Compton Effect: Selected Solutions

Q1/. Initial ~~total~~ total energy = $\boxed{h\nu + m_0c^2}$

Final total energy = $\boxed{h\nu' + \gamma m_0 c^2} \to (1)$

Energy Conservation: $\boxed{h\nu + m_0c^2 = h\nu' + \gamma m_0 c^2}$

Initial total momentum = $\boxed{h\nu/c}$ (due to the photon only)

Final total momentum = $\boxed{-h\nu'/c + \gamma m_0 v}$

The negative sign is because after the head-on collision the photon goes back in the opposite direction. The electron goes on straight ahead.

Momentum Conservation: $\boxed{h\nu/c = -h\nu'/c + \gamma m_0 v}$

$\Rightarrow \boxed{h\nu = -h\nu' + \gamma m_0 v c} \to (2)$. Add (1) and (2) to get.

$\Rightarrow \boxed{2h\nu + m_0c^2 = \gamma m_0 v c + \gamma m_0 c^2}$.

$\Rightarrow \boxed{\frac{2h\nu}{m_0c^2} + 1 = \gamma\left(1 + \frac{v}{c}\right)}$ Now $\boxed{\gamma = \frac{1}{\sqrt{1-(v/c)^2}}}$

$\Rightarrow \boxed{\frac{2h\nu}{m_0c^2} + 1 = \sqrt{\frac{(1+v/c)\cancel{(1+v/c)}}{(1-v/c)\cancel{(1+v/c)}}} = \sqrt{\frac{1+(v/c)}{1-(v/c)}}}$

$\left(1+\frac{v}{c}\right) = \left(1-\frac{v}{c}\right)\left[1 + \frac{2h\nu}{m_0c^2}\right]^2$ Now $\boxed{h\nu = 0.3\ \text{MeV}}$

$\boxed{m_0c^2 = 0.51\ \text{MeV}}$ for an electron (rest energy)

$\Rightarrow \left(1+\frac{v}{c}\right) = \left(1-\frac{v}{c}\right)\left[1 + \frac{2\times0.3}{0.51}\right]^2$ $\left(\frac{2h\nu}{m_0c^2} = 1.176\right)$

$\Rightarrow \left(1+\frac{v}{c}\right) = \left(1-\frac{v}{c}\right)\times(4.735)$

$\Rightarrow 1 + \frac{v}{c} = 4.735 - 4.735\frac{v}{c}$ $\Rightarrow 5.735\frac{v}{c} = 3.735$

$\Rightarrow \boxed{\frac{v}{c} = \frac{3.735}{5.735} = 0.65}$

Answer $\downarrow$ $\boxed{v = 0.65c}$

Q3/. $\boxed{\Delta\lambda = \lambda_c(1-\cos\phi)}$. $\boxed{\text{Maximum energy} \Rightarrow \phi = 180^\circ}$

$\Rightarrow \boxed{\Delta\lambda = \lambda_c[1-(-1)] = 2\lambda_c}$ $\therefore \boxed{\cos\phi = -1}$.

$\lambda_c \rightarrow$ Compton Wavelength

$\boxed{\Delta\mathcal{E} = \frac{hc}{\lambda'} - \frac{hc}{\lambda}}$ and $\boxed{\lambda' = \lambda + \Delta\lambda}$

$\Rightarrow \boxed{\Delta\mathcal{E} = hc\left(\frac{1}{\lambda+\Delta\lambda} - \frac{1}{\lambda}\right)}$

$\Rightarrow \frac{\Delta\mathcal{E}}{hc} = \frac{\cancel{\lambda} - \cancel{\lambda} - \Delta\lambda}{\lambda(\lambda+\Delta\lambda)}$ $\Rightarrow \lambda(\lambda+\Delta\lambda) = -\frac{hc}{\Delta\mathcal{E}}\Delta\lambda$

$\Rightarrow \boxed{\lambda^2 + (\Delta\lambda)\lambda + \frac{hc}{\Delta\mathcal{E}}(\Delta\lambda) = 0}$ $\boxed{\Delta\lambda = 2\lambda_c = \frac{2h}{m_e c}}$

$\Rightarrow \Delta\lambda = 2(2.4\,pm)$

$\Rightarrow \boxed{\lambda^2 + (2\lambda_c)\lambda + \frac{hc(2\lambda_c)}{\Delta\mathcal{E}} = 0}$

$\Rightarrow \lambda = \frac{-2\lambda_c \pm \sqrt{4\lambda_c^2 - 4\left(\frac{hc}{\Delta\mathcal{E}}2\lambda_c\right)}}{2}$

$\Rightarrow \boxed{\lambda = -\lambda_c \pm \sqrt{\lambda_c^2 - \left(\frac{hc}{\Delta\mathcal{E}}\right)2\lambda_c}}$ Since wavelength is a positive quantity choose the positive sign.

Also $\boxed{\Delta\mathcal{E} = -45\,keV}$, the negative sign implying that the photon has lost its energy to the electron.

$\therefore -\frac{hc}{\Delta\mathcal{E}} = \frac{6.626\times10^{-34}\times3\times10^{8}}{45\times10^{3}\times1.6\times10^{-19}} = \frac{6.626\times3}{45\times1.6}\times10^{-10} = \boxed{27.61\,pm}$

Also $\boxed{\lambda_c = 2.4\,pm}$ $\Rightarrow \lambda = -2.4 + \sqrt{2.4^2 + 2\times2.4\times27.61}$

$\Rightarrow \boxed{\lambda = -2.4 + 11.76 = 9.36\,pm}$ ← Answer

Note: This solution method is the same as in Q9. To find the energy, apply $\boxed{\mathcal{E} = hc/\lambda}$ λ is the answer above

Q4/ $\boxed{\Delta\lambda = \lambda_c(1-\cos\phi)}$ for maximum shift, $\boxed{\phi = 180°}$.
& $\boxed{\cos\phi = -1}$.

$\therefore \boxed{\Delta\lambda = 2\lambda_c} \Rightarrow$ Fractional shift $= \boxed{\frac{\Delta\lambda}{\lambda} = \frac{2\lambda_c}{\lambda}}$

$\boxed{\lambda_c = \frac{h}{mc}}$ If $m = m_e$ (mass of electron) $\boxed{\lambda_c = 2.4\,pm}$

If $m = m_p$ (mass of proton), $\boxed{\lambda_c = \frac{2.4\,pm}{1836}}$ Compton wavelength

Hence, when $\lambda = 0.1\,nm.$,

$\frac{\Delta\lambda}{\lambda} = \frac{2.4\,pm \times 2}{1836 \times 0.1 \times 10^3\,pm} = \boxed{2.61 \times 10^{-5}}$ Answer

---

Q5/. Energy required $= \boxed{2m_ec^2 = hc/\lambda_{pe}}$

$\Rightarrow \boxed{2\lambda_{pe} = \frac{h}{m_ec} = \lambda_c}$ Now $\boxed{\lambda' - \lambda = \lambda_c(1-\cos\phi)}$

$\Rightarrow \lambda' - \lambda = 2\lambda_{pe}(1-\cos\phi) \Rightarrow \boxed{\lambda' = \lambda + 2\lambda_{pe}(1-\cos\phi)}$

$\Rightarrow \boxed{\lambda' > 2\lambda_{pe}(1-\cos\phi)}$ Since the scattered photon can create a positron-electron pair,

$\boxed{\lambda' = \lambda_{pe}} \Rightarrow \boxed{\cancel{\lambda_{pe}} > 2\cancel{\lambda_{pe}}(1-\cos\phi)}$ Answer

$\Rightarrow 1-\cos\phi < 1/2 \Rightarrow \boxed{\cos\phi > 1/2} \Rightarrow \boxed{\phi < 60°}$

Q6/. $\boxed{E' = E/2} \Rightarrow \frac{\cancel{hc}}{\lambda'} = \frac{\cancel{hc}}{2\lambda} \Rightarrow \boxed{\lambda' = 2\lambda}$

$\therefore \boxed{\lambda' - \lambda = 2\lambda - \lambda = \lambda = \lambda_c(1-\cos\phi)}$ $\boxed{\phi = 45°}$ Answer

$\Rightarrow \boxed{\lambda = 2.4\,pm\left(1-\frac{1}{\sqrt{2}}\right) = 0.703\,pm}$, $\boxed{E = \frac{hc}{\lambda} = 1.77\,MeV}$

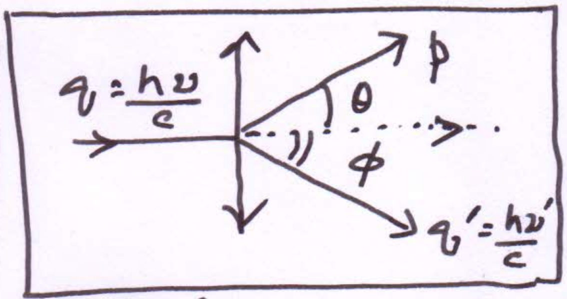
Q8/. $\boxed{\Delta\lambda = \lambda_c\,(1-\cos\phi)}$ $\quad \Delta\lambda = (10.5 - 10)\ \text{pm} \Rightarrow \boxed{\Delta\lambda = 0.5\ \text{pm}}$

$\boxed{\lambda_c = 2.4\ \text{pm}}$ $\quad \boxed{\cos\phi = 1 - \dfrac{\Delta\lambda}{\lambda_c}}$ $\Rightarrow \cos\phi = 1 - \dfrac{0.5}{2.4}$ $\quad \boxed{\cos\phi = 0.7917}$

$\boxed{q' = \dfrac{h\nu'}{c}}$ $\quad \boxed{p\sin\theta = \dfrac{h\nu'}{c}\sin\phi}$

$\Rightarrow \sin\theta = \dfrac{h}{\cancel{c}p}\,\dfrac{\cancel{c}}{\lambda'}\sin\phi$

$\Rightarrow \boxed{\sin\theta = \dfrac{h}{p\lambda'}\sin\phi}$ — (1)



Balance vertical components of momentum

Now balancing horizontal components of momentum

$\boxed{\dfrac{h\nu}{c} = p\cos\theta + \dfrac{h\nu'}{c}\cos\phi}$ $\Rightarrow \cos\theta = \dfrac{h}{p}\left(\dfrac{\nu}{c} - \dfrac{\nu'}{c}\cos\phi\right)$

$\Rightarrow \boxed{\cos\theta = \dfrac{h}{p}\left(\dfrac{1}{\lambda} - \dfrac{\cos\phi}{\lambda'}\right)}$ — (2) Dividing (1) by (2) and get,

$$\boxed{\frac{\sin\theta}{\cos\theta} = \tan\theta = \frac{\sin\phi}{\lambda'\left(\frac{1}{\lambda} - \frac{\cos\phi}{\lambda'}\right)} = \frac{\sin\phi}{(\lambda'/\lambda) - \cos\phi}}$$

$\boxed{\lambda = 10\ \text{pm},\ \lambda' = 10.5\ \text{pm}}$ $\quad \boxed{\cos\phi = 0.7917}$ $\quad \boxed{\sin\phi = \sqrt{1-\cos^2\phi}}$

$$\Rightarrow \tan\theta = \frac{\sqrt{1-\cos^2\phi}}{1.05-\cos\phi} = \frac{\sqrt{1-0.7917^2}}{1.05-0.7917} = \frac{0.6109}{0.2583}$$

$\Rightarrow \boxed{\tan\theta = 2.3651} \Rightarrow \boxed{\theta = 67.1^\circ}$ Direction of the recoil electron. Answer

From (1) $\boxed{p = \dfrac{h}{\lambda'\sin\theta}\sin\phi}$

$$p = \frac{6.626\times10^{-34}\times\sqrt{1-0.7917^2}}{10.5\times10^{-12}\times\sin 67.1^\circ} = \boxed{4.2\times10^{-23}\ \text{kg}\,\text{m}\,\text{s}^{-1}}$$

Magnitude of momentum

Q10/. $\boxed{\mathcal{E} = hc/\lambda}$ $\Rightarrow$ $\boxed{\lambda = \frac{hc}{\mathcal{E}}}$ and $\boxed{\lambda' = \frac{hc}{\mathcal{E}'}}$

$$\boxed{\lambda' - \lambda = hc\left(\frac{1}{\mathcal{E}'} - \frac{1}{\mathcal{E}}\right) = \lambda_c(1-\cos\phi)}$$

$$\Rightarrow \boxed{\cos\phi = 1 - \frac{hc}{\lambda_c}\left(\frac{\mathcal{E}-\mathcal{E}'}{\mathcal{E}\mathcal{E}'}\right)} \quad \boxed{\begin{array}{l}\mathcal{E} = 100\ \text{keV}\\ \mathcal{E}' = 90\ \text{keV}\end{array}}$$

$$\Rightarrow \cos\phi = 1 - \frac{6.626\times10^{-34}\times3\times10^{8}}{2.4\times10^{-12}}\left(\frac{100-90}{100\times90}\right)\frac{\times10^{3}\times10^{19}}{10^{6}\times1.6}$$

$$\Rightarrow \cos\phi = 1 - 0.5746 = 0.4253 \Rightarrow \boxed{\phi = 64.83^\circ}$$

← Answer

Q11/. $\boxed{\mathcal{E} = \frac{hc}{\lambda}}$ $\Rightarrow$ $\boxed{\lambda = \frac{hc}{\mathcal{E}}}$ $\Rightarrow \lambda = \frac{6.626\times10^{-34}\times3\times10^{8}}{210\times10^{6}\times1.6\times10^{-19}}$

$\Rightarrow \boxed{\lambda = 5.91\times10^{-15}\,\text{m}}$

$\boxed{\Delta\lambda = \lambda_c(1-\cos\phi)}$ → For maximum energy to be imparted, $\boxed{\Delta\lambda = 2\lambda_c = 2\times2.4\times10^{-12}\,\text{m} = 4.8\times10^{-12}\,\text{m}}$

Hence $\boxed{\Delta\lambda \gg \lambda}$ Now $\boxed{\Delta\mathcal{E} = \mathcal{E}-\mathcal{E}' = hc\left(\frac{1}{\lambda} - \frac{1}{\lambda'}\right)}$

$\boxed{\lambda' = \lambda + \Delta\lambda \simeq \Delta\lambda}$ $\therefore$ $\boxed{\Delta\mathcal{E} \simeq hc\left(\frac{1}{\lambda} - \frac{1}{\Delta\lambda}\right) \simeq \frac{hc}{\lambda}}$

Hence $\boxed{\Delta\mathcal{E} \simeq 210\,\text{MeV}}$ $\Rightarrow$ Almost all of the energy is lost to the target electron.

Answer

When the target is a proton, $\lambda_c = \frac{2.4\,\text{pm}}{1836}$

$\boxed{\lambda_c = 1.3\times10^{-15}\,\text{m}}$

$\therefore \boxed{\Delta\mathcal{E} = hc\left(\frac{1}{\lambda} - \frac{1}{\lambda+\Delta\lambda}\right)}$ $\Rightarrow \boxed{\Delta\mathcal{E} = \left(\frac{hc}{\lambda}\right)\cdot\left(\frac{\Delta\lambda}{\lambda+\Delta\lambda}\right)}$

$$\Delta\mathcal{E} = 210\,\text{MeV}\times\frac{2\lambda_c}{\lambda+2\lambda_c} = 210\times\frac{2\times1.3\cancel{\times10^{-15}}}{(5.9+2\times1.3)\cancel{\times10^{-15}}}$$

$\Rightarrow \boxed{\Delta\mathcal{E} = 64.3\,\text{MeV}}$ ← Answer (Because PROTON is heavier)