Integration of Scalar and Vector functions.

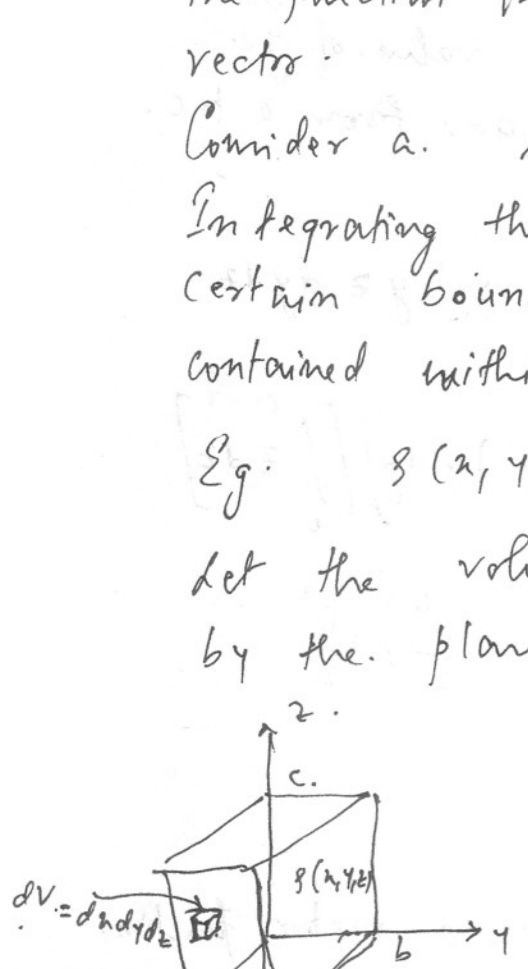
We can discuss integrals of scalar and vector functions depending upon the infinitesimal elements, line element, surface element and volume element.

Volume integral: Integration over volume element. The function to be integrated can be scalar or vector.

Consider a scalar function, like density, $\rho(x,y,z)$. Integrating this over a specified volume bounded by certain boundaries will give the ~~mass~~ total mass contained within the volume.

Eg. $\rho(x,y,z) = xyz$

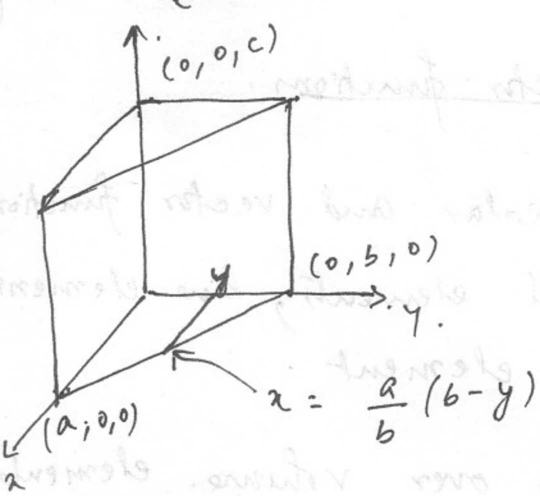
Let the volume V be the volume bounded by the planes $x=0$, $x=a$, $y=0$, $y=b$, $z=0$, $z=c$.



$$\int_V \rho \, dV = \int_0^c\int_0^b\int_0^a (xyz)\, dx\, dy\, dz$$

$$= \left[\frac{x^2}{2}\right]_0^a \left[\frac{y^2}{2}\right]_0^b \left[\frac{z^2}{2}\right]_0^c = \frac{1}{8}a^2b^2c^2$$

If the volume over which we integrate is more complicated we have to work out proper limits of the ~~for~~ variables. Consider the volume shown in the following figure.



$$\int_V \rho \, dx\,dy\,dz$$

$$= \int_0^c \int_0^b \int_0^{\frac{a}{b}(b-y)} (xyz)\, dx\,dy\,dz$$

For a particular value of y, x goes from 0 to $\frac{a}{b}(b-y)$. Hence we got the upper limit of x for every value of y. y goes from 0 to b and z goes from 0 to c.

So $$\int_V \rho \, dx\,dy\,dz = \int_0^c\int_0^b \frac{1}{2}\left[\frac{a}{b}(b-y)\right]^2 y z \, dy\,dz$$

$$= \left[\int_0^b \frac{1}{2}\frac{a^2}{b^2}(b^2-2yb+y^2)\,y\,dy\right]\left[\int_0^c z\,dz\right]$$

$$= \frac{a^2}{2b^2}\left(\frac{b^2 b^2}{2} - 2b\frac{b^3}{3} + \frac{b^4}{4}\right)\frac{c^2}{2}$$

$$= \frac{a^2 b^2 c^2}{48}$$

Vector function: Consider integrating a vector function over a volume: for e.g. integrating infinitesimal dipole moments to calculate the total dipole moment or integrating forces over infinitesimal elements over a volume to find the total force.

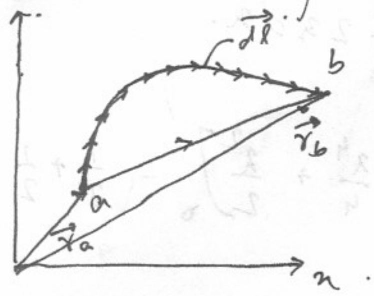
If $\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$ then.

$$\int_V \vec{A}\,dV = \hat{i}\int_V A_x\,dV + \hat{j}\int_V A_y\,dV + \hat{k}\int_V A_z\,dV$$

Individual integrals can be done as above.

Line integral.

Integrates a scalar or a vector functions along a curve over infinitesimal line elements.



$$\int_a^b \rho(x,y)\, \vec{dl}$$

The result of this integral is a vector quantity.

If $\rho(x,y)$ is a constant function then.

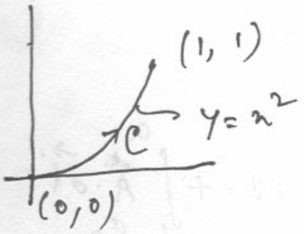
$$\int_a^b \rho(x,y)\,\vec{dl} = \rho \int_a^b \vec{dl} = \rho\left(\vec{r}_b - \vec{r}_a\right).$$

Vector function: Consider the following line integral of a vector function.

$$\int_a^b \vec{A}\cdot\vec{dl} \qquad . \quad \text{Here. } \vec{dl} = \hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz.$$

This kind of line integral have to be evaluated to compute the work done by a force in moving a particle from point a to point b.

Eg 1). Let $\vec{A} = x\hat{i} + y\hat{j}$. Let us integrate along a curve $y = x^2$ from $(0,0)$ to $(1,1)$.



$$\int_a^b \vec{A}\cdot\vec{dl} = \int_a^b x\,dx + y\,dy$$

Along the curve $y = x^2$ and $dy = 2x\,dx$.

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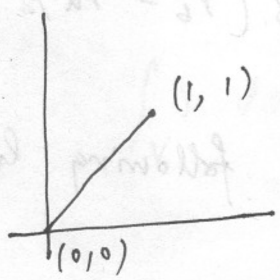
So the line integral gets converted into a single integral, in this case only over x.

$$\therefore \int_a^b \vec{A}\cdot\vec{dl} = \int_0^1 x\,dx + \int_0^1 x^2 \cdot 2x\,dx.$$

$$= \int_0^1 (2x^3 + x)\,dx = \left[2\frac{x^4}{4} + \frac{x^2}{2}\right]_0^1 = \frac{1}{2} + \frac{1}{2} = 1$$

Instead of the curve $y = x^2$ if we consider the curve $y = x$ from $(0,0)$ to $(1,1)$.



then $dy = dx$. We will have.

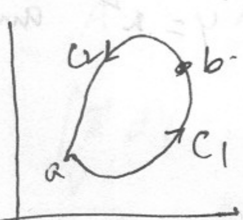
$$\int_a^b \vec{A}\cdot\vec{dl} = \int_0^1 x\,dx + \int_0^1 x\,dx$$

$$= 2\left[\frac{x^2}{2}\right]_0^1 = 2\times\frac{1}{2} = 1$$

The line integral is same along both the paths. This is not a coincidence. For the given function the line integral is independent of the path. ~~Thus~~

~~Thus~~ Now $\displaystyle\int_{a\,(\text{along } C)}^{b} \vec{A}\cdot\vec{dl} = -\int_{b\,(\text{along } C)}^{a} \vec{A}\cdot\vec{dl}$ (This is true for any curve C)

So if we have a closed curve then the line integral will be zero. ~~A~~ Such integrals are denoted as $\oint \vec{A}\cdot\vec{dl}$.



$$\oint \vec{A}\cdot\vec{dl} = \int_{a\,C_1}^{b} \vec{A}\cdot dl + \int_{b\,C_2}^{a} \vec{A}\cdot\vec{dl}$$

$$= \int_{a\,C_1}^{b} \vec{A}\cdot dl - \int_{a\,C_2}^{b} \vec{A}\cdot\vec{dl} = 0$$

Eg. 2 Let $\vec{A} = y\hat{i} - x\hat{j}$. The line integral with this function is dependent on the path taken. Along the path $y = x^2$ we will have.

$$\int_a^b \vec{A}\cdot\vec{dl} = \int_a^b y\,dx - x\,dy = \int_0^1 x^2\,dx - 2x^2\,dx$$

$$= \int_0^1 -x^2\,dx = -\frac{1}{3}$$

Along the path $y = x$

$$\int_a^b \vec{A}\cdot dl = \int_0^1 x\,dx - x\,dx = 0$$

So this integral is path dependent.

Occasionally we do have line integrals of the type $\int_a^b \vec{A}\times\vec{dl}$ along a curve C from point a to point b. The result of this integration is evidently a vector quantity. In the cartesian co-ordinate system we can write this integral as

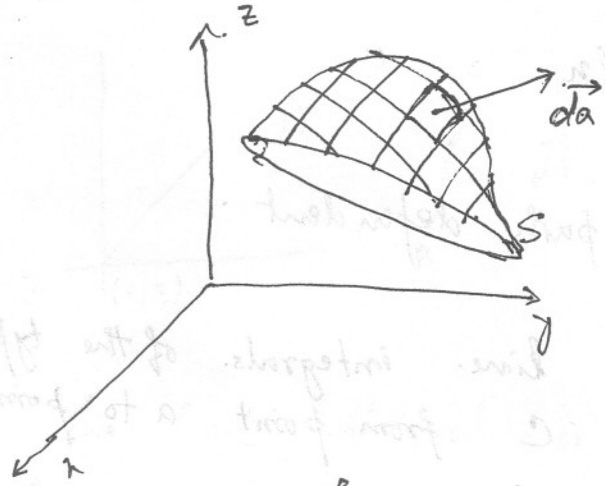
$$\hat{i}\int_a^b (A_y\,dz - A_z\,dy) + \hat{j}\int_a^b (A_z\,dx - A_x\,dz) + \hat{k}\int_a^b (A_x\,dy - A_y\,dx)$$

The individual integrals can be evaluated with the procedure stated above along a specified curve from point a to point b.

Surface Integral:

~~For three dimensional space.~~

Here a scalar or a vector function is integrated over ~~a~~ infinitesimal surface elements. In three dimension infinitesimal surface elements are vector quantities. Given an infinitesimal surface element, we associate a vector whose magnitude is the area of the surface and which is perpendicular to the infinitesimal surface.



The surface integral that we will generally encounter in Electrodynamics is given as

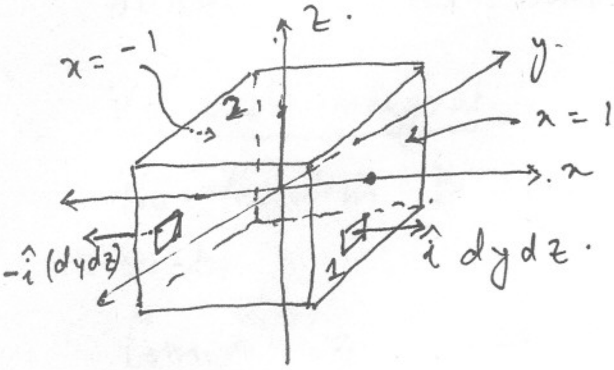
$$\int_S \vec{A} \cdot \vec{da}$$

where $\vec{A}$ is the vector field to be integrated over a given surface S. $\vec{da}$ is an infinitesimal element of the surface. S.

If the surface over which we integrate is closed, i.e. encloses a volume, then the surface integral is denoted as

$$\oint_S \vec{A} \cdot \vec{da}$$

Eg: Let $\vec{A} = x\hat{i} + y\hat{j} + z\hat{k}$. Let us integrate this over a closed surface of a cube formed by the planes $x = 1$, $x = -1$, $y = 1$, $y = -1$, $z = 1$, $z = -1$ as shown.



There are six surfaces enclosing a cubical volume. Consider the surface $x = 1$. At every point on this surface $\vec{da} = \hat{i}\,dydz$.

Let us call this surface as surface 1.

So $\displaystyle\int_1 \vec{A}\cdot\vec{da} = \int_{-1}^{1}\int_{-1}^{1} x\,dy\,dz$

Since on this surface $x = 1$ we have

$$\int_1 \vec{A}\cdot\vec{da} = \int_{-1}^{1}\int_{-1}^{1} dy\,dz = \int_{-1}^{1} dy \int_{-1}^{1} dz = 2\times 2 = 4$$

On the opposite surface $x = -1$ (call surface 2) we have

$\vec{da} = -\hat{i}\,dydz$ and $x = -1$

So $\vec{A}\cdot\vec{da} = (-\hat{i} + y\hat{j} + z\hat{k})\cdot(-\hat{i}\,dydz) = dydz$

$$\therefore \int_2 \vec{A}\cdot\vec{da} = \int_{-1}^{1}\int_{-1}^{1} dy\,dz = 4$$

So the total contribution from this pair of surfaces is $4 + 4 = 8$. We have 3 such pair. And by symmetry of the function we have

$\therefore$ the

$$\oint_S \vec{A}\cdot\vec{da} = 8\times 3 = 24.$$