Dimensions, Scales and Fundamental Constants

We consider the three basic dimensions of mass (M), length (L) and time (T).

In these dimensions, the scales that we can appreciate by our senses are:

M	$\longrightarrow$	10^{-3} kg $-$ 10^{2} kg
L	$\longrightarrow$	10^{-4} m $-$ 10^{4} m
T	$\longrightarrow$	10^{-1} sec $-$ 10^{3} secs

$\rightarrow$ These are orders-of-magnitude estimates in the S.I. system.

These scales are hard-wired in us by the process of natural evolution for our survival.

Example: $\boxed{h = \frac{1}{2} g t^2}$ $\quad g \sim 10\ \mathrm{m\,s^{-2}}$.

If $h \sim 10$ m (safe distance from predators)

then $t = \sqrt{\frac{2h}{g}} = \sqrt{\frac{2 \times 10}{10}}$ sec ~ 1 sec.

For safety our reflexes respond on a time scale of 1 sec to events on a length scale of 1 metre. Our speeds are $\sim h/t \sim 10\ \mathrm{m\,s^{-1}}$

Nature operates on vastly larger scales, determined by c, h, G (the fundamental constants)

Planck length and Planck time

In terms of M, L, T, the dimensions of the fundamental constants, c, h, G are:

i.) $[c]$ (speed of light in vacuum) $\equiv LT^{-1}$

ii.) $[h]$ (Planck's constant) $\equiv MLT^{-2}LT$
$\equiv ML^{2}T^{-1}$ (energy x time)

iii.) $[G]$ (Newton's universal gravitational constant)

$$\equiv \frac{MLT^{-2}L^{2}}{M^{2}} \qquad \left[\because G = -\frac{Fr^{2}}{m_1 m_2}\right] \text{ Newton's Law of Gravitation}$$

$$\Rightarrow [G] \equiv M^{-1}L^{3}T^{-2}.$$

A length dimension in terms of c, h, G is

$$M^{0}L^{1}T^{0} \equiv [G]^{\alpha}[h]^{\beta}[c]^{\gamma}$$

$$\Rightarrow M^{0}L^{1}T^{0} \equiv [M^{-1}L^{3}T^{-2}]^{\alpha}[ML^{2}T^{-1}]^{\beta}[LT^{-1}]^{\gamma}$$

$$\equiv M^{\beta-\alpha}\, L^{3\alpha+2\beta+\gamma}\, T^{-2\alpha-\beta-\gamma}$$

Comparing the left and right hand sides, we get $\beta-\alpha=0 \Rightarrow \boxed{\beta=\alpha}$, $-2\alpha-\beta-\gamma=0$.

$\Rightarrow \boxed{\gamma = -2\alpha-\beta = -3\alpha}$ and $3\alpha+2\beta+\gamma=1$

$\Rightarrow 3\alpha+2\alpha-3\alpha=1 \Rightarrow 2\alpha=1 \Rightarrow \boxed{\alpha=1/2}$

Hence, $\boxed{\beta=1/2}$ and $\boxed{\gamma=-3/2}$. With these exponents we get $\boxed{L \equiv G^{1/2}h^{1/2}c^{-3/2}}$ ~~e[illegible]~~.

(P.T.O.)

Planck length, $\boxed{l_{Pl} \sim \sqrt{\frac{hG}{c^3}}}$ By dimension-al argument

$\boxed{\begin{array}{l} G = 6.67 \times 10^{-11} \text{ S.I. unit}, \quad c = 3 \times 10^{8} \text{ S.I. unit} \\ \text{and } h = 6.626 \times 10^{-34} \text{ S.I. unit.} \end{array}}$

$$\therefore l_{Pl} \sim \sqrt{\frac{6.67 \times 10^{-11} \times 6.626 \times 10^{-34}}{(3 \times 10^{8})^3}} \sim 10^{-35} \text{ m}$$

Planck time, $\boxed{t_{Pl} \sim \frac{l_{Pl}}{c}} \sim \frac{10^{-35}}{3 \times 10^{8}} \text{ s} \sim 10^{-43} \text{ s}$

l_{Pl} probably gives the non-zero quantum of space, and t_{Pl} probably gives the non-zero quantum of time. In that sense, l_{Pl} and t_{Pl} are Nature's least count for space and time.

Mass of an electron, $\boxed{m_e \sim 10^{-30} \text{ kg}}$.

Taken together, m_e, l_{Pl} and t_{Pl}, set the lower limit of Nature's scale.

The upper limit of Nature's scale is set by the mass of the Universe, the age of the Universe and the spatial extent of the Universe. (All in dimensions M, L, T).

The number of protons and neutrons in the Universe is estimated to be $\sim 10^{80}$, each with a mass of $\sim 10^{-27}$ kg.

$\therefore$ Mass of the Universe $\sim 10^{80} \times 10^{-27}$ kg
$\sim 10^{53}$ kg

The age of the Universe $\simeq 14 \times 10^{9}$ years
$\sim 10^{17}$ sec

The span of the Universe $\sim c \times 10^{17} \sim 10^{25}$ m

The full scale of the Universe

M:	10^{-30} kg	10^{-3} kg $- 10^{2}$ kg	10^{53} kg (About 80 orders)
L:	10^{-35} m (Planck length)	10^{-4} m $- 10^{4}$ m	10^{25} m (60 orders)
T:	10^{-43} s (Planck time)	10^{-1} s $- 10^{3}$ s	10^{17} s (60 orders)

The middle scales in which we live.

The physical laws (eg. Newton's laws) by which we understand the Universe in the middle scales, need not be applicable either on very small or very large scales.

Physical implications of the fundamental constants

1/. The speed of light in vacuum, c:

i.) It is invariant through all reference frames, regardless of their relative velocities. It is a universally fixed quantity.

If a reference frame moves with a velocity v (eg. a train coach), with respect to ~~the~~ a fixed frame, and if an object moves with a velocity u', with respect to the moving frame, then the velocity of the same object measured in the fixed (ground) frame is

$$\boxed{u = \frac{v+u'}{1+u'v/c^2}}$$

→ The relativistic velocity addition formula. All velocities are in the same direction.

When $u', v \ll c$, then $\boxed{u \simeq v+u'}$, which is our usual experience. If c had a smaller value we would experience relativistic velocity addition more usually.

Example: Train travels with velocity v. The train headlamp sends out light at velocity $u'=c$.
A ground observer records $u = \frac{v+c}{1+vc/c^2} = \frac{c(\cancel{v+c})}{\cancel{c+v}}$
$\Rightarrow \boxed{u=c} \Rightarrow$ Light can never rest.

ii) Mass, length and time are not invariant in various reference frames.

If m_0 is the mass measured in a frame where ~~the~~ an object is at rest, l_0 is the length in a frame where the object is at rest, then in another reference frame moving with a relative velocity v, m and l are

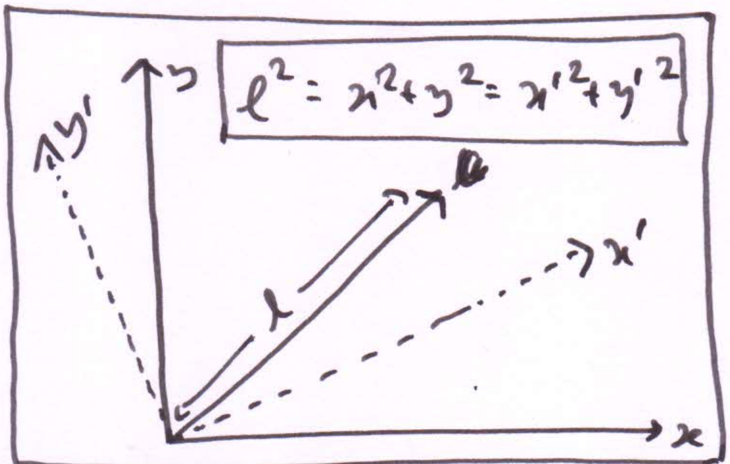
$$\boxed{m = \frac{m_0}{\sqrt{1-(v/c)^2}}}\ , \text{ and } \boxed{l = l_0\sqrt{1-(v/c)^2}} \rightarrow \text{Length Contraction.}$$

$m_0 \rightarrow$ rest mass, $l_0 \rightarrow$ proper length.

If t_0 is the time measured between two events, that have occurred at the same point in space, then in another frame moving with a relative velocity v,

$$\boxed{t = \frac{t_0}{\sqrt{1-(v/c)^2}}} \rightarrow \text{time dilation, } t_0 \rightarrow \text{proper time.}$$

iii) The scalar length of an object in the three spatial dimensions is given



by $\boxed{l^2 = x^2+y^2+z^2 = x'^2+y'^2+z'^2}$.

(Same in both coordinate systems.)

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When objects move close to the speed of light, the invariant scalar length measured is

$$\boxed{l^2 = x^2 + y^2 + z^2 - c^2t^2 = x'^2 + y'^2 + z'^2 - c^2t'^2},$$

in a four-dimensional coordinate system (x, y, z, ict) or (x', y', z', ict'). Time and space are unified in this four-dimensional ~~space~~ coordinate system. The unification is possible due to the invariance of c (same in all frames).

iv) Since c is a universal constant, it establishes an equivalence between mass and energy, by $\boxed{\mathcal{E} = mc^2}$.

2/. Planck's constant, h:

i) With h, Planck length and Planck time, $\boxed{l_p \sim \sqrt{\frac{Gh}{c^3}}}$ and $\boxed{t_p \sim \frac{l_p}{c}}$, respectively, quantise space and time. ~~are quantised~~

ii) With $h \neq 0$, the wave-particle duality is established. Wave-particle duality extends the wave nature of light to material particles, because light can

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be both a wave and a particle (photon). Since photons have no rest, the wave-particle duality also is associated with moving material particles. The de-Broglie wavelength of a moving particle is $\boxed{\lambda = \frac{h}{p} = \frac{h}{mv}}$. The non-zero value of h connects the wave-like property, through the wavelength λ, to the particle-like property, through the momentum $p = mv$. Since h has a small value of $\simeq 6 \times 10^{-34}$ Js on our scales, we do not experience it readily. (wave-particle duality) The wave-particle duality becomes appreciable for an electron, with a small mass. Hence, wave-particle duality applies well to the atomic and nuclear particles.

iii.) Photons have no rest and, therefore, no rest mass. Nevertheless, by the wave-particle duality, an effective mass of photons can be found. $\boxed{m_{eff} = \frac{\mathcal{E}}{c^2}}$

But $\boxed{\mathcal{E} = h\nu = \frac{hc}{\lambda}}$. Hence, $\boxed{m_{eff} = \frac{h\nu}{c^2} = \frac{h}{\lambda c}}$.

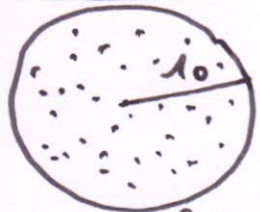
3/. Newton's gravitational constant, G:

i.) Escape velocity, v_{Esc}, is needed to escape ~~from~~ gravity. We write

$$m\frac{dv}{dt} = m\frac{dv}{dr}\frac{dr}{dt} = mv\frac{dv}{dr} = -\frac{GMm}{r^2}$$

Travel to infinity

$$\Rightarrow \boxed{\int_{v_{Esc}}^{0} v\,dv = -GM\int_{r_0}^{\infty} r^{-2}\,dr}$$



$r_0 \rightarrow$ radius of the planet/star.

$$\Rightarrow \left.\frac{v^2}{2}\right|_{v_{Esc}}^{0} = -\frac{GM}{-1}\left. r^{-1}\right|_{r_0}^{\infty}$$

$$\Rightarrow \frac{v_{Esc}^2}{2} = \frac{GM}{r_0} \Rightarrow \boxed{v_{Esc}^2 = 2G\left(\frac{M}{r_0}\right)} \rightarrow \text{Escape Velocity needed.}$$

When $\boxed{v_{Esc} = c}$ (maximum possible velocity), we write $\boxed{r_0 = R_s}$, the Schwarzschild radius of a spherical Black Hole.

$$\therefore \boxed{\frac{M}{R_s} = \frac{c^2}{2G}} \Rightarrow \boxed{R_s = \frac{2GM}{c^2}} \rightarrow \text{Condition to trap light by gravity.}$$

i.) Massive (large mass). ii.) Compact (small radius)

R_s (Schwarzschild radius) $\rightarrow$ Radius of the event horizon.
↳ (of a black hole)

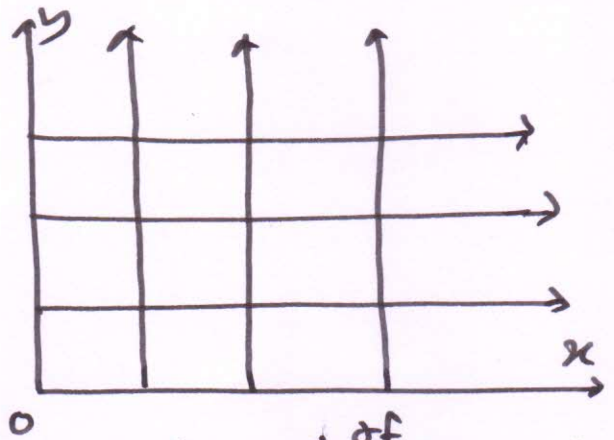
Example: For the earth, $M = 6 \times 10^{24}$ kg.

$$R_s = \frac{2 \times 6.67 \times 10^{-11} \times 6 \times 10^{24}}{(3 \times 10^8)^2} \simeq 9 \text{ mm}$$

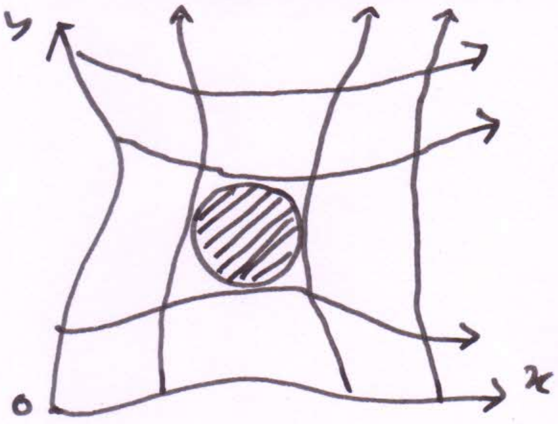
(to be a black hole)

ii.) Gravity as geometry

Gravity
- Field (Newton)
- Geometry (Einstein)



Geometry of space-time in gravity-free condition

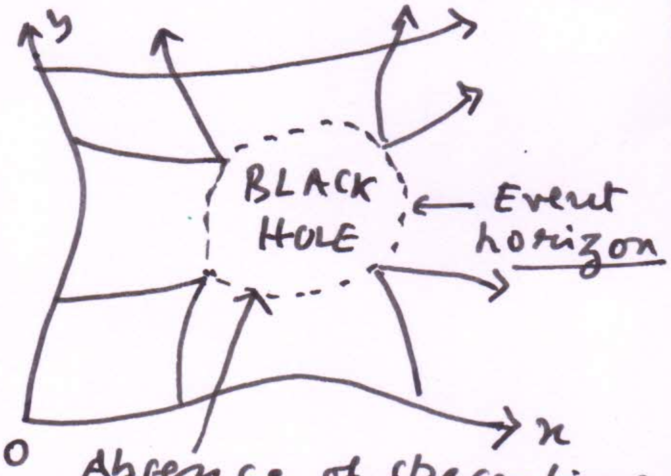


Warping of geometry due to gravity. (presence of mass)

a.) The geometry of space-time is physically influenced by the presence of matter.

b.) Light follows paths known as geodesics (eg. longitudes) in the geometric grid.

c.) Near massive compact objects the light path is bent and deviated.



I/. A black hole is a rupture in the geometric fabric of space-time.

II/. Light does not have a continuous passage through the rupture.

d.) The Gravitational constant G connects Einstein's geometric view of gravity, with Newton's field view of gravity.

Large-aggregate systems

The Boltzmann Constant $\boxed{K_B = \frac{R}{N_A}}$

i.) The ideal gas equation $\boxed{PV = nRT}$

$$PV = \frac{\text{Force}}{\text{Area}} \times \text{Volume} = \text{Force} \times \text{Displacement}$$

(Work → Energy)

PV → Work done on the gas to confine it.

n → No. of moles ⇒ $\boxed{n = \frac{N}{N_A} = \frac{\text{No. of particles}}{\text{Avogadro's number}}}$

PV → Energy contained.

$$\therefore \boxed{\frac{PV}{N} \sim \frac{R}{N_A} T} \Rightarrow \boxed{\langle \text{Energy} \rangle \sim K_B T}$$

Average energy is a microscopic measure.
Temperature, T, is a macroscopic measure.

$\boxed{\text{Boltzmann's constant, } K_B \text{, connects the two}}$.

ii.) Entropy $\boxed{S = k_B \ln \Omega}$, Ω → Multiplicity
S → Entropy.

2 Coin tossing

H	H
H	T
T	H
T	T

a.) Two heads and two tails are highly ordered but have low probability.
b.) One head and one tail are disordered but have high probability.

A head and tail combination has higher disorder and higher multiplicity (2). Hence, this disordered state has a greater probability of occurrence. Entropy is the measure of that disorder.