Mathematics Theory

- **Numerical integration of differential equations (20-01-2020):**

  Integrate the following differential equations by Euler's method and Taylor's method of the second order. Provide your results correct to 4 places of decimal. The first 4 differential equations are autonomous, and the rest are non-autonomous. In all of them $x \equiv x(t)$ and $\dot{x} \equiv \mathrm{d}x/\mathrm{d}t$.

  1. $\dot{x} = \cos^2 x$, $0 \leq t \leq 1$, $x(0) = 0$, $\Delta t = 0.2$.
  2. $\dot{x} = -x^2$, $1 \leq t \leq 2$, $x(1) = 1$, $\Delta t = 0.2$.
  3. $\dot{x} = 0.25x(1 - 0.05x)$, $0 \leq t \leq 2$, $x(0) = 1$, $\Delta t = 0.2$.
  4. $\dot{x} = 1 - 2x$, $x(0) = 2$, $\Delta t = 0.2$. Find $x(1)$.
  5. $\dot{x} = (1 + t^2)^{-1} - 2x^2$, $0 \leq t \leq 1$, $x(0) = 0$, $\Delta t = 0.2$.
  6. $\dot{x} = x \sin(t)/t$, $x(0) = 2$, $\Delta t = 0.25$. Find $x(1)$.
  7. Consider $\dot{x} = x(1 + e^{2t})$, $x(0) = 1$. Integrate up to $n = 3$, with a step-size of $\Delta t = 0.1$.
  8. Given $\dot{x} = -x + \sin(4\pi t)$, $x(0) = 0.5$, integrate up to $n = 2$, with $\Delta t = 0.1$.
  9. Given $\dot{x} = -2x + \exp(-2t)$, $x(0) = 0.1$, integrate up to $n = 2$, with $\Delta t = 0.1$.
  10. Given $\dot{x} = t^2 - x$, $x(0) = 1$, integrate up to $n = 3$, with $\Delta t = 0.1$.

- **Phase plots and linear stability of first-order autonomous systems (06-02-2020):**

  1. Analyze the following equations graphically. In each case, plot x–$\dot{x}$ by hand (which you can also support by plotting on a computer), find all the fixed points and classify their stability. Verify your conclusion about stability by a linear stability analysis.

     A. $\dot{x} = x^2 - 1$ B. $\dot{x} = -x^3$ C. $\dot{x} = x^3$ D. $\dot{x} = x^2$ E. $\dot{x} = 4x^2 - 16$ F. $\dot{x} = 1 - x^{14}$
     G. $\dot{x} = x - x^3$ H. $\dot{x} = e^{-x} \sin x$ I. $\dot{x} = 1 + 0.5 \cos x$ J. $\dot{x} = 1 - 2 \cos x$
     K. $\dot{x} = e^x - \cos x$ (Hint: Plot e^x and $\cos x$ as separate functions, and look for intersections.)

  2. Use linear stability analysis to classify the fixed points of the following equations. If linear stability analysis fails because $f'(x_{\mathrm{c}}) = 0$, use a graphical argument to decide the stability.

     A. $\dot{x} = x(1 - x)$ B. $\dot{x} = x(1 - x)(2 - x)$ C. $\dot{x} = \tan x$ D. $\dot{x} = x^2(6 - x)$
     E. $\dot{x} = 1 - \exp(-x^2)$ F. $\dot{x} = \ln x$ G. $\dot{x} = ax - x^3$ (Check for $a < 0$, $a = 0$ and $a > 0$.)

- **Conservative and reversible systems (13-02-2020):**

  1. For each of the following equations/systems, identify whether it is conservative or not. If it is conservative, then obtain the equation of conservation.

     (a) $\ddot{x} = x - x^2$ [conservative, $(\dot{x}^2/2) - (x^2/2) + (x^3/3) = C$]
     (b) $\dot{x} = -2\cos x - \cos y$, $\dot{y} = -2\cos y - \cos x$ [reversible but not conservative]
     (c) $\dot{x} = xy$, $\dot{y} = -x^2$ [conservative, $x^2 + y^2 = C$]