$$x_{cq}(t) = A_c \sin \phi(t) = A_c\left[\phi(t) - \frac{1}{3!}\phi^3(t) + \cdots\right]$$

Now we impose the simplifying condition

$$|\phi(t)| \ll 1 \text{ rad} \qquad [11a]$$

so that

$$x_{ci}(t) \approx A_c \qquad x_{cq}(t) \approx A_c\phi(t) \qquad [11b]$$

Then it becomes an easy task to find the spectrum $X_c(f)$ of the modulated wave in terms of an arbitrary message spectrum $X(f)$.

Specifically, the transforms of Eqs. (9) and (11*b*) yield

$$X_c(f) = \frac{1}{2}A_c\delta(f - f_c) + \frac{j}{2}A_c\Phi(f - f_c) \qquad f > 0 \qquad [12a]$$

in which

$$\Phi(f) = \mathscr{F}[\phi(t)] = \begin{cases} \phi_\Delta X(f) & \text{PM} \\ -jf_\Delta X(f)/f & \text{FM} \end{cases} \qquad [12b]$$

The FM expression comes from the integration theorem applied to $\phi(t)$ in Eq. (6).

Based on Eq. (12), we conclude that if $x(t)$ has message bandwidth $W \ll f_c$, then $x_c(t)$ will be a bandpass signal with bandwidth $2W$. But this conclusion holds only under the conditions of Eq. (11). For larger values of $|\phi(t)|$, the terms $\phi^2(t)$, $\phi^3(t)$, . . . cannot be ignored in Eq. (10) and will *increase* the bandwidth of $x_c(t)$. Hence, Eqs. (11) and (12) describe the special case of **narrowband** phase or frequency modulation (NBPM or NBFM).

EXAMPLE 5.1–1

An informative illustration of Eq. (12) is provided by taking $x(t) = \text{sinc } 2Wt$, so $X(f) = (1/2W)\Pi(f/2W)$. The resulting NBPM and NBFM spectra are depicted in Fig. 5.1–3. Both spectra have carrier-frequency impulses and bandwidth $2W$. However, the lower sideband in NBFM is 180° out of phase (represented by the negative sign), whereas both NBPM sidebands have a 90° phase shift (represented by j). Except for the phase shift, the NBPM spectrum looks just like an AM spectrum with the same modulating signal.

EXERCISE 5.1–2

Use the second-order approximations $x_{ci}(t) \approx A_c[1 - \frac{1}{2}\phi^2(t)]$ and $x_{cq}(t) \approx A_c\phi(t)$ to find and sketch the components of the PM spectrum when $x(t) = \text{sinc } 2Wt$.

Tone Modulation

The study of FM and PM with tone modulation can be carried out jointly by the simple expedient of allowing a 90° difference in the modulating tones. For if we take