First-order nonlinear autonomous systems

The Logistic Equation $\boxed{\frac{dx}{dt} = ax - bx^2}$ $a, b > 0$

The right hand side of $\boxed{\frac{dx}{dt} = f(x)}$ is nonlinear.

To rescale variables, we write $\frac{dx}{dt} = ax\left(1 - \frac{bx}{a}\right)$.

$\Rightarrow \frac{dx}{d(at)} = x\left(1 - \frac{x}{a/b}\right)$. Define $\boxed{K = a/b}$.

Thereafter, define $\boxed{X = x/K}$ and $\boxed{T = at}$.

Hence, we get $\frac{\cancel{K}\, dX}{dT} = \cancel{K} X(1-X)$, | Cancel K on both sides

$\Rightarrow \boxed{\frac{dX}{dT} = X(1-X)}$ This is a parameter-free nonlinear equation.

Separating variables, we get $\int \frac{dX}{X(1-X)} = \int dT$.

Partial fractions: $\boxed{\frac{1}{X(1-X)} \equiv \frac{A}{X} + \frac{B}{1-X}}$.

$\Rightarrow 1 \equiv A(1-X) + BX$. When $X = 0$, $\boxed{A = 1}$ and, when $X = 1$, $\boxed{B = 1}$. The integral is then

$$\int \frac{dX}{X} + \int \frac{dX}{1-X} = \int dT \Rightarrow \int \frac{dX}{X} - \int \frac{d(-X)}{1-X} = \int dT$$

$$\Rightarrow \boxed{\ln X - \ln(1-X) = \ln c + \ln e^T} \Rightarrow \boxed{\frac{X}{1-X} = ce^T}$$

$$\therefore X = ce^T - Xce^T \Rightarrow X(1 + ce^T) = ce^T$$

$$\boxed{X = \frac{ce^T}{1+ce^T}} \Rightarrow \boxed{X = \frac{1}{1 + c^{-1}e^{-T}}}$$

Initial condition is $T = 0$ ($t = 0$) and $X = X_0$ ($x = x_0$).

The initial value is NOT zero at $t=0$

Hence $X_0 = \frac{1}{1+c^{-1}} \Rightarrow 1+c^{-1} = \frac{1}{X_0}$

$\Rightarrow \frac{1}{c} = \frac{1}{X_0} - 1 \Rightarrow \boxed{\frac{1}{c} = \frac{1-X_0}{X_0}} \therefore \boxed{c = \frac{X_0}{1-X_0}}$

$\Rightarrow \boxed{c = \frac{x_0/K}{1-x_0/K} = \frac{x_0}{K-x_0}}$ since $\boxed{X = \frac{x}{K}}$ by rescaling.

Returning to the variables x and t, we get

$X = \frac{x}{K} = \frac{1}{1+c^{-1}e^{-at}} \Rightarrow \boxed{x = \frac{K}{1+c^{-1}e^{-at}}}$.

i) When $t \to \infty$, for any initial value, $x \to K$.
or $x \to a/b$.

ii.) We now recast the solution as $x = \frac{Ke^{at}}{e^{at} + \left(\frac{K-x_0}{x_0}\right)}$.

$\Rightarrow x = \frac{Kx_0e^{at}}{x_0e^{at}+(K-x_0)} \Rightarrow \boxed{x = \frac{x_0 e^{at}}{1+\frac{x_0}{K}(e^{at}-1)}}$

When $t=0$, $x = x_0$ (as expected). For $\boxed{t \ll \frac{1}{a}}$,

i.e. when $t \to 0$, we expand the series

$\boxed{e^{at} = 1 + at + \frac{(at)^2}{2!} + \cdots}$. For $\frac{t}{a^{-1}} \ll 1$, we

approximate $e^{at} \simeq 1 + at \Rightarrow \boxed{e^{at} - 1 \simeq at}$.

Hence, $\boxed{x \simeq \frac{x_0 e^{at}}{1+\frac{x_0}{K}at}}$. The dynamics is dominated by $\boxed{e^{at}}$

in the numerator, and not $\boxed{e^{at}-1}$ in the denominator. (P.T.O.)

Hence, the early growth is approximately exponential, $\boxed{x \simeq x_0 e^{at}}$, and must have an initial value $x_0 \neq 0$. The exponential early growth is also simulated by $K \to \infty$. In the early stages K appears large, especially for small values of b in $\boxed{\frac{dx}{dt} = ax - bx^2}$. If $\boxed{b=0}$ then $\boxed{\frac{dx}{dt} = ax}$ will show an exponential growth.

Going back to $\frac{dx}{dT} = x - x^2$, we get

$$\frac{d^2x}{dT^2} = \frac{dF}{dx}\frac{dx}{dT} = F\frac{dF}{dx} \quad (F = x - x^2)$$

Now $\boxed{\frac{dF}{dx} = 1-2x}$. Hence when $\frac{dF}{dx} = 0$, also $\frac{d^2x}{dT^2} = 0$. However when $\frac{dx}{dT} \neq 0$, we can still have $\frac{dF}{dx} = 0$, to make $\frac{d^2x}{dT^2} = 0$.

Now $\frac{dx}{dT} > 0$, starting from $\boxed{x = x_0}$ to $\boxed{x \to 1}$.

Hence $\frac{d^2x}{dT^2} = (1-2x)\frac{dx}{dT} = 0$ when $\frac{dx}{dT} \neq 0$ and $1-2x=0$.

$\Rightarrow \boxed{x = 1/2}$, where $F(x)$ has a turning point.

i.) when $x < 1/2$, $\frac{dF}{dx} = 1-2x > 0$

ii.) when $x > 1/2$, $\frac{dF}{dx} = 1-2x < 0$

At $x = 1/2$ $\frac{dF}{dx} = 0$

Since $\frac{dx}{dT} > 0$ for any finite value of x,

for $x < 1/2$, $\frac{d^2x}{dT^2} > 0$, i.e. growth is at an increasing rate. For $x > 1/2$, $\frac{d^2x}{dT^2} < 0$, i.e. growth occurs at a decreasing rate. Hence, before $x = 1/2$, growth is exponential, and beyond $x = 1/2$, growth saturates as $x \longrightarrow 1$.

At $x = 1/2$ the nonlinear effect $-bx^2$ starts to take effect. We find its time by writing,

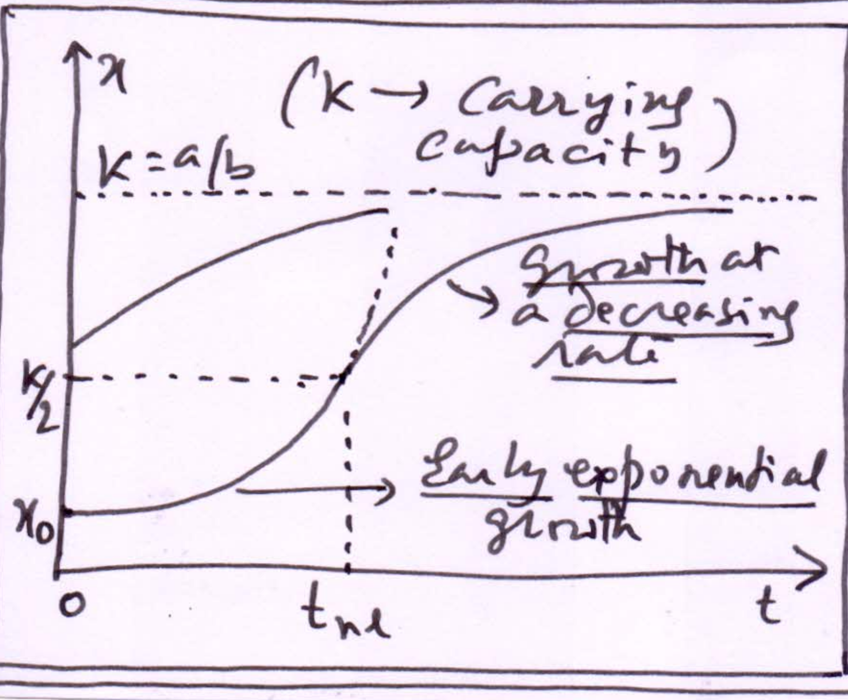
$$\boxed{x = \frac{1}{2} = \frac{1}{1 + c^{-1}e^{-T_{nl}}}} \Rightarrow 2 = 1 + c^{-1}e^{-T_{nl}} \Rightarrow \boxed{c\,e^{T_{nl}} = 1}.$$

$$\therefore\ T_{nl} = \ln\left(\frac{1}{c}\right) = \ln\left(\frac{1 - x_0}{x_0}\right) \Rightarrow a t_{nl} = \ln\left(\frac{K - x_0}{x_0}\right)$$

$$\Rightarrow \boxed{t_{nl} = \frac{1}{a}\ln\left(\frac{K}{x_0} - 1\right)}.$$ Since $t_{nl} > 0$ realistically,

$$\boxed{\frac{K}{x_0} - 1 > 1} \Rightarrow (K/x_0) > 2 \Rightarrow \boxed{x_0 < K/2}$$

$\Rightarrow$ The initial value $\boxed{x_0 < K/2}$ for a strong exponential growth at early times.



i.) After t_{nl} growth slows down.

ii.) If $\boxed{x_0 < K/2}$, then early growth is exponential

iii.) If $\boxed{\frac{K}{2} < x_0 < K}$, then growth happens only at a decreasing rate.

iv.) Growth stops at $\boxed{x = K}$. $\boxed{\frac{dx}{dt} = 0}$, $\boxed{x = a/b}$

Consider a nonlinear equation

$\boxed{\frac{dx}{dt} = a - bx^2}$. $a, b > 0$. We rescale it

as $\boxed{\frac{1}{a}\frac{dx}{dt} = 1 - \frac{x^2}{a/b}}$ Define, $\boxed{X = \frac{x}{\sqrt{a/b}}}$.

Defining further, $\boxed{T = \sqrt{ab}\, t}$,

$\boxed{\frac{dX}{dT} = 1 - X^2}$. Separating variables, we get

$\boxed{\int \frac{dX}{1-X^2} = \int \frac{dX}{(1-X)(1+X)} = \int dT}$. Now by using the method of

partial fractions $\frac{1}{(1-X)(1+X)} \equiv \frac{A}{1-X} + \frac{B}{1+X}$

$\Rightarrow \boxed{1 \equiv A(1+X) + B(1-X)}$. When $X = 1$, $\boxed{A = 1/2}$ and when $X = -1$, $\boxed{B = 1/2}$.

Hence we get,

$$\frac{1}{2}\int \frac{dX}{1-X} + \frac{1}{2}\int \frac{dX}{1+X} = \int dT \Rightarrow \int \frac{dX}{1+X} - \int \frac{d(-X)}{1-X} = 2\int dT$$

$$\Rightarrow \ln(1+X) - \ln(1-X) = \ln c + 2T = \ln c + \ln e^{2T}$$

$$\Rightarrow \ln\left(\frac{1+X}{1-X}\right) = \ln\left(ce^{2T}\right) \Rightarrow \boxed{\frac{1+X}{1-X} = ce^{2T}}$$

When $t = 0$, or $T = 0$, and $x = 0$, $X = 0$, $\Rightarrow \boxed{c = 1}$

$$\Rightarrow 1 + X = e^{2T} - Xe^{2T} \Rightarrow \boxed{X\left(1 + e^{2T}\right) = e^{2T} - 1}$$

$$\Rightarrow \boxed{X = \frac{e^{2T} - 1}{e^{2T} + 1} = \frac{\left(e^{T} - e^{-T}\right)/2}{\left(e^{T} + e^{-T}\right)/2}}$$

Now $\boxed{\sinh(T) = \frac{e^{T} - e^{-T}}{2}}$

(P. T. O.)

Further $\boxed{\cosh(T) = (e^{T} + e^{-T})/2}$.

Hence, $\boxed{X = \tanh(T)}$ since $\boxed{\tanh(T) = \frac{\sinh(T)}{\cosh(T)}}$.

$\Rightarrow$ $\boxed{x = \sqrt{\frac{a}{b}}\tanh(\sqrt{ab}\,t)}$ by returning to x and t.

i.) When $T \ll 1$ (or $t \ll (ab)^{-1/2}$), we write

$\boxed{e^{T} = 1 + T + T^2/2! + \cdots}$, $\boxed{e^{-T} = 1 - T + T^2/2! + \cdots}$

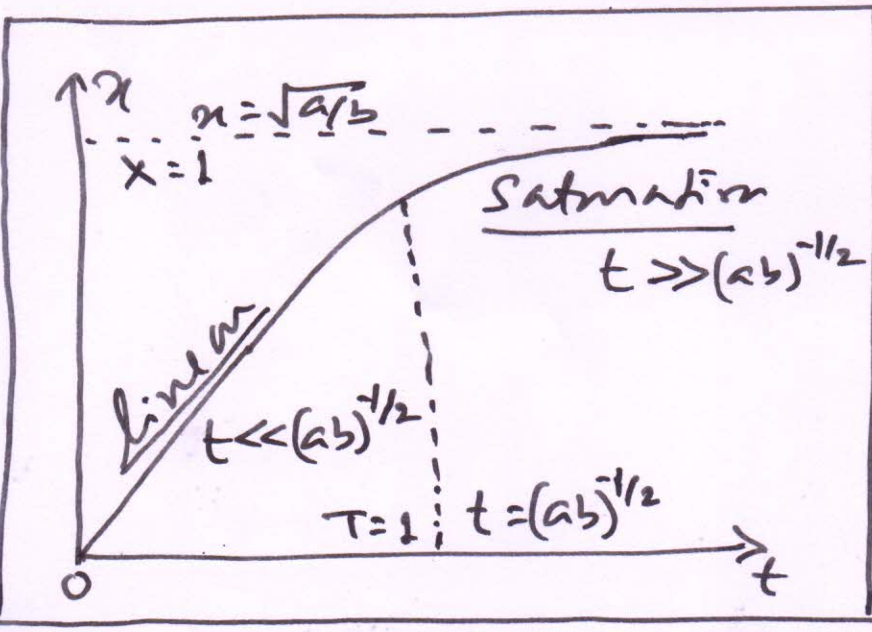
$$\therefore X = \frac{(\cancel{1} + T + \cancel{T^2/2!} + \cdots) - (\cancel{1} - T + \cancel{T^2/2!} + \cdots)}{(1 + \cancel{T} + T^2/2! + \cdots) + (1 - \cancel{T} + T^2/2! \cdots)}$$

$$\Rightarrow X \simeq \frac{2T}{2 + T^2} \simeq T \Rightarrow x \simeq \sqrt{\frac{a}{\cancel{b}}}\cdot\sqrt{a\cancel{b}}\,t$$

$\Rightarrow$ $\boxed{X \simeq at}$ i.e. early growth is linear.

ii.) $\boxed{X = \frac{e^{2T} - 1}{e^{2T} + 1} = \frac{1 - e^{-2T}}{1 + e^{-2T}}}$ $\Rightarrow$ When $T \to \infty$ (i.e. $T \gg 1$) $\boxed{X \longrightarrow 1}$.

$\therefore$ for $t \to \infty$, $\boxed{x \longrightarrow \sqrt{a/b}}$ (for $t \gg (ab)^{-1/2}$).



i/. For $\boxed{\frac{dx}{dt} = a - bx}$, saturation is for $\boxed{x = \frac{a}{b}}$.

ii/. For $\boxed{\frac{dx}{dt} = a - bx^2}$. Saturation happens earlier at $\boxed{x = \sqrt{a/b}}$.

iii/. At saturation $\frac{dx}{dt} = 0$.
$\Rightarrow$ $\boxed{x = \sqrt{a/b}}$.

Numerical errors while integrating

Consider a simple differential equation $\boxed{\frac{dx}{dt} = \lambda x}$ with the initial condition $\boxed{x(0) = 1}$.

$\Rightarrow \int \frac{dx}{x} = \int \lambda\, dt \Rightarrow \ln x = \ln x_0 + \ln e^{\lambda t}$

$\Rightarrow \boxed{x = x_0 e^{\lambda t}}$ when $\boxed{t = 0, x = 1}$.

$\Rightarrow \boxed{x = e^{\lambda t}}$. If $\lambda < 0$, $t \to \infty \Rightarrow x \to 0$.

Now to apply the Euler method, $\boxed{\Delta x = \lambda x \Delta t}$.

$\Rightarrow x_{n+1} - x_n = \lambda x_n \Delta t \Rightarrow \boxed{x_{n+1} = x_n (1 + \lambda \Delta t)}$

$\therefore x_1 = x_0 (1 + \lambda \Delta t)$, $x_2 = x_1 (1 + \lambda \Delta t) = x_0 (1 + \lambda \Delta t)^2$

By the same principle, $\boxed{x_n = x_0 (1 + \lambda \Delta t)^n}$.

The step size Δt is fixed. $\Rightarrow \boxed{t_n = t_0 + n \Delta t}$.

$\Rightarrow$ t_n follows an arithmetic progression and x_n follows a geometric progression.

The initial values are $\boxed{t_0 = 0}$ and $\boxed{x_0 = 1}$.

Hence, $\boxed{x_n = (1 + \lambda \Delta t)^n}$, $\boxed{t_n = n \Delta t}$.

Since, $\lambda < 0$, when $t_n \to \infty$, $n \to \infty$ and as a result $x_n \to 0$. This can ONLY occur when $\boxed{|1 + \lambda \Delta t| < 1}$. This convergence condition is violated when $\boxed{\begin{array}{l} \text{i) } \lambda \Delta t > 0 \text{ and} \\ \text{ii } \lambda \Delta t < -2 \end{array}}$

Hence, for stable convergence the criterion is $\boxed{-2 < \lambda \Delta t < 0}$. ($\lambda < 0$)

$\Rightarrow 2 > (-\lambda)\Delta t > 0 \Rightarrow \boxed{0 < \Delta t < 2/(-\lambda)}$.

Example: $\boxed{\lambda = -100}$, $\boxed{t_n = 0.2}$ $\boxed{\lambda^{-1} = \tau = -0.01}$.

The actual solution at $t_n = 0.2$ $\boxed{x_n = e^{-\lambda t_n} = e^{-100 \times 0.2} = 2.06 \times 10^{-9}}$

t_n is constant, $\boxed{t_n = n\Delta t}$, $\boxed{x_n = [1 + (\Delta t/\tau)]^n}$

Δt	0.1	0.05	0.02	0.01	0.001	Remarks
$n = \frac{t_n}{\Delta t}$	2	4	10	20	200	i) Small $\Delta t/\tau$ increases accuracy, but reduces efficiency.
$\frac{\Delta t}{\tau}$	-10	-5	-2	-1	-0.1	ii) Optimise between efficiency and accuracy (stability)
x_n	81	256	1	0	7.06×10^{-10}	

Binomial expansion of $\boxed{x_n = (1+z)^n}$, $\boxed{z = \frac{\Delta t}{\tau}}$.

i) $n = 2$: ($z = -10$)

$$x_2 = \frac{2!}{0!\,2!} 1^1 z^{2-0} + \frac{2!}{1!\,1!} 1^1 z^{2-1} + \frac{2!}{2!\,0!} 1^2 z^{2-2}$$

$$\Rightarrow x_2 = z^2 + 2z + 1 = 100 - 20 + 1 = 81$$

ii) $n = 4$: ($z = -5$)

$$x_4 = \frac{4!}{0!\,4!} 1^0 z^{4-0} + \frac{4!}{1!\,3!} 1^1 z^{4-1} + \frac{4!}{2!\,2!} 1^2 z^{4-2} + \frac{4!}{3!\,1!} 1^3 z^{4-3} + \frac{4!}{4!\,0!} 1^4 z^{4-4}$$

$$\Rightarrow x_4 = z^4 + 4z^3 + 6z^2 + 4z + 1 = 625 - 4 \times 125 + 6 \times 25 - 4 \times 5 + 1 = 256$$

$\boxed{x_n = (1+z)^n \sim z^n}$ for $|z| > 1$ and $n > 0$ (even value).

Numerical integration of non-Autonomous Equations

Consider $\boxed{\dfrac{dx}{dt} = f(t,x)}$ with $\boxed{x(t_0) = x_0}$

A Taylor expansion of the foregoing non-autonomous equation about (t_0, x_0) is

$$\boxed{x = x_0 + \left.\frac{dx}{dt}\right|_{t_0}(t-t_0) + \frac{1}{2!}\left.\frac{d^2x}{dt^2}\right|_{t_0}(t-t_0)^2 + \cdots}$$

higher orders

$$\boxed{\left.\frac{dx}{dt}\right|_{t_0} = f(t_0, x_0)} \quad \text{and} \quad \boxed{\left.\frac{d^2x}{dt^2}\right|_{t_0} = \left.\frac{df}{dt}\right|_{t_0}}$$

Now $\boxed{df = \dfrac{\partial f}{\partial t}dt + \dfrac{\partial f}{\partial x}dx}$ $\therefore$ $\boxed{\dfrac{df}{dt} = \dfrac{\partial f}{\partial t} + f\dfrac{\partial f}{\partial x}}$

Hence, we can recast the Taylor expansion as

$$\boxed{x = x_0 + f(t_0, x_0)(t-t_0) + \frac{1}{2!}\left[\frac{\partial f}{\partial t} + f\frac{\partial f}{\partial x}\right]_{t_0, x_0} \times (t-t_0)^2 + \cdots}$$

By Euler's method we retain only the linear term to get $\boxed{x \simeq x_0 + f(t_0, x_0)(t-t_0)}$, which can be generalised for $\Delta t = t - t_0$ as

$$\boxed{x_{n+1} = x_n + f(t_n, x_n)\Delta t} \quad \text{with} \quad \boxed{\Delta t = t_{n+1} - t_n}.$$

If we account for the second-order term, we get

$$\boxed{x_{n+1} = x_n + f(t_n, x_n)\Delta t + \frac{1}{2}\left[\frac{\partial f}{\partial t} + f\frac{\partial f}{\partial x}\right]_{t_n, x_n}(\Delta t)^2}$$

$\boxed{\Delta t = t_{n+1} - t_n}$ is a constant step size. All the values on the right hand side are fixed at (t_n, x_n).