Section 11 – Expected Utility

ECON 3030: Intermediate Microeconomic Theory

November 22, 2024

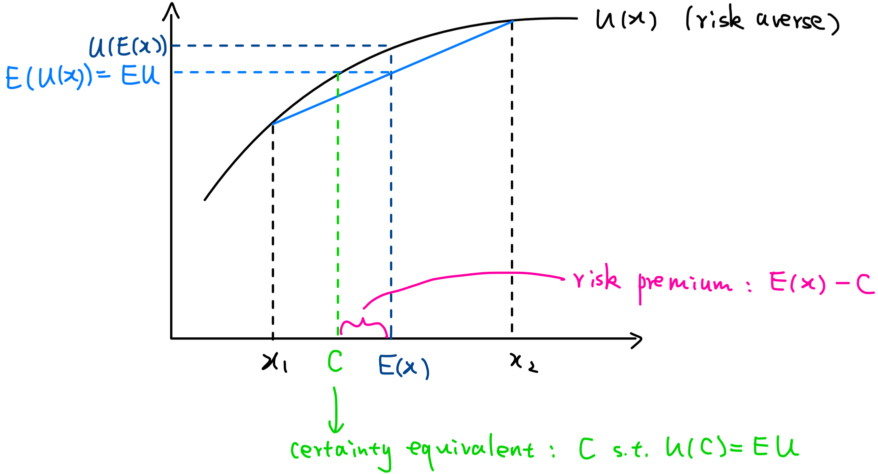
* **Expected utility**: $EU = \pi_1 u(x_1) + \pi_2 u(x_2)$

  - $\pi_{1,2}$: probability of each state $(\pi_1 + \pi_2 = 1)$
  - $x_{1,2}$: outcome of each state
  - $u(x)$: utility function $\Rightarrow$ "felicity function"
    - linear $\Rightarrow$ risk neutral $\Rightarrow E(u(x)) = u(E(x))$
    - concave : $u''(x) < 0 \Rightarrow$ risk averse $\Rightarrow E(u(x)) < u(E(x))$
    - convex : $u''(x) > 0 \Rightarrow$ risk loving $\Rightarrow E(u(x)) > u(E(x))$

* **Expected wealth** : $E(x) = \pi_1 x_1 + \pi_2 x_2$

* In general, expected utility $\neq$ utility of expected wealth

$$E(u(x)) \neq u(E(x))$$


u(x) (risk averse)
u(E(x))
E(u(x)) = EU
x1
C
E(x)
x2
risk premium : E(x) - C
certainty equivalent : C s.t. u(C) = EU

Expected Utility

Imagine Kevin owns his house and has no other wealth. His house is worth 100. There is a 50% probability that his house will be damaged by a flood, making it worth nothing. His utility over wealth is given by $u(x) = \sqrt{x}$.

a. Is Kevin risk-averse, risk-neutral, or risk-loving? Justify your answer formally.

b. Given the scenario described, what is Kevin's expected wealth.

c. What is Kevin's expected utility? How does this compare to the utility level he would achieve if he received the expected value for sure?

d. Draw a graph of Kevin's felicity function. Mark and label his expected wealth and expected utility on the graph.
$u(x)=\sqrt{x}$

e. What is his certainty equivalent? What is his risk premium? Mark each of these on your graph.

a. $u(x)=\sqrt{x} \Rightarrow u'(x)=\frac{1}{2}x^{-\frac{1}{2}}$, $u''(x)=-\frac{1}{4}x^{-\frac{3}{2}}<0$

$\Rightarrow u(x)$ is concave

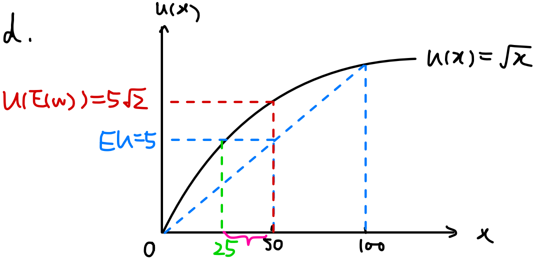
$\Rightarrow$ Kevin is risk-averse

b. Expected wealth: $E(w)=\frac{1}{2}\times 100+\frac{1}{2}\times 0=50$

c. Expected utility: $EU=\frac{1}{2}\times u(100)+\frac{1}{2}\times u(0)=\frac{1}{2}\sqrt{100}=5$

Utility of expected wealth: $u(E(w))=u(50)=\sqrt{50}=5\sqrt{2}$

d.



e. certainty equivalent = 25

risk premium = 50 - 25 = 25

Expected Utility, cont.

Kevin's uncle, Rich, is an insurance agent and offers Kevin a policy that will pay out in the event his house is flooded. The policy he offers costs p per dollar of insurance purchased (z) but he receives the full amount if he experiences the flood.

f. If Rich offers Kevin actuarially fair pricing, what is p?

g. Set up Kevin's expected utility maximization problem with insurance. Do not assume a specific price p.

h. Derive Kevin's demand for insurance, $z(p)$.

i. At what price would Kevin demand no insurance?

j. How much insurance does Kevin demand at the actuarially fair price? What do we call this amount? What is Kevin's wealth in each state of the world in this case?

f. actuarially fair price = expected payoff

= probability of payoff × payoff

$$p = \frac{1}{2} \times 1 + \frac{1}{2} \times 0 = \frac{1}{2}$$

g. choose: z (amount of insurance to buy)

① no flood: $100 - pz$

② flood: $0 + z - pz = z - pz$

$$\Rightarrow \max_{z} EU = \frac{1}{2} \times \sqrt{100 - pz} + \frac{1}{2} \times \sqrt{z - pz}$$

h. $$\frac{\partial EU}{\partial z} = \frac{1}{2} \times \frac{1}{2} \times (-p) \times (100 - pz)^{-\frac{1}{2}} + \frac{1}{2} \times \frac{1}{2} \times (1-p) \times (z - pz)^{-\frac{1}{2}}$$

$$= \frac{-p}{4\sqrt{100 - pz}} + \frac{1-p}{4\sqrt{(1-p)z}} = 0$$

$$\frac{\sqrt{1-p}}{\cancel{4}\sqrt{\cancel{(1-p)}z}} = \frac{p}{\cancel{4}\sqrt{100 - pz}},$$

$$\sqrt{(1-p)(100-pz)} = p\sqrt{z}$$

$$(1-p)(100-pz) = p^2 z$$

$$100 - pz - 100p + \cancel{p^2 z} = \cancel{p^2 z}$$

$$pz = 100 - 100p$$

$$\Rightarrow z(p) = \frac{100(1-p)}{p}$$

i. Let $z(p) = 0 \Rightarrow p = 1$

j. $p = \frac{1}{2} \Rightarrow z(p) = \frac{100 \times \frac{1}{2}}{\frac{1}{2}} = 100$. $\Rightarrow$ full coverage

① no flood: $100 - zp = 100 - \frac{1}{2} \times 100 = 50$

② flood: $z - zp = 100 - 100 \times \frac{1}{2} = 50$

equal!

(actuarially fair + full coverage)

Expected Utility, cont.

Rich is not a big fan of his nephew. Instead of offering Kevin the actuarially fair price, he's going to try to extract as much money has possible from him. Assume that Rich is risk-neutral, and the only insurance agent in town.

$u_R(x) = x$

k. How much should Rich charge for the insurance?

l. How much insurance will Kevin buy in this scenario? What is the expected profit for Rich?

m. Why doesn't Rich fully insure Kevin ($z = 100$) and charge him $p = \frac{3}{4}$ (this insurance scheme is equivalent to the CE)? What would happen to the p and $z(p)$ as more insurance agents came into town?

n. Will Kevin's mom invite Rich this year for Thanksgiving?

k. For Rich:

① no flood : zp

② flood : $zp - z$

$$\max_p \; z(p)\cdot p - \frac{1}{2} z(p) \quad \text{where } z(p) = \frac{100(1-p)}{p}$$

$$\Rightarrow \max_p \; 100(1-p) - \frac{100(1-p)}{2p}$$

$$\text{FOC: } -100 - \frac{-100\cdot 2p - 200(1-p)}{4p^2} = -100 + \frac{200p + 200(1-p)}{4p^2}$$

$$= \frac{200}{4p^2} - 100$$

$$= 0$$

$$\Rightarrow p^* = \sqrt{\frac{1}{2}} = \frac{\sqrt{2}}{2}$$

l. $z(p^*) = \frac{100(1-p^*)}{p^*} = 100 \times \frac{1-\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = 100(\sqrt{2}-1)$

$$E(w_R) = z(p^*) \times p^* - \frac{1}{2} z(p^*)$$

$$= 100(\sqrt{2}-1) \times \frac{\sqrt{2}}{2} - \frac{1}{2} \times 100(\sqrt{2}-1)$$

$$= 100(\sqrt{2}-1) \times \frac{\sqrt{2}-1}{2}$$

$$= 50(\sqrt{2}-1)^2$$

$$= 50(3-2\sqrt{2})$$

m. Bcz Kevin can choose diff. amount of z at $p = \frac{3}{4}$

n. No. (I guess)