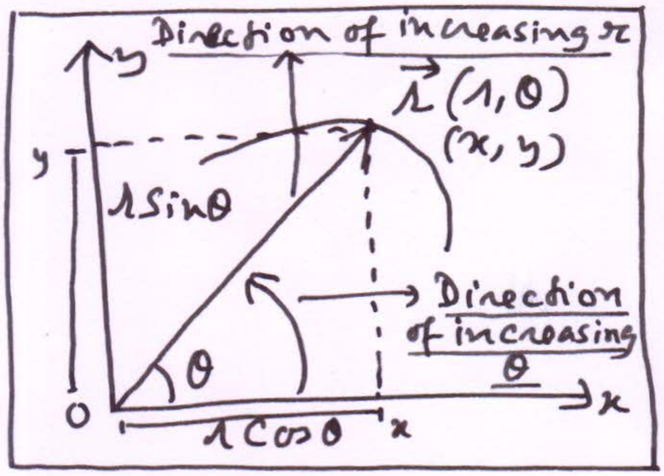
Particle motion on a plane

A particle is constrained to move on the x-y plane. Its position from the origin of coordinates



is $\vec{r} = r\hat{r}$. In the x-y coordinate system, the position is given by (x, y), and in the polar r-θ coordinate system, the position is given by (r, θ). The two coordinate systems are related by

$$\boxed{x = r\cos\theta}, \quad \boxed{y = r\sin\theta}, \quad \boxed{r^2 = x^2 + y^2}, \quad \boxed{\tan\theta = \frac{y}{x}}$$

Now, vectorially $\boxed{\vec{r} = r\hat{r} = x\hat{x} + y\hat{y}}$

$\Rightarrow r\hat{r} = r\cos\theta\,\hat{x} + r\sin\theta\,\hat{y}$.

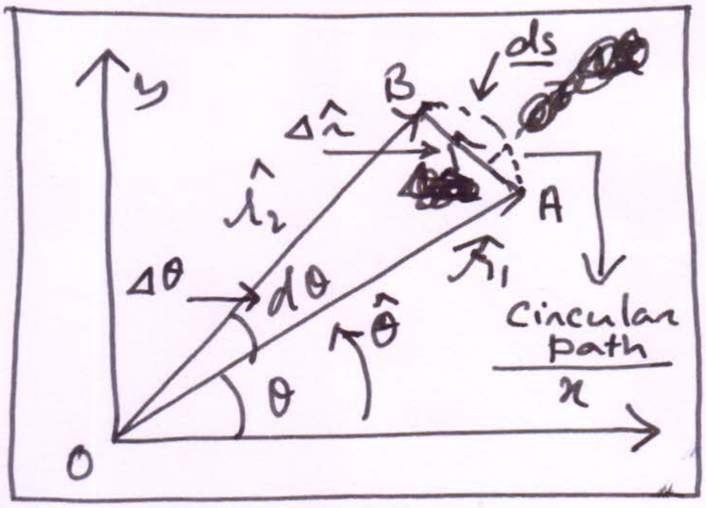
$\Rightarrow \boxed{\hat{r} = \cos\theta\,\hat{x} + \sin\theta\,\hat{y}}$. The unit radial vector.

In the Cartesian coordinate system, $\hat{x}$ and $\hat{y}$ are fixed unit vectors, i.e. their directions do not vary. However, the unit vector $\hat{r}$ has a varying direction, depending on the value of θ. Eg. when $\theta = 0$, $\hat{r} = \hat{x}$ and when $\theta = \pi/2$, $\hat{r} = \hat{y}$.

Now, velocity, $\boxed{\vec{v} = \frac{d\vec{r}}{dt} = \frac{d(r\hat{r})}{dt}}$, Since, $\hat{r}$ also varies, its derivative is to be taken.

$$\therefore \boxed{\vec{v} = \frac{d\vec{r}}{dt} = \dot{r}\hat{r} + r\frac{d\hat{r}}{dt}}$$ where, $\boxed{\dot{r} = \frac{dr}{dt}}$.

Since $\hat{r}$ is a unit vector, as it varies with θ, the tip of $\hat{r}$ traces out a circular path.



The vector $\boxed{\vec{AB} = \Delta\hat{r}}$. $\boxed{\hat{r}_2 = \hat{r}_1 + \Delta\hat{r}}$.

The arclength $\boxed{\overset{\frown}{AB} = ds = |\hat{r}_1|\, d\theta}$.

Now as $\boxed{\Delta\theta \to 0}$, the arclength $\overset{\frown}{AB}$, becomes equal to the length of the vector $|\vec{AB}| = |\Delta\hat{r}|$. Hence, $\boxed{\lim_{\Delta\theta\to 0} ds = |\Delta\hat{r}|}$.

Since, $\vec{AB}$ increases in the increasing direction of θ, for $\Delta\theta \to 0$, i.e. in $\hat{\theta}$ direction,

$\therefore \boxed{\Delta\hat{r} = |\hat{r}_1|\Delta\theta\, \hat{\theta}}$. Since $\boxed{|\hat{r}_1| = |\hat{r}_2| = 1}$, we get

$\boxed{\frac{\Delta\hat{r}}{\Delta\theta} = \hat{\theta}} \Rightarrow$ when $\Delta\theta \to 0$, $\boxed{\frac{d\hat{r}}{d\theta} = \hat{\theta}}$.

Now, $\boxed{\frac{d\hat{r}}{dt} = \frac{d\hat{r}}{d\theta}\frac{d\theta}{dt} = \dot{\theta}\hat{\theta}}$ where $\boxed{\dot{\theta} = \frac{d\theta}{dt}}$.

$$\therefore \boxed{\vec{v} = \frac{d\vec{r}}{dt} = \dot{r}\hat{r} + r\frac{d\hat{r}}{dt} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}}.$$

i) When r is constant (circular path), $\boxed{\dot{r} = 0}$. $\Rightarrow$ $\boxed{\vec{v} = r\dot{\theta}\hat{\theta}}$ $\rightarrow$ Circular velocity.

ii) When θ is constant (linear path), $\boxed{\dot{\theta} = 0}$. $\Rightarrow$ $\boxed{\vec{v} = \dot{r}\hat{r}}$ $\rightarrow$ Linear velocity.

In general $\vec{v}$ has both the linear ($\hat{r}$) and circular ($\hat{\theta}$) components.

Since $\boxed{\hat{r} = \cos\theta\,\hat{x} + \sin\theta\,\hat{y}}$ and $\boxed{\hat{\theta} = \frac{d\hat{r}}{d\theta}}$

by taking the derivative

$\boxed{\hat{\theta} = \frac{d\hat{r}}{d\theta} = -\sin\theta\,\hat{x} + \cos\theta\,\hat{y}}$. We also

make use of the scalar products

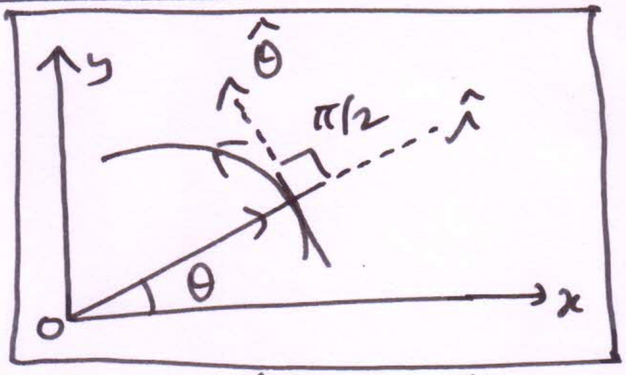
$\boxed{\hat{x}\cdot\hat{x} = \hat{y}\cdot\hat{y} = 1}$ and $\boxed{\hat{x}\cdot\hat{y} = \hat{y}\cdot\hat{x} = 0}$.

$$\therefore \boxed{\begin{aligned}\hat{r}\cdot\hat{r} &= (\cos\theta\,\hat{x} + \sin\theta\,\hat{y})\cdot(\cos\theta\,\hat{x} + \sin\theta\,\hat{y}) \\ &= \cos^2\theta + \sin^2\theta = 1\end{aligned}}$$

and
$$\boxed{\begin{aligned}\hat{\theta}\cdot\hat{\theta} &= (-\sin\theta\,\hat{x} + \cos\theta\,\hat{y})\cdot(-\sin\theta\,\hat{x} + \cos\theta\,\hat{y}) \\ &= \sin^2\theta + \cos^2\theta = 1\end{aligned}}$$

$$\boxed{\begin{aligned}\hat{r}\cdot\hat{\theta} &= (\cos\theta\,\hat{x} + \sin\theta\,\hat{y})\cdot(-\sin\theta\,\hat{x} + \cos\theta\,\hat{y}) \\ &= -\sin\theta\cos\theta + \sin\theta\cos\theta = 0\end{aligned}}$$

From $\boxed{\hat{r}\cdot\hat{r} = \hat{\theta}\cdot\hat{\theta} = 1}$ and $\boxed{\hat{r}\cdot\hat{\theta} = 0}$ we see that $\underline{\hat{r} \text{ and } \hat{\theta}}$ are <u>unit</u> <u>orthogonal vectors</u> (mutually <u>perpendicular</u>)



$$\therefore\; v^2 = \vec{v}\cdot\vec{v} = (\dot{r}\,\hat{r} + r\dot{\theta}\,\hat{\theta})\cdot(\dot{r}\,\hat{r} + r\dot{\theta}\,\hat{\theta})$$

$$\Rightarrow \boxed{v^2 = \dot{r}^2 + r^2\dot{\theta}^2} \text{ or } \boxed{v^2 = \left(\frac{dr}{dt}\right)^2 + \left(r\frac{d\theta}{dt}\right)^2}.$$

<u>Acceleration</u>, $\boxed{\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}(\dot{r}\,\hat{r}) + \frac{d}{dt}(r\dot{\theta}\,\hat{\theta})}$.

$$\Rightarrow \boxed{\vec{a} = \dot{r}\frac{d\hat{r}}{dt} + \ddot{r}\,\hat{r} + \dot{r}\dot{\theta}\,\hat{\theta} + r\ddot{\theta}\,\hat{\theta} + r\dot{\theta}\frac{d\hat{\theta}}{dt}}$$

But $\boxed{\frac{d\hat{r}}{dt} = \dot{\theta}\,\hat{\theta}}$ and $\boxed{\frac{d\hat{\theta}}{dt} = \frac{d\hat{\theta}}{d\theta}\frac{d\theta}{dt} = \dot{\theta}\frac{d\hat{\theta}}{d\theta}}$

$$\boxed{\hat{\theta} = -\sin\theta\,\hat{x} + \cos\theta\,\hat{y}} \Rightarrow \boxed{\frac{d\hat{\theta}}{d\theta} = -\cos\theta\,\hat{x} - \sin\theta\,\hat{y} = -\hat{r}}$$

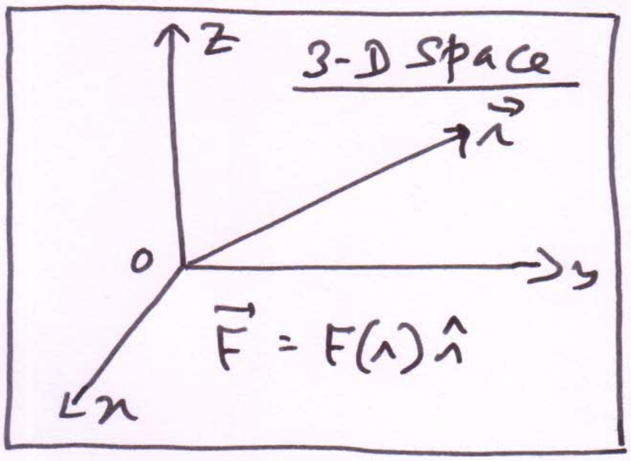
$$\therefore\; \vec{a} = \ddot{r}\,\hat{r} + \dot{r}\dot{\theta}\,\hat{\theta} + \dot{r}\dot{\theta}\,\hat{\theta} + r\ddot{\theta}\,\hat{\theta} + r\dot{\theta}^2(-\hat{r})$$

$$\Rightarrow \boxed{\vec{a} = \frac{d\vec{v}}{dt} = (\ddot{r} - r\dot{\theta}^2)\,\hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\,\hat{\theta}}$$ with both <u>radial ($\hat{r}$)</u> and <u>angular ($\hat{\theta}$)</u> <u>components</u>.

Central Forces

[In 3-dimensional space (x, y, z)]

A central force is one that is always directed towards or away from a centre (at O), along the radial vector ($\hat{r}$), and its magnitude depends only on the scalar distance r.



Hence, it is $\boxed{\vec{F} = F(r)\hat{r}}$. ($F(r)$ is any general function of scalar r)

i/. If $\boxed{F(r) < 0}$, the force is attractive.

Eg. Gravity, force field of an electron, etc.

ii/. If $\boxed{F(r) > 0}$, the force is repulsive.

Eg. Force field of a positive charge.

Now $\boxed{\vec{F} = F(r)\hat{r} = \frac{F(r)}{r}\vec{r} = g(r)\left[x\hat{x} + y\hat{y} + z\hat{z}\right]}$

where $\boxed{g(r) = \frac{F(r)}{r}}$

and $\boxed{\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}}$

The radial vector in 3-dimensions

Hence, $\vec{\nabla} \times \vec{F} =$ (Curl of $\vec{F}$)

$$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ g(r)x & g(r)y & g(r)z \end{vmatrix}$$

$$\boxed{\begin{aligned}\Rightarrow \vec{\nabla}\times\vec{F} &= \hat{x}\left[\frac{\partial}{\partial y}(\xi_s z) - \frac{\partial}{\partial z}(\xi_s y)\right] \\ &- \hat{y}\left[\frac{\partial}{\partial x}(\xi_s z) - \frac{\partial}{\partial z}(\xi_s x)\right] + \hat{z}\left[\frac{\partial}{\partial x}(\xi_s y) - \frac{\partial}{\partial y}(\xi_s x)\right]\end{aligned}}$$

In the $\underline{\hat{x} \text{ term}}$ above $\frac{\partial}{\partial y}(\xi_s z) = \underbrace{\xi_s \frac{\partial z}{\partial y}}_{\text{(zero)}} + z\frac{\partial \xi_s}{\partial r}\frac{\partial r}{\partial y}$

$\Rightarrow \frac{\partial}{\partial y}(\xi_s z) = z\left(\frac{\partial \xi_s}{\partial r}\right)\left(\frac{\partial r}{\partial y}\right)$ Similarly we can write $\frac{\partial}{\partial z}(\xi_s y) = \underbrace{\xi_s \frac{\partial y}{\partial z}}_{\text{(zero)}} + y\left(\frac{\partial \xi_s}{\partial r}\right)\cdot\left(\frac{\partial r}{\partial z}\right) = y\left(\frac{\partial \xi_s}{\partial r}\right)\left(\frac{\partial r}{\partial z}\right)$

Hence, $\vec{\nabla}\times\vec{F}$ can be written in full as

$$\vec{\nabla}\times\vec{F} = \hat{x}\left[z\frac{\partial \xi_s}{\partial r}\frac{\partial r}{\partial y} - y\frac{\partial \xi_s}{\partial r}\frac{\partial r}{\partial z}\right]$$

$$-\hat{y}\left[z\frac{\partial \xi_s}{\partial r}\cdot\frac{\partial r}{\partial x} - x\frac{\partial \xi_s}{\partial r}\cdot\frac{\partial r}{\partial z}\right] + \hat{z}\left[y\frac{\partial \xi_s}{\partial r}\frac{\partial r}{\partial x} - x\frac{\partial \xi_s}{\partial r}\frac{\partial r}{\partial y}\right]$$

Now $\boxed{r^2 = x^2+y^2+z^2}$ $\therefore 2r\frac{\partial r}{\partial x} = 2x \Rightarrow \boxed{\frac{\partial r}{\partial x} = \frac{x}{r}}$

Similarly $2r\frac{\partial r}{\partial y} = 2y \Rightarrow \boxed{\frac{\partial r}{\partial y} = \frac{y}{r}}$ and $2r\frac{\partial r}{\partial z} = 2z \Rightarrow \boxed{\frac{\partial r}{\partial z} = \frac{z}{r}}$

$$\therefore \vec{\nabla}\times\vec{F} = \frac{\partial \xi_s}{\partial r}\left[\cancel{\frac{zy}{r}} - \cancel{\frac{yz}{r}}\right]\hat{x} - \frac{\partial \xi_s}{\partial r}\left[\cancel{\frac{zx}{r}} - \cancel{\frac{xz}{r}}\right]\hat{y}$$

$$+\frac{\partial \xi_s}{\partial r}\left[\cancel{\frac{yx}{r}} - \cancel{\frac{xy}{r}}\right]\hat{z} = \underline{0\hat{x} + 0\hat{y} + 0\hat{z} = \vec{0}}\ \text{null vector.}$$

$\Rightarrow \boxed{\vec{\nabla}\times\vec{F} = \vec{\nabla}\times F(r)\hat{r} = \vec{0}}$ $\Rightarrow \underline{F(r)\hat{r}}$ is an irrotational vector.

$\because \boxed{\vec{\nabla}\times\vec{\nabla}\psi = \vec{0}}$ by comparison $\boxed{F(r) = \vec{\nabla}\psi}$ (gradient of a scalar)