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Homework 2

1] Consider $\mathbb{R}^3$ with the inner product

$$\langle x, y \rangle := x^T \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} y$$

Furthermore, we define e_1, e_2, e_3 as the standard canonical basis in $\mathbb{R}^3$

a] Determine the orthogonal projection $\pi_U(e_2)$ of e_2 onto $U = \text{span}[e_1, e_3]$

To find : $\pi_U(e_2) = ?$

Approach 1 :

soln

Let us consider,

Orthogonal projection $p = \pi_U(e_2)$

Since $p \in U$, consider $A = (\lambda_1, \lambda_3) \in \mathbb{R}^2$ such that $p = UA$

$$p = [\lambda_1 \; \lambda_3] \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$p = \begin{bmatrix} \lambda_1 \\ 0 \\ \lambda_3 \end{bmatrix}$$

We know that Orthogonal projection can be written as

$$p = \pi_U(e_2) \implies (p - e_2) \perp U$$

$$\implies \begin{bmatrix} \langle p - e_2, e_1 \rangle \\ \langle p - e_2, e_3 \rangle \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \langle p, e_1\rangle - \langle e_2, e_1\rangle \\ \langle p, e_3\rangle - \langle e_2 e_3\rangle \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{—— (1)}$$

Calculating each component as bellow

i] $\langle p, e_1\rangle = [\lambda_1 \; 0 \; \lambda_3] \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$= [\lambda_1 \; 0 \; \lambda_3] \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \boxed{2\lambda_1}$

ii] $\langle p, e_3\rangle = [\lambda_1 \; 0 \; \lambda_3] \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$= [\lambda_1 \; 0 \; \lambda_3] \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} = \boxed{2\lambda_3}$

iii] $\langle e_2, e_1\rangle = [0 \; 1 \; 0] \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$= [0 \; 1 \; 0] \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \boxed{1}$

iv] $\langle e_2, e_3\rangle = [0 \; 1 \; 0] \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$= [0 \; 1 \; 0] \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} = \boxed{-1}$

Substituting values in eq 1

$2\lambda_1 - 1 = 0 \rightsquigarrow \lambda_1 = 1/2$

$2\lambda_3 - (-1) = 0 \quad\quad \lambda_3 = -1/2$

∴ The orthogonal projection

$$p = \pi_U(e_2) = \begin{bmatrix} \lambda_1 \\ 0 \\ \lambda_3 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 0 \\ -1/2 \end{bmatrix}$$

(b) Compute the distance $d(e_2, U)$

sol^n The distance of $d(e_2, U)$ is the distance between e_2 and its Orthogonal projection onto U

$$\therefore \quad d(e_2, U) = \sqrt{\langle p-e_2, p-e_2 \rangle}$$

computing $\langle p-e_2, p-e_2 \rangle$

$$= \begin{bmatrix} 1/2 & -1 & -1/2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1/2 \\ -1 \\ -1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 1/2 & -1 & -1/2 \end{bmatrix} \begin{bmatrix} 1-1+0 \\ 1/2 - 2 + 1/2 \\ 0+1-1 \end{bmatrix}$$

$$= \begin{bmatrix} 1/2 & -1 & -1/2 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} = 1$$

∴ Distance $d(e_2, U) = \sqrt{1} = 1$ //

2 Let W be the subspace of $\mathbb{R}^3$ spanned by

$u_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $u_2 = \begin{bmatrix} 7 \\ -2 \\ 11 \end{bmatrix}$ $\left(\text{i.e, } W = \text{span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 7 \\ -2 \\ 11 \end{pmatrix} \right\} \right)$

Find the vector(s) that span(s) the Orthogonal complement $W^\perp$ of W

To Find: $X = \{x_1, x_2, x_3\}$ with $U.X = W.X = 0$

sol^n We know that,

$W^\perp$ can be written as a set of all vectors in X

i.e, $\{ X$ such that $X.u_1 = 0$ and $x_2.u_2 = 0 \}$

We need to find solution to above eqn

i.e. $x_1 + x_2 + x_3 = 0$

$7x_1 - 2x_2 + 11x_3 = 0$

The augmented matrix can be written as

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 7 & -2 & 11 & 0 \end{array}\right]$$

Reducing above matrix to RREF form

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 7 & -2 & 11 & 0 \end{array}\right] \rightsquigarrow R_2 \to 7R_1 - R_2 \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 16 & -10 & 0 \end{array}\right]$$

$$\rightsquigarrow R_2 \to 1/16\, R_2 \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 5/8 & 0 \end{array}\right]$$

$$\rightsquigarrow^{R_1 \to R_1 - R_2} \left[\begin{array}{ccc|c} 1 & 0 & 7/4 & 0 \\ 0 & 1 & 5/8 & 0 \end{array}\right]$$

We have RREF form as

$$\left[\begin{array}{ccc|c} 1 & 0 & 7/4 & 0 \\ 0 & 1 & 5/8 & 0 \end{array}\right]$$

Let say $x_3 = s$

$x_1 + 7/4 s = 0$, $x_1 = -7/4 s$

$x_2 + 5/8 s = 0$ $x_2 = -5/8 s$

Let express vectors X in vector form

$$X = \begin{bmatrix} -7/4 \\ -5/8 \\ 1 \end{bmatrix} s$$

Let $s = 4$ then $X = \begin{bmatrix} -7 \\ -5/2 \\ 4 \end{bmatrix}$

Therefore, The vector that span the orthogonal complement $W^\perp$ of W is

$$W^\perp = \left\{ \begin{pmatrix} -7 \\ -5/2 \\ 4 \end{pmatrix} \right\}$$

3] Let W be the subspace of $\mathbb{R}^5$ spanned by $u = \begin{bmatrix} 1 \\ 2 \\ 3 \\ -1 \\ 2 \end{bmatrix}$, $v = \begin{bmatrix} 2 \\ 4 \\ 7 \\ 2 \\ -1 \end{bmatrix}$. Find a basis of the orthogonal complement $W^\perp$ of W.

To find = $X = \{x_1, x_2, x_3, x_4, x_5\}$ with $W.X = 0$

sol^n We know that,

$W^\perp$ can be written as a set of all vectors in X

i.e., { X such that $x.u=0$ and $x.v=0$ }

we need to find sol^n to the above eq^n

i.e., $x_1 + 2x_2 + 3x_3 - x_4 + 2x_5 = 0$

$2x_1 + 4x_2 + 7x_3 + 2x_4 - x_5 = 0$

The augmented matrix can be written as

$$\left[\begin{array}{ccccc|c} 1 & 2 & 3 & -1 & 2 & 0 \\ 2 & 4 & 7 & 2 & -1 & 0 \end{array}\right]$$

Reducing above matrix to RREF form

$$R_2 \to R_2 - 2R_1 \quad \left[\begin{array}{ccccc|c} 1 & 2 & 3 & -1 & 2 & 0 \\ 0 & 0 & 1 & 4 & -5 & 0 \end{array}\right]$$

$$R_1 \to R_1 - 3R_2 \quad \left[\begin{array}{ccccc|c} 1 & 2 & 0 & -13 & 17 & 0 \\ 0 & 0 & 1 & 4 & -5 & 0 \end{array}\right]$$

Let $x_2 = s$, $x_4 = t$, $x_5 = u$

then $x_1 = -2s + 13t - 17u$

$x_3 = -4t + 5u$

Expressing X in vector form

$$X = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} s + \begin{bmatrix} 13 \\ 0 \\ -4 \\ 1 \\ 0 \end{bmatrix} t + \begin{bmatrix} -17 \\ 0 \\ 5 \\ 0 \\ 1 \end{bmatrix} u$$

Let s, t, u = 1

Therefore, Basis of the orthogonal complement $W^\perp$ of W are

$$W^\perp = \left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 13 \\ 0 \\ -4 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -17 \\ 0 \\ 5 \\ 0 \\ 1 \end{pmatrix} \right\}$$

4] Find the orthonormal basis for the subspace U of $\mathbb{R}^4$ spanned by the vectors

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 4 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 2 \\ -4 \\ -3 \end{bmatrix}$$

To Find, $\|u_1\|$, $\|u_2\|$, $\|u_3\|$

where, u_1, u_2, u_3 are orthogonal basis

sol^n First we need to find u_1, u_2, u_3

The formula is given by

$$u_1 = v_1$$

$$u_2 = v_2 - \frac{\langle v_2 \cdot u_1 \rangle}{\|u_1\|^2} \cdot u_1$$

$$u_3 = v_3 - \frac{\langle v_3 \cdot u_1 \rangle}{\|u_1\|^2} \cdot u_1 - \frac{\langle v_3 \cdot u_2 \rangle}{\|u_2\|^2} \cdot u_2$$

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$$u_1 = v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$u_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 4 \end{bmatrix} - \frac{\begin{bmatrix} 1 & 1 & 2 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}}{\left(\sqrt{\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}}\right)^2} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 \\ 2 \\ 4 \end{bmatrix} - \frac{\cancel{8}\,2}{\cancel{4}\,1}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 0 \\ 2 \end{bmatrix}$$

$$u_3 = \begin{bmatrix} 1 \\ 2 \\ -4 \\ -3 \end{bmatrix} - \frac{\begin{bmatrix} 1 & 2 & -4 & -3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}}{\left(\sqrt{\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}}\right)^2} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{\begin{bmatrix} 1 & 2 & -4 & -3 \end{bmatrix}\begin{bmatrix} -1 \\ -1 \\ 0 \\ 2 \end{bmatrix}}{\left(\sqrt{\begin{bmatrix} -1 & -1 & 0 & 2 \end{bmatrix}\begin{bmatrix} -1 \\ -1 \\ 0 \\ 2 \end{bmatrix}}\right)^2} \cdot \begin{bmatrix} -1 \\ -1 \\ 0 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 2 \\ -4 \\ -3 \end{bmatrix} - \frac{-\cancel{4}}{\cancel{4}}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{-\cancel{9}\,3}{\cancel{6}\,2}\begin{bmatrix} -1 \\ -1 \\ 0 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 3 \\ -3 \\ -2 \end{bmatrix} + \begin{bmatrix} -3/2 \\ -3/2 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 3/2 \\ -3 \\ 1 \end{bmatrix}$$

Now calculating Orthonormal Basis

$$||u_1|| = \sqrt{(1)^2+(1)^2+(1)^2+(1)^2} = \sqrt{4} = 2$$

$$||u_2|| = \sqrt{(-1)^2+(-1)^2+0+2^2} = \sqrt{1+1+4} = \sqrt{6}$$

$$||u_3|| = \sqrt{(1/2)^2+(3/2)^2+(-3)^2+1^2} = \sqrt{1/4+9/4+9+1}$$

$$= \frac{1+9+36+4}{4} = \frac{\sqrt{50}}{2}$$

Orthonormal Basis are

$$B = \text{span}\left\{ \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}, \begin{bmatrix} -1/\sqrt{6} \\ -1/\sqrt{6} \\ 0 \\ 2/\sqrt{6} \end{bmatrix}, \begin{bmatrix} +1/\sqrt{50} \\ 3/\sqrt{50} \\ -6/\sqrt{50} \\ 2/\sqrt{50} \end{bmatrix} \right\}$$

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5] Let

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 3 & 4 \\ 7 & -5 & 2 \end{bmatrix}$$

a] Are the rows of A orthogonal?

To prove: $\langle r_1 . r_2 \rangle$ OR $\langle r_2 . r_3 \rangle$ OR $\langle r_1 . r_3 \rangle = 0$

Sol^n Let say rows of given matrix A are

$r_1 = [1 \quad 1 \quad -1]$

$r_2 = [1 \quad 3 \quad 4]$

$r_3 = [7 \quad -5 \quad 2]$

Dot product can be calculated as bellow

i] $\langle r_1 . r_2 \rangle = [1 \quad 1 \quad -1] \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} = 1 + 3 - 4 = 0$

ii] $\langle r_2 . r_3 \rangle = [1 \quad 3 \quad 4] \begin{bmatrix} 7 \\ -5 \\ 2 \end{bmatrix} = 7 - 15 + 8 = 0$

iii] $\langle r_1 . r_3 \rangle = [1 \quad 1 \quad -1] \begin{bmatrix} 7 \\ -5 \\ 2 \end{bmatrix} = 7 - 5 - 2 = 0$

Since the dot products of all distinct row pair are zero
Hence, The rows of

matrix $A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 3 & 4 \\ 7 & -5 & 2 \end{bmatrix}$ are Orthogonal

(b) Is A an Orthogonal Matrix

To prove: $A \cdot A^T = I$ OR $A^{-1} = A^T$

sol^n Let us Find $A \cdot A^T$

$$= \begin{bmatrix} 1 & 1 & -1 \\ 1 & 3 & 4 \\ 7 & -5 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 7 \\ 1 & 3 & -5 \\ -1 & 4 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1+1+1 & 1+3-4 & 7-5-2 \\ 1+3-4 & 1+9+16 & 7-15+8 \\ 7-5-2 & 7-15+8 & 49+25+4 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 & 0 \\ 0 & 26 & 0 \\ 0 & 0 & 78 \end{bmatrix} \neq I$$

Hence, given Matrix A is <u>Not</u> Orthogonal Matrix

[C] Are the columns of A orthogonal?

To prove: $\langle c_1 \cdot c_2 \rangle$ OR $\langle c_1 \cdot c_3 \rangle$ OR $\langle c_2 \cdot c_3 \rangle = 0$

sol^n Let say columns of given matrix A are

$$c_1 = \begin{bmatrix} 1 \\ 1 \\ 7 \end{bmatrix}, \quad c_2 = \begin{bmatrix} 1 \\ 3 \\ -5 \end{bmatrix}, \quad c_3 = \begin{bmatrix} -1 \\ 4 \\ 2 \end{bmatrix}$$

Dot product can be calculated as bellow

i] $\langle c_1 . c_2 \rangle = \begin{bmatrix} 1 & 1 & 7 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ -5 \end{bmatrix} = 1 + 3 - 35 = -31$

ii] $\langle c_1 . c_3 \rangle = \begin{bmatrix} 1 & 1 & 7 \end{bmatrix} \begin{bmatrix} -1 \\ 4 \\ 2 \end{bmatrix} = -1 + 4 + 14 = 17$

iii] $\langle c_2 . c_3 \rangle = \begin{bmatrix} 1 & 3 & -5 \end{bmatrix} \begin{bmatrix} -1 \\ 4 \\ 2 \end{bmatrix} = -1 + 12 - 10 = 1$

Since the dot products of all distinct columns are zero

Hence, The columns of

matrix $A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 3 & 4 \\ 7 & -5 & 2 \end{bmatrix}$ are Not Orthogonal