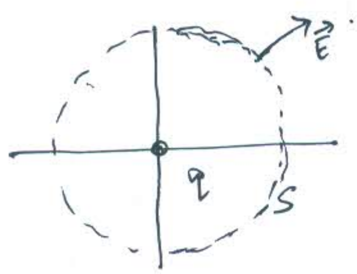
Dirac delta function

Consider the simplest electrostatic configuration.
A point charge q at the origin.
The electric field due to this charge is given as

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r}$$



The flux of $\vec{E}$ over a sphere of radius a will be

$$\oint_S \vec{E}\cdot\hat{n}\,da = \oint \frac{1}{4\pi\epsilon_0}\frac{q}{a^2}\cdot 4\pi a^2 = \frac{q}{\epsilon_0}$$

This is consistent with the integral form of the Gauss' law.
Now $\vec{\nabla}\cdot\vec{E} = \frac{q}{4\pi\epsilon_0}\vec{\nabla}\cdot\frac{\hat{r}}{r^2} = 0$ for $r>0$.

By divergence theorem

$$\int_V \vec{\nabla}\cdot\vec{E}\,dV = \oint_S \vec{E}\cdot\hat{n}\,da = \frac{q}{\epsilon_0}$$

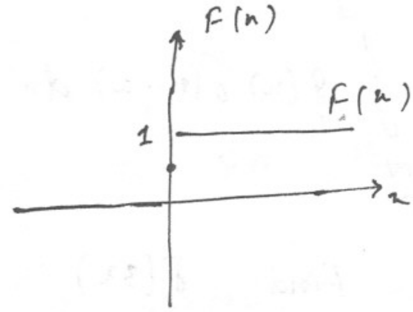
The contribution to the volume integral only comes from the origin since $\vec{\nabla}\cdot\vec{E} = 0$ at all other points. If $\vec{\nabla}\cdot\vec{E}$ is finite at $r=0$ then its contribution to the volume integral on the l.h.s. is 0 since the volume tends ~~to~~ 0. So $\vec{\nabla}\cdot\vec{E}\to\infty$ as $r\to 0$.

$$\int_V \vec{\nabla}\cdot\vec{E}\,dV = 0 \quad \text{if } V \text{ doesn't include the origin.}$$
$$= \frac{q}{\epsilon_0} \quad \text{if } V \text{ includes origin within it.}$$

$\vec{\nabla}\cdot\vec{E}$ is described by a function called the Dirac delta function.

Dirac delta function.

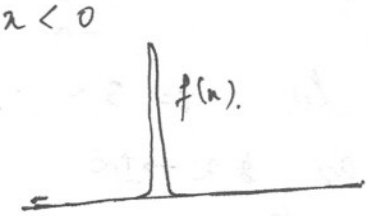
Consider the following function



$$F(x) = 1 \text{ for } x > 0$$
$$= 0 \text{ for } x < 0$$
$$= \frac{1}{2} \text{ at } x = 0$$

Let $f(x) = \dfrac{dF}{dx}$

Now $f(x) = 0$ for $x > 0$ and $x < 0$



$$\lim_{x \to 0} f(x) \to \infty$$

The integral $\displaystyle\int_{-a}^{+a} f(x). = \Big[F(x)\Big]_{-a}^{+a} = 1 - 0 = 1$

The value of this integral is 1 if the region includes ~~x~~ $x = 0$. Otherwise the value of the integral is zero.

Functions with such integral properties as that of $f(x)$ are called dirac delta function denoted as $\delta(x)$. This can be defined only in terms of its integral property as seen above. Another important property which is also sometimes considered as its defining property is the following:

For any continuous function $g(x)$ at the origin,

$$\int_{-a}^{+a} g(x)\,\delta(x)\,dx = g(0)$$

We say that a delta function fires only at $x = 0$ and selects the value of $g(x)$ at $x = 0$. We can also have delta function firing at $x = a$ denoted as $\delta(x-a)$

For such a function we will have.

$$\int_{-\infty}^{\infty} g(x)\,\delta(x-a)\,dx = g(a)$$

Eg1) Find $\delta(3x)$ in terms of $\delta(x)$

Consider $\displaystyle\int_{-\infty}^{\infty} \delta(3x)\,dx$

Let $t = 3x$. Then $dt = 3\,dx$

as $\cancel{t}\, x \to \pm\infty$ $\quad t \to \pm\infty$

$$\therefore \int_{-\infty}^{\infty} \delta(3x)\,dx = \frac{1}{3}\int_{-\infty}^{\infty}\delta(t)\,dt = \frac{1}{3}$$

$$\therefore \delta(3x) = \frac{1}{3}\,\delta(x)$$

Now let us find $\delta(-3x)$

Let $t = -3x$. Then $dt = -3\,dx$.

$\therefore \cancel{\int_{-\infty}^{\infty}\delta(-3x)\,dx}$. as $x \to \pm\infty$, $t \to \mp\infty$

$$\therefore \int_{-\infty}^{\infty} \delta(-3x)\,dx = \int_{\infty}^{-\infty} \delta(t)\left(-\frac{1}{3}\right)dt = \frac{1}{3}\int_{-\infty}^{\infty}\delta(t)\,dt$$

$$= \frac{1}{3}$$

$\therefore \delta(-3x)$ is also $\frac{1}{3}\,\delta(x)$

So we see that $\delta(kx) = \dfrac{1}{|k|}\,\delta(x)$

<u>In 3-dim</u> the deltafunction is denoted as $\delta^3(\vec{r})$

and defined as.

$$\int_{\text{all space}} \delta^3(\vec{r})\,dV = 1$$

In cartesian co-ordinate we can write it as.

$$\int_{\text{all space}} \delta^3(\vec{r})\,dV = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x)\,\delta(y)\,\delta(z)\,dx\,dy\,dz = 1$$

Just like the one-dim. δ- function. we have.

$$\int_V f(\vec{r})\, \delta^3(\vec{r}-\vec{a})\, dV = f(\vec{a})$$

where. the volume. V includes the point $\vec{a}$.

We. have. seen that $\vec{\nabla}\cdot\frac{\hat{r}}{r^2} = 0$ everywhere. except

the origin.

By divergance. theorem.

$$\int_V \vec{\nabla}\cdot\frac{\hat{r}}{r^2}\, dV = \oint_S \frac{\hat{r}}{r^2}\cdot\vec{da}$$

We. consider the volume. V as a sphere. of. radius R and S the surface of the. sphere.

Then $\vec{da} = \hat{r}\, R^2 \sin\theta\, d\theta\, d\phi$

$$\therefore \oint_S \frac{\hat{r}}{r^2}\cdot\vec{da} = \int_0^{2\pi}\int_0^{\pi} \frac{\hat{r}\cdot\hat{r}}{R^2} \cdot R^2 \sin\theta\, d\theta\, d\phi$$

$$= 4\pi$$

$$\therefore \int_V \left(\vec{\nabla}\cdot\frac{\hat{r}}{r^2}\right) dV = 4\pi$$ over any volume. enclosing the origin.

$$\therefore \vec{\nabla}\cdot\frac{\hat{r}}{r^2} = 4\pi\, \delta^3(\vec{r})$$

So we. have., if. $\vec{A}(\vec{r}) = 2x\hat{i} + 4\hat{j} + e^{-r}\hat{k}$

Then $$\int_{\text{all space}} \vec{A}(\vec{r})\; \vec{\nabla}\cdot\frac{\hat{r}}{r^2}\, dV = 4\pi\cdot\vec{A}(0) = 4\pi(4\hat{j}+\hat{k})$$

Eq: $\vec{E}(\vec{r}) = \frac{c a^2}{\epsilon_0} \frac{\hat{r}}{r^2}$; $r \geq a$.

$= 0$; $r < a$.

Find the charge density that causes this $\vec{E}$.

The volume charge density $\rho(\vec{r})$ is obtained from the differential form of the Gauss' law.

$$\frac{\rho(\vec{r})}{\epsilon_0} = \vec{\nabla} \cdot \vec{E}$$

for $r > a$ and $r < a$ $\vec{\nabla} \cdot \vec{E} = 0$

$\therefore \rho(\vec{r}) = 0$ for $r > a$ and $r < a$.

At $r = a$, $\vec{E}$ is discontinuous (check). So we can't compute $\vec{\nabla} \cdot \vec{E}$. We will have to work with the integral property of $\vec{\nabla} \cdot \vec{E}$. For a sphere with radius $R_1 < a$

$$\int_S (\vec{\nabla} \cdot \vec{E})\, dV = \int_0^{R_1} (\vec{\nabla} \cdot \vec{E})\, 4\pi r^2\, dr = 0$$

But if $R_2 \geq a$ then

$$\int (\vec{\nabla} \cdot \vec{E})\, dV = \int_0^{R_2} (\vec{\nabla} \cdot \vec{E})\, 4\pi r^2 dr = \oint_S (\vec{E} \cdot \hat{n})\, da$$

$$= \frac{4\pi c a^2}{\epsilon_0}.$$

So $\vec{\nabla} \cdot \vec{E} = k\, \delta(r - a)$

$$\therefore \int_0^{R_2} (\vec{\nabla} \cdot \vec{E}) \cdot 4\pi r^2\, dr = \int_0^{R_2} k\, \delta(r-a) \cdot 4\pi r^2\, dr = \frac{4\pi c a^2}{\epsilon_0}$$

$$\therefore k = \frac{c}{\epsilon_0} \Rightarrow \rho = \epsilon_0 \vec{\nabla} \cdot \vec{E} = c\, \delta(r - a)$$

From this the surface charge density is obtained as follows.

$$\sigma = \int_{a-\varepsilon}^{a+\varepsilon} \rho(r)\, dr = \int_{a-\varepsilon}^{a+\varepsilon} c\,\delta(r-a)\, dr = c$$