Homework 5

UID:
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1] From past experience, a stockbroker believes that under present economic conditions a customer will invest in tax-free bonds with a probability of 0.6, will invest in mutual funds with a probability of 0.3, and will invest in both tax-free bonds and mutual funds with a probability of 0.15. Find the probability that a customer will invest

a] in either tax-free bonds OR mutual funds.

sol^n Let's define the events
probability that a customer will invest in tax-free bonds
i.e., $p(A) = 0.6$
probability that a customer will invest in mutual funds
i.e., $p(B) = 0.3$
probability that a customer will invest in both tax-free and mutual funds
i.e., $p(A \cap B) = 0.15$

To find the probability that a customer will invest in either tax-free OR mutual funds is given by

$$p(A \cup B) = p(A) + p(B) - p(A \cap B)$$
$$= 0.6 + 0.3 - 0.15$$
$$p(A \cup B) = 0.75$$

(b) in neither tax-free bonds nor mutal funds

Sol^n To find the probability that customer will invest in neither tax-free bonds nor mutal funds is given by

$$P(\text{neither } A \text{ nor } B) = 1 - P(A \cup B)$$
$$= 1 - 0.75$$
$$= 0.25$$

2] A truth serum has the property that 90% of the guilty suspects are properly judged while 10% of the guilty suspects are improperly found innocent. On the other hand, innocent suspects are misjudged 1% of the time. If the suspect was selected from a group of suspects of which only 5% have ever committed a crime, and a serum indicates that he is guilty, what is the probability that he is innocent? (Hint: Bayes)

Soln

Let's define the events

G = The Suspect is guilty

I = The suspect is innocent

S = The serum indicates that the suspect is guity

T = The serum indicates that the Suspect is innocent

given the following probabilities

$p(G) = 0.05$

$p(I) = 1 - P(G) = 1 - 0.05 = 0.95$

$P(S|G) = 0.9$

$P(S|I) = 0.01$

We want to find the probability that the suspect is innocent given that the serum indicates guilt

i.e., $P(I|S) = ?$

Using Bayes theorem, we have

$$P(I|S) = \frac{P(S|I) * P(I)}{P(S|G) * P(G) + P(S|I) * p(I)}$$

$$= \frac{0.01 * 0.95}{(0.9 * 0.05) + (0.01 * 0.95)}$$

$$P(I|S) = 0.1743$$

so, $p(I \mid s) \approx 0.1743$

OR

$p(I/s) \simeq 17.43\%$

3] If a dealer's profit, in units of \$5000, on a new automobile is a random variable X having the density function

$$f(x) = \begin{cases} 2(1-x), & 0 < x < 1 \\ 0, & \text{elsewhere} \end{cases}$$

find the avarage profit per automobile

sol/n given,

$$f(x) = \begin{cases} 2(1-x), & 0 < x < 1 \\ 0 & \text{elsewhere} \end{cases}$$

To calculate the expected value of the profit per automobile.

$$E(x) = \int_0^1 2x(1-x)\,dx$$

$$= 2\int_0^1 x(1-x)\,dx$$

$$= 2\int_0^1 (x - x^2)\,dx$$

$$= 2\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1$$

$$= 2\left[\frac{1}{2} - \frac{1}{3}\right] = \frac{1}{\cancel{6}\,3} \times \cancel{2}\,1$$

$$E(x) = \frac{1}{3}$$

$\therefore$ the average profit per automobile
$= \frac{1}{3}$ units of \$5000

$\simeq \$1666.67$

4. If the proportion of a brand of television set requiring service during the first year of operation is a random variable having a beta distribution with $\alpha = 3$ and $\beta = 2$, what is the probability that at least 80% of the new models of this brand sold this year will require service during their first year of operation?

Soln

Let random variable X represent the proportion of brand of television set requiring service during the first year of operation

X follows a beta distribution with $\alpha = 3$ & $\beta = 2$

The probability density function of X is

$$f(x) = \left\{ \frac{1}{B(\alpha, \beta)} x^{(\alpha-1)} (1-x)^{(\beta-1)} \right.$$

$$f(x) = \begin{cases} \frac{1}{B(3,2)} x^2 (1-x)^{(2-1)} ; & 0 \le x \le 1 \\ 0 , & \text{elesewhere} \end{cases}$$

Let calculate the probability that at least 80% of the new models of sold this year will require serving their first year of operation

$$P(X) \geq 80\%) = P(X \geq 0.8)$$

$$= \int_{0.8}^{1} \frac{1}{B(3,2)} x^2 (1-x)\, dx$$

$$= \frac{1}{B(3,2)} \int_{0.8}^{1} x^2 (1-x)\, dx$$

$$= \frac{\Gamma(5)}{\Gamma(3)\Gamma(2)} \int_{0.8}^{1} (x^2 - x^3)\, dx$$

$$= \frac{4!}{2! \cdot 1!} \left[\frac{x^3}{3} - \frac{x^4}{4} \right]_{0.8}^{1}$$

$$= \frac{24}{2} \left[\frac{[1-0.8^3]}{3} - \frac{[1-0.8^4]}{4} \right]$$

$$= 12 \times (0.162667 - 0.1476)$$

$$= 0.1808$$

5] Service calls Come to a maintenance Center according to a Posisson process with λ calls per minute. A data set of 20 one-minute periods yields an average of 1.8 calls. If the prior for λ is an exponential distribution with mean 2, determine the posterior distribution of λ.

Sol^n From our Example we know

$X \sim P(\lambda)$ and $\lambda \sim Exp(2)$

The prior distribution, according to our example information, is

$\pi(\lambda) \propto e^{-\lambda/2}$, for $\lambda = 0$

we can obtain the posterior distribution of λ

$$\pi(\lambda/x) \propto f(x|\lambda)\pi(\lambda)$$

$$= \frac{e^{-n\lambda}\lambda^{\sum_{i=1}^{n} x_i}}{\prod_{i=1}^{n} x_i!} \propto e^{-(n+1)\lambda}\lambda^{\sum_{i=1}^{n} x_i}$$

Referring to the gamma distribution, we conclude that the posterior distribution of λ follows a gamma distribution with parameters $1+\sum_{i=1}^{n} x_i$ and $\frac{1}{n+1}$

$$\therefore \pi(\lambda|x) \propto e^{-(20+1/2)\lambda} \times \lambda^{(20)(1.8)}$$

$$\pi(\lambda|x) \propto e^{-(20.5)\lambda} \times \lambda^{36}$$

$\therefore$ we can concude that

$$\pi(\lambda|x) \sim \text{gamma}\left(1+\sum_{i=1}^{n} x_i, \frac{1}{n+1}\right)$$

$$\pi(\lambda|x) \sim \text{gamma}\left(1+36, \frac{1}{20.5}\right)$$

$$\pi(\lambda|x) \sim \text{gamma}(37, 0.0488)$$
