Stimulated Emission: Einstein's Coefficients

Consider a material medium (containing atoms) bathed in electromagnetic radiation (photon field) at temperature T.

1/. There is thermodynamic equilibrium between the atoms and the photons at a temperature T (thermalised).

2/. The energy distribution of the photons follows the blackbody energy distribution function (Planck).

3/. The energy distribution of the atoms follows the Boltzmann function.

The Boltzmann Function (OR Factor)

If a large aggregate of particles is in thermodynamic (pressure, density, temperature in equilibrium) equilibrium at a temperature T, then the energy distribution among the particles will follow a probability function,

(P.T.O.)

$\boxed{P_i = C e^{-\mathcal{E}_i/k_B T}}$ in which $\mathcal{E}_i$ is the energy of the i-th state. This is the Boltzmann function (factor).

Hence, $\boxed{\sum_i P_i = 1}$ → Summation over all states

gives a total probability of unity.

$$\Rightarrow \boxed{\sum_i C e^{-\mathcal{E}_i/k_B T} = 1}, \quad C \text{ is a constant}$$

$$\Rightarrow \boxed{C = \frac{1}{\sum_i e^{-\mathcal{E}_i/k_B T}}} \quad \therefore \boxed{P_i = \frac{e^{-\mathcal{E}_i/k_B T}}{\sum_i e^{-\mathcal{E}_i/k_B T}}}$$

$$\boxed{Z = \sum_i e^{-\mathcal{E}_i/k_B T}}$$ → Partition Function (Summation over states)

1/. Stimulated Absorption: The atoms interact with the photons. While doing so, some atoms in the state i absorb photons and jump to an excited state of higher energy j.

(P.T.O.)

The transition requires an energy change $\boxed{\mathcal{E}_j - \mathcal{E}_i = h\nu_{ji}}$.

($N_i \rightarrow$ No. of atoms in state i)

The rate of change of the number of atoms (in state i) is $\boxed{\frac{dN_i}{dt} \propto N_i\, u(\nu)}$ in which $u(\nu)$ is the energy density per (at the) frequency ν.

[Unit : $J\,m^{-3}/s^{-1} \equiv Js/m^3$].

Hence, $\boxed{\frac{dN_i}{dt} = -\,B_{ij}\, N_i\, u(\nu)}$ Negative sign implies decrease. (decrease of N_i)

($B_{ij} \rightarrow$ proportional constant)

2). Stimulated Emission : The excited atoms can interact with the photons, and release the excess energy to go down to the lower-energy state i. (Einstein)

($N_j \rightarrow$ No. of atoms in state j)

The rate of change of the number of atoms in state j is $\boxed{\frac{dN_j}{dt} \propto N_j\, u(\nu)}$.

Hence $\boxed{\frac{dN_j}{dt} = -\,B_{ji}\, N_j\, u(\nu)}$, in which B_{ji} is another proportional constant, and negative sign implies decrease. (decrease of N_j)

3/. Spontaneous Emission: The atoms in the excited state also lose the excess energy without the need of an external influence, and go back to the stable state. The excited state has a lifetime of $\sim 10^{-9}$ s.

The rate of change of the number of excited atoms (in the excited state) is $\boxed{\frac{dN_j}{dt} \propto N_j} \Rightarrow \boxed{\frac{dN_j}{dt} = -A_{ji} N_j}$.

A_{ji} is a proportional constant, and the negative sign implies decrease of N_j.

In equilibrium, the rate of upward transitions $(i \to j)$ equals the rate of downward transitions $(j \to i)$.

$$\therefore \boxed{\frac{dN_i}{dt} = \frac{dN_j}{dt} \text{ (from both stimulated and spontaneous emissions)}}$$

$$\Rightarrow \boxed{-B_{ij} N_i u(\nu) = -B_{ji} N_j u(\nu) - A_{ji} N_j}$$

(P.T.O.)

$$\Rightarrow \boxed{\frac{N_j}{N_i} = \frac{B_{ij}\,u(\nu)}{A_{ji} + B_{ji}\,u(\nu)}}$$

Both N_i and N_j follow the Boltzmann function

$$\therefore \boxed{\frac{N_j}{N_i} = \frac{N_0\, e^{-\mathcal{E}_j/k_B T}}{N_0\, e^{-\mathcal{E}_i/k_B T}}}$$

($N_0 \rightarrow$ Total no. of atoms) in which N_0 is a constant.

$$\Rightarrow \boxed{\frac{N_j}{N_i} = e^{-(\mathcal{E}_j - \mathcal{E}_i)/k_B T} = e^{-h\nu_{ji}/k_B T}}$$

$$\therefore \boxed{\frac{B_{ij}\,u(\nu)}{A_{ji} + B_{ji}\,u(\nu)} = e^{-h\nu_{ji}/k_B T}}$$

B_{ij}, A_{ji}, B_{ji} are Einstein's Coefficients

$$\Rightarrow B_{ij}\,u(\nu) = A_{ji}\, e^{-h\nu_{ji}/k_B T} + B_{ji}\,u(\nu)\, e^{-\frac{h\nu_{ji}}{k_B T}}$$

$$\Rightarrow \left(B_{ij} - B_{ji}\, e^{-h\nu_{ji}/k_B T}\right) u(\nu) = A_{ji}\, e^{-h\nu_{ji}/k_B T}$$

$$\Rightarrow \left(B_{ij}\, e^{h\nu_{ji}/k_B T} - B_{ji}\right) u(\nu) = A_{ji}$$

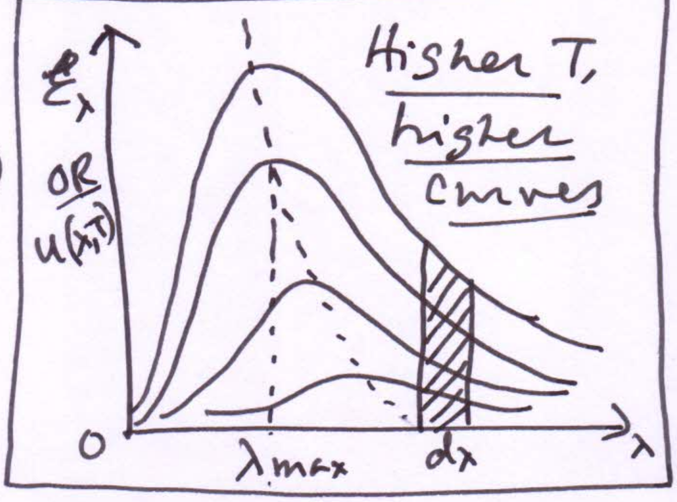
$$\Rightarrow \boxed{u(\nu) = \frac{A_{ji}/B_{ji}}{(B_{ij}/B_{ji})\, e^{h\nu_{ji}/k_B T} - 1}}$$

A_{ji}, B_{ji}, B_{ij} are Einstein's Coefficients

As $T \rightarrow \infty$, $\mathcal{E}_\lambda \rightarrow \infty$.
(from black body radiation)

$$\therefore \boxed{u(\lambda, T) = \frac{4}{c}\mathcal{E}_\lambda \longrightarrow \infty}$$

Hence, $u(\nu) \longrightarrow \infty$ when $T \rightarrow \infty$.



(P.T.O.)

When $T \to \infty$, $\boxed{e^{h\nu_{ji}/k_BT} \to e^0 = 1}$

Hence, for $u(\nu) \to \infty$, $\boxed{B_{ij} = B_{ji}}$.

Since B_{ij} and B_{ji} are constants, they are ~~[illegible]~~ the same for all T.

Hence, $\boxed{B_{ij} = B_{ji} = B}$ (constant independent of temperature.)

Writing $\boxed{A_{ji} = A}$, $\boxed{u(\nu) = \frac{A}{B}\left(\frac{1}{e^{h\nu_{ji}/k_BT} - 1}\right)}$.

Planck's black body distribution formula

$$\boxed{u(\lambda, T) = \frac{8\pi hc}{\lambda^5}\left(\frac{1}{e^{hc/\lambda k_BT} - 1}\right)}$$ Energy density function

Now $\boxed{u(\lambda, T)\, d\lambda = u(\nu, T)\, d\nu}$ (Same energy)

Since $\boxed{\nu\lambda = c}$ $\Rightarrow \boxed{d\lambda = -c\nu^{-2}\, d\nu}$ $\boxed{\nu_{ji} \equiv \nu}$

$$\therefore u(\lambda, T)\, d\lambda = \frac{8\pi hc\nu^5}{c^5} \cdot \frac{1}{e^{h\nu/k_BT} - 1} \cdot (-c\nu^{-2}\, d\nu)$$

$$\Rightarrow \boxed{u(\lambda, T)\, d\lambda = \frac{8\pi h\nu^3}{c^3} \cdot \frac{1}{e^{h\nu/k_BT} - 1}\, d\nu}$$

(Ignore negative sign)

$$\Rightarrow \boxed{\frac{A}{B} = \frac{8\pi h\nu^3}{c^3}}$$

[This theory has an important application in LASERS.]