Vector Analysis

1 Vectors and scalars

Vectors are physical quantities that have a magnitude and a direction while a scalar is a quantity that can be completely described by just a magnitude. Such a description give us a physical idea of what a vector and a scalar is but it is not sufficient to decide mathematically whether a given quantity behaves as a vector or a scalar. For e.g. Is the current in a circuit or the pressure at a point on a wall be considered a scalar or a vector? Angular momentum is a vector but does the triplet of moment of inertia (I_{xx}, I_{yy}, I_{zz}) about the three axes form a vector quantity? It is difficult to decide with this description. This description of scalars and vectors also don't give a way to generalize from scalars and vectors to more complex quantities. In particular we will be concerned with physical quantities that are functions of space. How do we decide whether a given quantity is a vector or a scalar function?

Mathematically scalars are just numbers like the real numbers or complex numbers. Vectors are quantities that form a mathematical structure called a vector space. These vectors can be combined and multiplied with numbers to form other vectors in the space. We don't need a separate definition of vectors in physics, but we need a criterion to identify a quantity as a vector or a scalar. This can be done by comparing a given quantity with the position vector which, self evidently, is a vector quantity. The description of a position vector depends upon the frame of reference. A change of the frame of reference (with the origin fixed) leads to a change of basis of the vector space. This changes the description of the position vector. Such changes are given by linear transformations on the position vector. Physical quantities that behave like position vectors under a change of frame of reference are called vector quantities. One common and important class of change is rotation of the frame of reference. The linear transformations in this case is described by the rotation matrices. In the cartesian coordinate system this is precisely what we mean by the 'components' of a vector, i.e if $\vec{v} = \hat{i}v_x + \hat{j}v_y + \hat{k}v_z$, then v_x, v_y and v_z behave exactly as x, y, z under rotation. Scalars are now defined as those physical quantities that don't change under rotation.

Position, velocity, momentum, force, electric and magnetic fields are vector quantities since their description changes like those of position as we rotate the coordinate axes. A force that is directed along $\hat{x}$ will be directed along $-\hat{y}$ under a rotation of the coordinate system by $90°$ about the z-axis. Physical quantities like charge, mass, temperature, pressure, current are scalar quantities since these quantities are numbers that don't change under the rotation of a coordinate system.

Eg.1:
Consider a rotation in two dimension as shown in the figure. The primed coordinate is
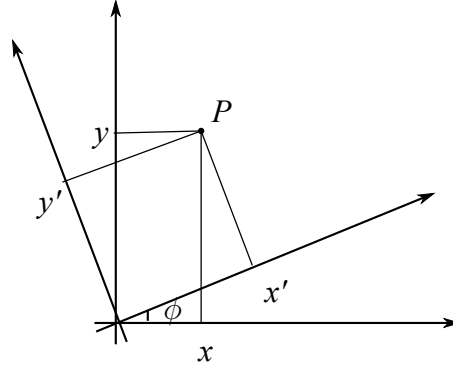
rotated by an angle ϕ with respect to the unprimed coordinate system. So the relation between the coordinates (x', y') and (x, y) of the point P is given by

$$
\begin{aligned}
x' &= \cos\phi\, x + \sin\phi\, y \\
y' &= -\sin\phi\, x + \cos\phi\, y
\end{aligned}
$$

This can be written as

$$
\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = R \begin{pmatrix} x \\ y \end{pmatrix}
$$

R is the rotation matrix.
Let $\vec{A}$ and $\vec{B}$ be two vectors. This means

$$
\begin{pmatrix} A'_x \\ A'_y \end{pmatrix} = R \begin{pmatrix} A_x \\ A_y \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} B'_x \\ B'_y \end{pmatrix} = R \begin{pmatrix} B_x \\ B_y \end{pmatrix}
$$

$\vec{A} + \vec{B}$ is a vector since

$$
\begin{pmatrix} A'_x + B'_x \\ A'_y + B'_y \end{pmatrix} = R \begin{pmatrix} A_x + B_x \\ A_y + B_y \end{pmatrix}
$$

The quantity $C = \begin{pmatrix} A_y \\ A_x \end{pmatrix}$ has two components. Under rotation

$$
C' = \begin{pmatrix} A'_y \\ A'_x \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} A_y \\ A_x \end{pmatrix} \neq R \begin{pmatrix} A_y \\ A_x \end{pmatrix}
$$

So these components don't transform like the coordinates under R. Hence it is not a vector. However the quantity $D = \begin{pmatrix} A_y \\ -A_x \end{pmatrix}$ is a vector since its components transforms like the coordinates, i.e. $D' = \begin{pmatrix} A'_y \\ -A'_x \end{pmatrix} = R \begin{pmatrix} A_y \\ -A_x \end{pmatrix} = RD$. Verify this.

2

Eg. 2:

Consider the following 3-dim rotation :

$$R = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Let $\vec{A}$ and $\vec{B}$ be two vectors in this 3-dimensional space.

Consider $\vec{C} = \vec{A} \times \vec{B} = (A_y B_z - A_z B_y)\hat{i} + (A_z B_x - A_x B_z)\hat{j} + (A_x B_y - A_y B_x)\hat{k}$.

Now, $\vec{C'} = \vec{A'} \times \vec{B'} = (A'_y B'_z - A'_z B'_y)\hat{i}' + (A'_z B'_x - A'_x B'_z)\hat{j}' + (A'_x B'_y - A'_y B'_x)\hat{k}'$.

Substituting for $\vec{A'}$ and $\vec{B'}$ in terms of $\vec{A}$ and $\vec{B}$ using the rotation matrix R we get $\vec{A'} \times \vec{B'} = R(\vec{A} \times \vec{B}) = R\vec{C}$. Verify this. Hence $\vec{C} = \vec{A} \times \vec{B}$ behaves as a vector.

Eg. 3:

Verify that the quantity $A_y B_z \hat{i} + A_x B_z \hat{j} + A_x B_y \hat{k}$ is not a vector.

Eg. 4:

Let $\vec{A} = A_x \hat{i} + A_y \hat{j}$ and $\vec{B} = B_x \hat{i} + B_y \hat{j}$ be two vectors. The dot product $\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y$ is a scalar.

If $\vec{A'} = R\vec{A}$ and $\vec{B'} = R\vec{B}$ then $\vec{A'} \cdot \vec{B'} = A'_x B'_x + A'_y B'_y = A_x B_x + A_y B_y = \vec{A} \cdot \vec{B}$.

The dot product doesn't change under rotation. Hence it is a scalar.

2 The Gradient

Let $F(x, y, z)$ be a function of three variables. If F is differentiable at (x, y, z) then a small change in F due to a small change in x, y and z is given by

$$\begin{aligned} dF &= \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy + \frac{\partial F}{\partial z}dz \\ &= \left(\vec{\nabla}F\right) \cdot \vec{dr} \end{aligned}$$

$$\text{where} \quad \vec{dr} = dx\hat{i} + dy\hat{j} + dz\hat{k} \quad \text{and} \quad \vec{\nabla}F = \frac{\partial F}{\partial x}\hat{i} + \frac{\partial F}{\partial y}\hat{j} + \frac{\partial F}{\partial z}\hat{k}$$

We can prove that $\vec{\nabla}F$ is indeed a vector quantity. It is called the gradient of the scalar function F at the point (x, y, z).

If $\vec{dr}$ is along the direction of $\vec{\nabla}F$ then

$$dF = |\vec{\nabla}F||\vec{dr}|$$

This is the maximum change in the function F.

If $\vec{dr}$ makes an angle θ with $\vec{\nabla}F$ then
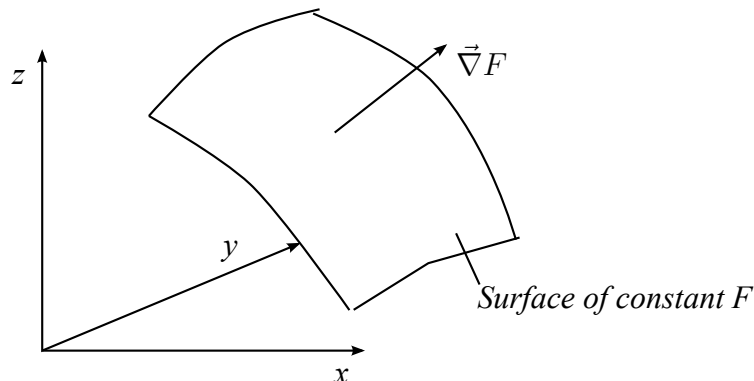
$$dF = |\vec{\nabla}F||\vec{dr}|\cos\theta$$

This is smaller in magnitude than the one seen above.

If $\theta = 90°$ then $dF = 0$.

So if we move orthogonal to the direction of $\vec{\nabla}F$ then the function doesn't change.

If we have a surface over which F is constant then $\vec{\nabla}F$ at any point on this surface is in the direction of the normal to the surface.
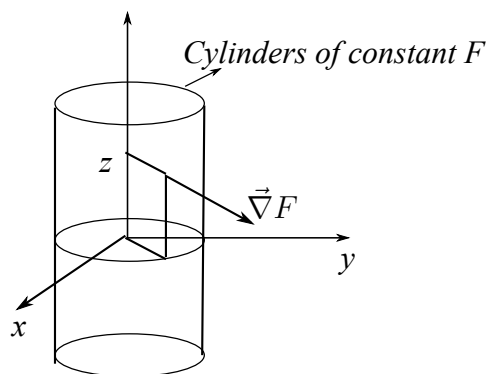

Surface of constant F

Eg.
$$F(x, y, z) = x^2 + y^2$$
$$\vec{\nabla}F = 2(x\hat{i} + y\hat{j}) = 2\vec{r}.$$
The function F is constant over surfaces of cylinders whose axis coincides with the z axis.

Cylinders of constant F

The function doesn't change over these cylindrical surfaces. It changes at the maximum rate along the normal to these surfaces. The normal to a cylindrical surface is a radial direction. Hence the gradient direction is along the radial direction $\vec{r}$.

2.1   The gradient vector

Now we will show that for a scalar function F, the gradient $\vec{\nabla}F$ is a vector quantity. We will do it for gradient in two dimension. The extension to three dimension is on similar lines though with some complexity.
We need to show that the components of $\vec{\nabla}F$ transforms like the coordinates under rotation.

A general rotational transformation in two dimension is given as

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{1}$$

$$\begin{aligned} \vec{\nabla}F &= \hat{i}\frac{\partial F}{\partial x} + \hat{j}\frac{\partial F}{\partial y} \\ \vec{\nabla}'F &= \hat{i}'\frac{\partial F}{\partial x'} + \hat{j}'\frac{\partial F}{\partial y'} \end{aligned}$$

We seek a relation between the components of $\vec{\nabla}'F$ and $\vec{\nabla}F$.

$$\frac{\partial F}{\partial x'} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial x'} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x'}$$

We need to calculate $\frac{\partial x}{\partial x'}$ and $\frac{\partial y}{\partial x'}$.
This is obtained by solving the linear eq.(1). This gives

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}$$

This gives $\frac{\partial x}{\partial x'} = \cos\phi$ and $\frac{\partial y}{\partial x'} = \sin\phi$.

$$\therefore \quad \frac{\partial F}{\partial x'} = \cos\phi\frac{\partial F}{\partial x} + \sin\phi\frac{\partial F}{\partial y}$$

Similarly we can obtain

$$\frac{\partial F}{\partial y'} = -\sin\phi\frac{\partial F}{\partial x} + \cos\phi\frac{\partial F}{\partial y}$$

So we have

$$\begin{pmatrix} \frac{\partial F}{\partial x'} \\ \frac{\partial F}{\partial y'} \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \frac{\partial F}{\partial x} \\ \frac{\partial F}{\partial y} \end{pmatrix}$$

$$\therefore \quad \vec{\nabla}'F = R\vec{\nabla}F$$

$\therefore \quad \vec{\nabla}F$ is a vector.

The computation only depended upon the chain rule of partial differentiation.

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial}{\partial y'}\frac{\partial y'}{\partial x}$$

The scalar function F just sat pretty in the whole derivation. In fact the partial differentiations are looked upon as operators that operate on any function on the right.

So, instead of considering $\vec{\nabla}F$ as the vector we can consider the differential operator $\vec{\nabla}$ as a vector operator. It is called the 'dell' operator or the 'grad' operator.
So we have

$$\vec{\nabla}' = R\vec{\nabla} \quad \text{or} \quad \begin{pmatrix} \frac{\partial}{\partial x'} \\ \frac{\partial}{\partial y'} \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix}$$

3   the Divergence

If we have a vector function $\vec{A}$ what kind of derivative of $\vec{A}$ can we construct?
Consider all possible partial differentiation of the components of $\vec{A}$. We write them in a matrix as follows.

$$D = \begin{pmatrix} \frac{\partial A_x}{\partial x} & \frac{\partial A_y}{\partial x} & \frac{\partial A_z}{\partial x} \\ \frac{\partial A_x}{\partial y} & \frac{\partial A_y}{\partial y} & \frac{\partial A_z}{\partial y} \\ \frac{\partial A_x}{\partial z} & \frac{\partial A_y}{\partial z} & \frac{\partial A_z}{\partial z} \end{pmatrix} \tag{2}$$

As we have seen earlier the meaningful quantities like vectors and scalars have specific way in which they transform under rotation. If R is the rotation matrix that transforms the coordinates then the matrix D transforms as RDR^T. It is interesting to note that the trace of this matrix remains invariant under rotation, i.e,

$$\frac{\partial A'_x}{\partial x'} + \frac{\partial A'_y}{\partial y'} + \frac{\partial A'_z}{\partial z'} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

This is called the divergence of $\vec{A}$. Since $\vec{\nabla}$ is a vector differential operator we can denote the divergence as
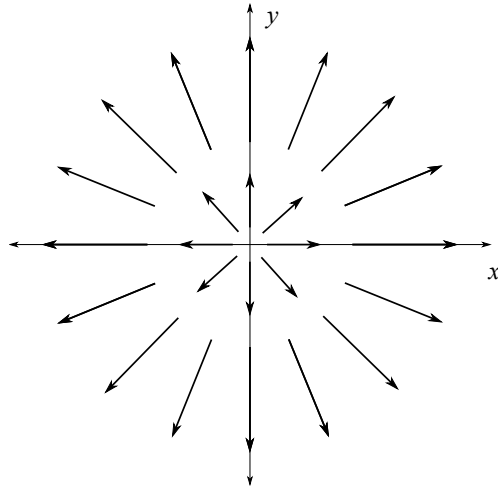
$$\vec{\nabla} \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

Just like we have shown that $\vec{\nabla}F$ to be a vector quantity, we can show that $\vec{\nabla} \cdot \vec{A}$ is a scalar quantity.

Eg.1
$\vec{A} = x\hat{i} + y\hat{j} + z\hat{k} = \vec{r}$

$$\vec{\nabla} \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = 1 + 1 + 1 = 3$$

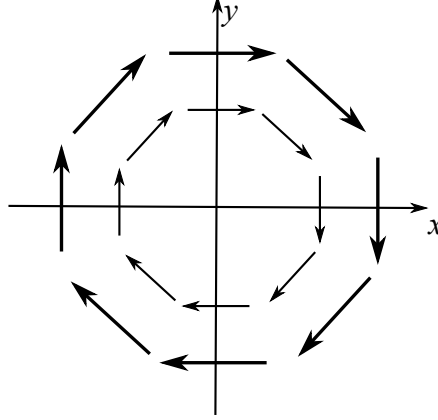If $\vec{A} = -\vec{r}$ then $\vec{\nabla} \cdot \vec{A} = -3$.
The physical interpretation of divergence is that it gives the amount of vector field diverging

from a point. A positive divergence means that the vectors are diverging out while a negative divergence means that it is falling in.

Eg.2

$\vec{v} = y\hat{i} - x\hat{j}.$

$$\vec{\nabla} \cdot \vec{v} = \frac{\partial y}{\partial x} + \frac{\partial(-x)}{\partial y} = 0$$

This vector field has no divergence at any point.

Eg.3

$\vec{v} = x\hat{i}/|x|.$

$$
\begin{aligned}
v_x &= 1 \quad \text{if} x > 0 \\
&= -1 \quad \text{if} x < 0 \\
&= 0 \quad \text{at } x = 0
\end{aligned}
$$

$$\vec{\nabla} \cdot \vec{V} = \frac{\partial v_x}{\partial x} = 0 \quad \text{for} \quad x \neq 0$$

For $x = 0$ we evaluate $\frac{\partial v_x}{\partial x}$ as follows:
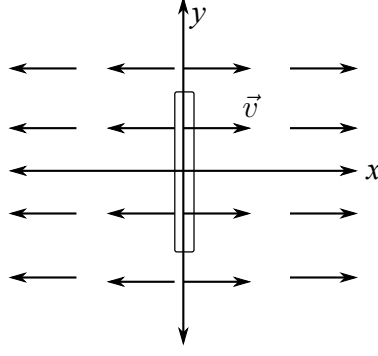
$$
\begin{aligned}
\left.\frac{\partial v_x}{\partial x}\right|_{x=0} &= \lim_{\epsilon \to 0} \frac{v_x(\epsilon) - v_x(-\epsilon)}{2\epsilon} = \frac{2}{2\epsilon} = \frac{1}{\epsilon} \\
&= \to \infty
\end{aligned}
\tag{3}
$$

$\therefore \ \vec{\nabla} \cdot \vec{v}$ is 0 everywhere except at points where $x = 0$. The following diagram shows this phenomenon.

4 the Curl

The offdiagonal elements of the matrix D in eq.(2) can be combined to form the following quantity denoted as $\vec{\nabla} \times \vec{A}$.
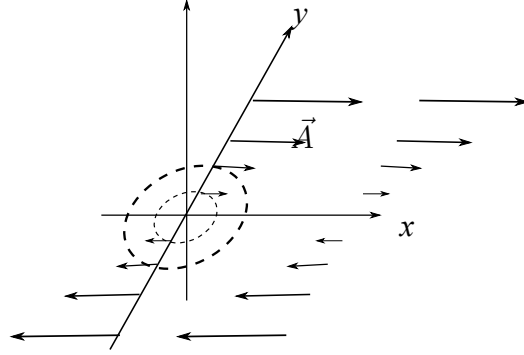
$$\vec{\nabla} \times \vec{A} = \hat{i}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{j}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{k}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

This is called the curl of $\vec{A}$. Just like the divergence, $\vec{\nabla} \times \vec{A}$ also has a very useful physical interpretation. It gives the amount by which the vector field $\vec{A}$ curls around a point. curl $\vec{A}$ is a vector quantity and the sign of this quantity specifies whether the curling around is clockwise or anticlockwise.

Eg.1

$\vec{A} = y\hat{i}$

$A_x = y, A_y = A_z = 0$

$$\vec{\nabla} \times \vec{A} = \hat{i}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{j}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{k}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) = -\hat{k} \qquad (4)$$

A rod along y will rotate clockwise if placed in a fluid that flows like this vector field. This rotation is represented as $-\hat{k}$ (right hand screw convention).

Eg. 2

$\vec{A} = y\hat{i} - x\hat{j}$

$$\vec{\nabla} \times \vec{A} = \hat{k}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) = -2\hat{k}$$

This field is plotted earlier.

5   Linearity of $\vec{\nabla}$

$\vec{\nabla}$ is a linear differential operator. So if f and g are two scalar functions then

$$\vec{\nabla}(f + g) = \vec{\nabla}f + \vec{\nabla}g$$

8

If $\vec{A}$ and $\vec{B}$ are two vector functions then

$$\vec{\nabla} \cdot (\vec{A} + \vec{B}) = \vec{\nabla} \cdot \vec{A} + \vec{\nabla} \cdot \vec{B}$$

and

$$\vec{\nabla} \times (\vec{A} + \vec{B}) = \vec{\nabla} \times \vec{A} + \vec{\nabla} \times \vec{B}$$

It is important to realize that $\vec{\nabla}$ is an operator and has meaning only when it operates on some field or a function. Though mostly we treat the operator $\vec{\nabla}$ as as any other vector quantity, divergence of $\vec{A}$ has to be written as $\vec{\nabla} \cdot \vec{A}$ and not $\vec{A} \cdot \vec{\nabla}$ as we could ordinarily do for dot products. It may be noted that sometimes we can have expressions like

$$(\vec{A} \cdot \vec{\nabla})f$$

where $\vec{A}$ is a vector function and f is a scalar function. Here $(\vec{A} \cdot \vec{\nabla})$ acts like an operator. Precisely, this differential operator is

$$\left(A_x \frac{\partial}{\partial x} + A_y \frac{\partial}{\partial y} + A_z \frac{\partial}{\partial z} \right) f$$

6 Product Rules

We have the following product rules which can be derived from the basic rule of derivative of product of two functions.

Let f and g be scalar functions and $\vec{A}$ and $\vec{B}$ be vector functions. Then

1. **Gradient**

   (a) $\vec{\nabla}(fg) = f\vec{\nabla}g + g\vec{\nabla}f$

   (b) $\vec{\nabla}(\vec{A} \cdot \vec{B}) = \vec{A} \times (\vec{\nabla} \times \vec{B}) + \vec{B} \times (\vec{\nabla} \times \vec{A}) + (\vec{A} \cdot \vec{\nabla})\vec{B} + (\vec{B} \cdot \vec{\nabla})\vec{A}$

2. **Divergence**

   (a) $\vec{\nabla} \cdot (f\vec{A}) = f(\vec{\nabla} \cdot \vec{A}) + \vec{A} \cdot (\vec{\nabla}f)$

   (b) $\vec{\nabla} \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\vec{\nabla} \times \vec{A}) - \vec{A} \cdot (\vec{\nabla} \times \vec{B})$

3. (a) $\vec{\nabla} \times (f\vec{A}) = f(\vec{\nabla} \times \vec{A}) - \vec{A} \times (\vec{\nabla}f)$

   (b) $\vec{\nabla} \times (\vec{A} \times \vec{B}) = (\vec{B} \cdot \vec{\nabla})\vec{A} - (\vec{A} \cdot \vec{\nabla})\vec{B} + \vec{A}(\vec{\nabla} \cdot \vec{B}) - \vec{B}(\vec{\nabla} \cdot \vec{A})$

7   Second Derivatives:

It is interesting to look at all the second derivatives constructed with the operator $\vec{\nabla}$.

If f is a scalar function then $\vec{\nabla} f$ is a vector function. We can find its divergence and curl.

1.
$$\vec{\nabla} \cdot (\vec{\nabla} f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \quad \text{Verify this}$$

This operator is denoted as $\nabla^2 f$. It is called the Laplacian.

2.
$$\vec{\nabla} \times (\vec{\nabla} f) = 0 \qquad \text{Verify}$$

The curl of a gradient is always zero.

For a vector function $\vec{A}$ we can find $\vec{\nabla} \cdot \vec{A}$ and $\vec{\nabla} \times \vec{A}$. The following second derivatives are possible.

1. $\vec{\nabla} \cdot \vec{A}$ is a scalar. We can find its gradient.

$$\vec{\nabla}(\vec{\nabla} \cdot \vec{A}) = \hat{i} \left(\frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_y}{\partial x \partial y} + \frac{\partial^2 A_z}{\partial x \partial z} \right) + \hat{j} \left(\frac{\partial^2 A_x}{\partial x \partial y} + \frac{\partial^2 A_y}{\partial y^2} + \frac{\partial^2 A_z}{\partial y \partial z} \right) + \hat{k} \left(\frac{\partial^2 A_x}{\partial x \partial z} + \frac{\partial^2 A_y}{\partial y \partial z} + \frac{\partial^2 A_z}{\partial z^2} \right)$$

2. $\vec{\nabla} \times \vec{A}$ is a vector. We can find its divergence and curl.

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0 \quad \text{Verify}$$

3.
$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A}$$

Here ∇^2 is the Laplacian which we saw earlier to operate on a scalar. On the vector $\vec{A}$ it means
$$\nabla^2 \vec{A} = \hat{i} \nabla^2 A_x + \hat{j} \nabla^2 A_y + \hat{k} \nabla^2 A_z$$