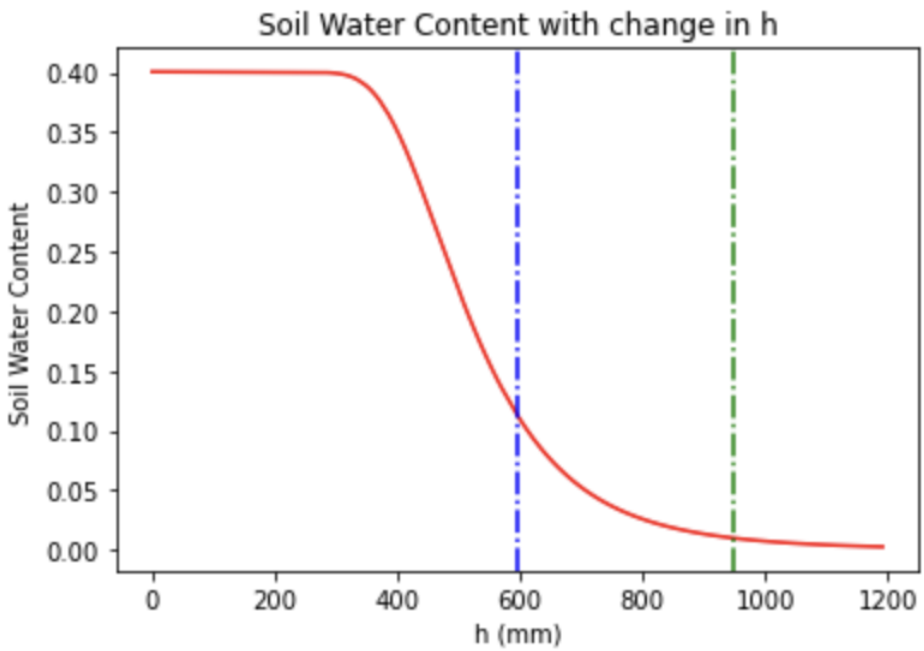


Fig 17: The Soil Water Content is found as a function of h. The blue line represents the height of rise corresponding to $d_{capillary} = Kd_{avg}$. The green line is the height beyond which $SWC \leq 0.01$.

$$h_{green-line} = 948.95\text{mm},\ h_{blue-line} = 595.66\text{mm}$$

Now, we note some inferences:

- Sand particles being larger, sand clearly has a much lower water holding capacity than silt. The plot that we have got shows this. The green line is at $h = \ 94cm$. Thus, beyond this, plants have no chance of growing.

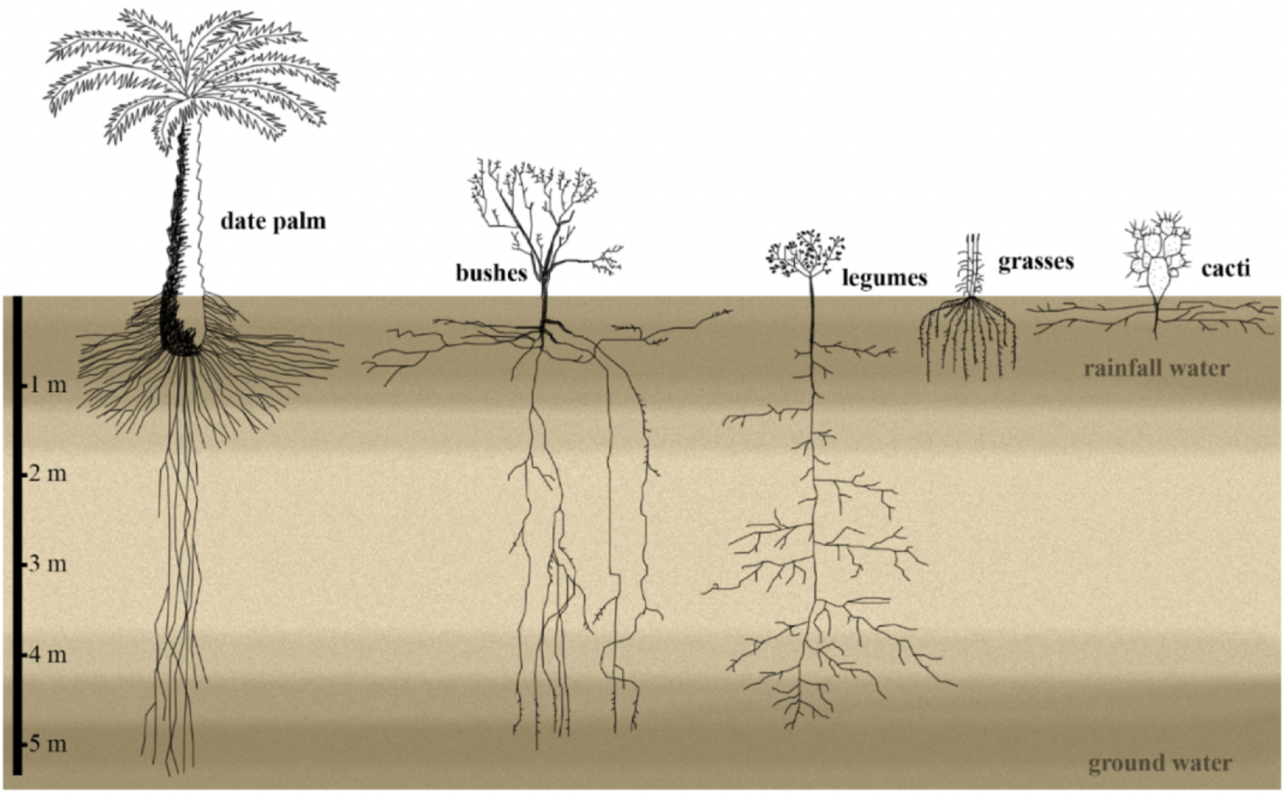
- We can also see that in this case, the SWC remains more constant in the initial stages (upto around 30cm). After this, the drop in SWC is much more steep, and within a meter, the Soil Water Content is less than 0.01.

The Sahara desert has a water bed typically at a depth of 100-250m below the land.[7] Thus, the capillary fringes which are less than a meter thick, are negligible. Hence, its rare for any slight amount of water to reach the surface. Additionally, the upper regions tend to be extremely dry due to low rainfall and evaporation by the Sun.

Thus, it's clear why the amount of vegetation in these regions is very scanty.

The characteristics of groundwater can also help us understand the adaptations of various desert plants. For example, the datepalm typically has fibrous roots that extend vertically as well as horizontally. The horizontal extensions can exploit the rainwater layers near the top as much as possible, while the vertical extensions exploit the groundwater.[11]

The cactus has fibrous roots, and can only exploit shallower regions.[11] Since rainfall is scanty and unpredictable, it needs other adaptations such as spikes instead of leaves in order to conserve its water. The datepalm understandably doesn't need these adaptations since it usually has a relatively more trustable source of water, the groundwater.



Source: https://www.mdpi.com/2073-4425/12/5/709/htm

Fig 18: Various plants of the desert have root adaptations to make the most of the water available to them.

The Sahara desert may have water up to a depth of 100-150m on average, so that's why in a majority of the regions of the desert, even these plants can hardly survive.[7] Typically, the datepalm's roots reach up to 6-7m or more, depending on the age of the plant.

As an application of our results, we can consider an oasis in the desert. Typically, you would find a larger concentration of vegetation nearer to the oasis, and a lower concentration of vegetation further away from it.

An oasis is probably formed because of some relatively higher amount of rainfall in a particular region, because of which the level of groundwater has risen above its normal levels. If the level of groundwater rises high enough, it then rises above the land, thereby forming a lake. This is the oasis.

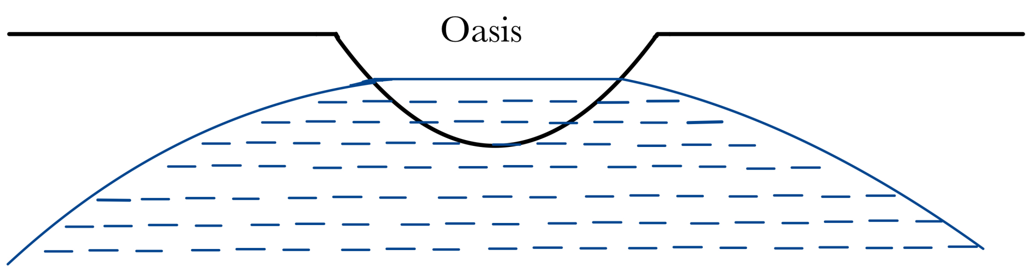


Fig 19: The ground water body gets deeper further away from the oasis.

We now consider the datepalm. We assume that all datepalms in the region have roots that are 6 m in length. Suppose we want to find out the maximum radius from the center of the oasis at which there's some chance of finding a datepalm.

The exact way in which the groundwater depth varies with distance from the centre of the oasis depends on specific factors, such as topography. We will assume the groundwater depth with distance from the center of the oasis to follow this profile:

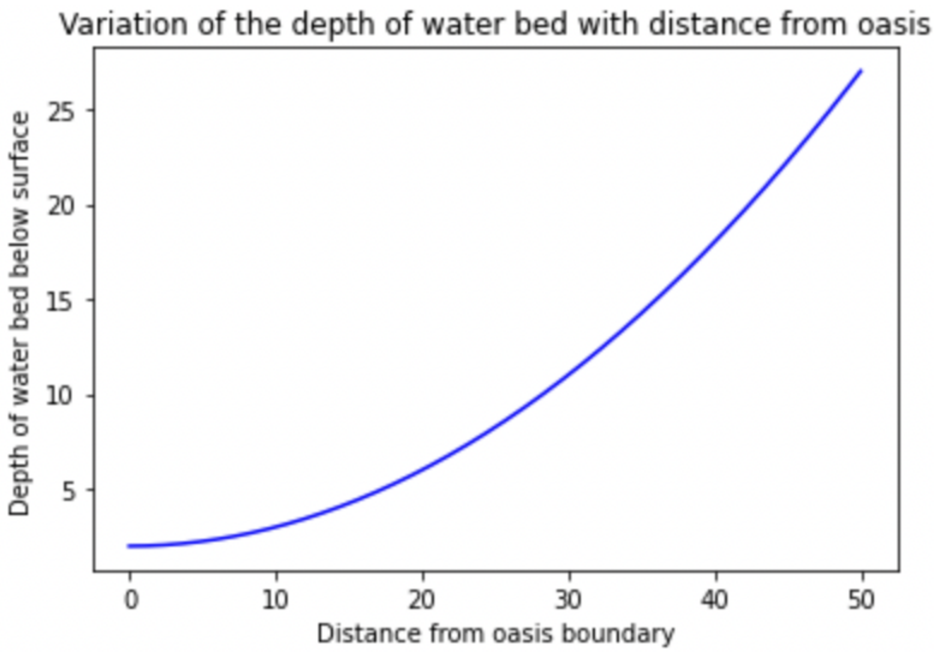


Fig 20: The variation of depth of the ground water body with distance from oasis.

This is the profile $D = 2 + 0.01x^2$ where D is the depth of the water bed and x is the distance from the boundary of the oasis.

Also, the maximum depth the datepalm can reach is 6m. Assuming the datepalm requires at least a Soil Water Content of 0.01,

$$SWC(h = D - 6) \geq 0.01$$

$$\Rightarrow D - 6 \leq 0.949\text{m}$$

$$\Rightarrow D \leq 6.949\text{m}$$
$$\Rightarrow 2 + 0.01x^2 \leq 6.949\text{m}$$
$$\Rightarrow 0.01x^2 \leq 4.949\text{m}$$
$$\Rightarrow x^2 \leq 494.9\text{m}$$
$$\Rightarrow x \leq 22.246\text{m}$$

Hence, this means that beyond 22.246m from the boundary of the oasis, we can expect a very low chance of finding any datepalms.

5 Further Avenues

A major gap in the current model for the soil water profile has been rainwater percolation. In most major types of soil, rainwater percolation affects the soil water profile in the upper strata and it is not negligible. I propose to model this using the 'Packed-Bed' concept and thereby calculate the velocity of flow of rainwater.

Rainwater may fall at some regular intervals, and due to its percolation into the soil, we can intuitively imagine a pulse of water flowing downwards into the soil, until after enough time, the upper layers become less wet again. We will attempt to model this percolation in a future study, by analysing the downward flow of water based on fundamental principles of transport phenomena. Once this has been achieved, we will have a reasonable estimate of the soil water profile, thereby finding a distribution of soil water content as a function of z and t.

Thus, on doing this model, we can add up the two models (the Packed Bed and the Capillary Fringe models) to get a net water profile in the soil. In certain cases, where the water bed is very deep and the capillary fringe is not too thick, these two profiles will be in totally independent zones. However, in other cases, they may overlap, and the way we sum them up would need to be analysed.

Once rainfall has been accounted for, additional factors such as water loss due to evaporation can also be accounted for in later models. Also, certain other cases such as depth-varying porosity, depth-varying d_{avg}, and variable angle-of-contact values could be accounted for in later models.

Once these things are accounted for, we would be able to predict an accurate soil water profile in a much larger variety of situations.

6 Acknowledgements

I would like to thank Prof. GK Suraishkumar for the opportunity to perform an analysis related to principles learnt in the Transport Phenomena course and apply those principles as a part of this report. Performing this analysis has let me explore the field of research and I was able to perform an academic analysis of my own, integrating domains of statistics and engineering to approach a certain relevant and important problem. Thus, this project has taught me a lot more than I had initially imagined.

All the images used in this report are original, unless otherwise stated. All graphs have been plotted using Python's matplotlib library. Conceptual diagrams have been created using Canva and/or GoodNotes.

7 References

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8 Supplementary

8.1 Complete Python code for Silty Soil SWC Prediction

```
## Silty Soil modelling

import numpy as np
import scipy.integrate as integrate
import matplotlib.pyplot as plt

def int_approx(f,a,b):
    return (b-a)*np.mean(f)

# Choose values realistically. But also make sure that 3Kn can at max equal
# d_av (otherwise, you'll have a math inconsistency)
K = 0.125
d_av = 0.01 # within the range of 'Silt'
n = d_av*0.2

# We need integral of r^2 * Gaussian distr, in ranges (-inf to d) as well as
# (-inf to inf)
def SCDF(r):
    a = 2*(K**2)*(n**2) # 2K^2n^2 = a
    d = K*d_av
    return (1/np.sqrt(np.pi*a))*np.exp((-1*(r-d)**2)/a)
```

```
def f(r):
    return (r**2)*SCDF(r)

h = np.linspace(d_av, 2*29.78322/(K*d_av), 10000)

a = 0 # Equivalent of -inf
b1 = 2*K*d_av # This is the equivalent of inf (If you increase b1, the value
# of the integral won't increase much further)

x_range1 = np.arange(a,b1+0.000011,.000001)
fx1 = f(x_range1)
SCDF1 = SCDF(x_range1)

plt.plot(x_range1, SCDF1, color='red')
plt.xlabel("r (mm)")
plt.ylabel("SCDF")
plt.title("SCDF Gaussian distribution")
plt.show()

plt.plot(x_range1, fx1, color='red')
plt.xlabel("r (mm)")
plt.ylabel("SCDF*r^2")
plt.title("Plot for SCDF*r^2")
plt.show()

Dr = int_approx(fx1,a,b1)
print("Dr: ",Dr)

SWCarr = []
h01 = -1
h_av = 29.78322/(K*d_av)
E = 0.4 # porosity

for hi in h:
    b2 = 29.78322/hi

    x_range2 = np.arange(a,b2+0.00011,.00001)
    fx2 = f(x_range2)
    Nr = int_approx(fx2,a,b2)

    # Calculating the SWC value

    SWC = E*Nr/Dr
    if(SWC<0.01):
        if(h01==-1):
            h01 = hi
    SWCarr.append(SWC)

plt.plot(h, SWCarr, color='red')
```

```
plt.axvline(linewidth = 1.5, x = h_av, color = 'b',linestyle="-.")
plt.axvline(linewidth = 1.5, x = h01, color = 'g',linestyle="-.")

print("h at SWC<0.01 = ",h01)
print("h_av = ",h_av)

plt.xlabel("h (mm)")
plt.ylabel("Soil Water Content")
plt.title("Soil Water Content with change in h")
plt.show()

plt.plot(h[:4200], SWCarr[:4200], color='red')
plt.axvline(linewidth = 1.5, x = 20000, color = 'y',linestyle="-.")

plt.xlabel("h (mm)")
plt.ylabel("Soil Water Content")
plt.title("Soil Water Content with change in h")
plt.show()
```

8.2 Complete Python code for Sandy Soil SWC Prediction

Note: This code is a continuation of Section 8.1.

```
## Sandy Soil Modelling

# Choose values realistically. But also make sure that 3Kn can at max equal
# d_av (otherwise, you'll have a math inconsistency)
K = 0.125
d_av = 0.4 # within the range of 'Silt'
n = d_av*0.3

h = np.linspace(d_av, 2*29.78322/(K*d_av), 10000)

a = 0 # Equivalent of -inf
b1 = 2*K*d_av # This is the equivalent of inf (If you increase b1,
# the value of the integral won't increase much further)

x_range1 = np.arange(a,b1+0.000011,.000001)
fx1 = f(x_range1)
SCDF1 = SCDF(x_range1)

plt.plot(x_range1, SCDF1, color='red')
plt.xlabel("r (mm)")
plt.ylabel("SCDF")
plt.title("SCDF Gaussian distribution")
plt.show()

plt.plot(x_range1, fx1, color='red')
plt.xlabel("r (mm)")
plt.ylabel("SCDF*r^2")
```

```
plt.title("Plot for SCDF*r^2")
plt.show()

Dr = int_approx(fx1,a,b1)
print("Dr: ",Dr)

SWCarr = []
h01 = -1
h_av = 29.78322/(K*d_av)
E = 0.4 # porosity

for hi in h:
    b2 = 29.78322/hi

    x_range2 = np.arange(a,b2+0.00011,.00001)
    fx2 = f(x_range2)
    Nr = int_approx(fx2,a,b2)

    # Calculating the SWC value

    SWC = E*Nr/Dr
    if(SWC<0.02):
        if(h01==-1):
            h01 = hi
    SWCarr.append(SWC)

plt.plot(h, SWCarr, color='red')
plt.axvline(linewidth = 1.5, x = h_av, color = 'b',linestyle="-.")
plt.axvline(linewidth = 1.5, x = h01, color = 'g',linestyle="-.")

print("h at SWC<0.01 = ",h01)
print("h_av = ",h_av)

plt.xlabel("h (mm)")
plt.ylabel("Soil Water Content")
plt.title("Soil Water Content with change in h")
plt.show()

## Plotting oasis groundwater density

def f(x):
    return (2+0.01*x**2)

xvalues = np.arange(0,50,0.01)
fx = f(xvalues)
plt.plot(xvalues,fx,color="b")
plt.xlabel("Distance from oasis boundary")
plt.ylabel("Depth of water bed below surface")
plt.title("Variation of the depth of water bed with distance from oasis")
plt.show()
```