Homework 1 UID: U01936040

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1] Compute the following matrix products, if possible. If the product is not possible, state why.

a] $\begin{bmatrix} 1 & 5 \\ 2 & 3 \\ 4 & 7 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

sol^n To prove : product of given two matrices is NOT possible.

Reason :

1] Size of Matrix A is,

Size(A) = Number of Rows of Matrix A $\times$ Number of columns of Matrix B

Size(A) = 3X2 → No of columns

i.e.,

$A = \begin{bmatrix} 1 & 5 \\ 2 & 3 \\ 4 & 7 \end{bmatrix}$

No. of Rows

Hence we represent,

$A = \begin{bmatrix} 1 & 5 \\ 2 & 3 \\ 4 & 7 \end{bmatrix}_{3\times 2}$ $B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}_{3\times 3}$

2] Matrix Multiplication Rule

Matrix multiplication of two matrices A & B is possible when, Number of columns of matrix A should match number of rows of Matrix B.

i.e.

$$\text{Size}(A) = 3 \times \underbrace{\boxed{2 \quad 3}}_{} \times 3 = \text{Size}(B)$$

Inner dimension of two Matrices A & B should be equal

In this case, $2 \neq 3$

Hence,

Product of Matrix A and B is <u>NOT</u> possible

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b]

$$A = \begin{bmatrix} 1 & 4 & 6 \\ 7 & 2 & 5 \\ 9 & 8 & 3 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

To prove: Product of two given Matrices is possible

sol^n :

i] Size of matrix:

$$A = \begin{bmatrix} 1 & 4 & 6 \\ 7 & 2 & 5 \\ 9 & 8 & 3 \end{bmatrix}_{3 \times 3} \quad B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}_{3 \times 3}$$

2) Rule:

No. of columns in Matrix A = No. of rows in Matrix B

i.e., 3 = 3

3] Given Matrices satisfy the matrix Multiplication Rule. Hence product of Matrix A and B is possible.

Now, A.B is calculated as follows,

$$A.B = \begin{bmatrix} 1 & 4 & 6 \\ 7 & 2 & 5 \\ 9 & 8 & 3 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1(1)+4(0)+6(1) & 1(0)+4(1)+6(1) & 1(1)+4(1)+6(0) \\ 7(1)+2(0)+5(1) & 7(0)+2(1)+5(1) & 7(1)+2(1)+5(0) \\ 9(1)+8(0)+3(1) & 9(0)+8(1)+3(1) & 9(1)+8(1)+3(0) \end{bmatrix}$$

$$= \begin{bmatrix} 1+6 & 4+6 & 1+4 \\ 7+5 & 2+5 & 7+2 \\ 9+3 & 8+3 & 9+8 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 10 & 5 \\ 12 & 7 & 9 \\ 12 & 11 & 17 \end{bmatrix} = A.B$$

c] $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 4 & 6 \\ 7 & 2 & 5 \\ 9 & 8 & 3 \end{bmatrix}$

To prove: Matrix Multiplication of given two matrices is possible

sol^n:

1] Size of Matrices are,

$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}_{3 \times 3}$ $B = \begin{bmatrix} 1 & 4 & 6 \\ 7 & 2 & 5 \\ 9 & 8 & 3 \end{bmatrix}_{3 \times 3}$

2] Rule:

No. of columns in Matrix A = No. of rows in Matrix B

$\text{Size}(A) = 3 \times \boxed{3 \quad 3} \times 3 = \text{Size}(B)$

Inner dimension of A & B is equal

Hence, Matrix Multiplication is possible

3] $A \cdot B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 4 & 6 \\ 7 & 2 & 5 \\ 9 & 8 & 3 \end{bmatrix}$

$= \begin{bmatrix} 1(1)+0(7)+1(9) & 1(4)+0(2)+1(8) & 1(6)+0(5)+1(3) \\ 0(1)+1(7)+1(9) & 0(4)+1(2)+1(8) & 0(6)+1(5)+1(3) \\ 1(1)+1(7)+0(9) & 1(4)+1(2)+0(8) & 1(6)+1(5)+0(3) \end{bmatrix}$

$$= \begin{bmatrix} 1+9 & 4+8 & 6+3 \\ 7+9 & 2+8 & 5+3 \\ 1+7 & 4+2 & 6+5 \end{bmatrix}$$

$$A \cdot B = \begin{bmatrix} 10 & 12 & 9 \\ 16 & 10 & 8 \\ 8 & 6 & 11 \end{bmatrix}$$

D] $\begin{bmatrix} 1 & 3 & 2 & 1 \\ 4 & 1 & -4 & -1 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ -1 & 1 \\ 2 & 2 \\ 4 & 2 \end{bmatrix}$

To prove: Matrix Multiplication for given two Matrices is possible.

sol^n:

i] Size of given matrices are,

$$A = \underset{2\times 4}{\begin{bmatrix} 1 & 3 & 2 & 1 \\ 4 & 1 & -4 & -1 \end{bmatrix}} \quad \underset{4\times 2}{\begin{bmatrix} 0 & 2 \\ -1 & 1 \\ 2 & 2 \\ 4 & 2 \end{bmatrix}}$$

Inner dimension of Matrix A & Matrix B is equal

Hence,

Matrix Multiplication is possible.

$$2]\ A.B = \begin{bmatrix} 1 & 3 & 2 & 1 \\ 4 & 1 & -4 & -1 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ -1 & 1 \\ 2 & 2 \\ 4 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1(0)+3(-1)+2(2)+1(4) & 1(2)+3(1)+2(2)+1(2) \\ 4(0)+1(-1)-4(2)-1(4) & 4(2)+1(1)-4(2)-1(2) \end{bmatrix}$$

$$= \begin{bmatrix} 0-3+4+4 & 2+3+4+2 \\ 0-1-8-4 & 8+1-8-2 \end{bmatrix}$$

$$A.B = \begin{bmatrix} 5 & 11 \\ -13 & -1 \end{bmatrix}$$

(e) $\begin{bmatrix} 0 & 2 \\ -1 & 1 \\ 2 & 2 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} 1 & 3 & 2 & 1 \\ 4 & 1 & -4 & -1 \end{bmatrix}$

To prove: Product of given matrices A and B is possible

sol^n: 1] Size of matrices A and B are

$$A = \begin{bmatrix} 0 & 2 \\ -1 & 1 \\ 2 & 2 \\ 4 & 2 \end{bmatrix}_{4\times 2} \qquad B = \begin{bmatrix} 1 & 3 & 2 & 1 \\ 4 & 1 & -4 & -1 \end{bmatrix}_{2\times 4}$$

2] Rule:

No. of columns of matrix A = No. of rows in matrix B

i.e $A = 4 \times 2 \qquad 2 \times 4 = \text{size}(B)$

inner dimension of A & B is equal

Hence,

Matrix Multiplication is possible

3] $$A \cdot B = \begin{bmatrix} 0 & 2 \\ -1 & 1 \\ 2 & 2 \\ 4 & 2 \end{bmatrix} \times \begin{bmatrix} 1 & 3 & 2 & 1 \\ 4 & 1 & -4 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 0(1)+2(4) & 0(3)+2(1) & 0(2)+2(-4) & 0(1)+2(-1) \\ -1(1)+1(4) & -1(3)+1(1) & -1(2)+1(-4) & -1(1)+1(-1) \\ 2(1)+2(4) & 2(3)+2(1) & 2(2)+2(-4) & 2(1)+2(-1) \\ 4(1)+2(4) & 4(3)+2(1) & 4(2)+2(-4) & 4(1)+2(-1) \end{bmatrix}$$

$$= \begin{bmatrix} 0+8 & 0+2 & 0-8 & 0-2 \\ -1+4 & -3+1 & -2-4 & -1-1 \\ 2+8 & 6+2 & 4-8 & 2-2 \\ 4+8 & 12+2 & 8-8 & 4-2 \end{bmatrix}$$

$$A \cdot B = \begin{bmatrix} 8 & 2 & -8 & -2 \\ 3 & -2 & -6 & -2 \\ 10 & 8 & -4 & 0 \\ 12 & 14 & 0 & 2 \end{bmatrix}$$

2] write $y = \begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix}$ as a linear Combination of

$$x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad x_3 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

so [n=1] we have to find α, β, γ, which satisfy the following equation

$$\alpha x_1 + \beta x_2 + \gamma x_3 = y$$

where, α, β, γ are Scalars

<u>OR</u>

we can also write

$$\alpha \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \gamma \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix}$$

Hence,

the problem is to solve the above matrix equation.

2] The augmented matrix can be written as

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 1 & 2 & -1 & -2 \\ 1 & 3 & 1 & 5 \end{array}\right]$$

Reducing above matrix into RRef form

$$R_1 \longleftrightarrow R_2 \quad \left[\begin{array}{ccc|c} 1 & 2 & -1 & -2 \\ 1 & 1 & 2 & 1 \\ 1 & 3 & 1 & 5 \end{array}\right]$$

$$\begin{array}{l} \\ R_2 \to R_1 - R_2 \\ R_3 \to R_1 - R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & -1 & -2 \\ 0 & 1 & -3 & -3 \\ 0 & -1 & -2 & -7 \end{array}\right]$$

$$R_3 \to R_2 + R_3 \quad \left[\begin{array}{ccc|c} 1 & 2 & -1 & -2 \\ 0 & 1 & -3 & -3 \\ 0 & 0 & -5 & -10 \end{array}\right]$$

$$R_3 \to \frac{-R_3}{5} \quad \left[\begin{array}{ccc|c} 1 & 2 & -1 & -2 \\ 0 & 1 & -3 & -3 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$R_1 \to -2R_2 + R_1 \quad \left[\begin{array}{ccc|c} 1 & 0 & 5 & 4 \\ 0 & 1 & -3 & -3 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$$\begin{array}{l} R_1 \to -5R_3 + R_1 \\ R_2 \to 3R_3 + R_2 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -6 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

$\therefore$ The value of

$$\alpha = -6, \ \beta = 3 \text{ and } \gamma = 2$$

and linear combination can be written as

$$(-6)\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 2\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix}$$

OR

$$-6x_1 + 3x_2 + 2x_3 = y.$$

3] Consider two subspace U_1 and U_2 where U_1 is the solution space of the homogeneous equations system $A_1x = 0$ and U_2 is the sol^n space of the homogeneous equations system $A_2x = 0$

$$A_1 = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 2 \\ 3 & 4 & 2 \\ 0 & 1 & 1 \end{bmatrix} \text{ and } A_2 = \begin{bmatrix} 2 & -2 & 0 \\ 1 & 2 & 2 \\ 6 & -4 & 2 \\ 2 & -1 & 3 \end{bmatrix}$$

Sol^n

1] Let us find $\cancel{X}_1$.

So the augmented matrix for $A_1x = 0$ can be written as

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 2 & -1 & 2 & 0 \\ 3 & +4 & 2 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

We know that $N(\text{rref}(A_1)) = N(A_1)$

so the above matrix can be reduced in RRef form as bellow

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 2 & -1 & 2 & 0 \\ 3 & 4 & 2 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \xrightarrow{\substack{R_2 \to R_2 - R_1(2) \\ R_3 \to R_3 - 3R_1}} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 4 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \rightarrow$$

$$\begin{array}{l} R_3 \to 4R_2 - R_3 \\ R_4 \to R_4 - R_2 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{R_1 \to R_1 - R_3} \begin{array}{l} \\ \\ \\ R_4 \to R_3 - R_4 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

As we observe, pivot in every row, so column span $\mathbb{R}^3$

$A_1 x = 0$ has No free Variable

$\therefore$ columns are linearly independent

Hence,

1] The dimension of $U_1 = 3$

2] Basis for $U_1 = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 2 \\ 3 & 4 & 2 \\ 0 & 1 & 1 \end{bmatrix}$

2] To find x_2, the augmented matrix for $Ax_2 = 0$ can be written as

$$\left[\begin{array}{ccc|c} 2 & -2 & 0 & 0 \\ 1 & 2 & 2 & 0 \\ 6 & -4 & 2 & 0 \\ 2 & -1 & 3 & 0 \end{array}\right]$$

The above matrix can be reduced in RRef form as

$$\left[\begin{array}{ccc|c} 2 & -2 & 0 & 0 \\ 1 & 2 & 2 & 0 \\ 6 & -4 & 2 & 0 \\ 2 & -1 & 3 & 0 \end{array}\right] \quad \begin{array}{c} \text{Replace} \\ R_1 \leftrightarrow R_2 \end{array} \quad \left[\begin{array}{ccc|c} 1 & 2 & 2 & 0 \\ 2 & -2 & 0 & 0 \\ 6 & -4 & 2 & 0 \\ 2 & -1 & 3 & 0 \end{array}\right]$$

$$\begin{array}{l} R_2 \to 2R_1 - R_2 \\ R_3 \to 6R_1 - R_3 \\ R_4 \to 2R_1 - R_4 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 2 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 8 & 10 & 0 \\ 0 & 3 & 1 & 0 \end{array}\right] \quad R_2 \to R_2/2 \quad \left[\begin{array}{ccc|c} 1 & 2 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 8 & 10 & 0 \\ 0 & 3 & 1 & 0 \end{array}\right]$$

$$R_3 \leftrightarrow R_4 \left[\begin{array}{ccc|c} 1 & 2 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 8 & 10 & 0 \end{array}\right] \quad \begin{array}{l} R_1 \to R_1 - 2R_2 \\ \\ R_3 \to 3R_2 - R_3 \\ R_4 \to 8R_2 - R_4 \end{array} \quad \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 6 & 0 \end{array}\right]$$

$$R_3 \to R_3/5 \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 6 & 0 \end{array}\right] \quad \begin{array}{l} R_1 \to R_1 + 2R_3 \\ R_2 \to R_2 - 2R_3 \\ \\ R_4 \to 6R_3 - R_4 \end{array} \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

As we observe, pivot in every row, so columns span $\mathbb{R}^3$

$A_2x = 0$ has NO free variable

$\therefore$ columns are linearly independent

Hence,

1] The dimension of $U_2 = 3$

2] Basis for $U_2 = \begin{bmatrix} 2 & -2 & 0 \\ 1 & 2 & 2 \\ 6 & -4 & -2 \\ 2 & -1 & 3 \end{bmatrix}$

c] Determine a basis for $U_1 \cap U_2$

We need to find

$$a_1 u_{11} + a_2 u_{12} + a_3 u_{13} = b_1 u_{21} + b_2 u_{22} + b_3 u_{23}$$

where $a_1\ a_2\ a_3\ b_1\ b_2\ b_3 \in R$

$$\Rightarrow a_1 \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix} + a_2 \begin{bmatrix} 0 \\ -1 \\ 4 \\ 1 \end{bmatrix} + a_3 \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix} - b_1 \begin{bmatrix} 2 \\ 1 \\ 6 \\ 2 \end{bmatrix} - b_2 \begin{bmatrix} -2 \\ 2 \\ -4 \\ -1 \end{bmatrix} - b_3 \begin{bmatrix} 0 \\ 2 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

augmented matrix is

$$\left[\begin{array}{cccccc|c} 1 & 0 & 1 & -2 & 2 & 0 & 0 \\ 2 & -1 & 2 & -1 & -2 & -2 & 0 \\ 3 & 4 & 2 & -6 & 4 & -2 & 0 \\ 0 & 1 & 1 & -2 & 1 & -3 & 0 \end{array}\right]$$

$$R_2 \to 2R_1 - R_2 \left[\begin{array}{cccccc|c} 1 & 0 & 1 & -2 & 2 & 0 & 0 \\ 0 & 1 & 0 & -3 & 6 & 2 & 0 \\ 0 & -4 & 1 & 0 & 2 & -2 & 0 \\ 0 & 1 & 1 & -2 & 1 & -3 & 0 \end{array}\right]$$

$$\begin{array}{l} R_3 \to 4R_2 + R_3 \\ R_4 \to R_2 + R_4 \end{array} \left[\begin{array}{cccccc|c} 1 & 0 & 1 & -2 & 2 & 0 & 0 \\ 0 & 1 & 0 & -3 & 6 & 2 & 0 \\ 0 & 0 & 1 & -12 & 26 & 6 & 0 \\ 0 & 0 & -1 & -1 & 5 & 5 & 0 \end{array}\right]$$

$$\begin{array}{l} R_1 \to R_1 - R_3 \\ \\ R_4 \to R_3 + R_4 \end{array} \left[\begin{array}{cccccc|c} 1 & 0 & 0 & 10 & -24 & -6 & 0 \\ 0 & 1 & 0 & -3 & 6 & 2 & 0 \\ 0 & 0 & 1 & -12 & 26 & 6 & 0 \\ 0 & 0 & 0 & -13 & 31 & 11 & 0 \end{array}\right]$$

$a_1 = -10s + 24u + 6w$, $a_2 = 3s - 6u - 2w$, $a_3 = 12s - 26u - 6w$

$b_1 = s$, $b_2 = u$. $b_3 = v$

$$\therefore U_1 \cap U_2 = \cancel{a_1 u_{11}}\ a_1 \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix} + a_2 \begin{bmatrix} 0 \\ -1 \\ 4 \\ 1 \end{bmatrix} + a_3 \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix} - b_1 \begin{bmatrix} 2 \\ 1 \\ 6 \\ 2 \end{bmatrix} - b_2 \begin{bmatrix} -2 \\ 2 \\ -4 \\ 1 \end{bmatrix} - b_3 \begin{bmatrix} 0 \\ 2 \\ 2 \\ 3 \end{bmatrix}$$

4] Suppose $S = \{v_1, v_2, \ldots v_m)$ spans a vector space V. Prove:

(a) If $w \in V$, then $\{w, v_1, v_2 \ldots v_m\}$ is linearly dependent and spans V

soln

Defination of linear dependent states that "A set of vectors $\{v_1, v_2 \ldots v_m\}$ is linearly dependent 'iff' at least one vector, in the set, can be written as a linear combination of the other vectors".

i] To prove: $\{w, v_1, v_2 \ldots v_m\}$ is linearly dependent

proof: Let us assume,

$\{w, v_1 v_2 \ldots v_m\}$ is linearly dependent —— ①

then, there exist co-efficients

$x_0, x_1, x_2 \ldots x_m$

where, X are not all zero i.e., atleast one should be nonzero value)

Such that,

$$x_0 w + x_1 v_1 + x_2 v_2 + \ldots + x_m v_m = 0 \quad \text{——} ②$$

eq ② is the non-trivial combination of set ①

Hence, assume that $x_0 \neq 0$

$\therefore$ eq ② can be written as

$$x_0 w = -x_1 v_1 - x_2 v_2 - \dots - x_m v_m$$

$$w = \frac{-x_1}{x_0} v_1 - \frac{x_2}{x_0} v_2 - \dots - \frac{-x_m}{x_0} v_m$$

Let $C_1 = \frac{-x_1}{x_0}$, $C_2 = \frac{-x_2}{x_0}$ ---- $C_m = \frac{-x_m}{x_0}$

where these are scalars

$\therefore \quad w = c_1 v_1 + c_2 v_2 + \dots + C_m v_m$

Hence, by defination we have shown that

$\{w, v_1, v_2 \dots v_m\}$ is linearly dependent
Hence proved.

b] To prove:
" if $w \in V$, then $\{v_1, v_1, w, \dots v_m\}$ spans V.

Soln

Let us assume,
$\{W, v_1, v_2, \dots v_m\}$ spans V

Let $v \in V$,
we can write v as a linear combination as

$v = a_0 w + a_1 v_1 + a_2 v_2 \dots + a_m v_m$ ——— ①

where, $a_0, a_1, \dots$ are coefficients

Since, $w \in V$ & S spans V
We can write w as a linear combination as

$w = b_1 v_1 + b_2 v_2 \dots + b_m v_m$ ——— ②

where, $b_1, b_2, \dots$ are coefficients

Substitute eq (2) in eq (1)

$$v = a_0(b_1v_1 + b_2v_2 + \cdots b_mv_m) + a_1v_1 + \cdots + a_mv_m$$

$$= (a_0b_1 + a_1)v_1 + (a_0b_2 + a_2)v_2 \pm \cdots \pm (a_0b_m + a_m)v_m$$

Let us say,

$C_1 = (a_0b_1 + a_1)$, $C_2 = (a_0b_2 + a_2) \cdots$ are scalars

$$\therefore v = C_1v_1 + C_2v_2 + \cdots + C_mv_m$$

Hence, v can be written as a linear combination of vectors in $\{w, v_1, v_2 \cdots v_m\}$

Therefore,

$$\boxed{\{w, v_1, v_2 \cdots v_m\} \text{ spans } V}$$

2] If v_i is a linear combination of $\{v_1, v_2 \ldots v_{i-1}\}$ then S without v_i spans V.

~~To prove~~

We know that, $v_i \in V$, v_i can be written as

$$v_i = a_1v_1 + a_2v_2 + \cdots a_{i-1}v_{i-1} \quad \text{——— (1)}$$

where, $a_1, a_2, \cdots a_{i-1}$ are coefficients.

Now, Let us assume that $v \in V$ & $S = \{v_1, v_2 \ldots v_m\}$

$\therefore$ The linear combination of v can be written as

$$v = c_1 v_1 + c_2 v_2 + \ldots + c_{i-1} v_{i-1} + c_i v_i + c_{i+1} v_{i+1} + \ldots + c_m v_m \longrightarrow (2)$$

where,

$c_1, c_2, c_{i-1}, c_i, c_{i+1}, \ldots c_m$ are coefficients

Substituting eq (1) in eq (2)

$$v = c_1 v_1 + c_2 v_2 + \ldots + c_{i-1} v_{i-1} + c_i (a_1 v_1 + a_2 v_2 + \ldots a_{i-1} v_{i-1}) + c_{i+1} v_{i+1} + \ldots + c_m v_m$$

Now consider,

$$w = c_i (a_1 v_1 + a_2 v_2 + \ldots + a_{i-1} v_{i-1})$$

$$\therefore v = c_1 v_1 + c_2 v_2 + \ldots + c_{i-1} v_{i-1} + w + c_{i+1} v_{i+1} + \ldots + c_m v_m$$

OR

$$v = w + c_1 v_1 + c_2 v_2 + \ldots + c_{i-1} v_{i-1} + c_{i+1} v_{i+1} + \ldots c_m v_m$$

$\therefore$ w is subspace & v without w spans V

The above equation concludes that v is an arbitrary vector in V and above equation is linear combination of vectors in $S\{v_i\}$. $\boxed{\text{Hence } S \text{ without } v_i \text{ spans } V}$

5] Consider the basis

$$B = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} \right\} \text{ of } \mathbb{R}^3$$

Find the change of bases matrix P from the standard basis $\{e_1, e_2, e_3\}$ to the basis B

$\underline{\underline{\text{sol}^n}}$ given

$$B = \left\{ \underset{b_1}{\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}}, \underset{b_2}{\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}}, \underset{b_3}{\begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}} \right\}$$

and

$$E = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

To Find:

$$P = ?$$

we know that,

$$P = BE \quad \text{——— (1)}$$

Substituting values in eq (1)

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 & 1 & 0 \\ 0 & 2 & 3 & 0 & 0 & 1 \end{array}\right] \quad \text{Reducing to RRef form}$$

$$R_2 \to 2R_1 - R_2 \quad \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 \\ 0 & 2 & 3 & 0 & 0 & 1 \end{array}\right]$$

$$R_2 \to -R_2 \quad \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 2 & 3 & 0 & 0 & 1 \end{array}\right]$$

$R_1 \to R_1 - R_2$

$R_3 \to 2R_2 - R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 3 & -1 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 0 & -1 & -4 & 2 & -1 \end{array}\right]$$

$R_3 \to -1R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 3 & -1 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & 4 & -2 & 1 \end{array}\right]$$

$R_1 \to R_1 + R_3$

$R_2 \to R_2 - R_3$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 7 & -3 & 1 \\ 0 & 1 & 0 & -6 & 3 & -1 \\ 0 & 0 & 1 & 4 & -2 & 1 \end{array}\right]$$

$$\therefore \quad P = \begin{bmatrix} 7 & -3 & 1 \\ -6 & 3 & -1 \\ 4 & -2 & 1 \end{bmatrix}$$