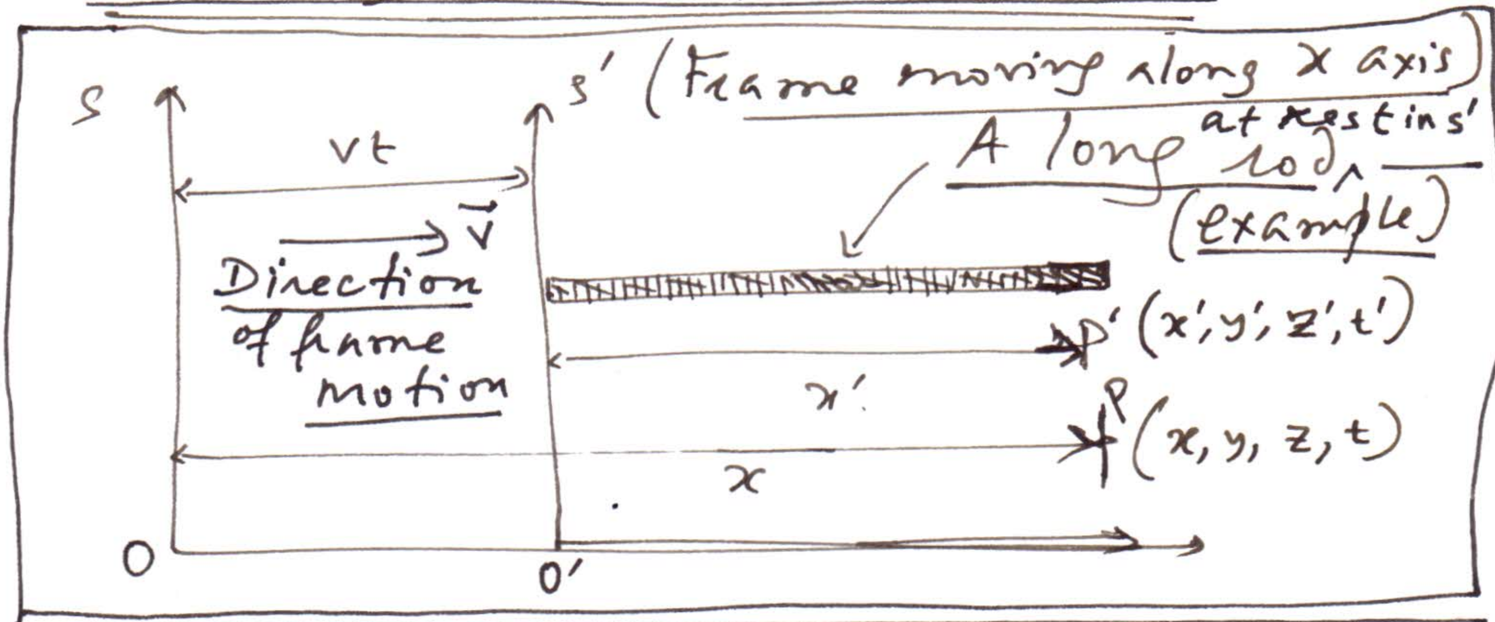
Lorentz Transformation



$$\boxed{x - vt = x'} \quad \text{(Galilean)}$$

x' is proper length.

$$\boxed{x - vt = \frac{x'}{\gamma}} \quad \text{(Lorentz)} \rightarrow \text{Applying length contraction}$$

$$\boxed{x' = \gamma\,(x - vt)} \qquad \boxed{\gamma = \frac{1}{\sqrt{1-(v/c)^2}}}$$

Exchange primes and $v \longrightarrow -v$

$$\boxed{x = \gamma\,(x' + vt')}$$

γ is not affected by the sign of v

for the inverse transform

Considering only x (in the direction of $\vec{v}$).

 Time transformation

$$x = \gamma x' + \gamma v t'$$

But $\boxed{x' = \gamma x - \gamma v t}$

$$\therefore \quad x = \gamma^2 x - \gamma^2 v t + \gamma v t'$$

$$\Rightarrow \gamma v t' = x(1-\gamma^2) + \gamma^2 v t$$

$$\Rightarrow t' = \gamma t - \frac{(\gamma^2 - 1)x}{\gamma v}$$

$$\Rightarrow t' = \gamma \left[t - \frac{\gamma^2 - 1}{\gamma^2} \cdot \frac{x}{v} \right]$$

$$\Rightarrow t' = \gamma \left[t - (1 - \gamma^{-2}) \frac{x}{v} \right]$$

But $\boxed{\gamma^2 = \frac{1}{1-\beta^2}}$ $\Rightarrow \gamma^{-2} = 1 - \beta^2$

$$\therefore \quad 1 - \gamma^{-2} = \beta^2 = (v/c)^2$$

$$\Rightarrow \boxed{t' = \gamma \left[t - \frac{v}{c^2} x \right]}$$ Time transfor-mation.

Inverse transform : $\boxed{t = \gamma \left[t' + \frac{v}{c^2} x \right]}$

Lorentz - Einstein Transformation

$$\boxed{\begin{aligned} x' &= \gamma\,(x - vt) \\ y' &= y\,, \quad z' = z \\ t' &= \gamma\left(t - \frac{v}{c^2}x\right) \end{aligned}}$$

Hendrik Lorentz + Albert Einstein

Applies ONLY to the direction of v (the direction of relative motion of frames)

Inverse Transform:

$$\boxed{\begin{aligned} x &= \gamma\,(x' + vt') \\ y &= y'\,, \quad z = z' \\ t &= \gamma\left(t' + \frac{v}{c^2}x'\right) \end{aligned}}$$

$$\gamma = \frac{1}{\sqrt{1-(v/c)^2}} = \frac{1}{\sqrt{1-\beta^2}} \quad \text{where } \boxed{\beta = \frac{v}{c}}$$

when $v \rightarrow -v$, γ remains invariant because of v^2

When $v \ll c$, $\gamma \simeq 1$, and Galilean transformation rules are obtained.

Motion ~~of a~~ ~~single~~ ~~particle~~ component along x

Relativistic Velocity Addition

$$\boxed{\begin{aligned} x' &= \gamma\,(x - vt) \\ t' &= \gamma\left(t - \frac{v}{c^2}x\right) \end{aligned}}$$

$$\boxed{\gamma = \frac{1}{\sqrt{1-(v/c)^2}}}$$

In the unprimed frame S,

$$\boxed{x = u_x t} \Rightarrow \boxed{u_x = x/t}$$

In the primed frame S',

$$\boxed{x' = u'_x t'} \Rightarrow \boxed{u'_x = x'/t'}$$

$$x - vt = \frac{x'}{\gamma} = \frac{u'_x t'}{\gamma}$$

$$\Rightarrow x - vt = \frac{u'_x}{\cancel{\gamma}} \cdot \cancel{\gamma}\left(t - \frac{v}{c^2}x\right)$$

$$\Rightarrow x = vt + u'_x t - \frac{u'_x v}{c^2}x$$

$$\Rightarrow x\left(1 + \frac{u'_x v}{c^2}\right) = (v + u'_x)\,t$$

$$\boxed{x = \left(\frac{v + u_x'}{1 + u_x' v/c^2} \right) t}$$

Compare with $\boxed{x = u_x t}$

$$\Rightarrow \frac{x}{t} = \frac{v + u_x'}{1 + u_x' v/c^2}$$

But $\boxed{x = u_x t} \Rightarrow \boxed{x/t = u_x}$

$$\boxed{u_x = \frac{v + u_x'}{1 + u_x' v/c^2}}$$

In the x-direction

Relativistic Velocity Addition.

When $\boxed{u_x' \ll c}$ and $\boxed{v \ll c}$

$\Rightarrow \left(\frac{u_x'}{c}\right)\left(\frac{v}{c}\right) \ll 1$

(Neglect the above)

$$\boxed{u_x \simeq v + u_x'}$$

Classical velocity Addition

Inverse Transform: $u_x' = \dfrac{u_x - v}{1 - u_x v/c^2}$

(Relativistic)

for $u_x v \ll c^2$

$\boxed{u_x' \simeq u_x - v}$

Rocket firing projectile

Examples: Rocket speed $v = 0.8c$

Eg 1. Firing speed of projectile $u_x' = 0.9c$

~~[illegible]~~

$$u_x = \frac{u_x' + v}{1 + v u_x'/c^2} = \frac{0.9c + 0.8c}{1 + 0.9 \times 0.8}$$

$$\Rightarrow \boxed{u_x = \frac{1.7}{1.72} c \simeq 0.99c}$$

Does not exceed c

Eg 2. Train switching on headlamp

$u_x' = c$

$v \rightarrow$ speed of train

$$u_x = \frac{v + c}{1 + vc/c^2} = \frac{v + c}{c/c + v/c}$$

$$\Rightarrow \boxed{u_x = c\left(\frac{v+c}{v+c}\right) = c}$$

Speed of light is invariant

Eg 3. Light on light

$u_x' = c$, $v = c$

$$u_x = \frac{c + c}{1 + c.c/c^2} = \frac{2c}{1+1} = \frac{2c}{2}$$

Speed of light is the same in all frames of reference.

$$\Rightarrow \boxed{u_x = c}$$

Speed of light is invariant

1/. No speed can exceed c.

2/. The speed of light c is invariant. (Einstein)

$P_2(x_2, y_2, z_2)$
OR (x_2', y_2', z_2')

$P_1(x_1, y_1, z_1)$
OR (x_1', y_1', z_1')

Velocity Addition in the orthogonal direction

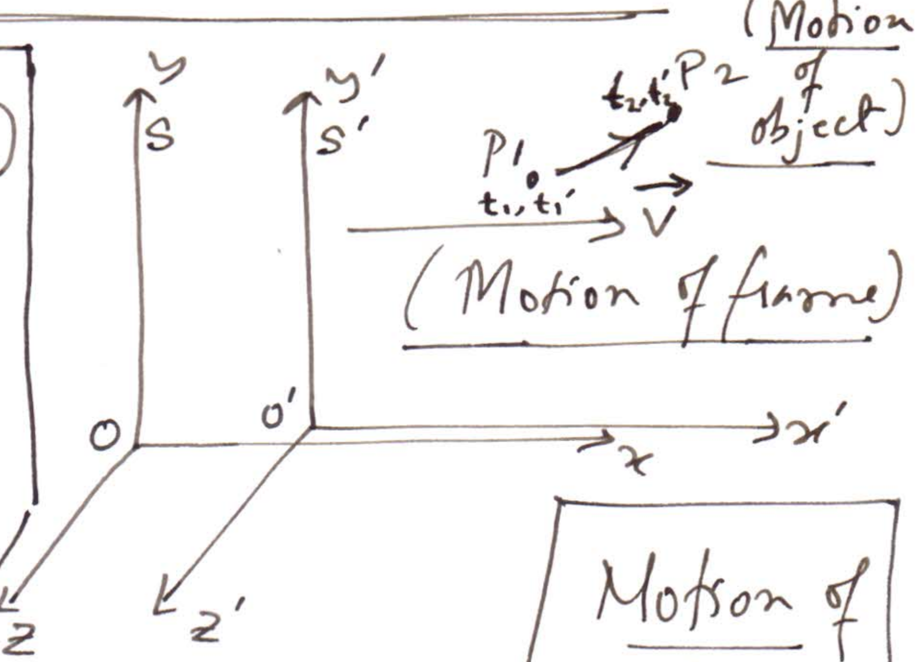
Lorentz Transformations

$$x' = \gamma(x - vt)$$

$$y' = y$$

$$z' = z$$

$$t' = \gamma\left(t - \frac{v}{c^2}x\right)$$



$x \to$ direction of relative motion of frames.

$\boxed{x = u_x t}$ and $\boxed{x' = u_x' t'}$

$$\therefore \quad \boxed{u_x = \frac{u_x' + v}{1 + u_x' v/c^2}}$$

in the x direction.

Motion of a particle in two frames.

Frame Motion with constant velocity.

In the y direction: $\boxed{y_1' = y_1}$

and $\boxed{y_2' = y_2}$

(Positions of two separate points)

From Lorentz transformations

Displacement along y

$$\boxed{\Delta y = y_2 - y_1 = y_2' - y_1' = \Delta y'}$$ and y'.

$$\boxed{t_1' = \gamma\left(t_1 - \frac{v}{c^2}x_1\right)}$$

$$\boxed{t_2' = \gamma\left(t_2 - \frac{v}{c^2}x_2\right)}$$

From Lorentz transformations

$$\therefore\ t_2' - t_1' = \Delta t' = \gamma(t_2 - t_1) - \frac{\gamma v}{c^2}(x_2 - x_1)$$

But $\boxed{t_2 - t_1 = \Delta t}$ and $\boxed{x_2 - x_1 = \Delta x}$

$$\therefore\ \boxed{\Delta t' = \gamma\Delta t - \frac{\gamma v}{c^2}\Delta x}$$

$$\boxed{\frac{\Delta y'}{\Delta t'} = u_y' = \frac{\Delta y}{\gamma\left(\Delta t - \frac{v}{c^2}\Delta x\right)}}$$

$$\Rightarrow\ u_y' = \frac{\Delta y/\Delta t}{\gamma\left(1 - \frac{v}{c^2}\frac{\Delta x}{\Delta t}\right)}$$

But $\boxed{u_x = \frac{\Delta x}{\Delta t}}$ and $\boxed{u_y = \frac{\Delta y}{\Delta t}}$

$$\boxed{u_y' = \frac{u_y}{\gamma\left(1 - \frac{v}{c^2} u_x\right)}}$$

$\boxed{u_y' \neq u_y}$ because $\boxed{\Delta t' \neq \Delta t}$ even though $\boxed{\Delta y' = \Delta y}$.

Exchanging primes and $v \longrightarrow -v$ for transforming.

$$\boxed{u_y = \frac{u_y'}{\gamma\left(1 + \frac{v}{c^2} u_x'\right)}}$$

Similarly in the z-direction:

$$\boxed{u_z' = \frac{u_z}{\gamma\left(1 - \frac{v}{c^2} u_x\right)}}$$

$\Delta z = z_2 - z_1$
$\Delta z' = z_2' - z_1'$
and $z_1 = z_1'$, $z_2 = z_2'$

and

$$\boxed{u_z = \frac{u_z'}{\gamma\left(1 + \frac{v}{c^2} u_x'\right)}}$$

Same derivation as in y

 (Alternative Derivation)

Velocity Addition in the x Direction:

$$\boxed{x' = \gamma(x - vt)} \quad \& \quad \boxed{t' = \gamma\left(t - \frac{v}{c^2}x\right)}$$

$$x_2' - x_1' = \gamma(x_2 - x_1) - \gamma v(t_2 - t_1)$$

$$\Rightarrow \boxed{\Delta x' = \gamma \Delta x - \gamma v \Delta t}$$

Considering the variation

Also $$\boxed{\Delta t' = \gamma \Delta t - \frac{\gamma v}{c^2}\Delta x}$$

$$\therefore \boxed{u_x' = \frac{\Delta x'}{\Delta t'} = \frac{\gamma(\Delta x - v\Delta t)}{\gamma\left(\Delta t - \frac{v}{c^2}\Delta x\right)}}$$

$$\Rightarrow u_x' = \frac{\Delta x/\Delta t - v}{1 - \frac{v}{c^2}\Delta x/\Delta t} = \frac{u_x - v}{1 - vu_x/c^2}$$

$\because$ ($u_x = \Delta x/\Delta t$)

$$\Rightarrow \boxed{u_x' = \frac{u_x - v}{1 - vu_x/c^2}}$$

and

$$\boxed{u_x = \frac{u_x' + v}{1 + vu_x'/c^2}}$$

$\rightarrow$ by exchanging primes and $v \rightarrow -v$

Time Dilation and Length Contraction from Lorentz Equations

$$\boxed{t = \gamma\left(t' + \frac{v}{c^2}x'\right)}$$

$$t_1 = \gamma\left(t_1' + \frac{v}{c^2}x_1'\right)$$

$$t_2 = \gamma\left(t_2' + \frac{v}{c^2}x_2'\right)$$

From Lorentz Transformation

$$\boxed{t_2 - t_1 = \gamma\left(t_2' - t_1'\right) + \frac{\gamma v}{c^2}\left(x_2' - x_1'\right)}$$

For proper time, $\boxed{x_1' = x_2'}$

Two events occurring at the same spatial position.

$$\therefore \quad \boxed{t_2 - t_1 = \gamma\left(t_2' - t_1'\right)}$$

Time Dilation

$$\boxed{\Delta t = \gamma \Delta t_0}$$

(Example 1.10, Beiser)

$$\boxed{x' = \gamma(x - vt)}$$

From Lorentz transformation

$$x_1' = \gamma(x_1 - vt_1)$$
$$x_2' = \gamma(x_2 - vt_2)$$

$$x_2' - x_1' = \gamma(x_2 - x_1) - \gamma v(t_2 - t_1)$$

$$\Rightarrow \boxed{\frac{x_2' - x_1'}{\gamma} = (x_2 - x_1) - v(t_2 - t_1)}$$

$x_2 - x_1$ is the length of the object measured AT THE SAME INSTANT in the unprimed frame.

$\therefore \boxed{t_1 = t_2}$ ($x_2' - x_1' \rightarrow$ Proper length)

$$\boxed{\frac{x_2' - x_1'}{\gamma} = x_2 - x_1}$$

Length Contraction

$$\boxed{l = \frac{l_0}{\gamma}}$$

(Example 1.9, Beiser)

Simultaneity

Two events happening at the same time.

$$\boxed{t' = \gamma\left(t - \frac{v}{c^2}x\right)}$$

From Lorentz transformation

$$t_1' = \gamma\left(t_1 - \frac{v}{c^2}x_1\right)$$

$$t_2' = \gamma\left(t_2 - \frac{v}{c^2}x_2\right)$$

In the static frame two events occur at x_1 and x_2 at the same time $\Rightarrow \boxed{t_1 = t_2}$ [$x_1 \neq x_2$. separate positions]

$$\therefore \boxed{t_2' - t_1' = \gamma\left(t_2 - t_1\right) - \gamma\frac{v}{c^2}\left(x_2 - x_1\right)}.$$

Since $t_1 = t_2$ (Simultaneous),

$$\boxed{t_2' - t_1' = \frac{\gamma v}{c^2}\left(x_1 - x_2\right)} \quad \therefore \boxed{t_1' \neq t_2'}$$

The two events are NOT simultaneous in a moving frame.

Acceleration from Lorentz transformation

$$\boxed{x' = \gamma(x - vt)},\ \boxed{y' = y},\ \boxed{z' = z}$$

$$\boxed{t' = \gamma\left(t - \frac{v}{c^2}x\right)} \Rightarrow \boxed{\Delta t' = \gamma\left(\Delta t - \frac{v}{c^2}\Delta x\right)}$$

$\Rightarrow$ Also $$\boxed{u_x = \frac{u_x' + v}{1 + u_x' v/c^2}} \Rightarrow \boxed{u_x' = \frac{u_x - v}{1 - u_x v/c^2}}$$

$$\boxed{a_x' = \frac{\Delta u_x'}{\Delta t'} = \frac{1}{\gamma\left(\Delta t - \frac{v}{c^2}\Delta x\right)}\Delta\left(\frac{u_x - v}{1 - u_x v/c^2}\right)}$$

$$\Rightarrow \Delta\left(\frac{u_x - v}{1 - u_x v/c^2}\right) = \frac{\Delta u_x}{1 - u_x v/c^2} + \frac{(u_x - v)\times -1}{(1 - u_x v/c^2)^2} \times \left(-\frac{v}{c^2}\right)\Delta u_x$$

$$\Rightarrow \Delta\left(\frac{u_x - v}{1 - u_x v/c^2}\right) = \frac{\Delta u_x\left(1 - u_x v/c^2\right) + (u_x - v)\frac{v}{c^2}\Delta u_x}{\left(1 - u_x v/c^2\right)^2}$$

$$\therefore a_x' = \frac{\Delta u_x'}{\Delta t'} = \frac{\Delta u_x}{(1 - u_x v/c^2)^2} \times \left[\frac{1 - \cancel{u_x v/c^2} + \cancel{\frac{u_x v}{c^2}} - v^2/c^2}{\gamma\left(\Delta t - \frac{v}{c^2}\Delta x\right)}\right]$$

$$\left[\because \gamma = (1 - v^2/c^2)^{-1/2}\right]$$

$$\Rightarrow a_x' = \frac{\left[1 - (v^2/c^2)\right]^{3/2}}{(1 - u_x v/c^2)^2} \cdot \frac{\Delta u_x}{\Delta t} \cdot \frac{1}{\left(1 - \frac{v}{c^2}\Delta x/\Delta t\right)}.$$

Since, $\boxed{a_x = \Delta u_x/\Delta t}$ and $\boxed{u_x = \Delta x/\Delta t}$,

$$\boxed{a_x' = a_x \frac{\left[1 - (v^2/c^2)\right]^{3/2}}{(1 - v u_x/c^2)^3}}$$

Applies to the motion along x-coordinate

The Invariance of the Lorentz Sphere

$$\boxed{x = \gamma(x' + vt')}, \quad \boxed{t = \gamma(t' + vx'/c^2)}, \quad \boxed{\begin{matrix} y = y' \\ z = z' \end{matrix}}.$$

$$\therefore x^2 - c^2t^2 = \gamma^2(x'+vt')^2 - \gamma^2\left(t' + \frac{vx'}{c^2}\right)^2 \times c^2$$

$$\Rightarrow x^2 - c^2t^2 = \gamma^2\left[x'^2 + \cancel{2x'vt'} + v^2t'^2 - c^2t'^2 - \cancel{2vx't} - v^2x'^2/c^2\right]$$

$$\Rightarrow \boxed{x^2 - c^2t^2 = \gamma^2\left[x'^2\left(1 - \frac{v^2}{c^2}\right) + t'^2(v^2 - c^2)\right]}$$

But $\boxed{\gamma^2 = \frac{1}{1 - v^2/c^2}}$. Hence $\boxed{\gamma^2\left(1 - \frac{v^2}{c^2}\right) = 1}$

$$\therefore x^2 - c^2t^2 = x'^2 + t'^2 \times -c^2\gamma^2\left(1 - \frac{v^2}{c^2}\right)$$

$$\Rightarrow \boxed{x^2 - c^2t^2 = x'^2 - c^2t'^2} \quad \because \boxed{\gamma^2\left(1 - \frac{v^2}{c^2}\right) = 1}.$$

Since $\boxed{y^2 = y'^2}$ and $\boxed{z^2 = z'^2}$

We get on adding $\boxed{x^2 + y^2 + z^2 - c^2t^2 = x'^2 + y'^2 + z'^2 - c^2t'^2}$

Hence $\boxed{x^2 + y^2 + z^2 - c^2t^2}$ is an <u>INVARIANT</u> <u>Quantity</u>.

(possible because of the invariance of c)

> A <u>Sphere</u> in <u>4-D spacetime</u> with <u>coordinates</u>, <u>(x, y, z, ict)</u> — <u>Lorentz Sphere</u>.