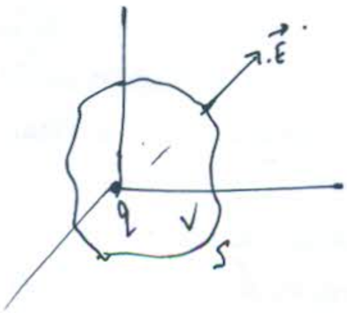
Gauss's law:



Consider a point charge q at the origin. Consider a volume V surrounded by a surface S.

The flux of $\vec{E}$ over the surface S is given as

$$\oint_S \vec{E}\cdot\hat{n}\,da.$$

If we increase the charge q then the Electric field will increase in the same proportion. So we may say the flux of the electric field across the closed surface is proportional to the charge q at the center. In fact we need not keep the charge q at the center but it can be kept anywhere within the surface S. The constant of proportionality is $\frac{1}{\epsilon_0}$. So

$$\oint_S \vec{E}\cdot\hat{n}\,da = \frac{q}{\epsilon_0}$$

We can have several charges $q_1, q_2, \ldots, q_n$ within the surface S. The total electric field due to all these charges is $E_1 + E_2 + \cdots + E_n = E$. It is clear that

$$\oint_V \vec{E}\cdot\hat{n}\,da = \frac{1}{\epsilon_0}\int(q_1+q_2+\cdots+q_n) = \frac{\text{total charge}}{\epsilon_0}.$$

This is the statement of the Gauss's law.

~~Now let us go back to the case of~~

If we have a continuous charge distribution $\rho(\vec{r})$ then the Gauss's law takes the form

$$\oint_S \vec{E}\cdot\hat{n}\,da = \frac{1}{\epsilon_0}\int_V \rho(\vec{r})\,dV$$

By divergence theorem $\oint_S \vec{E}\cdot\hat{n}\,da = \int_V \vec{\nabla}\cdot\vec{E}\,dV$

$$\therefore \int_V \vec{\nabla}\cdot\vec{E}\,dV = \frac{1}{\epsilon_0}\int_V \rho(\vec{r})\,dV$$

This is true for charge distribution over any closed volume V. So the integrands can be equated.

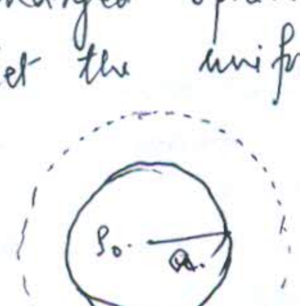
$$\therefore \vec{\nabla}\cdot\vec{E} = \frac{\rho(\vec{r})}{\epsilon_0}$$

This is the differential form of the Gauss law.

Eg: Find the electric field inside and outside a uniformly charged sphere of radius a.

Let the uniform charge density be ρ_0.

$$\rho(\vec{r}) = \rho_0 \quad : \quad 0 \le r \le a$$
$$= 0 \quad : \quad r > a$$



To find the electric field for outside the sphere, $r > a$, consider a spherical surface of radius r. Due to spherical symmetry of the problem the magnitude of the electric field $\vec{E}$ is same over the surface of this sphere and directed radially outward. Let $E(r)$ be the magnitude.

of this electric field. Then the flux of this field over the sphere of radius r is

$$\oint_S \vec{E}\cdot\hat{n}\,da = E(r)\times 4\pi r^2$$

According to Gauss' law this flux is equal to $\frac{q}{\epsilon_0}$ where q is the charge enclosed by the sphere. (we call this the imaginary sphere, the Gausian sphere.)

$$q = \frac{4}{3}\int_V \rho(\vec{r})\,dv = \rho_0\cdot\frac{4}{3}\pi a^3$$

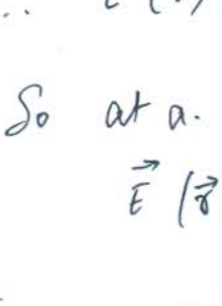
$$\therefore\ \cancel{By}\ E(r)\cdot 4\pi r^2 = \frac{1}{\epsilon_0}\frac{4}{3}\pi a^3\rho_0$$

$$\therefore\ E(r) = \frac{\rho_0}{3\epsilon_0}\frac{a^3}{r^2}$$

So at a point $\vec{r}$ the electric field outside the sphere is

$$\vec{E}(\vec{r}) = \frac{\rho_0 a^3}{3\epsilon_0 r^2}\hat{r}$$

To calculate the electric field inside the sphere we consider a Gausian surface ~~with as~~ a sphere with $r < a$



Then the total charge inside the sphere is

$$q = \frac{4}{3}\pi r^3\cdot\rho_0$$

By Gauss law then

$$E(r)\cdot 4\pi r^2 = \frac{\rho_0}{\epsilon_0}\cdot\frac{4}{3}\pi r^3$$

$$\therefore\ E(r) = \frac{\rho_0}{3\epsilon_0}\cdot r$$

$$\therefore\ \vec{E}(\vec{r}) = \frac{\rho_0}{3\epsilon_0}r\cdot\hat{r} = \frac{\rho_0}{3\epsilon_0}\vec{r} \quad \text{(inside)}.$$

Now we verify the differential form of the Gauss' law.

For outside the charge configuration.

$$\vec{\nabla}\cdot\vec{E}(\vec{r}) = \frac{\rho_0 a^3}{3\epsilon_0 \cancel{r}} \vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right)$$

We have seen that this divergence is 0 for $r > 0$.

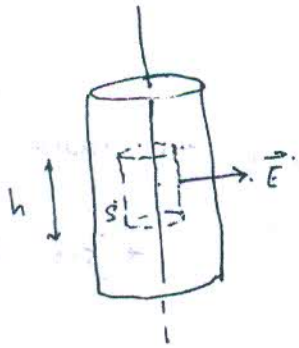
Hence $\vec{\nabla}\cdot\vec{E}(\vec{r}) = 0$ for $r > a$.

Inside the sphere.

$$\vec{\nabla}\cdot\vec{E}(\vec{r}) = \frac{\rho_0}{3\epsilon_0}\vec{\nabla}\cdot\vec{r} = \frac{\rho_0}{3\epsilon_0}\cdot 3 = \frac{\rho_0}{\epsilon_0}.$$

So this is consistent with the differential form of Gauss's law inside as well as outside the sphere.

Eg: A cylindrically symmetric charge distribution is given with $\rho(s) = ks$ from $s = 0$ to $s = R$. Find the electric field inside the cylinder of radius R and outside it.



By cylindrical symmetry $\vec{E}$ is along $\hat{s}$ every where.

Inside the charge distribution.

$$\oint_S \vec{E}\cdot\hat{n}\,da = E_s(s)\cdot 2\pi s h = \frac{q_{enc.}}{\epsilon_0}$$

The flux through the upper and lower flat surfaces of the Gaussian cylinder is zero since $\vec{E}$ is perpendicular to the normal to this surfaces.

$$q_{enc.} = \int_V \rho(s)\,dV = \int_0^h\int_0^{2\pi}\int_0^s ks\cdot s\,d\theta\,ds\,dz$$

$$= kh\cdot 2\pi\cdot\int_0^s s^2\,ds = \frac{2\pi k h s^3}{3}$$

$$\therefore E_s(s)\,2\pi s h = \frac{1}{\epsilon_0}\frac{2\pi k h s^3}{3}$$

$$\therefore E_s(s) = \frac{ks^2}{3\epsilon_0}$$

$$\therefore \vec{E}_{in} = \frac{ks^2}{3\epsilon_0}\hat{s} \quad (\text{proportional to } s^2)$$

Now we calculate electric field outside.

$$E_s \times 2\pi s h = \frac{1}{\epsilon_0}\frac{2\pi k h R^3}{3} \quad \left(\begin{array}{l}\text{The charge density exists}\\ \text{only upto } s = R.\end{array}\right)$$

$$\therefore E_s = \frac{1}{3\epsilon_0}\frac{kR^3}{s}$$

$$\therefore \vec{E}_{out} = \frac{kR^3}{3\epsilon_0}\frac{1}{s}\hat{s} \quad \left(\begin{array}{l}\text{Inversely}\\ \text{proportional to } s\end{array}\right)$$

We can calculate this even using the differential form of Gauss's law.

By symmetry of the problem we have the component $E_\phi = E_z = 0$. So only E_s is present. Using expression for divergence in cylindrical co-ordinates we get.

$$\vec{\nabla} \cdot \vec{E} = \frac{1}{\epsilon_0} \rho$$

$$\therefore \frac{1}{s} \frac{\partial}{\partial s}(s E_s) = \frac{1}{\epsilon_0} ks \quad \text{(inside the radius } R\text{)}$$

$$\therefore E_s = \frac{ks^2}{3\epsilon_0} \Rightarrow \vec{E} = \frac{ks^2}{3\epsilon_0} \hat{s} \quad * \text{(see below)}$$

Outside the radius R we have.

$$\vec{\nabla} \cdot \vec{E} = 0$$

$$\therefore \frac{1}{s} \frac{\partial}{\partial s}(s E_s) = 0 \Rightarrow s E_s = c \Rightarrow E_s = \frac{c}{s}$$

where c is some constant.

We demand $\vec{E}$ to be continuous at $s = R$ (not always true as we will see later)

$$\therefore \frac{c}{R} = \frac{kR^2}{3\epsilon_0} \Rightarrow c = \frac{kR^3}{3\epsilon_0}$$

$$\therefore \vec{E}_{out} = \frac{kR^3}{3\epsilon_0} \frac{1}{s} \hat{s}$$

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* In fact when we solve the differential Eq^n in s we will get $E_s = \frac{ks^2}{3\epsilon_0} + \frac{C_1}{s}$ where C_1 is an arbitrary constant.

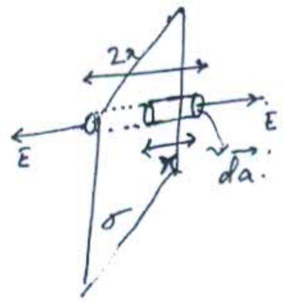
Let us check by calculating the divergence of this field at ~~the~~ $s = 0$. This we will do carefully.

$$\vec{\nabla} \cdot \vec{E}\Big|_{s=0} = \lim_{s \to 0} \frac{1}{(\pi s^2 h)} (E_s \cdot 2\pi s h) = \lim_{s \to 0} \left(\frac{2ks}{3\epsilon_0} + \frac{2C_1}{s^2} \right)$$

If $C_1 \neq 0$ then $\vec{\nabla} \cdot \vec{E} \to \infty$ as $s \to 0$ which is not consistent with the given charge density at $s = 0$.

Gauss' law Eg. 2.

Electric field due to an infinite plane of charge with uniform surface charge density σ



Let us find the $\vec{E}$ at a distance x from the plane.

Due to symmetry $\vec{E}$ is directed perpendicular to the plane.

The Gausian surface we consider is a cylinder whose length is $2x$ and as shown in the figure extends symmetrically on both sides of the charged plane.

There is no flux from the side walls of the cylinder since $\vec{E}$ is ~~norm~~ orthogonal to the normals to the surface. However the flux from the 'lid' of the cylinder is

$$\vec{E}\cdot\vec{da} + \vec{E}\cdot\vec{da} = 2E\,da.$$

By Gauss's law $2E\,da = \frac{\sigma}{\epsilon_0}$.

$$\therefore E = \frac{\sigma}{2\epsilon_0}.$$

$\therefore \vec{E} = \frac{\sigma}{2\epsilon_0}\hat{n}$ where $\hat{n}$ is the normal to the plane.

Note that $\hat{n}$ is directed opposite on the two sides of the plane. This electric field is independent of distance x and extends upto ∞. Ofcourse this is practically not possible. this is a hypothetical problem.