Work and Energy:

Let $V(\vec{r})$ be the potential in a region due to a charge distribution. Now if we move a charge q from ∞ to a point $\vec{a}$ then the work done on the charge is

$$W = qV(\vec{a})$$

If we are given a charge distribution consisting of n ~~at~~ point charges $q_1, q_2, \ldots, q_n$ ~~there~~ situated at position $\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_n$, then how much work is spent in making this configuration?

We start with q_1 and bring q_2 from ∞ and place it at $\vec{r}_2$. This will need an energy

$$W_2 = \frac{q_2}{4\pi\epsilon_0} \frac{q_1}{r_{12}}$$

where $r_{12} = |\vec{r}_2 - \vec{r}_1|$

Now we bring q_3 from ∞ to $\vec{r}_3$. The work done for this is

$$W_3 = \frac{q_3}{4\pi\epsilon_0}\left(\frac{q_1}{r_{13}} + \frac{q_2}{r_{23}}\right)$$

So the energy required to accumulate these three charges is

$$W_2 + W_3 = \frac{1}{4\pi\epsilon_0}\left[\frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}}\right]$$

$$= \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{2} \sum_{\substack{j=2 \\ j>i}}^{3} \frac{q_i q_j}{r_{ij}}$$

When we have accumulated n charges this becomes

$$W = \sum_{i=1}^{n-1} \sum_{\substack{j=2 \\ j>i}}^{n} \frac{1}{4\pi\epsilon_0} \frac{q_i q_j}{r_{ij}}$$

If we want to treat the indices i and j symmetrically then we can write this summation as

$$W = \frac{1}{2} \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{1}{4\pi\epsilon_0} \frac{q_i q_j}{r_{ij}} \quad \text{———} \ (1)$$

This can be written as

$$W = \frac{1}{2} \sum_{i=1}^{n} q_i \, V(\vec{r}_i) \quad \text{———} \ (2)$$

where $V(\vec{r}_i) = \frac{1}{4\pi\epsilon_0} \sum_{j\neq i} \frac{q_j}{r_{ij}}$ is the potential at the point $\vec{r}_i$ due to the other $n-1$ charges.

The energy stored in a charge distribution as expressed in the form (2) can be extended to a continuous charge distribution as follows.

$$W = \cancel{\frac{1}{4\pi\epsilon_0}\oint} \ \frac{1}{2} \int_\tau \rho(\vec{r}) \, V(\vec{r}) \, d\tau \quad \text{———} \ (3)$$

where τ is the region within which the charge density $\rho(\vec{r})$ exist.

~~Note: We cannot extend this expression to the work done to accumulate a surface charge distribution and line charge distribution.~~

By Gauss's law $\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}$. So we have.

in

$$W = \frac{1}{2}\int_{\tau} \epsilon_0 (\vec{\nabla}\cdot\vec{E})\, V(\vec{r})\, d\tau$$

$$\vec{\nabla}\cdot(\vec{E}\,V) = (\vec{\nabla}\cdot\vec{E})\,V + \vec{E}\cdot\vec{\nabla}V$$

$$\therefore\ (\vec{\nabla}\cdot\vec{E})\,V = \vec{\nabla}\cdot(\vec{E}\,V) - \vec{E}\cdot\vec{\nabla}V$$

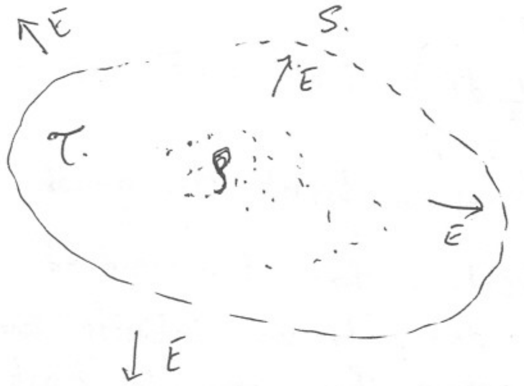
$$= \vec{\nabla}\cdot(\vec{E}\,V) + E^2$$

$$\therefore\ W = \frac{\epsilon_0}{2}\left[\int_{\tau} E^2\, d\tau + \int_{\tau} \vec{\nabla}\cdot(\vec{E}\,V)\, d\tau\right]$$

$$= \frac{\epsilon_0}{2}\left[\int_{\tau} E^2\, d\tau + \int_{S} \vec{E}\,V\cdot\hat{n}\, d\tau\right] \text{ --- } (4)$$

where ~~$\hat{n}$~~ S is the surface enclosing τ.

We started with the energy stored in a charge configuration within a volume τ in Eq (3). In Eq (4) we see that this is equal to just integral over a volume and a surface for electric fields and electric potentials $\vec{E}$ and V.



In Eq (4) the integral over τ extends over region where there may not be any charge density, but we may have non-zero electric field $\vec{E}$.

we can push the surface S to infinity and make
the integral over volume to be over the whole
space. If the charge configuration is confined
in space then $\vec{E}$ and V goes to 0 at infinity.
In this case the surface integral $\longrightarrow 0$ and
the energy of the charge configuration is

$$W = \frac{\epsilon_0}{2}\int_{\text{whole space}} E^2 \, d\tau \qquad \text{——— (5)}$$

Eq (5) says that if we have electric field $\vec{E}$ in a
region then we have an energy density $\frac{\epsilon_0}{2}E^2$ in the
region. This implies that electric field is not just a
mathematical convenience, but a physical entity carrying
energy. In fact we can have electric field without charges
which can carry energy as in electromagnetic waves.

When we started calculating the energy of a charge
configuration we started accumulating point charges.
We didn't bother what is the energy needed to make
these point charges. Eq (5) gives a way to calculate
the electrostatic energy of a point charge.

Since $E = \frac{q}{4\pi\epsilon_0 r^2}$

$$W = \frac{\epsilon_0}{2}\frac{q^2}{(4\pi\epsilon_0)^2}\int_{\text{whole space}} \frac{1}{r^4}\, r^2 \sin\theta \, d\theta \, d\phi \, dr$$

$$= \frac{q^2}{32\pi^2\epsilon_0}\left(\int_0^\infty \frac{1}{r^2}\, dr\right)\cdot 4\pi \longrightarrow \infty$$

We can calculate the energy of a point charge also by
other methods and convince ourselves that it is indeed ∞.
Point charge is a fiction that demands an infinite energy
to be created. Typically we ignore this infinite background
reference energy and live our humble life by considering
only the variations around it.

Superposition:

Suppose we have two charge configurations in a region. Due to one configuration the electric field is $\vec{E}_1$. Due to the other the electric field is $\vec{E}_2$. So due to the combined ~~configuration~~ charge distribution of the two configurations we have

$$\vec{E} = \vec{E}_1 + \vec{E}_2$$

The electrostatic energies of the individual configurations are

$$W_1 = \int E_1^2 \, d\tau \quad \text{and} \quad W_2 = \int E_2^2 \, d\tau.$$

Now due to the combined configuration $\vec{E}_1 + \vec{E}_2$ the total energy is

$$W_{tot} = \frac{\epsilon_0}{2}\int (\vec{E}_1 + \vec{E}_2)^2 \, d\tau$$

$$= \frac{\epsilon_0}{2}\left[\int E_1^2 \, d\tau + \int E_2^2 \, d\tau + 2\int (\vec{E}_1 \cdot \vec{E}_2) \, d\tau\right]$$

$$= W_1 + W_2 + \epsilon_0 \int (\vec{E}_1 \cdot \vec{E}_2) \, d\tau$$

In the special case when the individual configuration are point charges q_1 and q_2, W_1 and W_2 are infinities and the energy of the configuration W_{tot} is considered to be only the term

$$W_{tot} = \epsilon_0 \int \vec{E}_1 \cdot \vec{E}_2 \, d\tau.$$

We can show that this is equal to $\frac{q_1 q_2}{4\pi\epsilon_0 r_{12}}$.