# Homework-4

UID:

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1] Compute the derivate $f'(x)$ for
$$f(x) = e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$
where $\mu$ and $\sigma$ can be treated as constants.

Sol^n Some related derivates can be written as

$$f(x) = e^{x} \longrightarrow f'(x) = e^{x}$$

$$f(x) = e^{2x} \longrightarrow f'(x) = 2e^{x}$$

$$f(x) = e^{-2x} \longrightarrow -2e^{-2x}$$

$$f(x) = e^{(x-4)^2} \longrightarrow 2(x-4).e^{(x-4)^2}$$

$$f(x) = e^{x^2} \longrightarrow 2xe^{x^2}$$

$$f(x) = x^2 = 2x$$

$$\therefore f(x) = e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

$$= +\cancel{2}(x-\mu) \times \frac{-1}{\cancel{2}\sigma^2} \times e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

$$\boxed{f'(x) = \frac{-(x-\mu)}{\sigma^2} \times e^{-(x-\mu)^2/2\sigma^2}}$$

2] Compute the derivatives $\frac{df}{dx}$ of the following functions. Describe your steps in detail.

a] Use the chain rule. Provide the dimensions of every single partial derivative

$$f(z) = e^{-1/2 z}$$
$$z = g(y) = y^T S^{-1} y$$
$$y = h(x) = (x - \mu) \quad \text{where } x, \mu \in \mathbb{R}^{D \times D}$$

Soln

$$h(x) = (x - \mu) \longrightarrow h'(x) = I_D, \in \mathbb{R}^{D \times D}$$
$$g(y) = y^T S^{-1} y \longrightarrow g'(y) = 2y^T S^{-1}, \in \mathbb{R}^{1 \times D}$$
$$= y^{2T} S^{-1}$$
$$f(z) = e^{-1/2 z} \longrightarrow -\frac{1}{2} \times e^{\left(-\frac{1}{2} z\right)} \quad \in \mathbb{R}^{1 \times 1}$$

The desired derivative can be computed using the chain rule:

$$\boxed{\frac{df}{dx} = \frac{\partial f}{\partial z} \times \frac{\partial g}{\partial y} \times \frac{\partial h}{\partial x} \in \mathbb{R}^{1 \times D}} \left[\text{i.e } \underbrace{\frac{\partial f}{\partial z}}_{1 \times 1}, \underbrace{\frac{\partial g}{\partial y}}_{1 \times D}, \underbrace{\frac{\partial h}{\partial x}}_{D \times D}\right]$$

$$= -\frac{1}{\cancel{2}} e^{(-1/2 z)} \times \cancel{2} y^T S^{-1} \times 1$$

putting values of z and y in above eq

$$\boxed{\frac{df}{dx} = -e^{-\frac{1}{2} \times \left[(x-\mu)^T S^{-1} (x-\mu)\right]} \times (x-\mu)^T \cdot S^{-1}}$$

The dimension of $\frac{df}{dx}$ is $1 \times D$

[b] $f(x) = tr(xx^T + \sigma^2 I)$, $x \in \mathbb{R}^D$

[Hint: $xx^T$ is the outer produce, so you perform the outer product operation explicitly first to make it easier)

sol^n Let us start with outer product operation

Let, define $X = xx^T$ with $X_{ij} = x_i x_j$

Trace of matrix is the sum of all the diagonal elements

$$\therefore tr(X) = \sum_{i=1}^{D} x_{ii} \quad x \in \mathbb{R}$$

$$\therefore \frac{\partial}{\partial x_j} tr(X - \sigma^2 I) = \sum_{i=1}^{D} \frac{\partial X_{ii} + \sigma^2}{\partial x_j}$$

$$= 2x_j \quad \text{for } j = 1, \ldots D$$

$$\boxed{\therefore \frac{\partial}{\partial x} tr(xx^T + \sigma^2 I) = 2x^T \in \mathbb{R}^{1 \times D}}$$

The dimension is $1 \times D$

c] Use the chain rule. Provide the dimensions of every single partial derivative. You do not need to compute the product of the partial derivatives explicitly.

$$f = \sin(z) \in \mathbb{R}^M$$

$$z = Ax + b, \; x \in \mathbb{R}^N, A \in \mathbb{R}^{M \times N}, b \in \mathbb{R}^M$$

Here, sin is applied to every component of z

Sol$^n$ $\cancel{\frac{\partial f}{\partial x}} = \cancel{\cos(z)} \cancel{\in \mathbb{R}}$

$$\frac{\partial f}{\partial z} = \mathrm{diag}(\cos(z)) \in \mathbb{R}^{M\times M}$$

Since, $Z$ is the result of the matrix multiplication $Ax$ and $b$ $\therefore Z \in \mathbb{R}^M$ and ~~Therefore the~~ $f$ is an $M$-dimensional vector the resulting partial derivative will have dimension $M \times M$

$$\frac{\partial z}{\partial x} = \frac{\partial Ax}{\partial x} = A \in \mathbb{R}^{M\times N}$$

since, $A$ is an $M\times N$ dimensional matrix

We get the latter result by defining $y = Ax$, such that

$$y_i = \sum_j A_{ij} x_j \Rightarrow \frac{\partial y_i}{\partial x_k} = A_{ik} \Rightarrow \frac{\partial y_i}{\partial x} = [A_{i1}, \ldots, A_{iN}]$$

$$\Rightarrow \frac{\partial y}{\partial x} = A$$

$\therefore$ The overall derivative is ~~of~~ an $M\times N$ matrix

3] Consider the following functions:

i) $f_1(x) = \sin(x_1).\cos(x_2)$, $x \in \mathbb{R}^2$

ii) $f_2(x,y) = x^T y$ , $x, y \in \mathbb{R}^n$

iii) $f_3(x) = xx^T$ , $x \in \mathbb{R}^n$

a] What are the dimensions of $df_i/dx$?

i] consider $f_1(x) = \sin(x_1).\cos(x_2)$

derivative of $f_1(x)$ with respect to $x_1$

$$\frac{\partial f_1}{\partial x_1} = \cos(x_1)\cos(x_2)$$

derivative of $f_1(x)$ with respect to $x_2$ can be written as

$$\frac{\partial f_1}{\partial x_2} = -\sin(x_1).\sin(x) \in \mathbb{R}^2$$

Since $x \in \mathbb{R}^2$ ∴ Dimension is 2

The Jocobians can be written as

$$J = \left[\frac{\partial f_1}{\partial x_1} \quad \frac{\partial f_1}{\partial x_2}\right] \in \mathbb{R}^{\cancel{1}\times 2}$$

$$\Rightarrow \left[\cos(x_1).\cos(x_2) \quad -\sin(x_1).\sin(x_2)\right] \in \mathbb{R}^{1\times 2}$$

The dimension is $1\times 2$

ii] Consider $f_2(x,y) = x^T y$

derivative of $f_2(x,y)$ w.r.t $x$ can be written as :-

$$\frac{\partial f_2}{\partial x} = \left[\frac{\partial f_2}{\partial x_1}, \ldots \frac{\partial f_2}{\partial x_n}\right] \quad \text{since } x \in \mathbb{R}^n$$

$x^T y$ can be written as

$$x^T y = \sum_i x_i y_i$$

$$\therefore \left[\frac{\partial f_2}{\partial x_1} \cdots \frac{\partial f_2}{\partial x_n}\right] = [y_1 \cdots y_n] = y^T \in \mathbb{R}^n$$

because $f(x) = x \rightarrow f'(x) = 1$

derivative of $f_2(x,y)$ w.r.t $y$ can be written as

$$\frac{\partial f_2}{\partial y} = \left[\frac{\partial f_2}{\partial y_1} \cdots \frac{\partial f_2}{\partial y_n}\right] = [x_1 \cdots x_n] = x^T \in \mathbb{R}^n$$

The dimention of $f_2'(x,y) = x^T y' + y x^{T'} = n$

$\therefore$ Jacobian series can be written as

$$J = \left[\frac{\partial f_2}{\partial x_1} \quad \frac{\partial f_1}{\partial x_2}\right] = [y^T \quad x^T] \in \mathbb{R}^{1 \times 2n}$$

The dimension is $1 \times 2N$

iii] consider, $f_3(x) = x x^T \quad \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$

$$x x^T = \begin{bmatrix} x_1 x^T \\ x_2 x^T \\ \vdots \\ x_n x^T \end{bmatrix} = [x x_1 \; x x_2 \cdots x x_n] \in \mathbb{R}^{n \times n}$$

$$\Rightarrow \frac{\partial f_3}{\partial x_1} = \begin{bmatrix} x^T \\ 0_n^T \\ \vdots \\ 0_n^T \end{bmatrix} = [x \; 0_n \cdots 0_n] \in \mathbb{R}^{n \times n}$$

$$\Rightarrow \frac{\partial f_3}{\partial x_i} = \overbrace{\begin{bmatrix} 0_{(i-1)\times n} \\ x^T \\ 0_{(n-i+1)\times n} \end{bmatrix}}^{\in \mathbb{R}^{n\times n}} + \overbrace{\begin{bmatrix} 0_{n\times(i-1)} & x & 0_{n\times(n-i+1)} \end{bmatrix}}^{\in \mathbb{R}^{n\times n}}$$

The dimension $= n\times n$

To get the jocobian, we need to concatenate all partial derivatives $\frac{\partial f_3}{\partial x_i}$ and obtain

$$J = \begin{bmatrix} \frac{\partial f_3}{\partial x_1} & \cdots\cdots & \frac{\partial f_3}{\partial x_n} \end{bmatrix} \in \mathbb{R}^{(n\times n)\times n}$$

The dimension $= n\times n\times n$

4] Let $f(x) = e^x$

a] Find the Maclaurin Series for $f(x)$

~~soln~~ To find the Maclaurin series expansion of $f(x) = e^x$,
we can differentiate $f(x)$ repeatedly and evaluate ~~the~~ the derivatives at $x=0$ to obtain the co-efficients of the series.
Let's begin by finding the first few derivatives.

$f(x) = e^x$
$f'(x) = e^x$
$f''(x) = e^x$
$f'''(x) = e^x$
$\vdots$
$f^n(x) = e^x$

Now, Let's evaluate these derivatives at $x=0$

$f(0) = 1$
$f'(0) = 1$
$f''(0) = 1$
$f'''(0) = 1$
$\vdots$
$f^n(0) = 1$

The Maclaurin Series Expansion of the function is given by

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$$= f(0) + f'(0)x + f''(0)\frac{x^2}{2!} - - - - -$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + - - - - -$$

4b] Use the result from 4a] to find the maclaurin series for $g(x) = xe^x$

so|n Let's begin by finding the first few derivatives and refering from 4 a]

$g(x) = x.e^x$

$g'(x) = \text{derivative}(x) \times e^x + x.\text{derivative}(e^x)$

$= 1e^x + x.e^x$

$g''(x) = e^x + (e^x + x.e^x) = 2e^x + x.e^x$

$g'''(x) = 2e^x + e^x + xe^x = 3e^x + xe^x$

$g^4(x) = 3e^x + e^x + xe^x = 4e^x + xe^x$

Now, Let's evaluate these derivatives at $x=0$

$g(0) = 0$

$g'(0) = 1 + 0 = 1$

$g''(0) = 2(1) + 0 = 2$

$g'''(0) = 3(1) + 0 = 3$

$g^4(0) = 4(1) + 0 = 4$

The maclaurian series Expansion of the function is given by

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$$= f(0) + f'(0).x + f''(0)\frac{x^2}{2!} + f'''\frac{x^3}{3!} - - - -$$

$$xe^x = g(x) = g(0) + g'(0).x + g''(0)\frac{x^2}{2!} + g'''\frac{x^3}{3!} + - - -$$

$$= 0 + 1x + 2\frac{x^2}{2!} + 3.\frac{3^3}{3!} + - - - -$$

Simplifying this we get

$$xe^x = x + x^2 + \frac{x^3}{2} + \cancel{\frac{x^4}{3!}} + \frac{x^4}{6} + - -$$