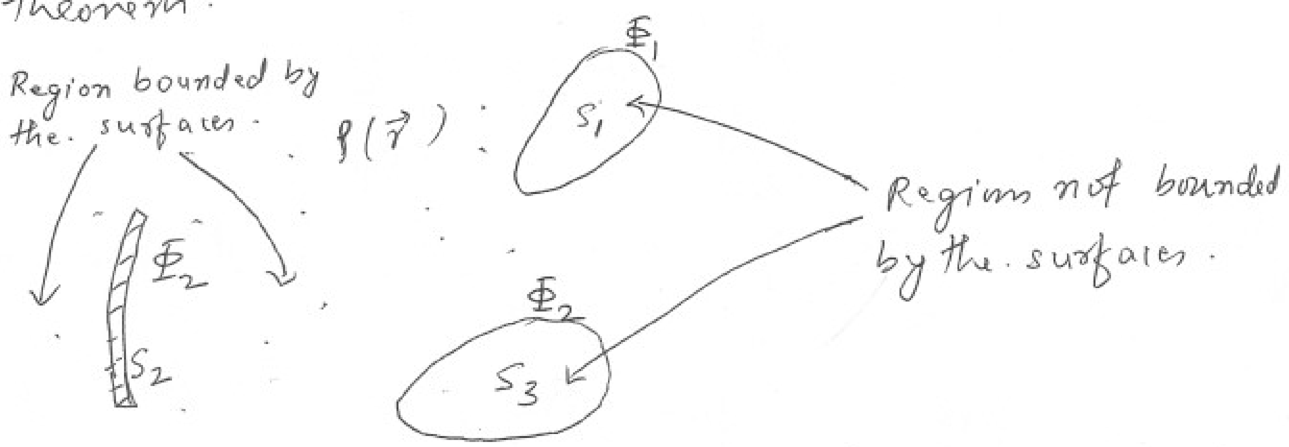
Uniqueness Theorems.

First uniqueness theorem:

Given a charge distribution in a region specified by the charge distribution $\rho(\vec{r})$, we know that the electric field $\vec{E}$ everywhere is uniquely ~~specified~~ determined. The potential function is specified up to a constant. Generally an electrostatic configuration is specified by the charge distribution $\rho(\vec{r})$ and certain surfaces $S_1, S_2, \ldots, S_n$ over which the potentials $\Phi_1, \Phi_2, \ldots, \Phi_n$ are specified. We know that with such specifications the electric field and the potential in the region bounded by these surfaces are uniquely ~~specified~~ determined and nature adopts this unique field configuration everywhere. This is the statement of the first uniqueness theorem.



Let us suppose that the solution to the Poisson's Eqn is not unique. Let Φ & Φ' be two solns. i.e $\nabla^2 \Phi = \nabla^2 \Phi' = -\frac{\rho}{\epsilon_0}$.

Moreover, both Φ and Φ' have the same values over the bounding surfaces $S_1, S_2, \ldots S_n$ as mentioned above. i.e.

~~Consider~~ $\Phi = \Phi' = \Phi_i$ on the surface S_i
$i = 1, 2, \ldots, n$.

Consider the function

$$\Phi'' = \Phi - \Phi'$$

$$\nabla^2 \Phi'' = \nabla^2 \Phi - \nabla^2 \Phi' = 0.$$

So Φ'' satisfies the Laplace's Eqn.
Since Φ & Φ' have the same values on the surfaces $S_1, S_2, \ldots S_n$, we have

$$\Phi'' = 0 \quad \text{on the surfaces } S_1, S_2 \ldots S_n.$$

This is equivalent to solving an electrostatic problem with no charge anywhere and all the bounding surfaces of the region is at 0 potential. The solution to this problem is obvious. There can't be any electric field anywhere and $\Phi'' = 0$ everywhere in the region.
This implies $\Phi = \Phi'$ everywhere in the region.
This proves the ~~p~~ uniqueness theorem of the first kind.
A few comments must be made about the 'naturally obvious' statements we made towards the end of the above arguments.

If we have no charge anywhere are we sure that electric field is ~~zero~~ 0 every where? We are looking for solution to the Laplace's Equation.

$$\nabla^2 \Phi = 0.$$

Certainly $\Phi = 0$ or $\Phi =$ constant is a solution which gives $\vec{E} = 0$ everywhere. But consider the following functions.

$$\Phi = kx + c \quad , \quad \Phi = \vec{k} \cdot \vec{r} + c$$

$$\Phi = e^{kx} \sin ky$$

Each of the above pottentials function satisfy the Laplace's equation. The electric fields calculated from these potentials are

$$\vec{E} = -k\hat{i} \quad , \quad \vec{E} = -\vec{k} \quad \text{and}$$

$$\vec{E} = k\left(\hat{i}\, e^{kx} \sin ky + \hat{j}\, e^{kx} \cos ky\right).$$

So we have non-zero electric fields produced by ~~non-zero~~ zero charge!

This controversy is resolved the moment we specify the potential $\Phi = 0$ on some fixed surface. To staisfy such conditions the only possible solutions is $\Phi = 0$ everywhere. If no such zero potential surfaces are specified we always 'assume' that $\Phi = 0$ at infinity.

or the electric field $\vec{E} = 0$ at infinity. This will rule out all the other non-zero solutions of the type written above.

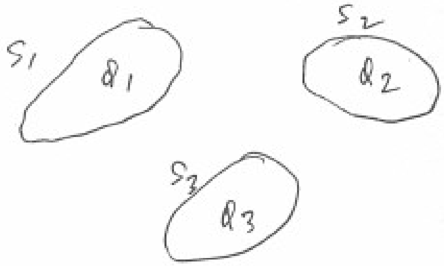
Some of the electrostatic problems we study, like an infinite plane of surface charge and infinite line charge, are ~~un~~ realistic. So we find the electric potential and electric field ~~extendin~~ being non-zero even at infinity. Such charge distribution can often lead to paradoxical situations. ref. Problem 2.50 in Griffiths.

Second Uniqueness Theorem :

Given a charge distribution and certain surfaces maintained at some known potential, the first uniqueness theorem guarantees a unique electric field and potential everywhere. Often the surfaces mentioned above are surfaces of conductors which are equipotential. One way to bring a conductor to a given potential is to put in a charge Q on it.

Sometime we know the charge put on the conductor but we don't know the potential that it comes to.

Now suppose the surfaces $S_1, S_2, \ldots, S_n$ are the surfaces of a conductor. Certain charges $Q_1, Q_2, \ldots, Q_n$ are thrown onto them. These charges will distribute themselves over the surfaces such that each of them are equipotential surfaces



However we don't know what will be the potential of $S_1, S_2, \ldots, S_n$. Will they arrive at some unique potentials $\Phi_1, \Phi_2, \ldots, \Phi_n$ or there can be more than one way to redistribute the charges over their respective surfaces, so that the potentials of the conductors are $\Phi_1', \Phi_2', \ldots, \Phi_n'$.

The second uniqueness theorem states that there is a unique way for the charges to spread over the conductors and the potentials attained by the conductors are unique.

The proof of the second uniqueness theorem is more physical than mathematical. We will make a reasonable assumption. If we have no charge in a region and we have conductor surfaces $S_1, S_2, \ldots, S_n$ with no charge on them; then we cannot have any electric field anywhere. So the potential everywhere will be 0 or constant. We assumed this also in the first uniqueness theorem.

Now suppose for the given problem with charge density $\rho(\vec{r})$ and charges $Q_1, Q_2, \ldots, Q_n$ on the surfaces $S_1, S_2, \ldots, S_n$, there are two kinds of distribution possible. One leading to potentials $\Phi_1, \Phi_2, \ldots, \Phi_n$ on the surfaces $S_1, S_2, \ldots, S_n$ and the other leading to potentials $\Phi_1', \Phi_2', \ldots, \Phi_n'$ on them. In the two situations let the potential functions be $\Phi(\vec{r})$ and $\Phi'(\vec{r})$. Then

$$\nabla^2 \Phi = -\frac{\rho}{\epsilon_0} \quad \text{and} \quad \nabla^2 \Phi' = -\frac{\rho}{\epsilon_0}$$

On the surfaces S_i,

$$\Phi = \Phi_i \quad \text{and} \quad \Phi' = \Phi_i'$$

Let $\vec{E}$ be the electric field corresponding to Φ and $\vec{E}'$ be the electric field corresponding to Φ'. Then

$$\vec{E} = -\vec{\nabla}\Phi \quad \text{and} \quad \vec{E}' = -\vec{\nabla}\Phi'$$

~~Consider the functions.~~

Consider the functions.

$$\Phi'' = \Phi - \Phi'.$$

$$\vec{E}'' = -\vec{\nabla}\Phi'' = -(\vec{\nabla}\Phi - \vec{\nabla}\Phi') = \vec{E} - \vec{E}'$$

Now $\nabla^2 \Phi'' = \nabla^2 \Phi - \nabla^2 \Phi'$

$$= -\frac{\rho}{\epsilon_0} - \left(-\frac{\rho}{\epsilon_0}\right) = 0 \quad \text{—— I.}$$

Over the surface S_i

$$\oint_{S_i} \vec{E}\cdot\hat{n}\,da = \oint_{S_i} \vec{E}'\cdot\hat{n}\,da = Q_i$$

since Q_i is the total charge on it.

$$\therefore \oint_{S_i} \vec{E}''\cdot\hat{n}\,da = \oint_{S_i}(\vec{E} - \vec{E}')\cdot\hat{n}\,da = 0 \quad \text{—— II.}$$

Eqn. I. and II. shows that Φ'' is the potential of the electrostatic problem with $\rho = 0$ everywhere and no charge on any of the conductors $S_1, S_2, \ldots, S_n$. The solution to this problem is $\vec{E}'' = 0$ or

$\Phi'' =$ constant.

$\therefore \vec{E} = \vec{E}'$ everywhere and hence the electric field can only be unique. Moreover on the surface of the conductors the charge density is given by

$$\sigma = \epsilon_0 \vec{E}\cdot\hat{n} = \epsilon_0 \vec{E}'\cdot\hat{n}.$$

Hence there is a unique way in which the charges will distribute themselves on the conductors.

We have presented a rather intuitive proof of the uniqueness theorem. For a more mathematically rigorous proof refer Griffiths sec. 3.1.5 and sec. 3.1.6.