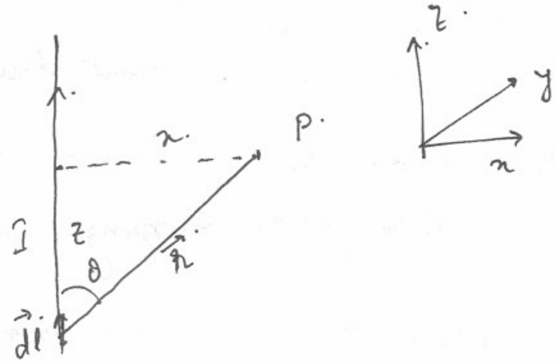
Magnetic field due to a long wire:



Consider a long ~~wire~~ straight wire along z-axis is carrying a current I along the +ve z-direction. Let us find the magnetic field $\vec{B}$ at a point P whose distance from the straight wire is x.

$$\vec{B} = \frac{\mu_0}{4\pi} I \int \frac{\vec{dl} \times \hat{r}}{r^2}$$

All along the wire $dl \times \hat{r}$ is along $\hat{y}$.

And $r^2 = x^2 + z^2$, $dl = dz$

$$\therefore \vec{B} = \frac{\mu_0}{4\pi} I \hat{y} \int_{-\infty}^{\infty} \frac{dz \sin\theta}{x^2 + z^2}$$

$$z = -x\cot(\theta)$$

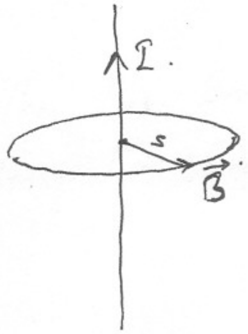
$$\therefore dz = x \operatorname{cosec}^2\theta \, d\theta$$

$$\therefore \vec{B} = \frac{\mu_0 I}{4\pi} \hat{y} \int_0^{\pi} \frac{\sin\theta}{x^2(1+\cot^2\theta)} x \operatorname{cosec}^2\theta \, d\theta$$

$$= \frac{\mu_0 I}{4\pi} \hat{y} \frac{1}{x} \int_0^{\pi} \sin\theta \, d\theta$$

$$= \frac{\mu_0 I}{2\pi x} \hat{y}$$

So the magnetic field due to an infinite long wire varies as $\frac{1}{s}$ where s is the distance from the wire. This field curls around the wire as shown in the figure below



This expression is equivalent to the field due to a point charge in electrodynamics. Any surface or volume current densities can be understood to be made up of a number of thin wires carrying current.

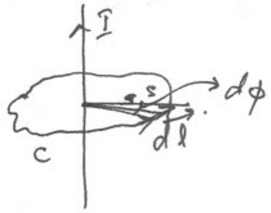
In cylindrical co-ordinates, when the current carrying wire is along the z-axis.

$$\vec{B}(s, \phi) = \frac{\mu_0 I}{2\pi s}\hat{\phi}$$

$$\therefore \quad B_s = 0, \quad B_\phi = \frac{\mu_0 I}{2\pi s}, \quad B_z = 0$$

$$\therefore \quad \vec{\nabla}\times\vec{B} = \hat{z}\frac{1}{s}\frac{\partial}{\partial s}(sB_\phi) = 0 \text{ if } s > 0$$

To find $\vec{\nabla}\times\vec{B}$ at $s = 0$ consider a loop around the wire.



We consider $\oint_C \vec{B}\cdot\vec{dl}$ along the curve C. Along the loop.

$$\vec{dl} = s\,d\phi\,\hat{\phi} + ds\,\hat{s}$$

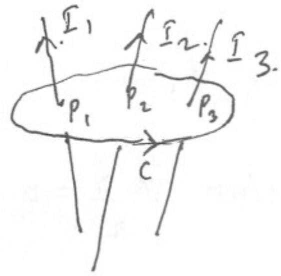
$$\therefore \quad \vec{B}\cdot\vec{dl} = \frac{\mu_0 I}{2\pi}\,d\phi$$

$$\therefore \quad \oint_C \vec{B}\cdot\vec{dl} = \int_0^{2\pi}\frac{\mu_0 I}{2\pi}\,d\phi = \mu_0 I$$

$\therefore$ $\vec{\nabla}\times\vec{B}$ is a two dimensional δ-function given by.

$$\vec{\nabla}\times\vec{B} = \mu_0 I\,\delta^{(2)}(x, y)$$

Now if we have a number of wires passing through a loop carrying currents $I_1, I_2, \ldots I_n$ then



$$\oint_C \vec{B}\cdot d\vec{l} = \cancel{\frac{\mu_0}{4\pi}} \mu_0 \left(I_1 + I_2 + \cdots + I_n\right)$$

~~By~~ Let S be a surface enclosed by the curve C. Then by Stokes' theorem we have

$$\int_S (\vec{\nabla}\times\vec{B})\cdot\hat{n}\,da = \oint_C \vec{B}\cdot d\vec{l} = \frac{\mu_0}{\cancel{4\pi}}\left(I_1 + I_2 + \cdots + I_n\right).$$

Now, instead of thin wires we have a current density $\vec{J}$ in the region enclosed by the loop then

$$\int_S \vec{\nabla}\times\vec{B}\cdot\hat{n}\,da = \mu_0 \int_S \vec{J}\cdot\hat{n}\,da$$

This implies $\quad \vec{\nabla}\times\vec{B} = \mu_0 \vec{J}$.

This is equivalent to the Gauss's law in electrostatics

i.e. $\quad \vec{\nabla}\cdot\vec{E} = \dfrac{\rho}{\epsilon_0}$

Static charge density ρ is the cause of the electric field $\vec{E}$ and steady current density $\vec{J}$ is the cause of static magnetic field $\vec{B}$.

From the Bio-Savart law, for a current density $\vec{J}$ we ~~get~~ have

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int_V \frac{\vec{J}\times\hat{\mathscr{r}}}{\mathscr{r}^2}\,d\tau'$$

We evaluate the divergence of $\vec{B}$

$$\vec{\nabla}\cdot\vec{B} = \frac{\mu_0}{4\pi}\int \vec{\nabla}\cdot\left(\vec{J}\times\frac{\hat{\imath}}{\imath^2}\right)d\tau'$$

$\vec{J}$ is only a function of $\vec{r}'$. So we have.

$$\vec{\nabla}\cdot\left(\vec{J}\times\frac{\hat{\imath}}{\imath^2}\right) = -\vec{J}\cdot\left(\vec{\nabla}\times\frac{\hat{\imath}}{\imath^2}\right) = 0 \quad \text{since } \vec{\nabla}\times\frac{\hat{\imath}}{\imath^2} = 0$$

$$\therefore\ \vec{\nabla}\cdot\vec{B} = \cancel{\frac{\mu_0}{4\pi}}\,0$$

So irrespective of the current density $\vec{\nabla}\cdot\vec{B} = 0$.
It may appear that we have proved $\vec{\nabla}\cdot\vec{B} = 0$. But this is not true. It is an experimental observation that $\vec{\nabla}\cdot\vec{B} = 0$. It is important to note that in electrostatic field $\vec{\nabla}\cdot\vec{E}$ given the charge density at a point. Everything starts from the existence of point charge which is an experimental fact. In magnetostatics we don't have any magnetic charges equivalent to electric charges. We only have magnetic dipoles (North & south poles). The dipoles are understood to be formed by current carrying loops. These fields can be evaluated using the Biot-Savart law. So the experimental fact is the Biot-Savart law which is equivalent to the Coulomb's law in electrostatics. This experimental law lead to the consequence that $\vec{\nabla}\cdot\vec{B} = 0$. So the possibility of the existence of magnetic monopoles is not ruled out theoretically.

Ampere's Law:

The Equation

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$

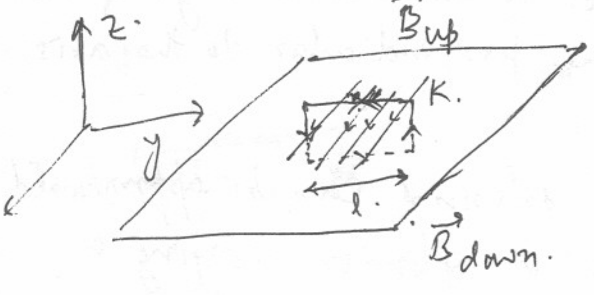
is called the Ampere's law in differential form. In integral form this equation is

$$\oint_C \vec{B} \cdot \vec{dl} = \mu_0 I_{enclosed}$$

where $I_{enclosed}$ is the total current passing through a surface enclosed by the loop C.

Like Gauss's law Amperes' law is useful in evaluating magnetic fields when there are certain symmetries in the problem.

Eg1) Let $\vec{K}$ be the surface current density (current per unit length) over an infinite plane. Find the magnetic field produced by this current.



Consider a loop (amperian loop) as shown. The loop has length l parallel to the plane. And has tiny elements piercing the plane. Let us traverse the loop anticlockwise and evaluate $\oint_C \vec{B} \cdot \vec{dl}$.

$$\oint_C \vec{B} \cdot \vec{dl} = B_{up}\, l + B_{down}\, l = \mu_0 K l.$$

B_{up} and B_{down} are the magnitude of the magnetic field above and below the plane respectively. Above, the field is along $-\hat{y}$ and below, the field is along $\hat{y}$. Now at the same distance above and below $B_{up} = B_{down} = B$.

$$\therefore \quad 2Bl = \mu_0 K l$$

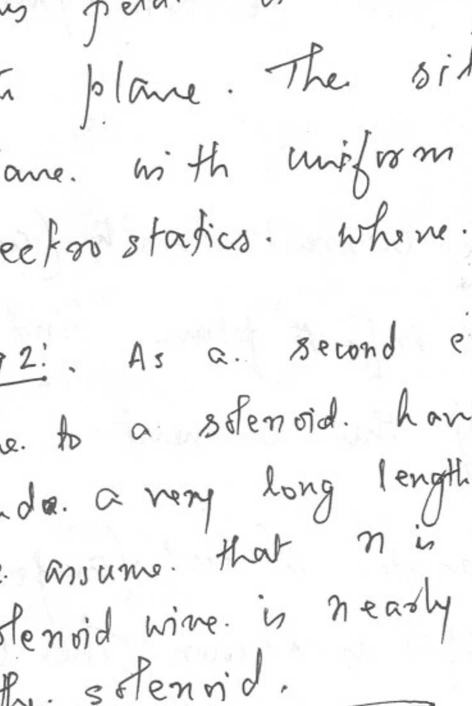
$$\therefore \quad B = \frac{\mu_0}{2} K.$$

$\therefore$ Above the plane $\vec{B} = -\frac{\mu_0}{2} K \hat{y}$

Below the plane $\vec{B} = \frac{\mu_0}{2} K \hat{y}$

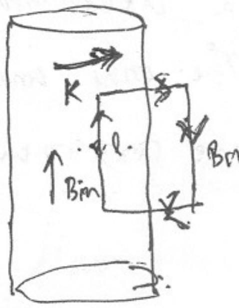
This field is independent of the distance form the plane. The situation is similar to an infinite plane with uniform surface charge density in electrostatics. where $E = \frac{\sigma}{2\epsilon_0}$

Eg 2: As a second example let us find the magnetic field due to a solenoid having n turns per unit length, radius R and a very long length. The current in the solenoid is I. We assume that n is very large so that the windings of the solenoid wire is nearly in a plane perpendicular to the axis of the solenoid.



The solenoid can be approximated with a cylinder carrying a surface current with density $\vec{K} = nI\,\hat{\phi}$. Since the solenoid has n-turns per unit length, the current flowing tangentially over the cylindrical surface per unit length is nI.

~~Now consider an Amperian loop as shown.~~



There can't be any magnetic field in the $\hat{\phi}$ direction. This is because if we consider an amperian loop which is circular and concentric with the cylinder, no current cuts the plane of the loop. By symmetry of the problem we have by Ampere's law

$$B_\phi \times 2\pi r = \mu_0 \times 0$$

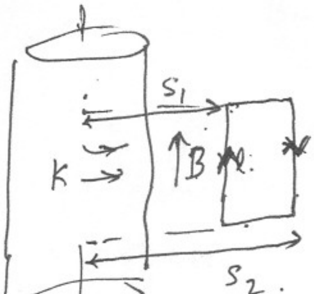
$$\therefore B_\phi = 0.$$

Consider an amperian loop as shown above traversed clockwise. ~~The upwards~~ ~~magnetic field inside the loop is vertically downwards~~ downward. ~~and outside it is vertically~~ ~~upwards~~ ~~downward~~.

$$\therefore (B_{in} - B_{out})\, l = \mu_0 K l.$$

$$\therefore B_{in} - B_{out} = \mu_0 K. \qquad \text{——— I.}$$

The contribution from the horizontal pieces of the loop is zero since $\vec{B}$ is perpendicular to the loop element. Now we claim that Bout is zero. For this



consider an amperian loop outside the solenoid. Then we have

$$[B(s_1) - B(s_2)]\, l = 0.$$

$$\therefore B(s_1) = B(s_2).$$

So if there is a magnetic field outside the solenoid it has to be constant everywhere. We have just seen that an infinite sheet of ~~current~~ current produces an magnetic field which is independent of distance from the sheet. The effect of the solenoid ~~is~~ has to be less pronounced than this. Moreover here we have the two sides of the solenoid creating opposite fields at any point outside. So we expect a decreasing magnetic field outside if it exists. The only constant magnetic field that is consistent with these arguments is $B_{out} = 0$ everywhere outside.

So from Eq I we have

$B_{in} = \mu_0 K = \mu_0 n I$, independent of the distance from the centre.

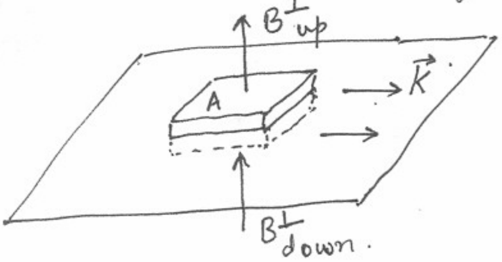
So a solenoid produces a uniform magnetic field inside along the axis of the solenoid.

$$\vec{B}_{in} = \mu_0 K \hat{z} = \mu_0 n I \hat{z}$$

$$\vec{B}_{out} = 0$$

Boundary Value Problems :

When magnetic fields are calculated in ~~chargefree~~ regions free of current density we have to suitably match the fields at the boundary surfaces that may have currents flowing over them. We have conditions on the components of magnetic fields, perpendicular, and parallel to the boundary surface



~~The~~ Consider a pill box parallel to the surface and lying on either side of the surface. Since $\vec{\nabla}\cdot\vec{B} = 0$ we have

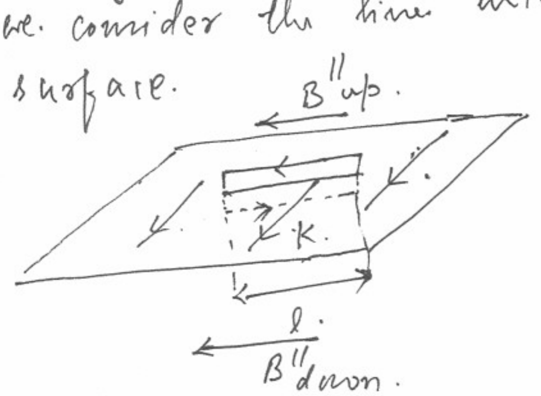
$$\oint_{\text{pill box}} \vec{B}\cdot\hat{n}\,da = 0$$

$$\therefore \int_A \left(B^{\perp}_{up} - B^{\perp}_{down}\right) da = 0$$

(the side walls of the box is infinitesimally small.)

$$\therefore B^{\perp}_{up} = B^{\perp}_{down} \qquad \text{——— I.}$$

To get conditions on the tangential (parallel) components we consider the line integral over a loop close to the surface.



~~Since $\vec{\nabla}\times\vec{B} = \mu_0 K$~~

$$\oint \vec{B}\cdot d\vec{l} = \left(B^{\parallel}_{up} - B^{\parallel}_{down}\right) l = \mu_0 K l.$$

$$\therefore B^{\parallel}_{up} - B^{\parallel}_{down} = \mu_0 K$$

The parallel components are perpendicular to the local direction of $\vec{K}$.

The two boundary conditions can be combined into a single vector eqn.

$$\vec{B}_{up} - \vec{B}_{down} = \mu_0 (\vec{K} \times \hat{n})$$

where $\hat{n}$ is normal to the surface on the 'up' side.