Electromagnetic Waves

The Maxwell equations of electrodynamics are $\boxed{\vec{\nabla}\cdot\vec{E} = \rho/\epsilon_0}$, $\boxed{\vec{\nabla}\cdot\vec{B} = 0}$

$\boxed{\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}}$ and $\boxed{\vec{\nabla}\times\vec{B} = \mu_0\left(\vec{J} + \epsilon_0\frac{\partial \vec{E}}{\partial t}\right)}$

In the $\boxed{\vec{\nabla}\times\vec{B} = \cdots}$ equation Maxwell introduced the displacement current $\boxed{\vec{J}_D = \epsilon_0\frac{\partial \vec{E}}{\partial t}}$ to correct Ampere's static ~~[illegible]~~ law.

The Faraday law, $\boxed{\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}}$ states that a changing magnetic field induces an electric field.

With Maxwell's correction of Ampere's law, the converse holds ~~true~~ true. It is a changing electric field induces a magnetic field. This can be seen clearly in free-space, where $\boxed{\rho = 0}$, $\boxed{\vec{J} = \vec{0}}$.

(P.T.O.)

This simplifies the Maxwell Equations as

$$\boxed{\boxed{\vec{\nabla}\cdot\vec{\mathcal{E}} = 0}\ ,\ \boxed{\vec{\nabla}\cdot\vec{B} = 0} \quad \boxed{\vec{\nabla}\times\vec{\mathcal{E}} = -\frac{\partial\vec{B}}{\partial t}}\ ,\ \boxed{\vec{\nabla}\times\vec{B} = \mu_0\epsilon_0\frac{\partial\vec{\mathcal{E}}}{\partial t}}}$$

The foregoing equations describe the electro ($\vec{\mathcal{E}}$) - magnetic ($\vec{B}$) fields in space that is free of charge and current (vacuum).

Now, we use the vector mathematical identity $\boxed{\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2\vec{A}}$, for a general vector field $\vec{A}$.

Setting $\boxed{\vec{A}\equiv\vec{\mathcal{E}}}$ (the electric field),

$$\boxed{\vec{\nabla}\times(\vec{\nabla}\times\vec{\mathcal{E}}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{\mathcal{E}}) - \nabla^2\vec{\mathcal{E}}}$$

Now $\boxed{\vec{\nabla}\cdot\vec{\mathcal{E}} = 0}$ and $\boxed{\vec{\nabla}\times\vec{\mathcal{E}} = -\frac{\partial\vec{B}}{\partial t}}$.

$\therefore \boxed{\vec{\nabla}\times\left(-\frac{\partial\vec{B}}{\partial t}\right) = -\nabla^2\vec{\mathcal{E}}}$ Exchanging $\boxed{\vec{\nabla}}$ and $\boxed{\frac{\partial}{\partial t}}$ in the left hand side, we get,

$\boxed{-\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{B}) = -\nabla^2\vec{\mathcal{E}}}$ But $\boxed{\vec{\nabla}\times\vec{B} = \mu_0\epsilon_0\frac{\partial\vec{\mathcal{E}}}{\partial t}}$

(P.T.O.)

[$\epsilon_0 \rightarrow$ Permittivity; $\mu_0 \rightarrow$ Permeability] of free space

$$-\frac{\partial}{\partial t}\left(\mu_0 \epsilon_0 \frac{\partial \vec{\mathcal{E}}}{\partial t}\right) = -\nabla^2 \vec{\mathcal{E}}$$

$$\Rightarrow \boxed{\mu_0 \epsilon_0 \frac{\partial^2 \vec{\mathcal{E}}}{\partial t^2} = \nabla^2 \vec{\mathcal{E}}}$$

μ_0 and ϵ_0 are universal constants

Similarly for $\boxed{\vec{A} \equiv \vec{B}}$ (the magnetic field)

$$\boxed{\vec{\nabla} \times (\vec{\nabla} \times \vec{B}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{B}) - \nabla^2 \vec{B}}$$

Now $\boxed{\vec{\nabla} \cdot \vec{B} = 0}$ and $\boxed{\vec{\nabla} \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{\mathcal{E}}}{\partial t}}$

$$\therefore \vec{\nabla} \times \left(\mu_0 \epsilon_0 \frac{\partial \vec{\mathcal{E}}}{\partial t}\right) = -\nabla^2 \vec{B}$$

$$\Rightarrow \boxed{\mu_0 \epsilon_0 \frac{\partial}{\partial t}\left(\vec{\nabla} \times \vec{\mathcal{E}}\right) = -\nabla^2 \vec{B}}$$

(Now, $\boxed{\vec{\nabla} \times \vec{\mathcal{E}} = -\frac{\partial \vec{B}}{\partial t}}$)

Hence, $\mu_0 \epsilon_0 \left(-\frac{\partial^2 \vec{B}}{\partial t^2}\right) = -\nabla^2 \vec{B}$

$$\Rightarrow \boxed{\mu_0 \epsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2} = \nabla^2 \vec{B}}$$

(Just like the $\vec{\mathcal{E}}$ equation)

i.) $\vec{\mathcal{E}}$-equation: $\boxed{\mu_0 \epsilon_0 \frac{\partial^2 \vec{\mathcal{E}}}{\partial t^2} = \frac{\partial^2 \vec{\mathcal{E}}}{\partial x^2} + \frac{\partial^2 \vec{\mathcal{E}}}{\partial y^2} + \frac{\partial^2 \vec{\mathcal{E}}}{\partial z^2}}$

ii.) $\vec{B}$-equation: $\boxed{\mu_0 \epsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2} = \frac{\partial^2 \vec{B}}{\partial x^2} + \frac{\partial^2 \vec{B}}{\partial y^2} + \frac{\partial^2 \vec{B}}{\partial z^2}}$

Both equations are hyperbolic second-order partial differential equations.

The Wave Equation

$V \rightarrow$ speed of wave propagation

1. Scalar: $$\boxed{\frac{1}{V^2}\frac{\partial^2 \psi}{\partial t^2} = \nabla^2 \psi}$$ $\psi \rightarrow$ Scalar function

2. Vector: $$\boxed{\frac{1}{V^2}\frac{\partial^2 \vec{A}}{\partial t^2} = \nabla^2 \vec{A}}$$ $\vec{A} \rightarrow$ Vector function

The $\vec{E}$ and $\vec{B}$ fields follow the latter.

$$\therefore \boxed{\frac{1}{V^2} = \mu_0 \epsilon_0} \Rightarrow V^2 = \frac{1}{\mu_0 \epsilon_0} \Rightarrow \boxed{V = \frac{1}{\sqrt{\mu_0 \epsilon_0}}}$$

$$\boxed{\begin{aligned} \mu_0 &= 4\pi \times 10^{-7} \text{ henry m}^{-1} \\ \epsilon_0 &= 8.85 \times 10^{-12} \text{ farad m}^{-1} \end{aligned}}$$ Electromagnetic Constants.

$$\therefore V = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = \frac{1}{\sqrt{11.1 \times 10^{-18}}} \text{ ms}^{-1} = \frac{10^9}{\sqrt{11.1}} \text{ ms}^{-1}$$

$$\Rightarrow \boxed{V \simeq 3 \times 10^8 \text{ ms}^{-1}} \rightarrow \text{The speed of light!}$$

Electromagnetic waves propagate at the speed of light. $\Rightarrow$ Light is an electromagnetic wave $\boxed{C = \frac{1}{\sqrt{\epsilon_0 \mu_0}}}$.

$C \rightarrow$ speed of light in vacuum.

(P.T.O.)

1/. Electricity, Magnetism and Optics are unified by $\boxed{c = \frac{1}{\sqrt{\epsilon_0 \mu_0}}}$.

$c \rightarrow$ Speed of light (optical constant)
$\epsilon_0 \rightarrow$ Permittivity of free space (electric constant)
$\mu_0 \rightarrow$ Permeability of free space (Magnetic constant)

2/. The unification was made possible by the displacement current $\boxed{\epsilon_0 \frac{\partial \vec{E}}{\partial t}}$ introduced by Maxwell.

3/. Electromagnetic waves are ripples in the electromagnetic field, propagating at the speed of light (c), over a wide range of wavelength (and frequency)

4/. Light has the range $\boxed{400\,nm < \lambda < 700\,nm}$

5/. In a transparent medium, $\boxed{v = \frac{c}{n}}$, $n \rightarrow$ Refractive index of the medium.

$\therefore \boxed{n = \frac{c}{v} = \frac{\sqrt{\epsilon \mu}}{\sqrt{\epsilon_0 \mu_0}}}$ $\boxed{\epsilon = \epsilon_r \epsilon_0}$ $\boxed{\mu = \mu_r \mu_0}$

$\boxed{n > 1}$ $\boxed{c > v}$ $\Rightarrow \boxed{n = \sqrt{\epsilon_r \mu_r}}$

$\epsilon_r \rightarrow$ Relative permittivity

$\mu_r \rightarrow$ Relative permeability

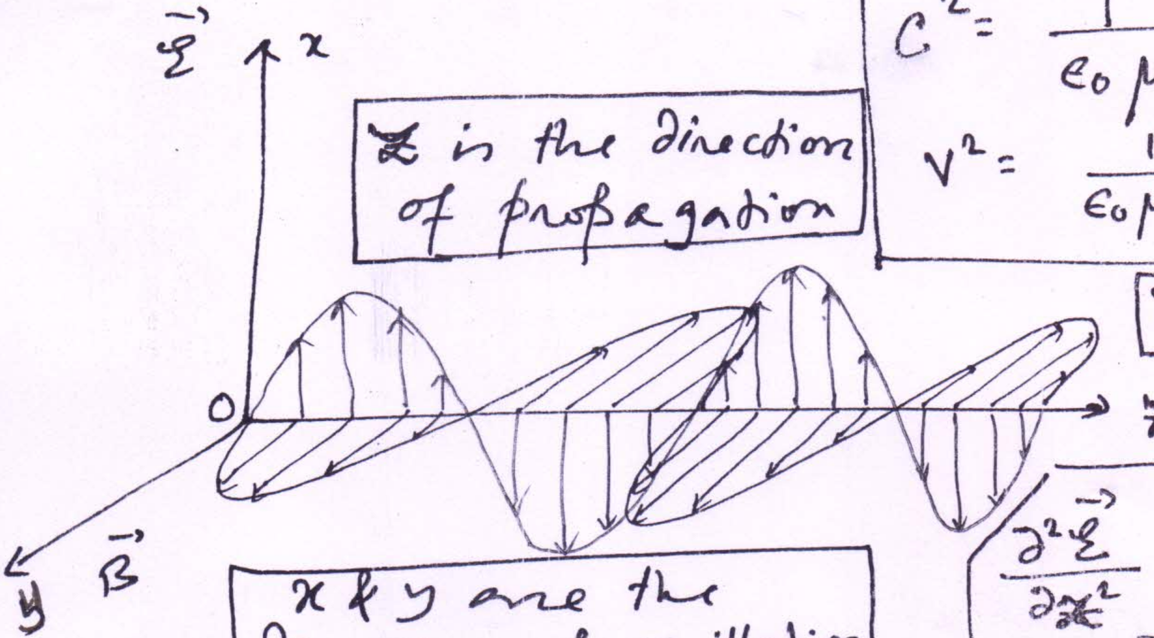
Light waves:

$$\boxed{\frac{c}{v} > 1} \qquad \boxed{n = \frac{c}{v} \Rightarrow n = \sqrt{\epsilon_r \mu_r}}$$

$$c^2 = \frac{1}{\epsilon_0 \mu_0}$$

$$v^2 = \frac{1}{\epsilon_0 \mu_0}\left(\frac{1}{\epsilon_r \mu_r}\right)$$

$$\boxed{n = \sqrt{\epsilon_r \mu_r}}$$



$$\frac{\partial^2 \vec{E}}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2}$$

OR $$\frac{\partial^2 \vec{B}}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 \vec{B}}{\partial t^2}$$

Waves: Non-localised (exists everywhere).

Disturbance moves, not the medium.

The ripple propagates, the field oscillates.

Ripples arise because accelerating charged particles radiate energy.
Ripples carry away the energy.

$$\boxed{\frac{\partial^2 \vec{E}}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} \quad \text{OR} \quad \frac{\partial^2 \vec{B}}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 \vec{B}}{\partial t^2}}$$

The Wave Equation

$$\boxed{\frac{\partial^2 \psi}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2}}$$

$\boxed{\psi \equiv \psi(x,t)}$

One-Dimensional

$\psi \rightarrow$ Displacement function

ψ_1 and ψ_2 are solutions.

$$\frac{\partial^2 \psi_1}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 \psi_1}{\partial t^2}$$

$$\frac{\partial^2 \psi_2}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 \psi_2}{\partial t^2}$$

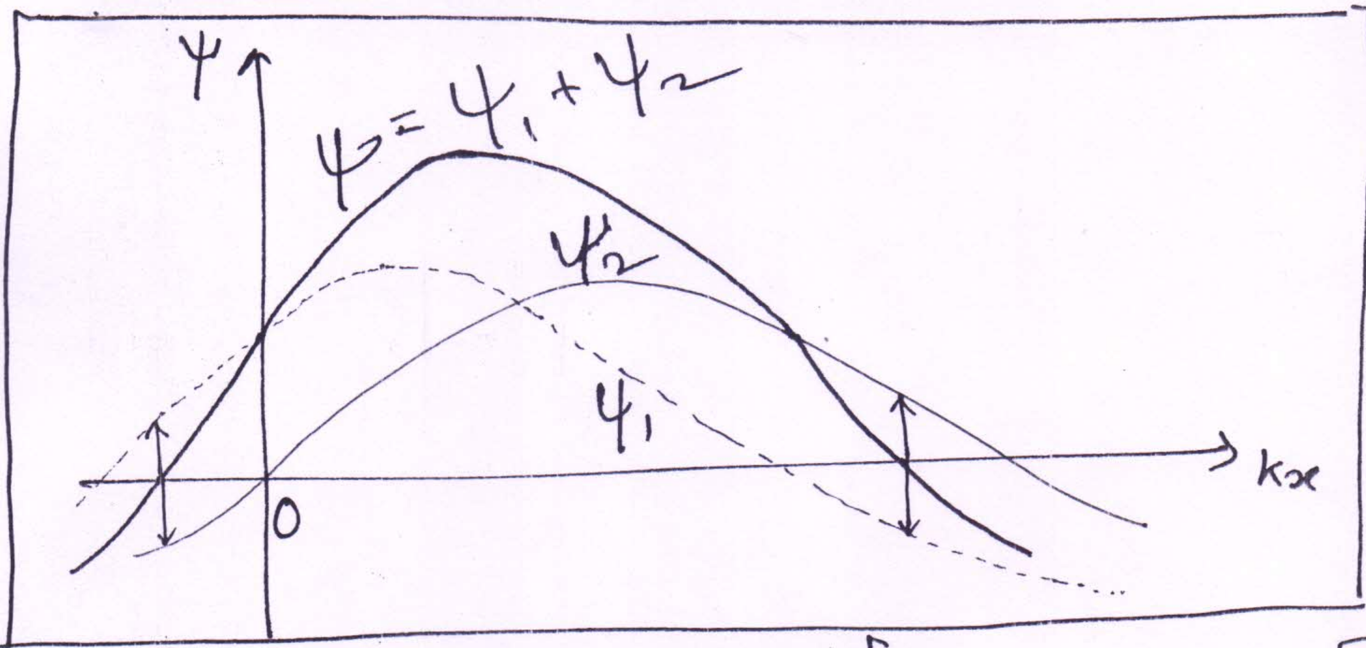
Add the two equations

$$\frac{\partial^2}{\partial x^2}\left(\psi_1 + \psi_2\right) = \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\left(\psi_1 + \psi_2\right)$$

$\boxed{\psi = \psi_1 + \psi_2}$ is also a solution.

(Initial and boundary conditions)

Principle of Superposition:



$$\boxed{\frac{\partial^2 \psi}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2}}$$

[Partial differential equation of second-order]

$$\boxed{\psi = f_1(x - vt) + f_2(x + vt)}$$

If $\boxed{v > 0}$

$f_1(x - vt)$ → Propagation along positive x

$f_2(x + vt)$ → Propagation along negative x

General Relativity

An Introduction for Physicists

M. P. HOBSON, G. P. EFSTATHIOU
and A. N. LASENBY

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for ever to do so. Nevertheless, the objects themselves cross the horizon in a finite proper time and still have an infinite lifetime ahead of them.

Appendix 1A: Einstein's route to special relativity

Most books on special relativity begin with some sort of description of the Michelson–Morley experiment and then introduce the Lorentz transformation. In fact, Einstein claimed that he was not influenced by this experiment. This is disputed by various historians of science and biographers of Einstein. One might think that these scholars are on strong ground, especially given that the experiment is referred to (albeit obliquely) in Einstein's papers. However, it may be worth taking Einstein's claim at face value.

Remember that Einstein was a *theorist* – one of the greatest theorists who has ever lived – and he had a *theorist's* way of looking at physics. A good theorist develops an intuition about how Nature works, which helps in the formulation of physical laws. For example, possible symmetries and conserved quantities are considered. We can get a strong clue about Einstein's thinking from the *title* of his famous 1905 paper on special relativity. The first paragraph is reproduced below.

On the Electrodynamics of Moving Bodies
by A. Einstein

It is known that Maxwell's electrodynamics – as usually understood at the present time – when applied to moving bodies, leads to asymmetries which do not appear to be inherent in the phenomena. Take, for example, the reciprocal electrodynamic action of a magnet and a conductor. The observable phenomenon here depends only on the relative motion of the conductor and the magnet, whereas the customary view draws a sharp distinction between the two cases in which either the one or the other of these bodies is in motion. For if the magnet is in motion and the conductor at rest, there arises in the neighbourhood of the magnet an electric field with a certain definite energy, producing a current at the places where parts of the conductor are situated. But if the magnet is stationary and the conductor in motion, no electric field arises in the neighbourhood of the magnet. In the conductor, however, we find an electromotive force, to which in itself there is no corresponding energy, but which gives rise – assuming equality of relative motion in the two cases discussed – to electric currents of the same path and intensity as those produced by the electric forces in the former case.

You see that Einstein's paper is not called 'Transformations between inertial frames', or 'A theory in which the speed of light is assumed to be a universal constant'. Electrodynamics is at the heart of Einstein's thinking; Einstein realized that Maxwell's equations of electromagnetism *required* special relativity.

Maxwell's equations are

$$\vec{\nabla}\cdot\vec{D}=\rho, \qquad \vec{\nabla}\cdot\vec{B}=0,$$

$$\vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}, \qquad \vec{\nabla}\times\vec{H}=\vec{j}+\frac{\partial\vec{D}}{\partial t},$$

where $\vec{D}=\epsilon_0\vec{E}+\vec{P}$ and $\vec{B}=\mu_0(\vec{H}+\vec{M})$, $\vec{P}$ and $\vec{M}$ being respectively the polarisation and the magnetisation of the medium in which the fields are present. In free space we can set $\vec{j}=\vec{0}$ and $\rho=0$, and we then get the more obviously symmetrical equations

$$\vec{\nabla}\cdot\vec{E}=0, \qquad \vec{\nabla}\cdot\vec{B}=0,$$

$$\vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}, \qquad \vec{\nabla}\times\vec{B}=\mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}.$$

Taking the curl of the equation for $\vec{\nabla}\times\vec{E}$, applying the relation

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E})=\vec{\nabla}(\vec{\nabla}\cdot\vec{E})-\nabla^2\vec{E}$$

and performing a similar operation for $\vec{B}$ in the equation for $\vec{\nabla}\times\vec{B}$, we derive the equations for electromagnetic waves:

$$\nabla^2\vec{E}=\mu_0\epsilon_0\frac{\partial^2\vec{E}}{\partial t^2}, \qquad \nabla^2\vec{B}=\mu_0\epsilon_0\frac{\partial^2\vec{B}}{\partial t^2}.$$

These both have the form of a wave equation with a propagation speed $c=1/\sqrt{\mu_0\epsilon_0}$. Now, the constants μ_0 and ϵ_0 are properties of the 'vacuum':

μ_0, the permeability of a vacuum, equals $4\pi\times10^{-7}\,\mathrm{Hm}^{-1}$,

ϵ_0, the permittivity of a vacuum, equals $8.85\times10^{-12}\,\mathrm{Fm}^{-1}$.

This relation between the constants ϵ_0 and μ_0 and the speed of light was one of the most startling consequences of Maxwell's theory. But what do we mean by a 'vacuum'? Does it define an absolute frame of rest? If we deny the existence of an absolute frame of rest then how do we formulate a theory of electromagnetism? How do Maxwell's equations appear in frames moving with respect to each other? Do we need to change the value of c? If we do, what will happen to the values of ϵ_0 and μ_0?

Einstein solves all of these problems at a stroke by saying that Maxwell's equations take the same mathematical form in all inertial frames. The speed of light c is thus the same in all inertial frames. The theory of special relativity (including amazing conclusions such as $E=mc^2$) follows from a generalisation of this simple and theoretically compelling assumption. *Maxwell's equations therefore require special relativity*. You see that for a master theorist like Einstein, the

Michelson–Morley experiment might well have been a side issue. Einstein could 'see' special relativity lurking in Maxwell's equations.

Exercises

1.1 For two inertial frames S and S' in standard configuration, show that the coordinates of any given event in each frame are related by the Lorentz tranformations (1.3).

1.2 Two events A and B have coordinates (t_A, x_A, y_A, z_A) and (t_B, x_B, y_B, z_B) respectively. Show that both the time difference $\Delta t = t_B - t_A$ and the quantity

$$\Delta r^2 = \Delta x^2 + \Delta y^2 + \Delta z^2$$

are separately invariant under any Galilean transformation, whereas the quantity

$$\Delta s^2 = c^2 \Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2$$

is invariant under any Lorentz transformation.

1.3 In a given inertial frame two particles are shot out simultaneously from a given point, with equal speeds v in orthogonal directions. What is the speed of each particle relative to the other?

1.4 An inertial frame S' is related to S by a boost of speed v in the x-direction, and S'' is related to S' by a boost of speed u' in the x'-direction. Show that S'' is related to S by a boost in the x-direction with speed u, where

$$u = c \tanh(\psi_v + \psi_{u'});$$

$\tanh \psi_v = v/c$ and $\tanh \psi_{u'} = u'/c$.

1.5 An inertial frame S' is related to S by a boost $\vec{v}$ whose components in S are (v_x, v_y, v_z). Show that the coordinates (ct', x', y', z') and (ct, x, y, z) of an event are related by

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta_x & -\gamma\beta_y & -\gamma\beta_z \\ -\gamma\beta_x & 1+\alpha\beta_x^2 & \alpha\beta_x\beta_y & \alpha\beta_x\beta_z \\ -\gamma\beta_y & \alpha\beta_y\beta_x & 1+\alpha\beta_y^2 & \alpha\beta_y\beta_z \\ -\gamma\beta_y & \alpha\beta_z\beta_x & \alpha\beta_z\beta_y & 1+\alpha\beta_z^2 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix},$$

where $\vec{\beta} = \vec{v}/c$, $\gamma = (1 - |\vec{\beta}|^2)^{-1/2}$ and $\alpha = (\gamma - 1)/|\vec{\beta}|^2$. *Hint: The transformation must take the same form if both S and S′ undergo the same spatial rotation.*

1.6 An inertial frame S' is related to S by a boost of speed u in the positive x-direction. Similarly, S'' is related to S' by a boost of speed v in the y'-direction. Find the transformation relating the coordinates (ct, x, y, z) and (ct'', x'', y'', z'') and hence describe how S and S'' are physically related.

1.7 The frames S and S' are in standard configuration. A straight rod rotates at a uniform angular velocity ω' about its centre, which is fixed at the origin of S'. If the rod lies along the x'-axis at $t' = 0$, obtain an equation for the shape of the rod in S at $t = 0$.