Homework 3

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1] Find the singular value decomposition (SVD) of

$$\text{martix } A = \begin{bmatrix} 4 & 2 \\ 2 & -7 \\ 4 & -2 \end{bmatrix}$$

To Find : $A = U \Sigma V^T$ (singular value decomposition of A)

Approch 1 :

Steps to be followed

i] Determine V and then V^T, where V is a matrix whose columns are unit eigenvectors of $A^T . A$

ii] Determine the singular values, σ_i and then Σ, where Σ is a matrix with the singular value of A on the main diagonal and all other entries of zero.

iii] Determine U, using $A = U \Sigma V^T$

sol^n

i] we know that matrix V is an orthogonal matrix whose columns are unit eigenvectors of $A^T A$

Let us determine $A^T A$

$$\begin{bmatrix} 4 & 2 & 4 \\ 2 & -7 & -2 \end{bmatrix} \begin{bmatrix} 4 & 2 \\ 2 & -7 \\ 4 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 4(4)+2(2)+4(4) & 4(2)+2(-7)+4(-2) \\ 2(4)-7(2)-2(4) & 2(2)-7(-7)-2(-2) \end{bmatrix}$$

$$= \begin{bmatrix} 16+4+16 & 8-14-8 \\ 8-14-8 & 4+49+4 \end{bmatrix} = \begin{bmatrix} 36 & -14 \\ -14 & 57 \end{bmatrix}$$

Determing eigenvalues

we know that,

An eigenvalue of A is a Scalar λ, such that $\det(A-\lambda I)=0$

$\therefore \det(B-\lambda I)=0$

where, matrix B is 2×2 matrix resulting from $A^T A$

$$\therefore \det\left(\begin{bmatrix} 36 & -14 \\ -14 & 57 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)=0$$

$$\det\left(\begin{vmatrix} (36-\lambda) & -14 \\ -14 & (57-\lambda) \end{vmatrix}\right)=0$$

$$(36-\lambda)(57-\lambda)-196=0$$

$$2052-57\lambda-36\lambda+\lambda^2-196=0$$

$$\lambda^2-93\lambda+1856$$

solving quadratic equation

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$= \frac{93 \pm \sqrt{(93)^2 - 4(1856}}{2} = \frac{93 \pm \sqrt{8649 - 7424}}{2}$$

$$\lambda = \frac{93 \pm \sqrt{1125}}{2} = \frac{93 \pm 35}{2}$$

$\therefore \lambda = 64, 29$

The next step is to find corresponding unit eigenvectors

Bigining with, $\lambda_1 = 64$

The eigenvectors of A corresponding to λ are the nonzero solutions of $(A - \lambda I)\vec{x} = \vec{0}$

$$\therefore \begin{bmatrix} (36-64) & -14 \\ -14 & 57-64 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Augmented matrix can be written as

$$\left[\begin{array}{cc|c} 28 & 14 & 0 \\ 14 & 7 & 0 \end{array}\right]$$

Reducing to RREF form

$$R_1 \to \tfrac{1}{2} R_1 \left[\begin{array}{cc|c} 1 & 1/2 & 0 \\ 14 & 7 & 0 \end{array}\right] \rightsquigarrow 14R_1 - R_2 \left[\begin{array}{cc|c} 1 & 1/2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

say,

$x_2 = s$

$x_1 = -1/2 s$

$$X_1 = \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} s \qquad |X_1| = \sqrt{\left(\frac{-1}{2}\right)^2 + 1} = \sqrt{\frac{1}{4} + 1} = \frac{\sqrt{5}}{2}$$

For $\lambda_2 = 29$

$$\begin{bmatrix} 36-29 & -14 \\ -14 & 57-29 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Augmented matrix can be written as

$$\left[\begin{array}{cc|c} 7 & -14 & 0 \\ -14 & 28 & 0 \end{array}\right]$$

Reducing it to RREF Form

$$R_1 \to \tfrac{1}{7} R_1 \quad \left[\begin{array}{cc|c} 1 & -2 & 0 \\ -14 & 28 & 0 \end{array}\right]$$

$$R_2 \to 14R_1 + R_2 \quad \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Say, $x_2 = s$

$x_1 = 2s$

$\therefore$

$$X_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} s \qquad |X_2| = \sqrt{2^2 + 1^2} = \sqrt{4+1} = \sqrt{5}$$

V_1 and V_2 Can be written as

$$V_1 = \frac{1}{\frac{\sqrt{5}}{2}} \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} = \frac{2}{\sqrt{5}} \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$$

$$V_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$$

$\therefore$

$$V = \begin{bmatrix} -1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} \quad V^T = \begin{bmatrix} -1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$$

ii] Determing Singular Values σ_i

$$\sigma_1 = \sqrt{\lambda_1} = \sqrt{64} = 8$$
$$\sigma_2 = \sqrt{\lambda_2} = \sqrt{29}$$

$$\therefore \Sigma = \begin{bmatrix} 8 & 0 \\ 0 & \sqrt{29} \end{bmatrix}$$

iii] Determing U using $A = U \Sigma V^T$

we know that,

$$Av_1 = \sigma_1 u_1$$

$$u_1 = \frac{1}{\sigma_1} A v_1 = \frac{1}{8} \begin{bmatrix} 4 & 2 \\ 2 & -7 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$$

$$= \frac{1}{8} \begin{bmatrix} 4 & 2 \\ 2 & -7 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} -1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix} = \frac{1}{8} \begin{bmatrix} 4\left(-1/\sqrt{5}\right) + 2\left(2/\sqrt{5}\right) \\ 2\left(-1/\sqrt{5}\right) - 7\left(2/\sqrt{5}\right) \\ 4\left(-1/\sqrt{5}\right) - 2\left(2/\sqrt{5}\right) \end{bmatrix}$$

$$= \frac{1}{8} \begin{bmatrix} \frac{-4}{\sqrt{5}} + \frac{4}{\sqrt{5}} \\ \frac{-2}{\sqrt{5}} - \frac{14}{\sqrt{5}} \\ -4/\sqrt{5} - 4/\sqrt{5} \end{bmatrix} = \frac{1}{8} \begin{bmatrix} 0 \\ -16/\sqrt{5} \\ -8/\sqrt{5} \end{bmatrix} = \begin{bmatrix} 0 \\ -2/\sqrt{5} \\ -1/\sqrt{5} \end{bmatrix}$$

$$u_2 = \frac{1}{\sigma_2} \cdot A \cdot v_2 = \frac{1}{\sqrt{29}} \begin{bmatrix} 4 & 2 \\ 2 & -7 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$$

$$= \frac{1}{\sqrt{29}} \begin{bmatrix} 4\left(2/\sqrt{5}\right) + 2\left(1/\sqrt{5}\right) \\ 2\left(2/\sqrt{5}\right) + (-7)\left(1/\sqrt{5}\right) \\ 4\left(2/\sqrt{5}\right) - 2\left(1/\sqrt{5}\right) \end{bmatrix} = \frac{1}{\sqrt{29}} \begin{bmatrix} 8/\sqrt{5} + 2/\sqrt{5} \\ 4/\sqrt{5} - 7/\sqrt{5} \\ 8/\sqrt{5} - 2/\sqrt{5} \end{bmatrix}$$

$$= \frac{1}{\sqrt{29}} \begin{bmatrix} 10/\sqrt{5} \\ -3/\sqrt{5} \\ 6/\sqrt{5} \end{bmatrix} = \begin{bmatrix} 10/\sqrt{145} \\ -3/\sqrt{145} \\ 6/\sqrt{145} \end{bmatrix}$$

$\therefore$

$$A = \begin{bmatrix} 0 & 10/\sqrt{145} \\ -2/\sqrt{5} & -3/\sqrt{145} \\ -1/\sqrt{5} & 6/\sqrt{145} \end{bmatrix} \begin{bmatrix} 8 & 0 \\ 0 & \sqrt{29} \end{bmatrix} \begin{bmatrix} -1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$$

1 question's answer.

Approach 2:

Defination of SVD of matrix

Let A be an mxn matrix. Then $A = U\Sigma V^T$ is the singular value decomposition of A. Such that,

* U is an mxm orthogonal matrix with Columns equal to the unit eigenvectors of $A.A^T$
* V is an nxn Orthogonal matrix whose columns are unit eigenvectors of $A^T.A$.
* Σ is an mxn matrix with the singular values of A on the main diagonal and all Other entries of zero.

$$\therefore A_{3\times 2} = U_{3\times 3} \Sigma_{3\times 2} V^T_{2\times 2}$$

i] Steps to be followed

i] Determine U

ii] Determine Σ

iii] Determine V, using $A = U\Sigma V^T$

Sol^n given Matrix

$$A = \begin{bmatrix} 4 & 2 \\ 2 & -7 \\ 4 & -2 \end{bmatrix}$$

Let us determine $A A^T$

$$\underset{3\times 2}{\begin{bmatrix} 4 & 2 \\ 2 & -7 \\ 4 & -2 \end{bmatrix}} \times \underset{2\times 3}{\begin{bmatrix} 4 & 2 & 4 \\ 2 & -7 & -2 \end{bmatrix}}$$

$$= \cancel{4(4)+2(2) \quad 2(4)-7(2) \quad 4(4)-2(2)}$$

$$= \begin{bmatrix} 4(4)+2(2) & 4(2)+2(-7) & 4(4)+2(-2) \\ 2(4)-7(2) & 2(2)-7(-7) & 2(4)-7(-2) \\ 4(4)-2(2) & 4(2)-2(-7) & 4(4)-2(-2) \end{bmatrix}$$

$$= \begin{bmatrix} 16+4 & 8-14 & 16-4 \\ 8-14 & 4+49 & 8+14 \\ 16-4 & 8+14 & 16+4 \end{bmatrix}$$

$$= \begin{bmatrix} 20 & -6 & 12 \\ -6 & 53 & 22 \\ 12 & 22 & 20 \end{bmatrix} = A \cdot A^T$$

Determine eigenvalues

We know that,

An eigen value of A is a scalar λ, such that $\det(A - \lambda I) = 0$

$\therefore \det(B - \lambda I) = 0$

where, matrix B is 3×3 matrix resulting from $A \cdot A^T$

$$\therefore \det\left(\begin{vmatrix} (20-\lambda) & -6 & 12 \\ -6 & (53-\lambda) & 22 \\ 12 & 22 & (20-\lambda) \end{vmatrix}\right) = 0$$

$$(20-\lambda)\cdot\begin{vmatrix} (53-\lambda) & 22 \\ 22 & (20-\lambda) \end{vmatrix} - (-6)\cdot\begin{vmatrix} -6 & 22 \\ 12 & (20-\lambda) \end{vmatrix} + 12\cdot\begin{vmatrix} -6 & (53-\lambda) \\ 12 & 22 \end{vmatrix}$$

$$= (20-\lambda)\left[(53-\lambda)(20-\lambda)-484\right] - (-6)\left[-6(20-\lambda)-264\right] + 12\left[-132-12(53-\lambda)\right]$$

$$= (20-\lambda)\left[(53-\lambda)(20-\lambda)-484\right] - (-6)\left[-120+6\lambda-264\right] + 12\left[-132-636+12\lambda\right]$$

$$= (20-\lambda)\left[(53-\lambda)(20-\lambda)-484\right] - \left[720-36\lambda+1584\right] + \left[-1584-7632+144\lambda\right]$$

$$= (20-\lambda)\left[(53-\lambda)(20-\lambda)-484\right] - 2304+36\lambda-9216+144\lambda$$

$$(20-\lambda)(53-\lambda)(20-\lambda)-484(20-\lambda)+180\lambda-11520$$

$$(1060-53\lambda-20\lambda+\lambda^2)(20-\lambda)-9680+484\lambda+180\lambda-11520$$

$$(\lambda^2-73\lambda+1060)(20-\lambda)+664\lambda-21200$$

$$20\lambda^2-1460\lambda+\cancel{21200}-\lambda^3+73\lambda^2-1060\lambda+664\lambda-\cancel{21200}$$

$$-\lambda^3+93\lambda^2-1856\lambda=0$$

$$-\lambda\left(\lambda^2 - 93\lambda + 1856\right) = 0$$

solving quadratic equation

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{93 \pm \sqrt{(93)^2 - 4(1856)}}{2}$$

$$= \frac{93 \pm \sqrt{8649 - 7424}}{2} = \frac{93 \pm \sqrt{1255}}{2}$$

$$\lambda = \frac{93 \pm 35}{2} = \frac{\cancel{128}^{\,64}}{\cancel{2}_{\,1}}, \frac{\cancel{58}^{\,29}}{\cancel{2}_{\,1}}$$

$$\boxed{\therefore \lambda = 64, 29, 0}$$

The next step is to find corresponding unit eigenvectors

Biginning with $\lambda_1 = 64$

The eigenvectors of A corresponding to λ are the nonzero solutions of $(A - \lambda I)\vec{x} = \vec{0}$.

$$\begin{bmatrix} (20-\lambda) & -6 & 12 \\ -6 & (53-\lambda) & 22 \\ 12 & 22 & (20-\lambda) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 20-64 & -6 & 12 \\ -6 & 53-64 & 22 \\ 12 & 22 & 20-64 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

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$$\left[\begin{array}{ccc|c} -44 & -6 & 12 & 0 \\ -6 & -11 & 22 & 0 \\ 12 & 22 & -44 & 0 \end{array}\right]$$

Reducing it to RREF Form

$R_1 \to \frac{R_1}{-44}$
$$\left[\begin{array}{ccc|c} 1 & 3/22 & -3/11 & 0 \\ -6 & -11 & 22 & 0 \\ 12 & 22 & -44 & 0 \end{array}\right] \begin{array}{l} R_2 \to 6R_1 + R_2 \\ R_3 \to R_3 - 12R_1 \end{array}$$

$$\begin{bmatrix} 1 & 3/22 & -3/11 \\ 0 & -112/11 & 224/11 \\ 0 & 224/11 & -448/11 \end{bmatrix} \quad R_2 \to -\frac{11 \times R_2}{112}$$

$$\begin{bmatrix} 1 & 3/22 & -3/11 \\ 0 & 1 & -2 \\ 0 & 224/11 & -448/11 \end{bmatrix} \quad \begin{array}{l} R_1 \to R_1 - 3/22\, R_2 \\ R_3 \to \frac{224}{11} R_{R_2} - R_3 \end{array}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

Say, $x_3 = s$
$x_1 = 0$
$x_2 = 2s$

$$u_1 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} s$$

Unit vector can be written as

$$|u_1| = \sqrt{0^2 + 2^2 + 1} = \sqrt{4+1} = \sqrt{5}$$

$$u_1 = \begin{bmatrix} 0 \\ 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$$

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$\lambda_2 = 29$

$$\begin{bmatrix} (20-29) & -6 & 12 \\ -6 & (53-29) & 22 \\ 12 & 22 & (20-29) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} -9 & -6 & 12 & 0 \\ -6 & 24 & 22 & 0 \\ 12 & 22 & -9 & 0 \end{array}\right] \quad R_1 \to \frac{R_1}{-9}$$

Reducing it to RREF form

$$\left[\begin{array}{ccc|c} 1 & 2/3 & -4/3 & 0 \\ -6 & 24 & 22 & 0 \\ 12 & 22 & -9 & 0 \end{array}\right] \quad \begin{array}{l} R_2 \to 6R_1 + R_2 \\ R_3 \to R_3 - 12R_1 \end{array}$$

$$\begin{bmatrix} 1 & 2/3 & -4/3 \\ 0 & 28 & 14 \\ 0 & 14 & 7 \end{bmatrix} \quad R_2 \to \frac{R_2}{28}$$

$$\begin{bmatrix} 1 & 2/3 & -4/3 \\ 0 & 1 & 1/2 \\ 0 & 14 & 7 \end{bmatrix} \quad \begin{array}{l} R_1 \to R_1 - 2/3\, R_2 \\ \\ R_3 \to R_2(14) - R_3 \end{array} \quad \begin{bmatrix} 1 & 0 & -5/3 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{bmatrix}$$

say = $x_3 = s$, $x_1 = 5/3\, s$, $x_2 = -1/2\, s$

$$u_2 = \begin{bmatrix} 5/3 \\ -1/2 \\ 1 \end{bmatrix} s$$

Unit vector can be written as

$$\|u_2\| = \sqrt{(5/3)^2 + (-1/2)^2 + 1^2}$$

$$= \sqrt{\frac{25}{9}^{\times 4} + \frac{1}{4}^{\times 9} + 1^{\times 36}} = \sqrt{\frac{145}{36}} = \frac{\sqrt{145}}{6}$$

$$u_2 = \frac{6}{\sqrt{145}} \cdot \begin{bmatrix} 5/3 \\ -1/2 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{\cancel{6}^{2}}{\sqrt{145}} \times \frac{5}{\cancel{3}_1} \\ \frac{\cancel{6}^{3}}{\sqrt{145}} \times \frac{-1}{\cancel{2}_1} \\ \frac{6}{\sqrt{145}} \end{bmatrix} = \begin{bmatrix} 10/\sqrt{145} \\ -3/\sqrt{145} \\ 6/\sqrt{145} \end{bmatrix}$$

$\lambda_3 = 0$

$$\begin{bmatrix} (20-0) & -6 & 12 \\ -6 & (53-0) & 22 \\ 12 & 22 & (20-0) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Writing it in augmented matrix and Reducing it in RREF form

$$\left[\begin{array}{ccc|c} 20 & -6 & 12 & 0 \\ -6 & 53 & 22 & 0 \\ 12 & 22 & 20 & 0 \end{array}\right] \quad R_1 \to R_1/20$$

$$\left[\begin{array}{ccc|c} 1 & -3/10 & 3/5 & 0 \\ -6 & 53 & 22 & 0 \\ 12 & 22 & 20 & 0 \end{array}\right] \begin{array}{l} R_2 \to 6R_1 + R_2 \\ R_3 \to 12R_1 - R_3 \end{array} \left[\begin{array}{ccc|c} 1 & -3/10 & 3/5 & 0 \\ 0 & 256/5 & 128/5 & 0 \\ 0 & -128/5 & -64/5 & 0 \end{array}\right]$$

$$R_2 \to \frac{5}{256} \times R_2 \left[\begin{array}{ccc|c} 1 & -3/10 & 3/5 & 0 \\ 0 & 1 & 1/2 & 0 \\ 0 & -128/5 & -64/5 & 0 \end{array}\right] \begin{array}{l} R_1 \to R_1 + 3/10\, R_2 \\ \\ R_3 \to 128/5\, R_2 + R_3 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 3/4 & 0 \\ 0 & 1 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

say, $x_3 = s$

$x_1 = -3/4\, s$

$x_2 = -1/2\, s$

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$$u_3 = \begin{bmatrix} -3/4 \\ -1/2 \\ 1 \end{bmatrix} s \qquad |u_3| = \sqrt{\left(\frac{-3}{4}\right)^2 + \left(\frac{1}{2}\right)^2 + 1}$$

$$= \sqrt{\frac{9}{16} + \frac{1}{4} + 1} = \sqrt{\frac{9+4+16}{16}}$$

unit vector can be written as

$$= \sqrt{\frac{29}{16}} = \sqrt{29}/4$$

$$u_3 = \frac{4}{\sqrt{29}} \times \begin{bmatrix} -3/4 \\ -1/2 \\ 1 \end{bmatrix} = \begin{bmatrix} -3/\sqrt{29} \\ -2/\sqrt{29} \\ 4/\sqrt{29} \end{bmatrix}$$

$\therefore$

$$U = \begin{bmatrix} 0 & 10/\sqrt{145} & -3/\sqrt{29} \\ 2/\sqrt{5} & -3/\sqrt{145} & -2/\sqrt{29} \\ 1/\sqrt{5} & 6/\sqrt{145} & 4/\sqrt{29} \end{bmatrix}$$

Determining Singular Values σ_i

$$\sigma_1 = \sqrt{\lambda_1} = \sqrt{64} = 8$$

$$\sigma_2 = \sqrt{\lambda_2} = \sqrt{29}$$

$$\sigma_3 = 0$$

$\therefore$

$$\Sigma = \begin{bmatrix} 8 & 0 & 0 \\ 0 & \sqrt{29} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

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iii) Determing V using $A = U \Sigma V^T$

We know that $v_i = \frac{1}{\sigma_i} A^T u_i$

$$v_1 = \frac{1}{8} \begin{bmatrix} 4 & 2 & 4 \\ 2 & -7 & -2 \end{bmatrix} \begin{bmatrix} 0 \\ 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$$

$$= \frac{1}{8} \begin{bmatrix} 0 + 4/\sqrt{5} + 4/\sqrt{5} \\ 0 - 14/\sqrt{5} - 2/\sqrt{5} \end{bmatrix} = \frac{1}{8} \begin{bmatrix} 8/\sqrt{5} \\ -16/\sqrt{5} \end{bmatrix}$$

$$v_1 = \begin{bmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \end{bmatrix}$$

$$v_2 = \frac{1}{\sqrt{29}} \times \begin{bmatrix} 4 & 2 & 4 \\ 2 & -7 & -2 \end{bmatrix} \begin{bmatrix} 10/\sqrt{145} \\ -3/\sqrt{145} \\ 6/\sqrt{145} \end{bmatrix}$$

$$= \frac{1}{\sqrt{29}} \begin{bmatrix} \frac{40}{\sqrt{145}} - \frac{6}{\sqrt{145}} + \frac{24}{\sqrt{145}} \\ \frac{20}{\sqrt{145}} + \frac{21}{\sqrt{145}} - \frac{12}{\sqrt{145}} \end{bmatrix} = \frac{1}{\sqrt{29}} \begin{bmatrix} \frac{58}{\sqrt{145}} \\ \frac{29}{\sqrt{145}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{29}} \times \frac{58}{\sqrt{29} \times \sqrt{5}} \\ \frac{29}{\sqrt{29} \times \sqrt{5}} \times \frac{1}{\sqrt{29}} \end{bmatrix} = \begin{bmatrix} \frac{\cancel{58}\,2}{\cancel{29}\sqrt{5}} \\ \frac{\cancel{29}}{\cancel{29}\sqrt{5}} \end{bmatrix} = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$$

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$$V^T = \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 10/\sqrt{145} & -3/\sqrt{29} \\ 2/\sqrt{5} & -3/\sqrt{145} & -2/\sqrt{29} \\ 1/\sqrt{5} & 6/\sqrt{145} & 4/\sqrt{29} \end{bmatrix} \begin{bmatrix} 8 & 0 \\ 0 & \sqrt{29} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$$

2] Decompose matrix

$$B = \begin{bmatrix} 11 & -8 & 4 \\ -8 & -1 & -2 \\ 4 & -2 & -2 \end{bmatrix}$$

Find the characteristic polynomial, the eigenvalues and eigenvectors.

soln i] characteristic polynomial

$$\det(A - \lambda I) = 0$$

$$\det\left(\begin{bmatrix} 11 & -8 & 4 \\ -8 & -1 & -2 \\ 4 & -2 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) = 0$$

$$\det\left(\begin{vmatrix} (11-\lambda) & -8 & 4 \\ -8 & (-1-\lambda) & -2 \\ 4 & -2 & (-2-\lambda) \end{vmatrix} \right) = 0$$

$$(11-\lambda) \begin{vmatrix} (-1-\lambda) & -2 \\ -2 & (-2-\lambda) \end{vmatrix} + 8 \begin{vmatrix} -8 & -2 \\ 4 & (-2-\lambda) \end{vmatrix} + 4 \begin{vmatrix} -8 & (-1-\lambda) \\ 4 & -2 \end{vmatrix}$$

$$= (11-\lambda)\left[(-1-\lambda)(-2-\lambda) - 4\right] + 8\left[-8(-2-\lambda) + 8\right] + 4\left[16 - 4(-1-\lambda)\right]$$

$$= (11-\lambda)\left[(-1-\lambda)(-2-\lambda) - 4\right] - 64(-2-\lambda) + 64 + 64 - 16(-1-\lambda)$$

$$= (11-\lambda)\left[(-1-\lambda)(-2-\lambda) - 4\right] + 128 + 64\lambda + 64 + 64 + 16 + 16\lambda$$

$$= (11-\lambda)\left[(-1-\lambda)(-2-\lambda) - 4\right] + 80\lambda + 272$$

$$= (11-\lambda)(-1-\lambda)(-2-\lambda) - 4(11-\lambda) + 80\lambda + 272$$

$= (-11-11\lambda+\lambda+\lambda^2)(-2-\lambda)-44+4\lambda+80\lambda+272$

$= (\lambda^2-10\lambda-11)(-2-\lambda)+84\lambda+228$

$= -2\lambda^2+20\lambda+22-\lambda^3+10\lambda^2+11\lambda+84\lambda+228$

$= -\lambda^3+8\lambda^2+115\lambda+250 = 0$

$\hookrightarrow$ Characteristic polynomial

ii] Determing eigenvalues

Solving above eqn

$-\lambda^3+8\lambda^2+115\lambda+250$

Substituting -5 in above eqn gives 0

$$250 \quad \wedge \quad \pm 1 \quad \pm 2 \quad \pm 5 \quad \pm 10 \quad \pm 25$$

$$\begin{array}{r|rrrr} -5 & -1 & 8 & 115 & 250 \\ & \downarrow & 5 & -65 & -250 \\ \hline & -1 & 13 & 50 & 0 \end{array}$$

$\therefore (-\lambda^2+13\lambda+50)(\lambda+5) = 0$

$\lambda = \dfrac{-b \pm \sqrt{b^2-4ac}}{2a} = \dfrac{-13 \pm \sqrt{(13)^2+4\times 50}}{2(-1)} = \dfrac{-13 \pm \sqrt{169+200}}{-2}$

$= \dfrac{-13 \pm \sqrt{369}}{-2} = \dfrac{-13+19.2094}{-2}, \dfrac{-13-19.2094}{2}$

$= \dfrac{6.2094}{-2}, \dfrac{-32.2094}{2}$

$\lambda = -5, -3.1047, 16.1047$

The next step is to find corresponding ~~unit~~ eigenvectors

Bigining with, $\lambda_1 = -5$

$$\begin{bmatrix} 11+5 & -8 & 4 \\ -8 & -1+5 & -2 \\ 4 & -2 & -2+5 \end{bmatrix} = \begin{bmatrix} 16 & -8 & 4 \\ -8 & 4 & -2 \\ 4 & -2 & 3 \end{bmatrix}$$

Reducing it to RREF Form

$$R_1 \to \frac{1}{16} R_1 \begin{bmatrix} 1 & -1/2 & 1/4 \\ -8 & 4 & -2 \\ 4 & -2 & 3 \end{bmatrix} \quad R_2 \to 8R_1 + R_2 \begin{bmatrix} 1 & -1/2 & 1/4 \\ 0 & 0 & 0 \\ 4 & -2 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1/2 & 1/4 \\ 4 & -2 & 3 \\ 0 & 0 & 0 \end{bmatrix} \quad R_2 \to 4R_1 - R_2 \begin{bmatrix} 1 & -1/2 & 1/4 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix} \quad R_2 \to \frac{1}{-2} R_2$$

$$\begin{bmatrix} 1 & -1/2 & 1/4 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad R_1 \to \frac{1}{-4} R_2 + R_1 \begin{bmatrix} 1 & -1/2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Say $x_2 = s$

$$X = \begin{bmatrix} +1/2 \\ 1 \\ 0 \end{bmatrix} s$$

$\lambda_2 = -3.1047$

$$\begin{bmatrix} 11+3.1047 & -8 & 4 \\ -8 & 3.1047-1 & -2 \\ 4 & -2 & 3.1047-2 \end{bmatrix} \begin{bmatrix} 14.1047 & -8 & 4 \\ -8 & 2.1047 & -2 \\ 4 & -2 & 1.1047 \end{bmatrix}$$

$$R_1 \rightarrow \frac{1}{14.1047} \times R_1 \begin{bmatrix} 1 & -0.5672 & 0.2836 \\ -8 & 2.1047 & -2 \\ 4 & -2 & 1.1047 \end{bmatrix} \quad \begin{matrix} R_2 \rightarrow 8R_1 + R_2 \\ R_3 \rightarrow 4R_1 - R_3 \end{matrix}$$

$$\begin{bmatrix} 1 & -0.5672 & 0.2836 \\ 0 & -2.4329 & 0.2688 \\ 0 & -0.2688 & 0.0297 \end{bmatrix} R_2 \rightarrow \frac{1}{-2.4329} R_2 \begin{bmatrix} 1 & -0.5672 & 0.2836 \\ 0 & 1 & -0.1105 \\ 0 & -0.2688 & 0.0297 \end{bmatrix}$$

$$\begin{matrix} R_1 \rightarrow 0.5672R_2 + R_1 \\ R_3 \rightarrow 0.2688R_2 + R_3 \end{matrix} \begin{bmatrix} 1 & 0 & 0.2209 \\ 0 & 1 & -0.1105 \\ 0 & 0 & 0.000 \end{bmatrix}$$

Say $x_3 = s$

$x_1 = -0.2209$

$x_2 = 0.1105$

$$\therefore \quad X_2 = \begin{bmatrix} -0.2209 \\ 0.1105 \\ 1 \end{bmatrix} s$$

$\lambda_3 = 16.1047$

$$\begin{bmatrix} 11-16.1047 & -8 & 4 \\ -8 & -16.1047-1 & -2 \\ 4 & -2 & -16.1047-2 \end{bmatrix} = \begin{bmatrix} -5.1047 & -8 & 4 \\ -8 & -17.1047 & -2 \\ 4 & -2 & -18.1047 \end{bmatrix}$$

Reducing it to RREF Form

$R_1 \to \frac{1}{-5.1047} \times R_1$ $\begin{bmatrix} 1 & 1.5672 & -0.7836 \\ -8 & -17.1047 & -2 \\ 4 & -2 & -18.1047 \end{bmatrix}$ $R_2 \to 8R_1 + R_2$

$R_3 \to 4R_1 - R_3$

$$\begin{bmatrix} 1 & 1.5672 & -0.7836 \\ 0 & -4.5671 & -8.2688 \\ 0 & 8.2688 & 14.9703 \end{bmatrix} R_2 \to \frac{R_2}{-4.5671} \begin{bmatrix} 1 & 1.5672 & -0.7836 \\ 0 & 1 & 1.81052 \\ 0 & 8.2688 & 14.9703 \end{bmatrix}$$

$R_1 \to R_1 - 1.5672R_2$

$R_3 \to 8.2688R_2 - R_3$

$$\begin{bmatrix} 1 & 0 & -3.6211 \\ 0 & 1 & 1.81052 \\ 0 & 0 & 0.0000 \end{bmatrix}$$

Say = $x_3 = s$

$x_1 = 3.6211$

$x_2 = -1.81052$

$$X_3 = \begin{bmatrix} 3.6211 \\ -1.8105 \\ 1 \end{bmatrix} s$$

i] characteristic polynomial

$-\lambda^3 + 8\lambda^2 + 115\lambda + 250 = 0$

ii] Eigenvalues $\to \lambda = -5, -3.1047, 16.1047$

iii] Eigenvectors $\to v_1 = \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} -0.2209 \\ 0.1105 \\ 1 \end{bmatrix} \begin{bmatrix} 3.6211 \\ -1.8105 \\ 1 \end{bmatrix}$

3] Find the determinant of the following matrix using LaPlace expansion.

$$C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 15 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Sol^n LaPlace expansion can be written as

$$\det(A) = \sum_{K=1}^{n} (-1)^{K+j} a_{Kj} \cdot \det(A_{Kj})$$

given, $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 15 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

$$\det(A) = (-1)^2 \cdot (1) \cdot \begin{vmatrix} 15 & 6 \\ 8 & 9 \end{vmatrix} + (-1)^3 \cdot 2 \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + (-1)^4 \cdot 3 \cdot \begin{vmatrix} 4 & 15 \\ 7 & 8 \end{vmatrix}$$

$$= 1\,(15\times 9 - (8\times 6)) - 2\left[(4\times 9) - (7\times 6)\right] + 3\left[(4\times 8) - (15\times 7)\right]$$

$$= (135-48) - 2(36-42) + 3(32-105)$$

$$= 87 - 2(-6) + 3(-73)$$

$$= 87 + 12 - 219$$

$$\det(A) = -120$$

4] Find the rank-1 approximation of

$$D = \begin{bmatrix} 2 & 4 & 3 \\ -2 & 3 & 2 \end{bmatrix}$$

Sol^n

Defination:

The best rank-one approximation is

$$A = \sigma_1 u_1 v_1^T$$

where,

σ_1 = First (largest) singular value,

u_1 = the first left singular vector,

v_1 = the first right singular vector

$$A = \begin{bmatrix} 2 & 4 & 3 \\ -2 & 3 & 2 \end{bmatrix}$$

Let Determine $A^T A$

$$\begin{bmatrix} 2 & -2 \\ 4 & 3 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 2 & 4 & 3 \\ -2 & 3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2(2)+2(2) & 2(4)-2(3) & 2(3)-2(2) \\ 4(2)+3(-2) & 4(4)+3(3) & 4(3)+3(2) \\ 3(2)+2(-2) & 3(4)+2(3) & 3(3)+2(2) \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 8-6 & 6-4 \\ 8-6 & 16+9 & 12+6 \\ 6-4 & 12+6 & 9+4 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 2 & 2 \\ 2 & 25 & 18 \\ 2 & 18 & 13 \end{bmatrix}$$

Determining the eigen values

we know that,

$$\det(B-\lambda I)=0$$

$$\det\left(\begin{bmatrix}(8-\lambda) & 2 & 2\\ 2 & (25-\lambda) & 18\\ 2 & 18 & (13-\lambda)\end{bmatrix}\right)$$

$$=(8-\lambda)\begin{vmatrix}(25-\lambda) & 18\\ 18 & (13-\lambda)\end{vmatrix}-2\begin{vmatrix}2 & 18\\ 2 & (13-\lambda)\end{vmatrix}+2\begin{vmatrix}2 & (25-\lambda)\\ 2 & 18\end{vmatrix}$$

$$=(8-\lambda)\left[(25-\lambda)(13-\lambda)-324\right]-2\left[2(13-\lambda)-36\right]+2\left[36-2(25-\lambda)\right]$$

$$=(8-\lambda)\left[(25-\lambda)(13-\lambda)-324\right]-4(13-\lambda)+72+72-4(25-\lambda)$$

$$=(8-\lambda)(25-\lambda)(13-\lambda)-324(8-\lambda)-4(13-\lambda)-4(25-\lambda)+72+72$$

$$=(200-8\lambda-25\lambda+\lambda^2)(13-\lambda)-2592+324\lambda-52+4\lambda-100+4\lambda+144$$

$$=(\lambda^2-33\lambda+200)(13-\lambda)+324\lambda+4\lambda+4\lambda-2592-52-100+144$$

$$=(\lambda^2-33\lambda+200)(13-\lambda)+332\lambda-2600$$

$$=13\lambda^2-429\lambda+\cancel{2600}-\lambda^3+33\lambda^2-200\lambda+332\lambda-\cancel{2600}$$

$$=-\lambda^3+13\lambda^2+33\lambda^2-429\lambda-200\lambda+332\lambda$$

$$=-\lambda^3+46\lambda^2-297\lambda$$

$$=-\lambda(\lambda^2-46\lambda+297)=0$$

$\lambda^2 - 46\lambda + 297$

Solving quadratic equation using formula

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{46 \pm \sqrt{(46)^2 - 4(1)(297)}}{2(1)}$$

$$= \frac{46 \pm \sqrt{2116 - 1188}}{2} = \frac{46 \pm \sqrt{928}}{2} = \frac{46 \pm \sqrt{58 \times 16}}{2}$$

$$= \frac{46 \pm 4\sqrt{58}}{2} = \frac{\cancel{2}(23 \pm 2\sqrt{58})}{\cancel{2}}$$

$\therefore \lambda = 23 + 2\sqrt{58},\ 23 - 2\sqrt{58},\ 0$

OR

$\lambda = 38.2316,\ 7.7685,\ 0$

i] Calculating σ_1

$$\sigma_1 = \sqrt{\lambda_1} = \sqrt{38.2316} = 6.1832$$

ii] Calculating v_1

$\lambda_1 = 38.2316$

$$\begin{bmatrix} 8 - 38.2316 & 2 & 2 \\ 2 & 25 - 38.2316 & 18 \\ 2 & 18 & 13 - 38.2316 \end{bmatrix}$$

$$\begin{bmatrix} -30.2316 & 2 & 2 \\ 2 & -13.2316 & 18 \\ 2 & 18 & -25.2316 \end{bmatrix}$$

Reducing it to RREF Form

$$R_1 \to \frac{R_1}{-30.2316} \begin{bmatrix} 1 & -0.0662 & -0.0662 \\ 2 & -13.2316 & 18 \\ 2 & 18 & -25.2316 \end{bmatrix} \begin{matrix} R_2 \to 2R_1 - R_2 \\ R_3 \to 2R_1 - R_3 \end{matrix}$$

$$\begin{bmatrix} 1 & -0.0662 & -0.0662 \\ 0 & 13.0992 & -18.1324 \\ 0 & -18.1324 & 25.0992 \end{bmatrix} \quad R_2 \to \frac{R_2}{13.0992}$$

$$\begin{bmatrix} 1 & -0.0662 & -0.0662 \\ 0 & 1 & -1.3843 \\ 0 & -18.1324 & 25.0992 \end{bmatrix} \quad \begin{matrix} R_1 \to R_1 + 0.0662 R_2 \\ R_3 \to R_3 + 18.1324 R_2 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & -0.1579 \\ 0 & 1 & -1.3843 \\ 0 & 0 & 0.000 \end{bmatrix}$$

say $x_3 = s$, $x_1 = 0.1572$, $x_2 = 1.3843$

$$V_1 = \begin{bmatrix} 0.1572 \\ 1.3843 \\ 1 \end{bmatrix} s$$

ii] Calculating U_1 using $A = U \Sigma V^T$

$$u_1 = \frac{1}{\sigma_1} A V_1$$

$$u_1 = \frac{1}{6.1832} \begin{bmatrix} 2 & 4 & 3 \\ -2 & 3 & 2 \end{bmatrix} \begin{bmatrix} 0.1572 \\ 1.3843 \\ 1 \end{bmatrix}$$

$$= \frac{1}{6.1832}\begin{bmatrix} 2(0.1573)+4(1.3843)+3(1) \\ -2(0.1573)+3(1.3843)+2(1) \end{bmatrix}$$

$$= \frac{1}{6.1832}\begin{bmatrix} 0.3146+5.5317+3 \\ -0.3146+4.1529+2 \end{bmatrix}$$

$$= \frac{1}{6.1832}\begin{bmatrix} 8.8459 \\ 5.8383 \end{bmatrix} = \begin{bmatrix} 1.4306 \\ 0.9443 \end{bmatrix}$$

Finally Calculating

$$D = \sigma_1 u_1 v_1^T$$

$$= 6.1832 \begin{bmatrix} 1.4306 \\ 0.9443 \end{bmatrix} \begin{bmatrix} 0.1572 & 1.3843 & 1 \end{bmatrix}$$

$$= 6.1832 \begin{bmatrix} 1.4306(0.1572) & 1.4306(1.3843) & 1.4306(1) \\ 0.9443(0.1572) & 0.9443(1.3843) & 0.9443(1) \end{bmatrix}$$

$$= 6.1832 \begin{bmatrix} 0.2249 & 1.9804 & 1.4306 \\ 0.1485 & 1.3072 & 0.9443 \end{bmatrix}$$

$$D = \begin{bmatrix} 1.3907 & 12.2453 & 8.84857 \\ 0.1983 & 8.0827 & 5.8388 \end{bmatrix}$$

$\hookrightarrow$ Rank-One approximation

5] Find the Choleky decomposition of

$$E = \begin{bmatrix} 4 & 2 & 6 \\ 2 & 10 & 7 \\ 6 & 7 & 21 \end{bmatrix}$$

To Find : $E = L \cdot L^T$

sol^n Choleky decomposition states that
If A is a symmetric positive definite matrix such that $x^T A x \geq 0$
then, A can be decomposed into $A = LL^T$

Given matrix

$$E = \begin{bmatrix} 4 & 2 & 6 \\ 2 & 10 & 7 \\ 6 & 7 & 21 \end{bmatrix}$$

By inspection, given matrix is symmetric positive definite matrix

∴ Formula to find elements of L and L^T can be written as

$$l_{ki} = \frac{a_{ki} - \sum_{j=1}^{k-1} l_{ij} \cdot l_{kj}}{l_{ii}}$$

where $l_{kk} = \sqrt{a_{kk} - \sum_{j=1}^{k-1} l_{kj}^2}$

i] $l_{11} = \sqrt{a_{11}} = \sqrt{4} = 2$

ii] $l_{21} = \dfrac{a_{21}}{l_{11}} = \dfrac{2}{2} = 1$

iii] $l_{22} = \sqrt{a_{22} - l_{21}^2} = \sqrt{10 - 1^2} = \sqrt{9} = 3$

v] $l_{32} = \dfrac{a_{32} - l_{31} \times l_{21}}{l_{22}} = \dfrac{7 - 3 \times 1}{3} = \dfrac{4}{3}$

vi] $l_{33} = \sqrt{a_{33} - l_{31}^2 - l_{32}^2} = \sqrt{21 - 3^2 - \left(\dfrac{4}{3}\right)^2}$

$= \sqrt{21 - 9 - \dfrac{16}{9}} = \sqrt{\dfrac{189 - 81 - 16}{9}} = \sqrt{\dfrac{92}{9}}$

$l_{33} = \dfrac{2\sqrt{23}}{3}$

Matrix L can be written as

$$L = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 3 & 0 \\ 3 & 4/3 & \dfrac{2\sqrt{23}}{3} \end{bmatrix} \quad L^T = \begin{bmatrix} 2 & 1 & 3 \\ 0 & 3 & 4/3 \\ 0 & 0 & \dfrac{2\sqrt{23}}{3} \end{bmatrix}$$

$\therefore A = L \cdot L^T$

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 3 & 0 \\ 3 & \dfrac{4}{3} & \dfrac{2\sqrt{23}}{3} \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 \\ 0 & 3 & 4/3 \\ 0 & 0 & \dfrac{2\sqrt{23}}{3} \end{bmatrix}$$