
NAME

MATH 461 - X13, Test 1, Spring 2015

March 9, 2015

Calculators, books, notes and extra papers are *not* allowed on this test

Please show all your work and explain all answers to qualify for full credit

1. (14 points) (a) Suppose that A and B are two events with $\mathbb{P}(A) = \mathbb{P}(B) = \frac{1}{2}$ and $P(A \cap B) = \frac{1}{3}$. Find $\mathbb{P}(A \cup B^c)$.

   (b) Suppose that E_1, E_2, E_2, E_4 are independent and that $\mathbb{P}(E_1) = \mathbb{P}(E_2) = \frac{1}{2}$, $\mathbb{P}(E_3) = \mathbb{P}(E_4) = \frac{1}{3}$. Find $\mathbb{P}(((E_1 \cap E_2) \cup E_3) \cap E_4)$.

   Solution: (a)

$$\begin{aligned}\mathbb{P}(A \cup B^c) &= \mathbb{P}(A) + \mathbb{P}(B^c) - \mathbb{P}(A \cap B^c) = \mathbb{P}(A) + \big(1 - \mathbb{P}(B)\big) - \big(\mathbb{P}(A) - \mathbb{P}(A \cap B)\big) \\ &= 1 - \mathbb{P}(B) + \mathbb{P}(A \cap B) = 1 - \frac{1}{2} + \frac{1}{3} = \frac{5}{6}\,.\end{aligned}$$

   (b)

$$\begin{aligned}&\mathbb{P}(((E_1 \cap E_2) \cup E_3) \cap E_4) = P((E_1 \cap E_2) \cup E_3)\mathbb{P}(E_4) \\ &= \Big(\mathbb{P}(E_1 \cap E_2) + \mathbb{P}(E_3) - \mathbb{P}((E_1 \cap_2) \cap E_3)\Big)\mathbb{P}(E_4) \\ &= \Big(\mathbb{P}(E_1)\mathbb{P}(E_2) + \mathbb{P}(E_3) - \mathbb{P}(E_1)\mathbb{P}(E_2)\mathbb{P}(E_3)\Big)\mathbb{P}(E_4) \\ &= \Big(\frac{1}{2}\frac{1}{2} + \frac{1}{3} - \frac{1}{2}\frac{1}{2}\frac{1}{3}\Big)\frac{1}{3} = \frac{1}{6}\end{aligned}$$

2. (16 points) If 4 married couples are arranged in a row, find the probability that no husband sits next to his wife.

   Solution: Let $E_i = \{i\text{-th couple sit together}\}$, $i = 1, 2, 3, 4$. Then

$$\begin{aligned}\mathbb{P}(E_i) &= \frac{2 \cdot 7!}{8!} = \frac{1}{4}\,, \\ \mathbb{P}(E_i \cap E_j) &= \frac{2^2 \cdot 6!}{8!} = \frac{1}{14}\,, \\ \mathbb{P}(E_i \cap E_j \cap E_k) &= \frac{2^3 \cdot 5!}{8!} = \frac{1}{42}\,, \\ \mathbb{P}(E_1 \cap E_2 \cap E_3 \cap E_4) &= \frac{2^4 \cdot 4!}{8!} = \frac{1}{7 \cdot 5 \cdot 3}\end{aligned}$$

By the inclusion-exclusion formula

$$\begin{aligned}
\mathbb{P}\left(\cap_{i=1}^4 E_i\right) &= \binom{4}{1}\mathbb{P}(E_i) - \binom{4}{2}\mathbb{P}(E_i\cap E_j) + \binom{4}{3}\mathbb{P}(E_i\cap E_j\cap E_k) - \mathbb{P}(E_1\cap E_2\cap E_3\cap E_4) \\
&= 4\frac{1}{4} - 6\frac{1}{14} + 4\frac{1}{42} - \frac{1}{7\cdot 5\cdot 3} = 1 - \frac{12}{35}.
\end{aligned}$$

Hence, the desired probability is $\frac{12}{35}$.

3. (10 points) Let X be a random variable whose distribution function F is given by

$$F(x) = \begin{cases} 0, & x < 0, \\ x/4, & 0 \le x < 1, \\ x/3 & 1 \le x < 2, \\ 1, & 2 \le x. \end{cases}$$

Find (a) $\mathbb{P}(\frac{1}{2} < X \le \frac{3}{2})$; (b) $\mathbb{P}(\frac{1}{2} \le X \le 1)$; (c) $\mathbb{P}(1 \le X < 2)$; (d) $\mathbb{P}(X > 3/2)$; (e) $\mathbb{P}(X = 1)$.

Solution:

(a) $\mathbb{P}(\frac{1}{2} < X \le \frac{3}{2}) = F(\frac{3}{2}) - F(\frac{1}{2}) = \frac{1}{3}\frac{3}{2} - \frac{1}{4}\frac{1}{2} = \frac{3}{8}$;
(b) $\mathbb{P}(\frac{1}{2} \le X \le 1) = F(1) - F(\frac{1}{2}-) = \frac{1}{3} - \frac{1}{4}\frac{1}{2} = \frac{5}{24}$;
(c) $\mathbb{P}(1 \le X < 2) = F(2-) - F(1-) = \frac{2}{3} - \frac{1}{4} = \frac{5}{12}$;
(d) $\mathbb{P}(X > 3/2) = 1 - \mathbb{P}(X \le \frac{3}{2}) = 1 - F(\frac{3}{2}) = 1 - \frac{1}{3}\frac{3}{2} = \frac{1}{2}$;
(e) $\mathbb{P}(X = 1) = F(1) - F(1-) = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$.

4. (15 points) There are three coins in a box. Coin No. 1 is a fair coin, coin No. 2 is a biased coin that comes up heads 25% of the time, and coin No. 3 is a biased coin that comes up heads 75% of the time. When a coin is randomly chosen from the box and flipped, it shows heads. Find the probability that the chosen coin was coin No. 2.

Solution: Let $E = \{\text{the flipped coin shows heads}\}$, $E_j = \{\text{Coin No. } j \text{ is chosen}\}$, $j = 1, 2, 3$. Then

$$\begin{aligned}
\mathbb{P}(E_2\big|H) &= \frac{\mathbb{P}(E_2)\mathbb{P}(H\big|E_2)}{\mathbb{P}(E_1)\mathbb{P}(H\big|E_1) + \mathbb{P}(E_2)\mathbb{P}(H\big|E_2) + \mathbb{P}(E_3)\mathbb{P}(H\big|E_3)} \\
&= \frac{\frac{1}{3}\frac{1}{4}}{\frac{1}{3}\frac{1}{2} + \frac{1}{3}\frac{1}{4} + \frac{1}{3}\frac{3}{4}} = \frac{\frac{1}{4}}{\frac{2}{4} + \frac{1}{4} + \frac{3}{4}} = \frac{1}{6}.
\end{aligned}$$

5. (15 points) A and B play a series of games. Each game is independently won by A with probability $\frac{2}{3}$ and by B with probability $\frac{1}{3}$. They stop when the total number of wins of one of the players is two greater than that of the other player. Find the probability that a total of 6 games were played.

Solution: Let $E = \{\text{a total of 6 games were played}\}$. Then E occurs if and only if each player wins one of the first two games, each player wins one of the next two games, and then one of the players wins the last two games. The probability of this happening is

$$\left(\frac{2}{3}\frac{1}{3} + \frac{1}{3}\frac{2}{3}\right)\left(\frac{2}{3}\frac{1}{3} + \frac{1}{3}\frac{2}{3}\right)\left(\frac{2}{3}\frac{2}{3} + \frac{1}{3}\frac{1}{3}\right) = \left(\frac{4}{9}\right)^2 \frac{5}{9} = \frac{80}{9^3}$$

Alternatively, A wins in the following outcomes: $ABABAA$, $ABBAAA$, $BAABAA$, $BABAAA$. Each of these outcomes has probability $\left(\frac{2}{3}\right)^4 \left(\frac{1}{3}\right)^2 = \frac{16}{3^6}$. Since there are four of them, the probability that A wins in 6 games is $\frac{64}{3^6}$. Similarly, the probability that B wins in 6 games is $\frac{4 \cdot 4}{3^6}$. By adding these two probabilities we get that the required probability is $\frac{80}{3^6}$.

6. (15 points) Five buses carrying 200 students from the same school arrive at the football stadium. The buses carry, respectively, 20, 30, 40, 50 and 60 students. One of the students is randomly selected. Let X denote the number of students who were on the bus carrying the randomly selected student. Compute $\mathbb{E}X$ and $\mathrm{Var}(X)$.

   Solution:

$$\begin{aligned}
\mathbb{E}X &= 20\mathbb{P}(X = 20) + 30\mathbb{P}(X = 30) + 40\mathbb{P}(X = 40) + 50\mathbb{P}(X = 50) + 60\mathbb{P}(X = 60) \\
&= 20\frac{20}{200} + 30\frac{30}{200} + 40\frac{40}{200} + 50\frac{50}{200} + 60\frac{60}{200} \\
&= \frac{400 + 900 + 1600 + 2500 + 3600}{200} = \frac{90}{2} = 45 \\
\mathbb{E}X^2 &= 20^2\mathbb{P}(X = 20) + 30^2\mathbb{P}(X = 30) + 40^2\mathbb{P}(X = 40) + 50^2\mathbb{P}(X = 50) + 60^2\mathbb{P}(X = 60) \\
&= \frac{20^3}{200} + \frac{30^3}{200} + \frac{40^3}{200} + \frac{50^3}{200} + \frac{60^3}{200} \\
&= \frac{80 + 270 + 640 + 1250 + 2160}{2} = 2200 \\
\mathrm{Var}(X) &= \mathbb{E}X^2 - (\mathbb{E}X)^2 = 2200 - (45)^2 = 2200 - 20125 = 175
\end{aligned}$$

7. (15 points) Four cards are drawn at random (without replacement) from an ordinary deck of 52 cards. Find the probability that there are exactly two aces in the 4 selected cards if it is known that there is at least one ace in those selected.

   Solution: Let $E = \{\text{there are exactly two aces}\}$, $F = \{\text{there is at least one ace}\}$. Then $\mathbb{P}(E) = \frac{\binom{4}{2}\binom{48}{2}}{\binom{52}{4}}$, $\mathbb{P}(F) = 1 - \frac{\binom{48}{4}}{\binom{52}{4}}$, and so

$$\mathbb{P}(E \big| F) = \frac{\mathbb{P}(E \cap F)}{\mathbb{P}(F)} = \frac{\mathbb{P}(E)}{\mathbb{P}(F)} = \frac{\binom{4}{2}\binom{48}{2}}{\binom{52}{4} - \binom{48}{4}}.$$