Dynamical Systems and Differential Equations

The integral equation of a straight line is $\boxed{y = mx + c}$, m and c being fixed parameters.

Taking derivatives, $\boxed{\frac{dy}{dx} = m}$ and $\boxed{\frac{d^2y}{dx^2} = 0}$.

The differential equation of a straight line is $\boxed{\frac{d^2y}{dx^2} = 0}$ (free of parameters).

i/. Successive derivatives reduce the number of fixed parameters. This implies greater generalisation and more universal relevance.

ii/ Derivatives capture changes (implying dynamics) and are necessary to follow evolving systems in time.

ADVANTAGES OF DIFFERENTIAL EQUATIONS

We consider differential equations to express changes of a dependent variable x, varying through time t, (independent variable). Hence $\boxed{x \equiv x(t)}$.

First-order autonomous systems

$$\boxed{\frac{dx}{dt} = f(x)} \rightarrow \text{Ordinary, autonomous differential equation of first order.}$$

$$\Rightarrow \boxed{x \equiv x(t)} \therefore \boxed{x(t_0) = x_0} \rightarrow \text{Initial condition.}$$

We carry out a Taylor expansion of $x(t)$ about the initial condition to get

$$\boxed{x = x_0 + \left.\frac{dx}{dt}\right|_{t_0}(t-t_0) + \frac{1}{2!}\left.\frac{d^2x}{dt^2}\right|_{t_0}(t-t_0)^2 + \frac{1}{3!}\left.\frac{d^3x}{dt^3}\right|_{t_0}(t-t_0)^3 + \dots(\text{higher orders})}$$

Now $\boxed{\left.\frac{dx}{dt}\right|_{t_0} = f(x_0)}$. In terms of $f(x)$,

$$\boxed{\frac{d^2x}{dt^2} = \frac{df}{dt} = \frac{df}{dx}\frac{dx}{dt} = f\frac{df}{dx}} \therefore \boxed{\left.\frac{d^2x}{dt^2}\right|_{t_0} = \left.f\frac{df}{dx}\right|_{x_0}}$$

$$\Rightarrow \frac{d^3x}{dt^3} = \frac{d}{dt}\left(\frac{d^2x}{dt^2}\right) = \frac{d}{dx}\left(f\frac{df}{dx}\right)\frac{dx}{dt} = \left[\frac{d}{dx}\left(f\frac{df}{dx}\right)\right]f$$

$$\Rightarrow \boxed{\frac{d^3x}{dt^3} = f\left[\left(\frac{df}{dx}\right)^2 + f\frac{d^2f}{dx^2}\right] = f\left(\frac{df}{dx}\right)^2 + f^2\frac{d^2f}{dx^2}}$$

Hence $\boxed{\left.\frac{d^3x}{dt^3}\right|_{t_0} = \left.\left[f\left(\frac{df}{dx}\right)^2 + f^2\frac{d^2f}{dx^2}\right]\right|_{x_0}}$. In the Taylor expansion

$$\boxed{x = x_0 + f(x_0)(t-t_0) + \frac{1}{2}\left.\left(f\frac{df}{dx}\right)\right|_{x_0}(t-t_0)^2 + \frac{1}{6}\left.\left[f\left(\frac{df}{dx}\right)^2 + f^2\frac{d^2f}{dx^2}\right]\right|_{x_0}(t-t_0)^3 + \dots(\text{higher orders})}$$

From the Taylor expansion, we get

$$\frac{x - x_0}{t - t_0} = \frac{\Delta x}{\Delta t} = \left.\frac{dx}{dt}\right|_{t_0} + \frac{1}{2}\left.\frac{d^2x}{dt^2}\right|_{t_0}(t - t_0) + \frac{1}{6}\left.\frac{d^3x}{dt^3}\right|_{t_0}(t - t_0)^2 + \cdots$$

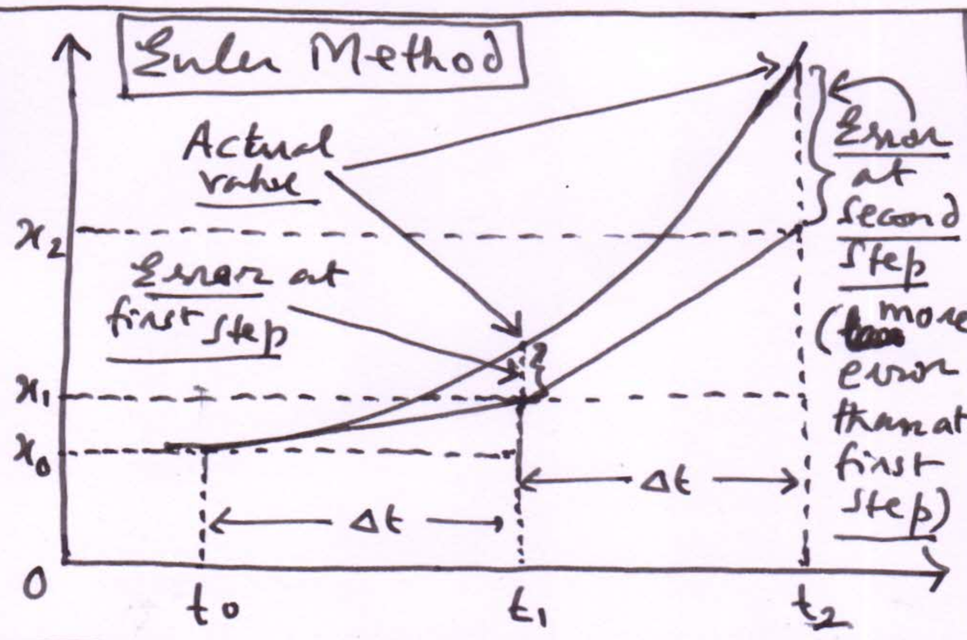
$$\Rightarrow \boxed{\frac{\Delta x}{\Delta t} = f(x_0) + \frac{1}{2}\left(f\frac{df}{dx}\right)_{x_0}(t - t_0) + \frac{1}{6}\left[f\left(\frac{df}{dx}\right)^2 + f^2\frac{d^2f}{dx^2}\right]_{x_0}(t - t_0)^2 + \cdots}$$

$$\boxed{\lim_{t \to t_0} \frac{\Delta x}{\Delta t} = \frac{df}{dt} = f(x_0)}$$ at $t = t_0$, $x = x_0$

In the Taylor Expansion keeping only the first order, we get approximately

$$\boxed{x \simeq x_0 + f(x_0)(t - t_0)} \rightarrow \text{The Forward Euler Method.}$$

(Euler Formula)



$$\int_{x_n}^{x_{n+1}} dx = \int_{t_n}^{t_{n+1}} \frac{dx}{dt}\, dt$$

$$\Rightarrow x_{n+1} - x_n = \int_{t_n}^{t_{n+1}} f(x)\, dt$$

$$\Rightarrow \boxed{x_{n+1} - x_n \simeq f(x_n)\,\Delta t}$$

If f is constant, the Euler method is EXACT.

$\Delta t \rightarrow$ Time steps (all equal) $\boxed{\Delta t = t_{n+1} - t_n}$

If $f(x)$ is not constant, then the Euler method gives errors in the numerical calculations.

Example: $\boxed{\frac{dx}{dt} = f(x) = -x}$ $\boxed{x(0) = 1}$ Initial Condition

$\Rightarrow \int \frac{dx}{x} = -\int dt \Rightarrow \boxed{\ln x = -t + \ln C}$

$\Rightarrow x = Ce^{-t}$ when $t = 0, x = 1 \Rightarrow C = 1$.

Hence $\boxed{x = e^{-t}}$ → The exact solution.

By Euler's formula $\boxed{x_{n+1} \simeq x_n + f(x_n)\Delta t}$.

Hence with $\boxed{f(x) = -x}$ we can write

$\boxed{x_{n+1} = x_n - x_n \Delta t}$ Consider $\boxed{\Delta t = 0.1}$ and $\boxed{x_0 = 1}$

$\Rightarrow x_1 = x_0 - x_0 \Delta t \Rightarrow x_1 = 1 - 1 \times 0.1 = 0.90$

$\Rightarrow x_2 = x_1 - x_1 \Delta t \Rightarrow x_2 = 0.9 - 0.9 \times 0.1 = 0.81$

Exact values: $x_1 = e^{-0.1} \simeq 0.9048$
$\boxed{x = e^{-t}}$ $x_2 = e^{-0.2} \simeq 0.8187$

Matches ~~Correct with~~ the numerical values upto 2 places of decimal

Percentage errors: At x_1, $\frac{0.9 - 0.9048}{0.9048} \simeq -0.5\%$

At x_2, $\frac{0.81 - 0.8187}{0.8187} \simeq -1.06\%$.

Negative sign implies falling short of actual values.

The error increases with successive steps.

First-order linear autonomous systems

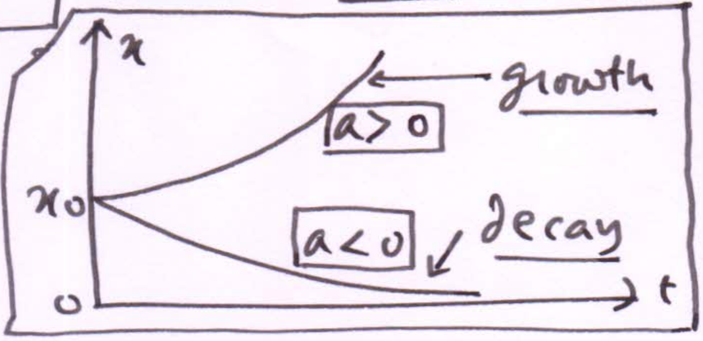
Rate $\propto$ state: $\boxed{\frac{dx}{dt} \propto x} \Rightarrow \boxed{\frac{dx}{dt} = \pm a x}$ $a > 0$

We rescale by $\boxed{\frac{dx}{d(at)} = \pm x}$ and setting a rescaled time $\boxed{T = at}$.

Separating variables, $\int \frac{dx}{x} = \pm \int dT$

$\Rightarrow \boxed{\ln x = \ln A \pm \ln e^{T}}$ when $t = 0$, $x = x_0$.

$\Rightarrow \boxed{x = x_0 e^{\pm at}}$ for exponential growth $x_0 \neq 0$.



We consider a more general, linear expression for $f(x)$, in $\boxed{\frac{dx}{dt} = f(x) = a \pm bx}$ $a, b > 0$

We consider the negative sign first.

$\Rightarrow \boxed{\frac{dx}{dt} = a - bx}$ We rescale this as follows.

$\boxed{\frac{dx}{d(bt)} = \frac{a}{b} - x}$ Next we set new rescaled variables as $\boxed{T = bt}$ and $\boxed{x_0 = a/b}$. Rescaling x as $\boxed{X = x/x_0}$ we can get an equation

(P.T.O.)

$$\Rightarrow \frac{dx}{dT} = x_0 - x \quad \Rightarrow \frac{d(x/x_0)}{dT} = 1 - \frac{x}{x_0}$$

$$\Rightarrow \boxed{\frac{dX}{dT} = 1 - X} \rightarrow$$ An equation free of the two parameters a, b.

$$\Rightarrow \int \frac{dX}{1-X} = \int dT$$ by separating variables.

$$\Rightarrow \int \frac{d(-X)}{1-X} = -\int dT \Rightarrow \ln(1-X) = \ln X_0 + \ln e^{-T}$$
$$\Rightarrow \ln(1-X) = \ln(X_0 e^{-T})$$

$$\Rightarrow \boxed{1 - X = X_0 e^{-T}}$$ Initial condition, $t=0, x=0$.

For $t>0$, we have $1 - 0 = X_0 \Rightarrow \boxed{X_0 = 1}$.

$$\Rightarrow \boxed{X = 1 - e^{-T}} \quad \Rightarrow \boxed{x = x_0(1 - e^{-bt}) = \frac{a}{b}(1 - e^{-bt})}.$$

i) When $T = 0$ $(t=0)$, $X = 0$ $(x = 0)$ (small time limit).

ii) When $T \rightarrow \infty$ $(t \rightarrow \infty)$, $X \rightarrow 1$ (or $x \rightarrow a/b$) (long time limit).

Now $\boxed{\frac{dX}{dT} = e^{-T} = 1 - X}$ $\therefore$ When $T \rightarrow \infty$,

and $X \rightarrow 1$, $\boxed{\frac{dX}{dT} \rightarrow 0}$. Also $\boxed{\frac{d^2X}{dT^2} = -e^{-T} \rightarrow 0}$

iii) Hence, there is no turning point at $X = 1$

For a first-order autonomous system, $\boxed{\frac{dx}{dt} = f(x)}$.

$\boxed{\frac{d^2x}{dt^2} = \frac{df}{dt} = \frac{df}{dx}\frac{dx}{dt}}$. Hence, when $\boxed{\frac{dx}{dt} = 0} \Rightarrow \boxed{\frac{d^2x}{dt^2} = 0}$.

Both first and second derivative vanish. NOT a turning point.

iv) When $\boxed{0 < T \ll 1}$, we can expand

$$\boxed{e^{-T} = 1 - T + \frac{T^2}{2!} - \frac{T^3}{3!} + \cdots}$$

, which we can use in

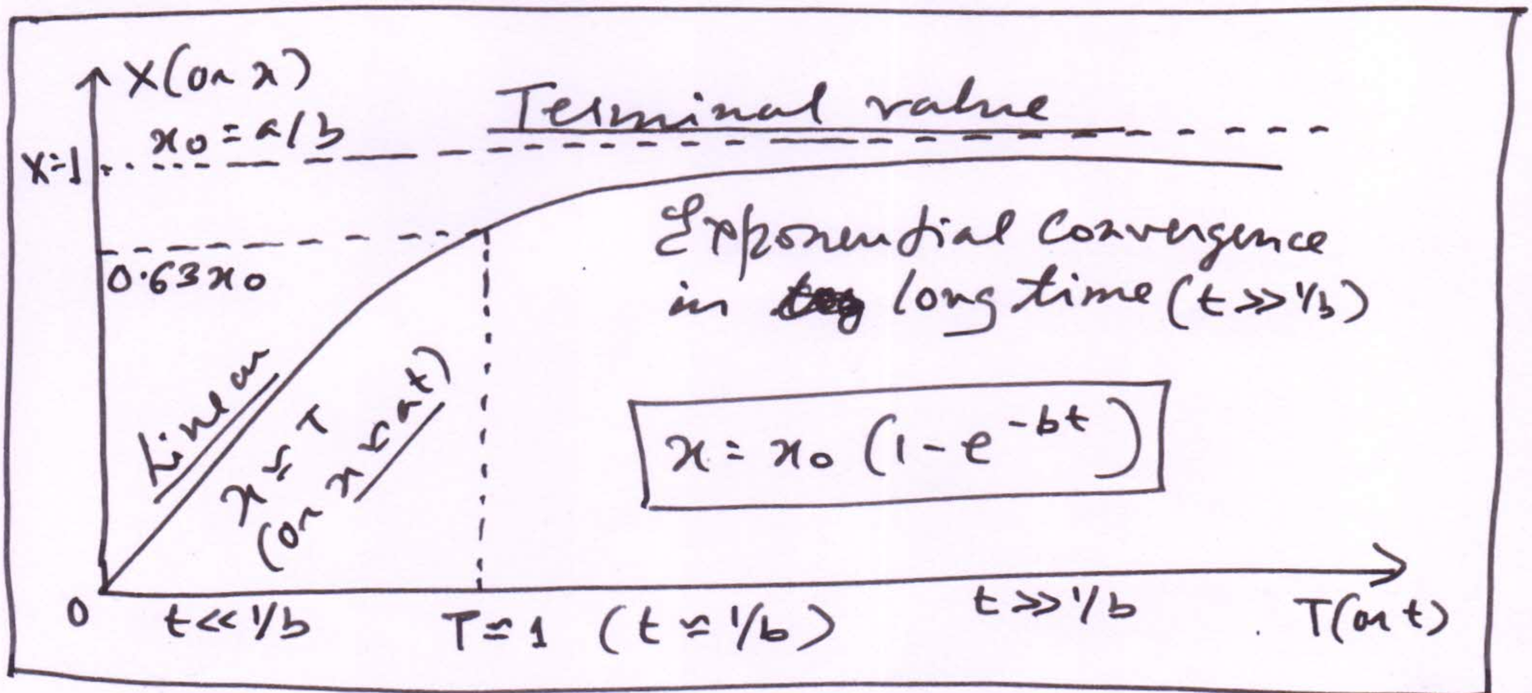
$$X = 1 - e^{-T} = 1 - \left(1 - T + \frac{T^2}{2!} - \frac{T^3}{3!} + \cdots\right)$$

Since $T \ll 1$, successive terms in this series diminish rapidly, and hence

$$\boxed{X \simeq 1 - (1 - T) \simeq T} \Rightarrow \frac{x}{x_0} \simeq bt.$$

$$\Rightarrow \boxed{x \simeq \frac{a}{\cancel{b}}\cancel{b}t \simeq at}$$ $\therefore$ Early growth is linear.

v) Since for $\boxed{T \gg 1}$ $(T \to \infty)$, there is a convergence towards $\boxed{X \to 1}$ (or $x \to a/b$), the transition takes place at $\boxed{T = 1}$ or $\boxed{t \simeq 1/b}$.



There are no turning points of $x \equiv x(t)$. At the transition, when $\boxed{t \simeq 1/b}$, $\boxed{x \simeq 0.63x_0} \to \boxed{x = x_0\left(1 - \frac{1}{e}\right)}$

When $\boxed{\frac{dx}{dt} = a + bx}$, $a, b > 0$, we consider it as

$\boxed{\frac{dx}{dt} = a - (-bx)}$ whose solution is simply found by taking $\boxed{b \to -b}$ in $\boxed{x = \frac{a}{b}\left(1 - e^{-bt}\right)}$

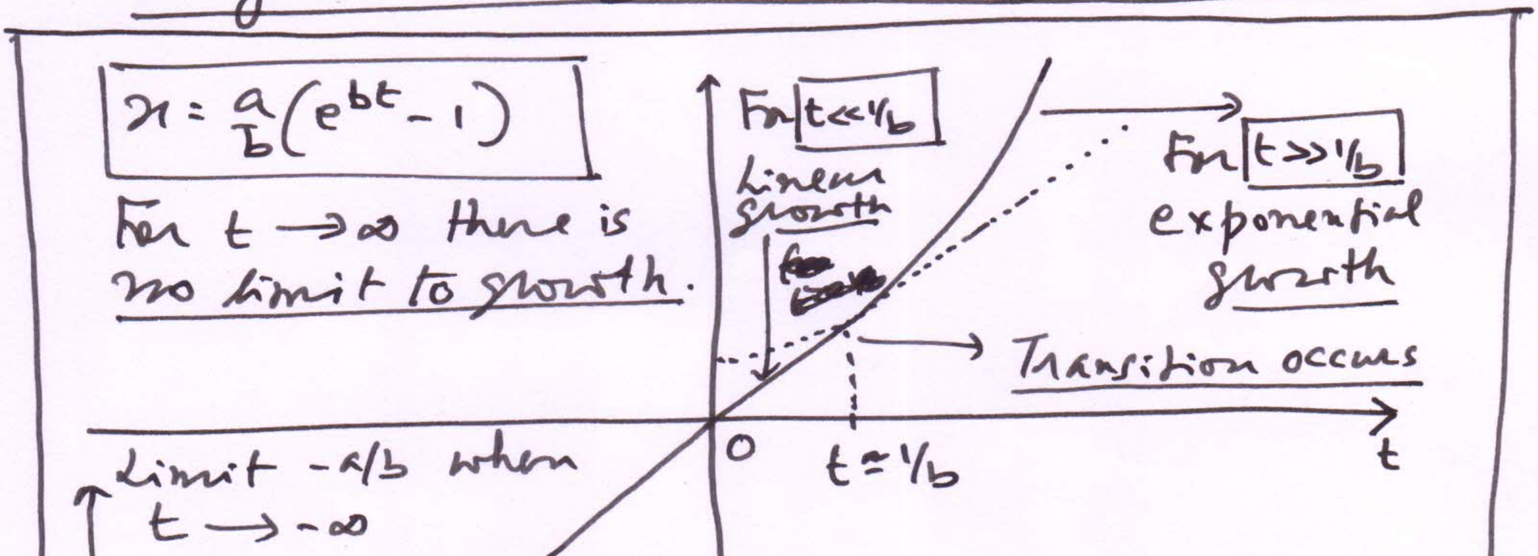
Hence $\boxed{x = \frac{a}{-b}\left(-e^{bt} + 1\right) = \frac{a}{b}\left(e^{bt} - 1\right)}$ is the solution of $\boxed{\frac{dx}{dt} = a + bx}$.

i.) When $t \ll 1/b$ $\boxed{x \simeq \frac{a}{b}\left(\cancel{1} + bt - \cancel{1}\right) \simeq at}$.

⇒ Early growth is linear.

ii.) When $t \gg 1/b$ $\boxed{e^{bt} - 1 \simeq e^{bt}}$ ∴ $\boxed{x \simeq \frac{a}{b} e^{bt}}$.

⇒ Late growth is exponential, without any limiting behaviour for $t \to \infty$.



Hypothetically, if $t \to -\infty$, then $x \to -a/b$. The third quadrant for $t < 0$, reflects the behaviour of $\frac{dx}{dt} = a - bx$, when $t > 0$.