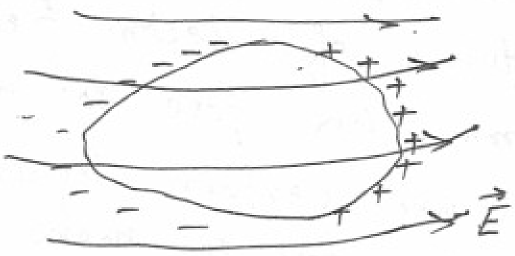
Conductors.

All material is electrically neutral. However there are a large (enormously large) number of point charges within a material with equal amount of positive and negative types. These are the electrons and the positive ions in the material. Generally these charges are tightly bound to each other and hence they don't move even under the influence of electric field. But in metals the electrons are free to wander around in the metal. There are certain liquids where positive and negative ions are free to move around. They are called electrolytes. These kind of substances which have enormously large ($\sim 10^{23}$) number of charged particles to freely move around are conductors. Ideally Conductors are considered as substances with an infinite supply of positive and negative charges.

When we place a conductor in a region which has no electric field, every part of the conductor is neutral i.e. +ve and −ve charges are found in equal amount in every part. Once we place the conductor in an electric field, the free charges start moving. They charges will keep moving till the net electric field everywhere within the conductor is 0. This is because, as long as there is even a tiny electric field somewhere, an infinite amount of electric charges will respond to it and move. The motion can stop only when every region in the conductor is devoid of any electric field.

Due to this movement of the electric ~~field~~ charges, the conductor no longer remains neutral everywhere. The +ve charges move along the direction of the electric field and the −ve charges move opposite to the electric field. This leads to accumulation of charges as shown in the figure.



Since the charges can't leave the surface of the conductor, ~~all~~ the charges accumulate on the surface and creates a surface charge distribution on the conductor. The final configuration of these surface charges is such that there is no electric field inside the conductor i.e. the configuration of ~~this~~ the surface charges on the conductor produces an electric field which is opposite to the external electric field in which the conductor is placed.

These surface charges that gets produced on the conductor ~~is call~~ are called induced charge on the conductor due to the electric field $\vec{E}$.

Now, since the electric field in a conductor is always zero, the conductor has a constant potential throughout. This is a very important characteristic of a conductor. The surface of a conductor becomes a very convenient surface to specify the boundary condition of an electrostatic problem. These surfaces are equipotential. Let a and b be two points within the conductor; then the potential difference between a and b is

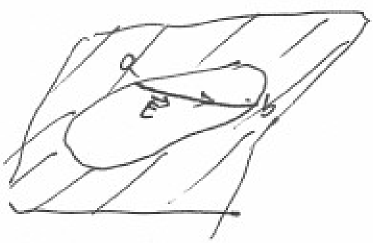
$$V_b - V_a = -\int_a^b \vec{E} \cdot \vec{dl} = 0 \quad \text{since } \vec{E} \text{ is } 0.$$

$\therefore \quad V_b = V_a.$

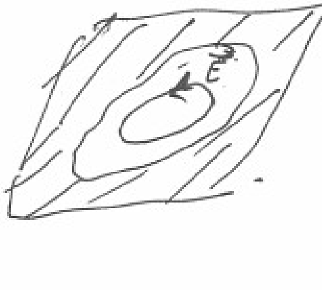
This shows that potential everywhere is same.

Cavity inside a conductor

What happens if we have a cavity inside a conductor? We can show that the electric field inside the cavity is also 0.



The whole conductor is at the same potential. The wall of the cavity is also at this potential. We cannot have any electric field lines starting from one point of the cavity wall to another point as shown in the figure above. Since $\int_a^b \vec{E}\cdot d\vec{l}$ will lead to a potential difference between point a and point b. The only way, if at all, we can have electric field inside the cavity is to have a loop of electric field line as shown below.

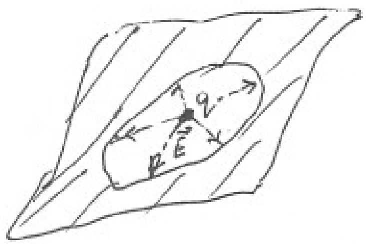


But this is impossible too. This is because in such a case

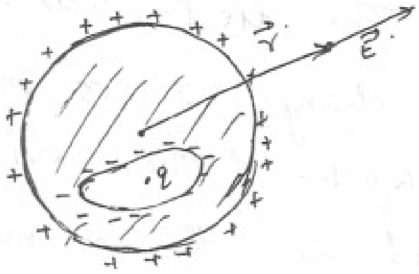
$$\oint \vec{E}\cdot d\vec{l} \neq 0$$

along the closed curve. This would mean $\vec{\nabla}\times\vec{E} \neq 0$ which is absurd for an electrostatic field. Hence we cannot have any electric field inside the conductor. So the cavity is also an equipotential region. The potential inside the cavity is the same as the potential of the conductor.

All this is true when we don't have charges inside the cavity. If we have charges isolated inside the cavity, we can have electric field in the cavity. These electric field is no more divergenceless. The fields can diverge from the charge sources and terminate on the walls of the cavity as shown.



Eg: Consider a solid spherical conductor with a cavity inside. A point charge q is situated inside the cavity. What will be the electric field outside the outer surface of the spherical conductor :



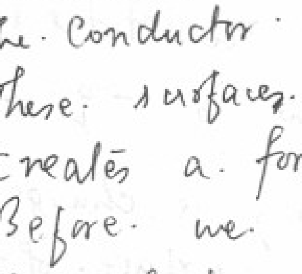
Due to the point charge q in the cavity, a charge $-q$ is induced on the inner surface of the cavity. This is because if we consider a Gausian surface enclosing the cavity but lying completely within the conductor then $\oint_S \vec{E}\cdot\hat{n}\,da = 0$ since $\vec{E} = 0$. So the charge enclosed in the Gausian surface is 0. The only charges are the point charge q and the surface charge on the inner surface of the cavity. So the total surface charge on the wall of the cavity is $-q$. Since the conductor was chargeless, a charge $+q$ will accumulate over the outer surface of the conductor. Since there is no electric field in the conductor, this charge $+q$ distributes uniformly over the whole sphere. The only electric field outside the conductor is due to this surface charge since the internal charges have cancelled each others effect. So the electric field will be

$$\vec{E} = \frac{q}{4\pi\epsilon_0 r^2}\hat{r}$$

So from outside the spherical conductor we only know the total charge q placed within the cavity. All other information about the position of this charge and the shape of the cavity is lost.

If we now place this ~~ele~~ conductor in an external electric field, only the outer surface charge $+q$ will readjust to maintain the electric field inside the conductor to be zero. No change occurs on the ~~inner~~ surface charge over the walls of the cavity. This phenomenon is ~~call~~ described as "the internal region of a conductor is shielded from any electric field in the outside world." Such an enclosure is called the Faraday's cage.

<u>Force on the surface of a conductor</u>: As the charges accumulate ~~on~~ over the surface of a conductor, the conductor achieves a surface charge density σ. These surfaces charges tends to move outward. This creates a force on the surface of the conductor. Before we calculate this force let us calculate the electric field near the surface of the conductor. This will be normal to the surface of the conductor. This is because the surface of the conductor is equipotential. So there cannot be any tangential electric field. To calculate the normal component we apply the ~~Gauss's law~~ boundary condition on $E_\perp$. Since $E_{in}=0$ inside we have



$$E_{out} - E_{in} = \frac{\sigma}{\epsilon_0}$$

$$\therefore \vec{E}_{out} = \frac{\sigma}{\epsilon_0}\hat{n}$$

If σ is +ve. $\vec{E}$ is along $\hat{n}$. If σ is -ve. $\vec{E}$ is along $-\hat{n}$

Due to this electric field outside the conductor, the surface charge over an element of area da experiences a force.

If we didn't have the rest of the charges over the surface of the conductor, the electric field on either side of the small element da would be $\frac{\sigma}{2\epsilon_0}$. Since the conductor had no electric field on the inside, we conclude that the effect of this rest of the charges is to produce exactly an equal and opposite electric field to that of the inside field due to the small element da. So the electric field due to the rest of the charges is

$$\vec{E}_{rest} = \frac{\sigma}{2\epsilon_0}\hat{n}.$$

This electric field due to the rest of the charges exerts a force over the element da. This force is given as

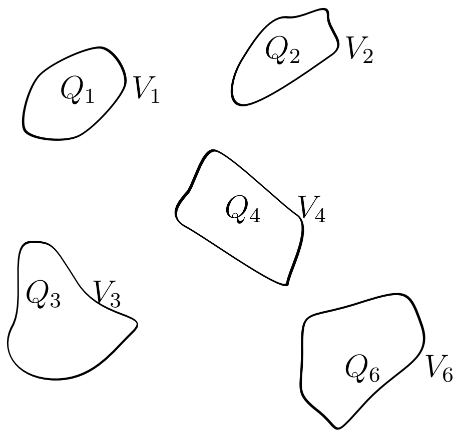
$$\vec{f} = \vec{E}_{rest} \times (\sigma\, da) = \frac{\sigma^2}{2\epsilon_0}\hat{n}$$

Note that this force is always towards the outward normal to the surface of the conductor since σ^2 is always +ve irrespective of whether σ is +ve or −ve.

Capacitance

When a charge Q is placed on a conductor it acquires a potential V. V is proportional to Q, $V = KQ$. The constant K depends on the geometrical shape of the conductor. The capacitance C of the conductor is defined as the amount of charge Q required to raise the potential of the conductor by 1 volt. So $C = Q/V = 1/K$.

Now let us consider a configuration of n conducting surfaces as shown in the figure.



If an amount of charge Q_j is placed on the j^{th} conductor then according to the linear superposition principle the potential on the i^{th} conductor is given as $V_i = \sum_{j=1}^{n} K_{ij} Q_j$. Here K_{ij} depends on the geometrical shapes and the position of the conductors. Further, the Uniqueness theorem states that if the potentials V_i of the i^{th} conductor is specified then the potential $V(\vec{r})$ at all points is completely specified. This uniquely specifies the electric field $\vec{E} = -\vec{\nabla} V$ in the region. The Gauss' law gives $Q_i = \oint_{S_i} \vec{E} \cdot \hat{n} da$. So specifying potentials V_i on conductor i, uniquely specifies the charges on them. This implies the matrix K is invertible.

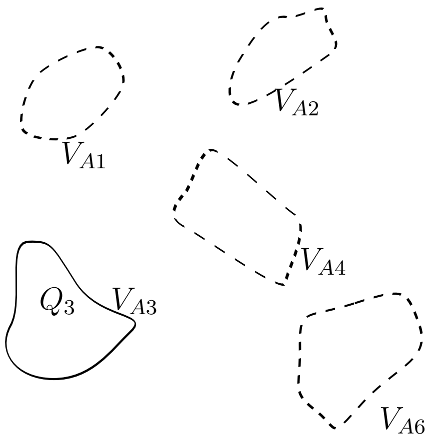
It can be shown that $K_{ij} = K_{ji}$, i.e, K is a symmetric matrix. This follows from the Green's reciprocity theorem. It states that in a region τ surrounded by surface S if we are given two different charge configurations, one consisting of volume charge density ρ_A and a surface charge density σ_A, and another consisting of volume charge density ρ_B and a surface charge density σ_B, then the potential function in the region V_A due to configuration A and V_B due to the configuration B satisfies the relation

$$\int_\tau \rho_A V_B + \oint_S \sigma_A V_B = \int_\tau \rho_B V_A + \oint_S \sigma_B V_A$$

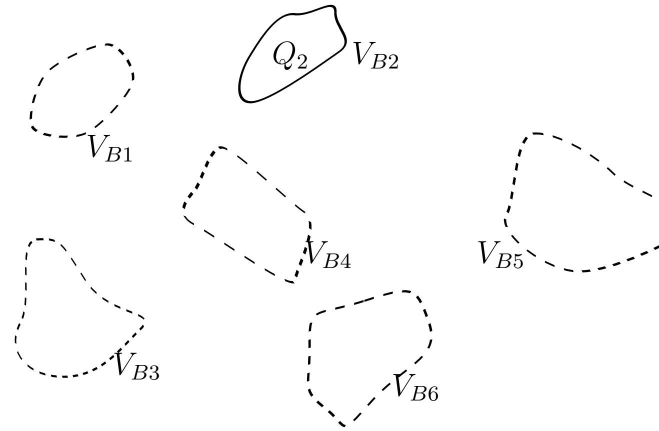
(Do try to prove this).
In our case there are no volume charges. So the Green's reciprocity theorem reduces to

$$\oint_S \sigma_A V_B = \oint_S \sigma_B V_A \tag{1}$$

Now let us define configuration A as one where only the i^{th} conductor has a charge Q_i while the others are uncharged while in configuration B only the j^{th} conductor has a charge Q_j while the others are uncharged. See figure.



Configuration A



Configuration B

Putting this in Eq. 1 we get $Q_i K_{ij} Q_j = Q_j K_{ji} Q_i$. This gives $K_{ij} = K_{ji}$.

An alternative proof of this statement comes from the consideration of total energy of a configuration as follows:

The electrostatic energy of the configuration is given as $\frac{1}{2}\sum_{i=1}^{n} Q_i V_i$. This can be written as $\frac{1}{2}Q^T K Q$ where the column matrix $Q = (Q_1, Q_2, ..., Q_n)^T$.

If only two of the conductors, say i and j are charged with surface charge Q_i and Q_j respectively, while all others are uncharged then the electrostatic energy of the configuration only in terms of Q_i, Q_j and the matrix elements of K is

$$E = \frac{1}{2}\left(Q_i^2 K_{ii} + Q_j^2 K_{jj} + Q_i Q_j (K_{ij} + K_{ji})\right)$$

This energy can be obtained by considering two configurations of the n conductors. One in which only conductor i is charged with an amount Q_i, while the other where only conductor j has charge Q_j. We referred to these configurations as configuration A and configuration B above. These two configurations have energy $Q_i^2 K_{ii}/2$ and $Q_j^2 K_{jj}/2$. So when these two configurations are far away the total energy of the system is $\frac{1}{2}(Q_i^2 K_{ii} + Q_j^2 K_{jj})$. When one configuration is moved towards the other, a work is done and this adds to the energy of the system. This will be either $Q_i Q_j K_{ij}$ or $Q_i Q_j K_{ji}$ depending upon which one is moved. Since the final configuration and the energy have to be the same either way, we conclude the interaction energy in either way will be the same. This gives $K_{ij} = K_{ji}$.

If $\lambda_1, \lambda_2, ..., \lambda_n$ are the eigenvalues of the matrix K. Let $\mathcal{Q}^i = (Q_1, Q_2, ..., Q_n)^T$ be the eigenvector corresponding to λ_i. Then $\mathcal{V}^i = \lambda_i \mathcal{Q}^i$ where $\mathcal{V}^i = (V_1, V_2, ..., V_n)^T$. In this charge configuration the potential of each conductor is directly proportional to the amount of charge on it. The capacitance in this configuration of charges may be defined as $1/\lambda_i$. We can define the inverse of the matrix K as the capacitance matrix C. $C_i = 1/\lambda_i$ is an eigenvalue of the matrix C. C_i's are the characteristic capacitance of the given configuration of conductors.

Now consder the case where we have only two conductors. 1 and 2. Then we have

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} K_{11} & K_{12} \\ K_{12} & K_{22} \end{pmatrix} \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}$$

We will concentrate on the special case when $K_{11} = K_{22}$. Here $\lambda_1 = \frac{K_{11}+K_{12}}{2}$ and $\lambda_2 = \frac{K_{11}-K_{12}}{2}$. The eigenvector corresponding to λ_1 is $(Q, Q)^T$ while that corresponding to λ_2 is $(Q, -Q)^T$. Here Q can be any arbitrary amount of charge. In the first case the potential of the two conductors are same given by $V_1 = V_2 = (K_{11} + K_{12})Q$. We may say that the capacitance of this configuration is $\frac{1}{K_{11}+K_{12}}$. This case is not very interesting since here the two conductors behaves like a single conductor charged to a potential V_1 due to an amount of charge $2Q$ on it. The second case gives $V_1 = -V_2 = (K_{11} - K_{12})Q$. According to the usual definition of capacitance this leads to $C = \frac{1}{2(K_{11}-K_{12})}$.

Note that the practical definition of capacitance with two conductors is related to the eigenvalues of the capacitance matrix but not equal to the eigenvalues. If we generalize the idea of capacitance to more than two conductors, the most natural and useful definition of capacitance is the various eigenvalues of the capacitance matrix.