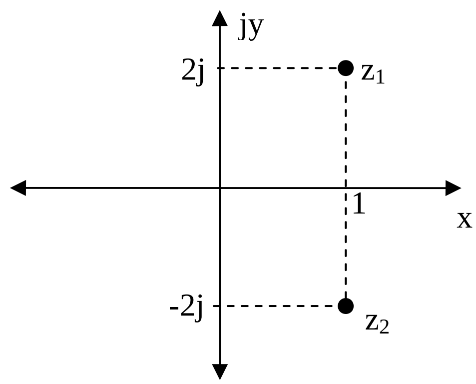
Signals and Systems (CT 203)

Tutorial Sheet-01 DA-IICT, Gandhinagar.

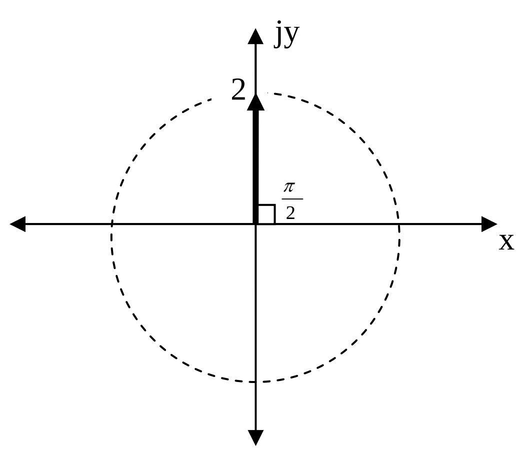
1. Represent following complex number in Z-plane for
   a) $z_1 = 1 + 2j$,
   b) $z_2 = 1 - 2j$,
   Sol: Problem (a) and (b) are shown in figure 1



   c) $z_3 = 2\cos\left(\frac{1}{2}\pi\right) + j2\sin\left(\frac{1}{2}\pi\right)$,

   Sol:

$$z_3 = 2\cos\left(\tfrac{\pi}{2}\right) + j2\sin\left(\tfrac{\pi}{2}\right)$$
$$z_3 = 2\cos\left(\tfrac{\pi}{2}\right) + j2\sin\left(\tfrac{\pi}{2}\right)$$
$$z = r\cos(\theta) + jr\sin(\theta)$$
$$x = 2\cos\left(\tfrac{\pi}{2}\right), y = 2\sin\left(\tfrac{\pi}{2}\right)$$
$$r = 2 \text{ and } \theta = \frac{\pi}{2}$$

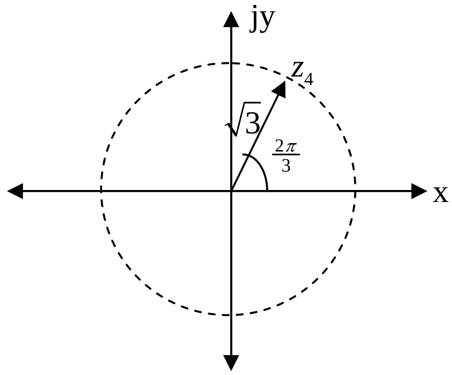


d) $z_4 = \sqrt{3}\cos\left(\frac{2\pi}{3}\right) + j\sqrt{3}\sin\left(\frac{2\pi}{3}\right)$.

Sol:

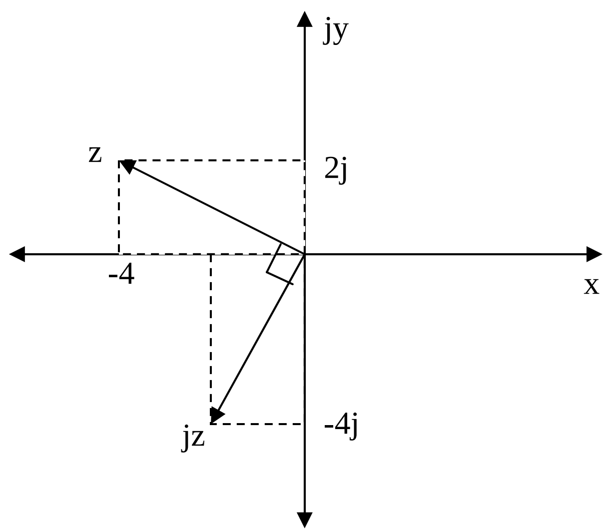
$$z_4 = \sqrt{3}\cos\left(\tfrac{2\pi}{3}\right) + j\sqrt{3}\sin\left(\tfrac{2\pi}{3}\right)$$

$$r = \sqrt{3} \text{ and } \theta = \left(\tfrac{2\pi}{3}\right)$$



2. Multiplication by j to a vector z is geometrically a counterclockwise rotation of z through $\pi/2$ .Verify this by plotting z and jz and the angle of rotation for

a) $z = -4 + 2j$,



Geometrically, z and jz makes and angle of 90^o, therefore we can say that they are orthogonal.

To check mathematically, Dot Product of orthogonal vectors is zero, which is proved below.

$$z = -4 + 2j$$
$$jz = j(-4 + 2j) = -4j + 2j^2 = -2 - 4j$$
$$\langle z, jz \rangle = \langle (-4 + 2j), (-2 - 4j)^* \rangle = (-4 \times -2) + (2j \times 4j) = 0$$

b) $z = 4 + j,$

c) $z = 5 - 3j$

Similarly one can easily show for (b) and (c).

3. Using Taylor's series, prove Euler's relation and hence deduce

a) $\cos(\theta) = \dfrac{e^{j\theta} + e^{-j\theta}}{2}$

b) $\sin(\theta) = \dfrac{e^{j\theta} - e^{-j\theta}}{2j}$

Sol:

Taylor's series expansion of the function $f(x)$ around $x = x_0$ is given by the following equation,

$$f(x) = \sum_{k=0}^{\infty} \frac{f^k(x_0)}{k!}(x - x_0)^k$$

Expanding around x=0 we get,

$$f(x) = \sum_{k=0}^{\infty} \frac{f^k(0)}{k!}(x)^k$$

$$f(x) = \frac{f^0(0)}{0!}x^0 + \frac{f^1(0)}{1!}x^1 + \frac{f^2(0)}{2!}x^2 \ldots\ldots + \frac{f^k(0)}{k!}x^k$$

Consider, $f(x) = e^x$

$$f(x) = e^x \Rightarrow f(0) = e^0 = 1$$
$$f'(x) = e^x \Rightarrow f'(0) = e^0 = 1$$
$$f''(x) = e^x \Rightarrow f''(0) = e^0 = 1$$

Put these values in above equation we get,

$$e^{x} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots\dots$$

Replacing x by $j\theta$ we get,

$$e^{j\theta} = 1 + j\theta + \frac{(j\theta)^2}{2!} + \frac{(j\theta)^3}{3!} + \frac{(j\theta)^4}{4!} + \dots\dots$$

Rearranging the terms,

$$e^{j\theta} = \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots\right) + j\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \dots\right) \dots\dots\dots\dots (3.1)$$

Similarly we can easily expand the following,

$$\cos\theta = \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \dots\right) \dots\dots\dots\dots (3.2)$$

$$\sin\theta = \left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \dots\right) \dots\dots\dots\dots (3.3)$$

From equation (3.1), (3.2) and (3.3), we write,

$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

Likewise,

$$e^{-j\theta} = \cos(\theta) - j\sin(\theta)$$

From the above results problem (a) and (b) can be proved.

4. Express each of the following complex numbers in Cartesian form

Sol:

$$re^{j\theta} = r\cos(\theta) + jr\sin(\theta)$$

(a) $\frac{1}{2}e^{j\pi}$

$$\frac{1}{2}e^{j\pi} = \tfrac{1}{2}\cos(\pi) + j\tfrac{1}{2}\sin(\pi) = -\frac{1}{2}$$

(b) $e^{j\pi/2}$ (Self study)

(c) $e^{-j\pi/2}$ (Self study)

(d) $e^{-j\pi}$ (Self study)

5. Express the following complex numbers in Polar form

Sol:

$$z = x + jy = re^{j\theta}\text{, Where } r = \sqrt{x^2 + y^2} \text{ and } \theta = \tan^{-1}\left(\tfrac{y}{x}\right)$$

(a) $1 + j$

Here $x = 1, y = 1$,

$$r = \sqrt{2}, \theta = \frac{\pi}{4}$$

(b) 5 (Self study)

(c) $\frac{1}{2} - j\frac{\sqrt{3}}{2}$ (Self study)

6. If z is a complex number, i.e., $z = x + jy = re^{j\theta}$ and $z^* = x - jy = re^{-j\theta}$ (complex conjugate), then prove that

Sol:

Given that $z = x + jy = re^{j\theta}$ and $z^* = x - jy = re^{-j\theta}$

As we know that $r = \sqrt{x^2 + y^2}$, $\theta = \tan^{-1}\left(\tfrac{y}{x}\right)$

(a) $zz^* = r^2$

$$zz^* = x^2 + y^2 = |r|^2$$

(b) $\frac{z}{z^*} = e^{j2\theta}$

$$\frac{z}{z^*} = \frac{x + jy}{x - jy} = \frac{re^{j\theta}}{re^{-j\theta}} = e^{2j\theta}$$

(c) $z + z^* = 2\operatorname{Re}\{z\}$

$$z + z^* = x + jy + x - jy = 2x = 2\operatorname{Re}\{z\}$$

(d) $z - z^* = 2j\operatorname{Im}\{z\}$ (Self study)

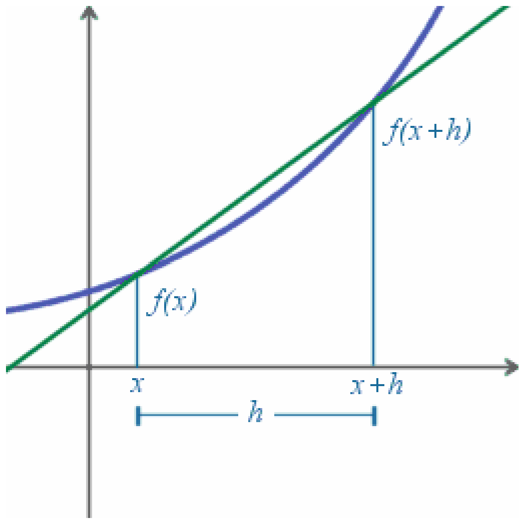
(e) $|z| = |z^*|$ (Self study)

(f) $|z_1 z_2| = |z_1||z_2|$ (Self study)

7. Understanding of derivative of function is geometrically equivalent to find the slope of a line or gradient of a curve

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Sol:



From basic rule of trigonometry, slope of the line is given by,

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x+h) - f(x)}{x+h-x} = \frac{f(x+h) - f(x)}{h} = \tan(\theta)$$

where, theta is the inclination of a line (i.e., angle made by a line with positive direction of X-axis).
Now as the strip width approaches to infinitesimally small then it can be approximated as the derivative of the function at that point of consideration. It is shown below,

$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = slope$ (Leibniz's definition of derivative by first principle)

8. Understanding fundamental theorem of integral calculus, *viz.*, definite integral is nothing but area under a curve bounded by lines defined by limits of integral

$\int_{x=a}^{x=b} f(x)\,dx = F(b) - F(a) = \lim_{h \to 0} \sum_{n=1}^{N} f(x_n)h$ =Area under a curve bounded by lines x=a and x=b.

Solution:- Let $y = f(x)$ be a function which is bounded by lines $x=a$ and $x=b$ as shown in Fig. A.8. Then the area under curve is calculated approximately as follows. Divide the total area into strips of width h and height $f(x_n)$. Then the area of strip would be $f(x_n).h$. Then total area would be summation of all strips which covers entire area under the curve. The error in this approximation will be minimized if the strip width is infinitely small, i.e.,

$$\int_a^b f(x)dx = \lim_{h\to 0} \sum_{<n>} f(x_n)h. \tag{A.34}$$

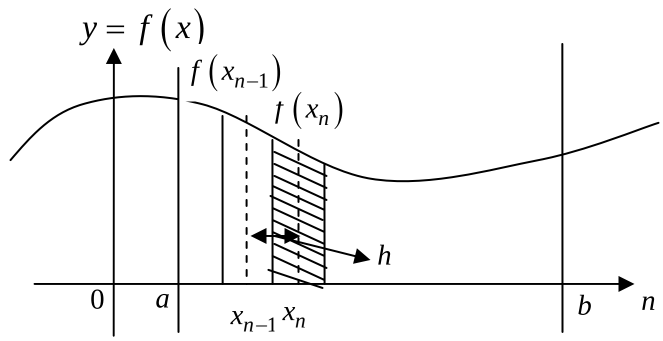


Figure A.8. Understanding geometrical interpretation of definite integral.

This results is nothing but Geometrical Interpretation of Definite Integral (Fundamental Theorem of Integral Calculus)

Note: This result is very useful in deriving shifting property and convolution integral for continuous-time LTI systems. In addition, it is also used to derive expression for continuous-time Fourier transform.

9. Derive sum of first *n* terms and *infinite* terms of geometric progression (G.P.).

Solution:- Consider a geometric sequence whose terms are $a, ar, ar^2, ..., ar^{n-1}$ where a is the first term and r (with $|r|<1$) is the common ratio of G.P. Let the sum of nth term of G.P. as

$$S_n = a + ar + ar^2 + ... + ar^{n-1}$$

$$\Rightarrow rS_n = ar + ar^2 + ... + ar^{n-1} + ar^n$$

$$\therefore S_n - rS_n = a - ar^n$$

$$\Rightarrow S_n = a\left(\frac{1-r^n}{1-r}\right) = \sum_{k=0}^{n-1} ar^k$$

$$\Rightarrow \sum_{k=0}^{n} ar^k = a\left(\frac{1-r^{n+1}}{1-r}\right)$$

$$\therefore \lim_{n\to\infty} S_n = \lim_{n\to\infty} a\left(\frac{1-r^n}{1-r}\right) = a\left(\frac{1-\lim\limits_{n\to\infty} r^n}{1-r}\right)$$

$$\because |r|<1, \quad S_\infty = \frac{a}{1-r}.$$

If $|r|>1, \ S_\infty = \infty$, i.e., geometric series diverges. These formulas are extensively used in convolution sum, DTFT, Z-transform.

CT 203: Signals and Systems
Tutorial # 1
(Date: August 02, 2011)

1. Prove that for a complex signal $f(t)$

$$\int_{-\infty}^{+\infty} f^*(t)dt = \left(\int_{-\infty}^{+\infty} f(t)dt\right)^* , \tag{1}$$

i.e., integration commutes with complex conjugation.

Solution: Consider RHS of (1): $\left(\int_{-\infty}^{+\infty} f(t)dt\right)^*$ with $f(t) = a(t) + ib(t)$ (such that $a(t),\ b(t) \in \mathbb{R}$). Then

$$\begin{aligned}
\left(\int_{-\infty}^{+\infty} f(t)dt\right)^* &= \left(\int_{-\infty}^{+\infty} (a(t) + ib(t))dt\right)^* \\
&= \left(\int_{-\infty}^{+\infty} a(t)dt + i\int_{-\infty}^{+\infty} b(t)dt\right)^* \\
&= \int_{-\infty}^{+\infty} a(t)dt - i\int_{-\infty}^{+\infty} b(t)dt \text{ (since } (X+iY)^* = X - iY) \\
&= \int_{-\infty}^{+\infty} \underbrace{(a(t) - ib(t))}_{=f^*(t)} dt \\
&= \text{LHS}
\end{aligned}$$

2. Prove that $|\int_{-\infty}^{+\infty} f(t)dt| \leq \int_{-\infty}^{+\infty} |f(t)|dt$ (you may interpret $|.|$ as the magnitude operator for complex functions and as the absolute value operator for real functions).

Solution: Consider LHS: $|\int_{-\infty}^{+\infty} f(t)dt|$

$$\begin{aligned}
\text{Note: } \int_{-\infty}^{+\infty} f(t)dt &= \lim_{\Delta t_n \to 0} \sum_{n=-\infty}^{+\infty} f(t_n)\Delta t_n \text{ (by fundamental theorem of calculus)} \\
\left| \int_{-\infty}^{+\infty} f(t)dt \right| &= \left| \lim_{\Delta t_n \to 0} \sum_{n=-\infty}^{+\infty} f(t_n)\Delta t_n \right| \\
&= \lim_{\Delta t_n \to 0} \left| \sum_{n=-\infty}^{+\infty} f(t_n)\Delta t_n \right| \\
&\leq \lim_{\Delta t_n \to 0} \sum_{n=-\infty}^{+\infty} |f(t_n)\Delta t_n| \text{ (by triangle inequality)} \\
&= \int_{-\infty}^{+\infty} |f(t)|dt
\end{aligned}$$