Signals and Systems (CT 203)

Solutions of Tutorial Sheet-04 DA-IICT, Gandhinagar.

1. Using the definition of linearity, show that the ideal delay system and moving average (MA) system are both linear systems.

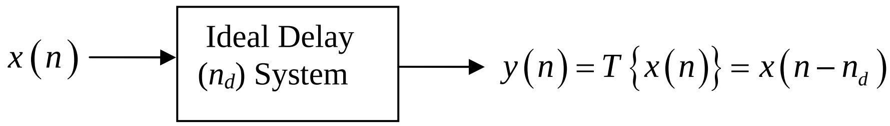


Fig.1a. Ideal Delay System

Ans:- Linear system must follow **superposition** and **homogeneity** property:-

Here, $y(n) = T\{x(n)\} = x(n - n_d)$ ….(1.1)

Let,

$y_1(n) = T\{x_1(n)\} = x_1(n - n_d)$ ….(1.2)

$y_2(n) = T\{x_2(n)\} = x_2(n - n_d)$ ….(1.3)

we need to prove : $y_1(n) + y_2(n) = T\{x_1(n) + x_2(n)\}$

Proof:

LHS= $y_1(n) + y_2(n) = x_1(n - n_d) + x_2(n - n_d)$ [From eq. (1.2) and (1.3)]

RHS= $T\{x_1(n) + x_2(n)\} = x_1(n - n_d) + x_2(n - n_d)$ [From eq. (1.1)]

So,

$y_1(n) + y_2(n) = T\{x_1(n) + x_2(n)\}$

For, **homogeneity** we need to prove $\alpha y(n) = T\{\alpha x(n)\}$

Proof:

LHS= $T\{\alpha x(n)\} = \alpha x(n - n_d)$ [From eq. (1.1)]

RHS= $T\{\alpha x(n)\} = \alpha x(n - n_d)$

So, $\alpha y(n) = T\{\alpha x(n)\}$

Hence, given system is **<u>linear system.</u>**

$$x(n) \longrightarrow \boxed{\text{Moving Average (MA) System}} \longrightarrow y(n)=T\{x(n)\}=\frac{1}{N_1+N_2+1}\sum_{k=-N_1}^{N_2} x(n-k)$$

Fig.1b. Moving Average (MA) System

Ans:- Linear system must follow **superposition** and **homogeneity** property:-

Here, $y(n)=T\{x(n)\}=\dfrac{1}{N_1+N_2+1}\displaystyle\sum_{k=-N_1}^{N_2} x(n-k)$ ….(1.4)

Let,

$$y_1(n)=T\{x_1(n)\}=\frac{1}{N_1+N_2+1}\sum_{k=-N_1}^{N_2} x_1(n-k) \quad \text{….(1.5)}$$

$$y_2(n)=T\{x_2(n)\}=\frac{1}{N_1+N_2+1}\sum_{k=-N_1}^{N_2} x_2(n-k) \quad \text{….(1.6)}$$

we need to prove : $y_1(n)+y_2(n)=T\{x_1(n)+x_2(n)\}$

Proof:

$$y_1(n)+y_2(n)$$

$$=\frac{1}{N_1+N_2+1}\sum_{k=-N_1}^{N_2} x_1(n-k)+\frac{1}{N_1+N_2+1}\sum_{k=-N_1}^{N_2} x_2(n-k)$$

$$=\frac{1}{N_1+N_2+1}\sum_{k=-N_1}^{N_2}\{x_1(n-k)+x_2(n-k)\}$$

$$=T\{x_1(n)+x_2(n)\}$$

For, **homogeneity** we need to prove $\alpha y(n)=\alpha T\{x(n)\}=T\{\alpha x(n)\}$

Proof:

$$T\{\alpha x(n)\}$$

$$=\alpha\frac{1}{N_1+N_2+1}\sum_{k=-N_1}^{N_2} x(n-k)$$

$$=\alpha\left\{\frac{1}{N_1+N_2+1}\sum_{k=-N_1}^{N_2} x(n-k)\right\}$$

$$=\alpha T\{x(n)\}$$

Hence, given system is **<u>linear system.</u>**

2. For each of the following systems, determine whether the system is

- Stable
- Causal
- linear
- time-invariant
- memoryless

$$x(n) \longrightarrow \boxed{T\{.\}} \longrightarrow y(n) = T\{x(n)\}$$

Fig.2. Discrete-time system

(a) $y(n) = T\{x(n)\} = g(n)x(n)$

(b) $y(n) = T\{x(n)\} = \sum_{k=n_o}^{n} x(k)$

(c) $y(n) = T\{x(n)\} = e^{x(n)}$

(d) $y(n) = T\{x(n)\} = ax(n) + b$

Ans:-
Ans(a):-

For **Stability** Bounded input should produce bounded output.
i.e., for $|x(n)| \le B_x < \infty, \ \forall n$ then $|y(n)| \le B_y < \infty, \ \forall n$

Here, $y(n) = T\{x(n)\} = g(n)x(n)$ $\quad \therefore |y(n)| \le |g(n)||x(n)| < \infty$

if $|g(n)| < \infty, \forall n$ then $|y(n)| \le B_y < \infty, \ \forall n$

So, the system is stable on the condition that $|g(n)| < \infty, \forall n$

$y(n) = g(n)x(n) = f\{x(n)\}$ i.e., $y(n)$ dependence on current sample of input $x(n)$.
So, the system is **causal**.

$$T\{ax_1(n) + bx_2(n)\} = g(n)\{ax_1(n) + bx_2(n)\}$$
$$= a\{g(n)x_1(n)\} + b\{g(n)x_2(n)\} = aT\{x_1(n)\} + bT\{x_2(n)\}$$

The system is **linear**.

For Time-invariant test:- we need to prove $y(n - n_0) = T\{x(n - n_0)\}$

proof:-

$y(n - n_0) = g(n - n_0)x(n - n_0)$

$T\{x(n - n_0)\} = g(n)x(n - n_0)$

So, $y(n - n_0) \ne T\{x(n - n_0)\}$ Hence, the system is **NOT time-invariant.**

As $y(n)$ dependence only on present sample of input $x(n)$ not past or future samples. Therefore, the system is **memoryless**.

Ans(b):-

For **Stability** Bounded input should produce bounded output.
i.e., for $|x(n)| \le B_x < \infty, \ \forall n$ then $|y(n)| \le B_y < \infty, \ \forall n$

Here, $y(n) = T\{x(n)\} = \sum_{k=n_o}^{n} x(k)$

$|y(n)| \le \sum_{k=n_o}^{n} |x(k)|$

$\therefore |x(n)| \le B_x < \infty$

$\therefore |y(n)| \le |n - n_0| B_x$

$\therefore \lim_{n\to\infty} |y(n)| \le \lim_{n\to\infty} |n - n_0| B_x = \infty$

The system is **unstable**.

For $n < n_0$ say, $n = n_0 - 1$

$y(n) = \sum_{k=n_o}^{n} x(k) \Rightarrow y(n_o - 1) = \sum_{k=n_o}^{n_o - 1} x(k) = x(n_o - 1) + x(n_o)$

i.e., $y(n)$ dependence on current sample of input $x(n)$ as well as future samples.
So, the system is **not causal**.

$$T\{ax_1(n) + x_2(n)\} = \sum_{k=n_o}^{n} \{ax_1(k) + x_2(k)\}$$

$$= \sum_{k=n_o}^{n} \{ax_1(k)\} + \sum_{k=n_o}^{n} \{x_2(k)\}$$

$$= a\sum_{k=n_o}^{n} \{x_1(k)\} + \sum_{k=n_o}^{n} \{x_2(k)\}$$

$$= aT\{x_1(n)\} + T\{x_2(n)\}$$

So the system is **linear**.

For Time-invariant test:- we need to prove $y(n - n_0) = T\{x(n - n_0)\}$
proof:-

$y(n - n_0) = \sum_{k=n_o}^{n-n_0} x(k)$

$$T\left\{x\left(n-n_0\right)\right\}=\sum_{k=n_o}^{n} x\left(k-n_0\right)=\sum_{k=0}^{n-n_0} x\left(k\right)$$

So, $y\left(n-n_0\right) \neq T\left\{x\left(n-n_0\right)\right\}$ Hence, the system is **NOT time-invariant.**

As $y\left(n\right)$ does not only dependence on present samples. So the system is **not memoryless**.

Ans(c):-

For **Stability** Bounded input should produce bounded output.
i.e., for $|x(n)| \leq B_x < \infty, \ \forall n$ then $|y(n)| \leq B_y < \infty, \ \forall n$

Here, $y\left(n\right)=T\left\{x\left(n\right)\right\}=e^{x\left(n\right)}$

$$\left|y\left(n\right)\right|=\left|e^{x\left(n\right)}\right| \leq e^{\left|x\left(n\right)\right|} < B_X < \infty$$

The system is **stable**.

$y\left(n\right)$ dependence only on the present sample of input $x\left(n\right)$, not future samples So, the system is **causal**.

$$T\left\{ax_1\left(n\right)+x_2\left(n\right)\right\}=e^{\left\{ax_1\left(k\right)+x_2\left(k\right)\right\}}=e^{\left\{ax_1\left(k\right)\right\}}e^{\left\{x_2\left(k\right)\right\}}$$

$$aT\left\{x_1\left(n\right)\right\}+T\left\{x_2\left(n\right)\right\}=ae^{\left\{x_1\left(k\right)\right\}}+e^{\left\{x_2\left(k\right)\right\}}$$

So the system is **not linear**.

For Time-invariant test:- we need to prove $y\left(n-n_0\right)=T\left\{x\left(n-n_0\right)\right\}$
proof:-

$$y\left(n-n_0\right)=e^{x\left(n-n_0\right)}$$

$$T\left\{x\left(n-n_0\right)\right\}=T\left\{x_1\left(n\right)\right\}=e^{x_1\left(n\right)}=e^{x\left(n-n_0\right)}$$

So, $y\left(n-n_0\right)=T\left\{x\left(n-n_0\right)\right\}$ Hence, the system is **Time-invariant.**

As $y\left(n\right)$ dependence only on present samples. So the system is **memoryless**.

Ans(d):-

For **Stability** Bounded input should produce bounded output.
i.e., for $|x(n)| \leq B_x < \infty, \ \forall n$ then $|y(n)| \leq B_y < \infty, \ \forall n$

Here, $y\left(n\right)=T\left\{x\left(n\right)\right\}=ax\left(n\right)+b$

$$\left|y(n)\right| = \left|ax(n)+b\right| \le a\left|x(n)\right| + b < \infty$$

The system is **stable** (provided a and b are finite).

$y(n)$ dependence only on the present sample of input $x(n)$, not future samples So, the system is **causal**.

$$T\left\{x_1(n)+x_2(n)\right\} = a\left\{x_1(n)+x_2(n)\right\}+b$$
$$T\left\{x_1(n)\right\}+T\left\{x_2(n)\right\} = \left\{ax_1(n)+b\right\}+\left\{ax_2(n)+b\right\}$$
$$T\left\{x_1(n)+x_2(n)\right\} \ne T\left\{x_1(n)\right\}+T\left\{x_2(n)\right\}$$

So the system is **not linear**.

For Time-invariant test:- we need to prove $y(n-n_0) = T\left\{x(n-n_0)\right\}$

proof:-

$$y(n-n_0) = a\left\{x(n-n_0)\right\}+b$$
$$T\left\{x(n-n_0)\right\} = T\left\{x_1(n)\right\} = ax_1(n)+b = a\left\{x(n-n_0)\right\}+b$$

So, $y(n-n_0) = T\left\{x(n-n_0)\right\}$ Hence, the system is **time-invariant.**

As $y(n)$ dependence only on present samples. So the system is **memoryless**.

3. Let $x(n) = \delta(n)+2\delta(n-1)-\delta(n-3)$ and $h(n) = 2\delta(n+1)+2\delta(n-1)$.
Compute and plot each of the following convolutions.

(a) $y_1(n) = x(n)*h(n)$

(b) $y_2(n) = x(n+2)*h(n)$

(c) $y_3(n) = x(n)*h(n+2)$

Ans.

Here we need to use some properties of convolution.

$\delta(n)*x(n) = x(n)$ **(Impulse has void effect under convolution) …. (3.1)**

$\delta(n\pm n_0)*x(n) = x(n\pm n_0)$ **(Time shifting of impulse) …. (3.2)**

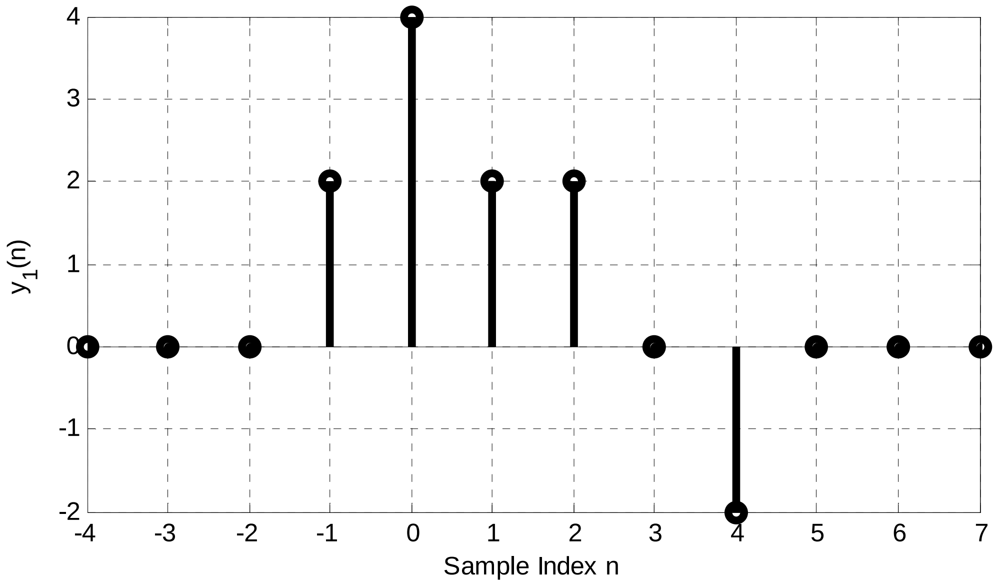
$\{x_1(n)+x_1(n)\}*h(n) = x_1(n)*h(n)+x_1(n)*h(n)$

(distributive property of impulse operator) ……(3.3)

(a)

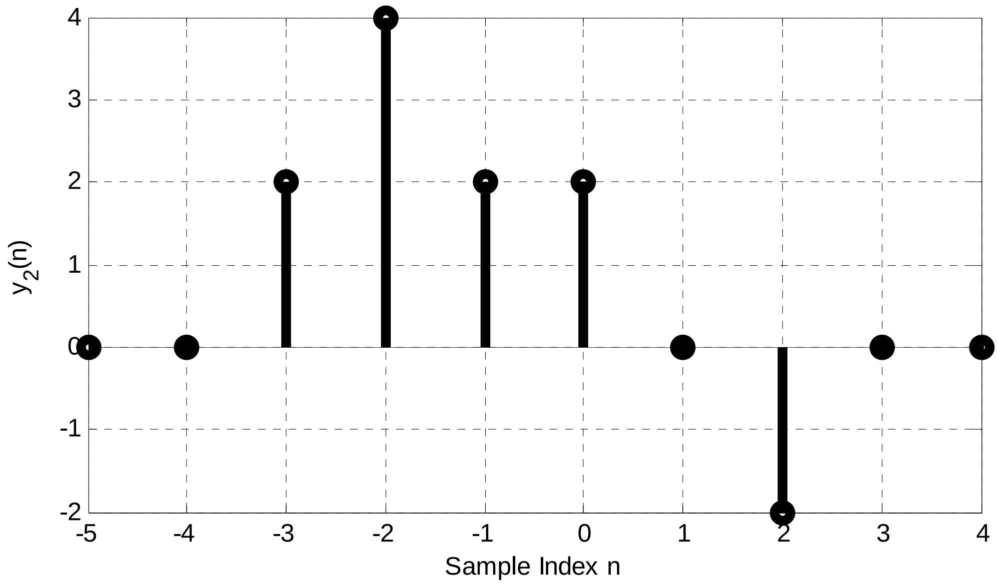
$$y_1(n) = x(n)*h(n)$$
$$= \left\{\delta(n)+2\delta(n-1)-\delta(n-3)\right\}*\left\{2\delta(n+1)+2\delta(n-1)\right\}$$

$= \{\delta(n) * 2\delta(n+1)\} + \{2\delta(n-1) * 2\delta(n+1)\} - \{\delta(n-3) * 2\delta(n+1)\} +$
$\{\delta(n) * 2\delta(n-1)\} + \{2\delta(n-1) * 2\delta(n-1)\} - \{\delta(n-3) * 2\delta(n-1)\}$ $(\because\ 3.3)$
$= 2\delta(n+1) + 4\delta(n+1-1) - 2\delta(n+1-3) + 2\delta(n-1) + 4\delta(n-1-1) - 2\delta(n-1-3)$ $(\because 3.1\text{-}3.2)$
$= 2\delta(n+1) + 4\delta(n) + 2\delta(n-1) + 2\delta(n-2) - 2\delta(n-4)$



(b)

$x(n+2) = \delta(n+2) + 2\delta(n-1+2) - \delta(n-3+2) = \delta(n+2) + 2\delta(n+1) - \delta(n-1)$
$y_2(n) = x(n+2) * h(n)$
$= \{\delta(n+2) + 2\delta(n+1) - \delta(n-1)\} * \{2\delta(n+1) + 2\delta(n-1)\}$
$= \{\delta(n+2) * 2\delta(n+1)\} + \{2\delta(n+1) * 2\delta(n+1)\} - \{\delta(n-1) * 2\delta(n+1)\} +$
$\{\delta(n+2) * 2\delta(n-1)\} + \{2\delta(n+1) * 2\delta(n-1)\} - \{\delta(n-1) * 2\delta(n-1)\}$ $(\because (3.3))$
$= 2\delta(n+1+2) + 4\delta(n+1+1) - 2\delta(n+1-1) + 2\delta(n-1+2) + 4\delta(n-1+1) - 2\delta(n-1-1)$
$= 2\delta(n+3) + 4\delta(n+2) - 2\delta(n) + 2\delta(n+1) + 4\delta(n) - 2\delta(n-2)$
$= 2\delta(n+3) + 4\delta(n+2) + 2\delta(n+1) + 2\delta(n) - 2\delta(n-2)$



(c)

$h(n+2)=2\delta(n+1+2)+2\delta(n-1+2)=2\delta(n+3)+2\delta(n+1)$

$y_3(n)=x(n)*h(n+2)$

$=\{\delta(n)+2\delta(n-1)-\delta(n-3)\}*\{2\delta(n+3)+2\delta(n+1)\}$

$=\{\delta(n)+2\delta(n-1)-\delta(n-3)\}*2\delta(n+3)+\{\delta(n)+2\delta(n-1)-\delta(n-3)\}*2\delta(n+1)\}$

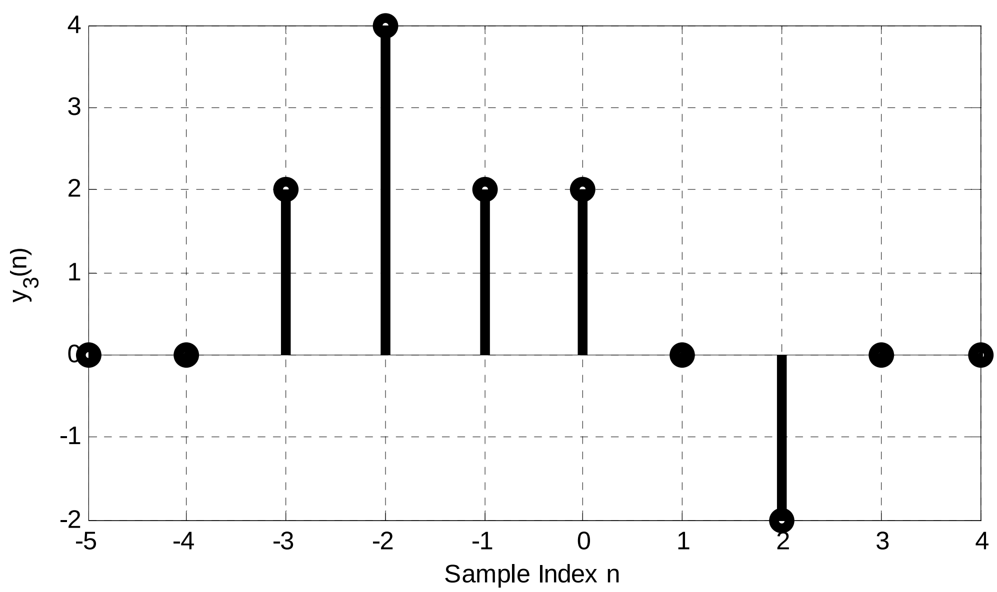
$=\delta(n)*2\delta(n+3)+2\delta(n-1)*2\delta(n+3)-\delta(n-3)*2\delta(n+3)+$

$\{\delta(n)*2\delta(n+1)+2\delta(n-1)*2\delta(n+1)-\delta(n-3)*2\delta(n+1)$

$=2\delta(n+3)+4\delta(n+3-1)-2\delta(n+3-3)+2\delta(n+1)+4\delta(n+1-1)-2\delta(n+1-3)$

$=2\delta(n+3)+4\delta(n+2)-2\delta(n)+2\delta(n+1)+4\delta(n)-2\delta(n-2)$

$=2\delta(n+3)+4\delta(n+2)+2\delta(n+1)+2\delta(n)-2\delta(n-2)$



Note:- $y_2(n)=y_3(n)=y_1(n+2)$

4. Evaluate the integral $\int_0^\infty \delta(t+\frac{3}{4})e^{-t}dt$ (Hint:- Use properties of impulse signal)

Ans:- $\delta(t+\frac{3}{4})=0$, when $t+\frac{3}{4}\neq 0 \Rightarrow t\neq \frac{-3}{4}$, $(t=\frac{-3}{4})\notin[0,\infty)$

so, $\int_0^\infty \delta(t+\frac{3}{4})e^{-t}dt=0$

5. For each of the following input-output relationships, determine weather the corresponding system is linear, time invariant or both.

A) $y(t)=t^2x(t-1)$

B) $y[n]=x^2[n-2]$

Sol.

A) Linear and time variant

$y_1(t)=T\{x_1(t)\}=t^2x_1(t-1)$

$y_2(t)=T\{x_2(t)\}=t^2x_2(t-1)$

Now giving both the input together

$T\{ax_1(t)+bx_2(t)\}=t^2\{ax_1(t)+bx_2(t)\}$

$=at^2\{x_1(t)\}+bt^2\{x_2(t)\}=aT\{x_1(t)\}+bT\{x_2(t)\}$

Therefore given system is **linear**.

For Time-invariant test:- we need to prove $y(t-t_0)=T\{x(t-t_0)\}$

proof:-

$y(t-t_0)=(t-t_0)^2\{x(t-1-t_0)\}$

$T\{x(t-t_0)\}=T\{x_1(t)\}=t^2x_1(t-1)=t^2x(t-1-t_0)$

So, $y(t-t_0)\neq T\{x(t-t_0)\}$ Hence, the system is **NOT time-invariant.**

B) Non- linear (as there is a second order in input signal).

$T\{\alpha x[n]\}=\{\alpha x[n-2]\}^2\neq \alpha T\{x[n]\}$

For Time-invariant test:- we need to prove $y(n-n_0)=T\{x(n-n_0)\}$

proof:-

$y(n-n_0)=\{x(n-n_0-2)\}^2$

$T\{x(n-n_0)\}=T\{x_1(n)\}=\{x_1(n-2)\}^2=\{x(n-n_0-2)\}^2$

So, $y(n-n_0)=T\{x(n-n_0)\}$ Hence, the system is **time-invariant.**

6. Prove that the system given by the following input-output (I/O) is nonlinear.

$$y[n] = T\{x[n]\} = x^*[n]$$

Sol:

For linearity is to be proved two conditions those are additivity and homogeneity should be satisfied

Additivity:

Let's assume two different inputs $x_1[n]$ and $x_2[n]$ lead two different outputs $y_1[n]$ and $y_2[n]$ respectively.

$$y_1[n] = T\{x_1[n]\} = x_1^*[n]$$

$$y_2[n] = T\{x_2[n]\} = x_2^*[n]$$

The sum of the two inputs gives

$$T\{x_1[n] + x_2[n]\} = \{x_1[n] + x_2[n]\}^* = x_1^*[n] + x_2^*[n] = y_1[n] + y_2[n]$$

Therefore additivity is satisfied.

Homogeneity:

Let's say input is scaled by some arbitrary complex number α ,

$$T\{\alpha x[n]\} = \{\alpha x[n]\}^* = \alpha^* x^*[n] = \alpha^* y[n] \neq \alpha y[n]$$

Therefore homogeneity is not satisfied so we can say that this system is non linear.