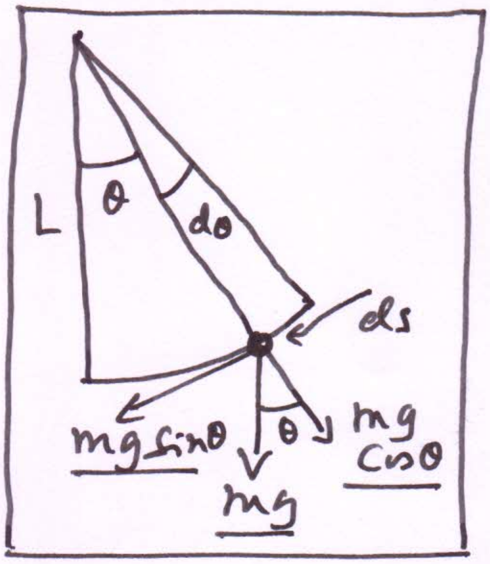
Physical Systems of the Second-Order

The Pendulum (Oscillator)



$\boxed{ds = L\,d\theta} \Rightarrow \frac{ds}{dt} = L\frac{d\theta}{dt}$

Now, $\boxed{v = \frac{ds}{dt} = L\frac{d\theta}{dt} = L\dot{\theta}}$

$\boxed{\text{Force} = m\frac{dv}{dt} = mL\frac{d^2\theta}{dt^2} = mL\ddot{\theta}}$

Hence $\boxed{mL\ddot{\theta} = -mg\sin\theta} \rightarrow$ The nonlinear pendulum equation

$\Rightarrow \boxed{\ddot{\theta} = -\frac{g}{L}\sin\theta}$ Now, $\boxed{\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots}$

For small oscillations, $\boxed{\sin\theta \simeq \theta}$ (linear order)

$\Rightarrow \boxed{\ddot{\theta} = -\omega^2\theta}$ where $\boxed{\omega^2 = g/L}$.

To integrate by Euler's method, we define.

$\boxed{\frac{d\theta}{dt} = \Omega}$ and $\boxed{\frac{d\Omega}{dt} = -\frac{g}{L}\theta}$ or $\boxed{\frac{d\Omega}{dt} = -\omega^2\theta}$

$\boxed{\theta_{i+1} - \theta_i = \Omega\Delta t} \Rightarrow \boxed{\theta_{i+1} = \theta_i + \Omega\Delta t}$.

And $\boxed{\Omega_{i+1} - \Omega_i = -\frac{g}{L}\theta_i\Delta t} \Rightarrow \boxed{\Omega_{i+1} = \Omega_i - \frac{g}{L}\theta_i\Delta t}$

Numerically integrate two first-order equations with initial values of θ_0 and Ω_0.

The Damped Oscillator

If the oscillator undergoes damping and its amplitude is restored by a driving force, we write the full equation as

$$\boxed{mL\ddot{\theta} = -mg\sin\theta + (F - D\dot{\theta})}$$ $F \to$ Driving Force

and $D\dot{\theta} \to$ Damping force (in the first order).

Define (rescale) $\boxed{t = \tau T}$ (T is the time scale)

$$\Rightarrow \boxed{\frac{mL}{T^2}\frac{d^2\theta}{d\tau^2} + \frac{D}{T}\frac{d\theta}{d\tau} + mg\sin\theta = F}$$

($\tau \to$ dimensionless time)

$$\Rightarrow \frac{L}{gT^2}\frac{d^2\theta}{d\tau^2} + \frac{D}{mgT}\frac{d\theta}{d\tau} + \sin\theta = \frac{F}{mg}$$

Define $\boxed{t_{os}^2 = L/g}$, $\boxed{t_d = \frac{D}{mg}}$ and $\boxed{f = \frac{F}{mg}}$

$$\Rightarrow \boxed{\frac{t_{os}^2}{T^2}\frac{d^2\theta}{d\tau^2} + \frac{t_d}{T}\frac{d\theta}{d\tau} = f - \sin\theta}.$$

If the damping time, $t_d \gg t_{os}$ (oscillatory time scale), and it dominates, then $\boxed{T = t_d}$. Hence, $\boxed{t_{os}^2/t_d^2 \ll 1}$. The second-order term can be neglected, $\Rightarrow \boxed{\frac{d\theta}{d\tau} \simeq f - \sin\theta}$ (no longer oscillatory). Damping time is large when i.) D is large, and ii.) m is small in $\boxed{t_d = \frac{D}{mg}}$

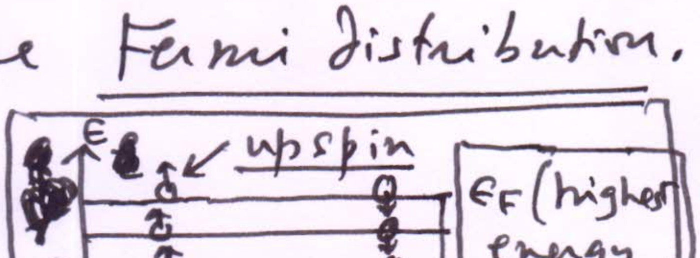
Basic Concepts of Superconductivity

1/. Absence of d.c. electrical resistivity below a very low critical temperature.

2/. In a normal metallic conductor (such as copper), conduction happens due to the movement of free electrons in an electron gas. The free electrons are given up by the ions in the metallic lattice.

3/. Electrons obey Pauli's exclusion principle, and the Fermi distribution.

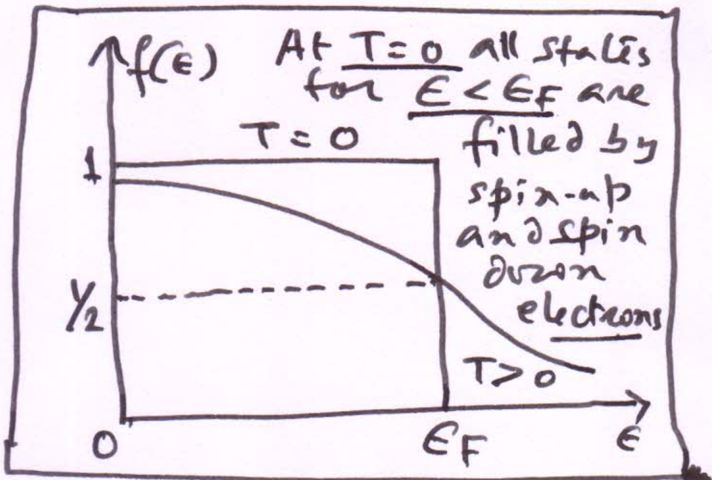
An energy level ~~can~~ can be occupied by only two electrons, one spin up and the other spin down. Each electron has a half-integral spin, $\boxed{+1/2 \text{ or } -1/2}$.



4/. At 0K (absolute zero), the highest-occupied energy level is ϵ_F → the Fermi energy level. All levels below it are fully occupied, and all levels above it are vacant.

5/. The occupation probability is given by the Fermi function

$$\boxed{f(\epsilon) = \frac{1}{1 + e^{(\epsilon - \epsilon_F)/k_B T}}}$$



When $T = 0$, for $\epsilon < \epsilon_F$

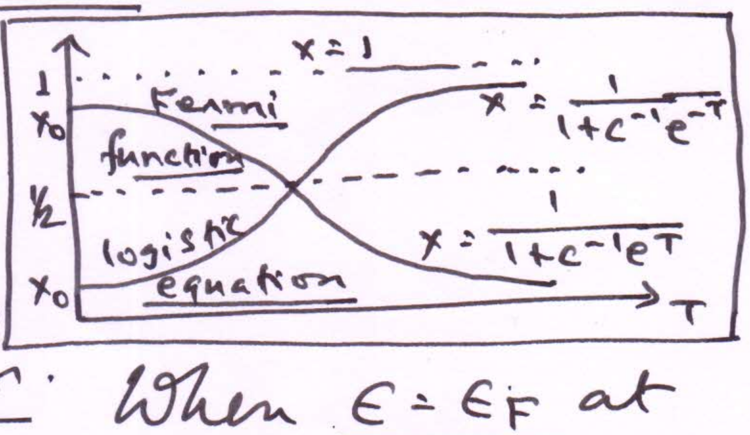
$$\boxed{f(\epsilon) = \frac{1}{1+e^{-\infty}} = 1}$$, i.e

all states are occupied. But for $\epsilon > \epsilon_F$

$$\boxed{f(\epsilon) = \frac{1}{1+e^{\infty}} = 0}$$, i.e all states are vacant.

This distribution generally follows an equation $\boxed{\frac{dx}{dT} = -x(1-x)}$. Compared with the logistic equation, $\boxed{\frac{dx}{dT} = x(1-x)}$, whose solution is $\boxed{x = \frac{1}{1+c^{-1}e^{-T}}}$, the negative sign on the right hand side makes $T \longrightarrow -T$.

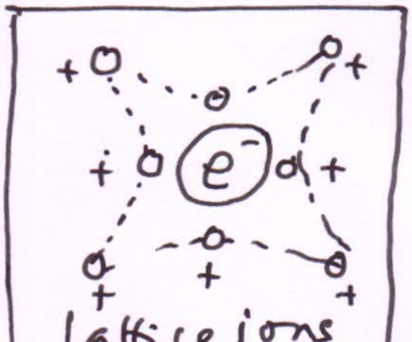
$\therefore \boxed{x = \frac{1}{1+c^{-1}e^{T}}}$. When $T \to \infty$, $x \longrightarrow 0$. This is the inversion of the logistic equation.



When $\epsilon = \epsilon_F$ at $T = 0$, $\boxed{f(\epsilon) = \frac{1}{1+e^{0/0}}}$. This is indeterminate and is given by the vertical line at $\epsilon = \epsilon_F$.

6/. In a superconductor the current carriers are a pair of spin-up and spin-down electrons — Cooper pair.

7/. Cooper pairs are formed due to an electron-lattice interaction.



An electron deforms the lattice of positively-charged ions. A second electron interacts with the deformed lattice, and gets into a lower energy state, resulting in an attractive interaction with the first electron. The two electrons are attracted to each other via the lattice to form a bound state, with an energy that is less than what it is in the normal state. This superconducting state is due to the mediation of the lattice. (Bardeen, Cooper and Schrieffer theory). The scale of binding energy is $\boxed{\sim 10^{-3}\,eV}$. $\Rightarrow$ $\boxed{T = \frac{10^{-3}\times 1.6\times 10^{-19}}{1.38\times 10^{-23}} \sim 10K}$

scale of superconductivity temperature.

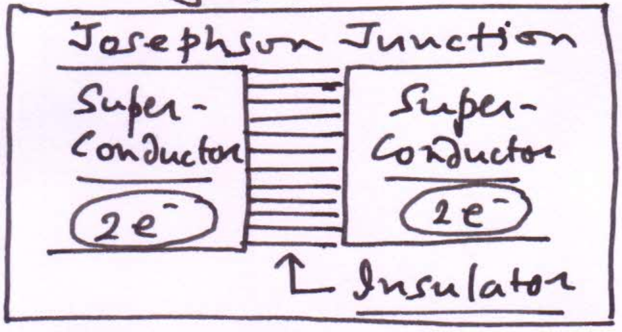
8/. Cooper pairs are super electrons with a charge of $2e^-$ each. Since the pairs are formed by two electrons of opposite spin, Cooper pairs are in a spin zero state. They are in a bosonic state (does not obey Pauli's exclusion principle). At the critical temperature for superconductivity, the pair breaks into two fermions (electrons)

9/. At low temperatures the lattice vibrations reduce and in a normal conductor (metal), the electrons ~~[illegible]~~ suffer less collisions with the lattice ions. Hence conductivity increases. However, the lattice is not easily distorted in good conductors. In poor conductors the ease of lattice distortion is greater at low ~~[illegible]~~ temperatures. Hence, poor conductors are ~~[illegible]~~ more likely to be superconductors. Good conductors, on the other hand, do not exhibit superconductivity.

 (Brian Josephson)

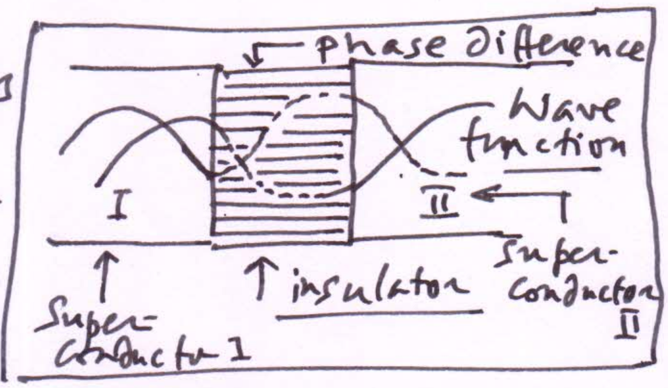
Josephson Effect (D.C. and A.C.)

1/. It is a quantum tunnelling phenomenon, whereby the Cooper pairs can pass ~~[illegible]~~ from one superconductor to another through the insulating barrier. This happens because the Cooper pairs (made of two electrons) are like waves, in keeping with the wave-particle duality $\boxed{\lambda = \frac{h}{mv}}$.



Waves can easily pass through a barrier, regardless of their energy. Particles, on the other hand, must have more energy than the barrier to pass through. Due to quantum tunnelling, it is possible for a current to pass through the insulator.

The insulator introduces a phase difference between the two waves. The phase difference ϕ is ϕ_0.



P.T.O.

2/. The superconducting current is $\boxed{I = I_c \sin\phi_0}$, where I_c is a constant depending on the width of the insulator and its material. This current does not require a voltage and is due to the D.C. Josephson Effect.

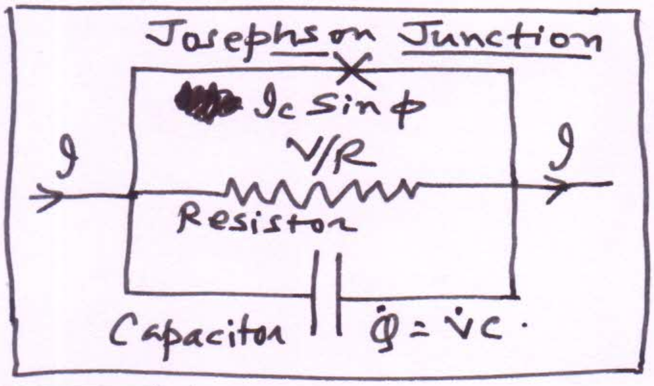
3/. If a voltage V is applied across the insulator, the equivalent energy (work done) is $\boxed{2eV}$. The frequency is found from $\boxed{\frac{2eV}{h}}$ ($\because \boxed{E = hf} \rightarrow$ Planck's law).

Angular frequency, $\boxed{\omega = 2\pi f = \frac{2\pi 2eV}{h}}$

Writing $\boxed{\hbar = h/2\pi}$, $\boxed{\omega = \frac{2eV}{\hbar}} \Rightarrow \boxed{V = \frac{\hbar\omega}{2e}}$.

Now phase $\boxed{\phi = \omega t}$. Hence the modified current will be due to $\boxed{V = \frac{\hbar\dot{\phi}}{2e}}$ $\boxed{\dot{\phi} = \omega}$, given as $\boxed{I = I_c \sin\left(\phi_0 + \frac{2eV}{\hbar}t\right)}$. This is an alternating current, due to a static potential V — A.C. Josephson Effect.

4/. In a <u>parallel circuit</u> with a <u>Josephson junction</u>, a <u>capacitor</u> and a <u>resistor</u>,



$$\boxed{I = I_c \sin\phi + \frac{V}{R} + \dot{V}C} \longrightarrow \text{Total Current.}$$

Now $\boxed{V = \frac{\hbar\dot{\phi}}{2e}}$ and $\boxed{\dot{V} = \frac{\hbar\ddot{\phi}}{2e}}$. In terms of ϕ,

$$\boxed{\frac{\hbar C}{2e}\ddot{\phi} + \frac{\hbar}{2eR}\dot{\phi} + I_c \sin\phi = I}$$

. Hence, we can write after rescaling $\boxed{t = \tau T}$ and dividing by I_c

$$\boxed{\frac{1}{T^2}\left(\frac{\hbar C}{2eI_c}\right)\frac{d^2\phi}{d\tau^2} + \left(\frac{\hbar}{2eRI_c}\right)\frac{1}{T}\frac{d\phi}{d\tau} = \frac{I}{I_c} - \sin\phi}.$$

[Compare with $\boxed{\frac{1}{T^2}\left(\frac{L}{g}\right)\frac{d^2\theta}{d\tau^2} + \left(\frac{D}{mg}\right)\frac{1}{T}\frac{d\theta}{d\tau} = f - \sin\theta}$].

The <u>circuit</u> is a <u>superconducting analogue of a mechanical damped-driven oscillator</u>.

<u>Comparison</u>: $\boxed{mg \longrightarrow I_c}$ $\boxed{\frac{\hbar C}{2eI_c} \longrightarrow \frac{L}{g}}$.

$\boxed{\frac{\hbar}{2eI_cR} \longrightarrow \frac{D}{mg}}$ $\Rightarrow$ $\boxed{\frac{1}{R} \longrightarrow D}$, $\boxed{m \longrightarrow I_c}$

and $\boxed{C \longrightarrow mL}$. <u>Electrical and mechanical equivalence</u>

The Linear Oscillator $(m, k > 0)$

Undamped: $\boxed{m\frac{d^2x}{dt^2} = -kx}$, in which the restoring force is given by Hooke's law

$$\Rightarrow \boxed{\frac{d^2x}{dt^2} = \ddot{x} = \frac{-k}{m}x} \Rightarrow \boxed{\ddot{x} = -\omega^2 x} \quad \boxed{\omega^2 = k/m}$$

Decomposing into a coupled system of two first-order equations,

$\boxed{\frac{dx}{dt} = v}$ and $\boxed{\frac{dv}{dt} = -\omega^2 x}$ in the general form $\boxed{\dot{x} = f(x,y)}$ and $\boxed{\dot{y} = g(x,y)}$.

Now $\frac{dv/dt}{dx/dt} = \frac{dv}{dx} = -\frac{\omega^2 x}{v}$.

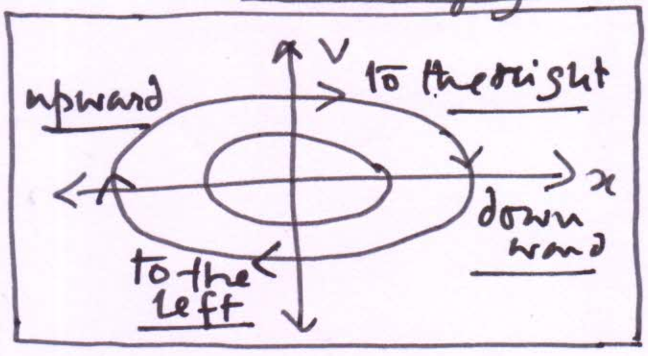
$$\Rightarrow \boxed{v\,dv + \omega^2 x\,dx = 0} \Rightarrow \int v\,dv + \omega^2\int x\,dx = \text{constant.}$$

$$\therefore \boxed{\frac{v^2}{2} + \frac{\omega^2 x^2}{2} = C} \Rightarrow \boxed{\frac{mv^2}{2} + \frac{kx^2}{2} = \mathcal{E}}$$

Equation of a ellipse — Conservation of Energy ↑

$$\Rightarrow \boxed{\frac{v^2}{2C} + \frac{x^2}{2C/\omega^2} = 1}$$

A family of ellipses for different values of C.



The initial condition sets the value of c.

Now $\boxed{(\dot{x}, \dot{v}) = (v, -\omega^2 x)}$. When $v = 0$,

$\boxed{(\dot{x}, \dot{v}) = (0, -\omega^2 x)}$ ∴ When $x > 0$, $\dot{v} < 0$ and v is directed downwards. When $x < 0$, $\dot{v} > 0$ and v is directed upwards.

When $x = 0$, $\boxed{(\dot{x}, \dot{v}) = (v, 0)}$. ∴ When $v > 0$, $\dot{x} > 0$ and x is directed to the right. When $v < 0$, $\dot{x} < 0$, and x is directed to the left. (All are shown in the figure)

With Damping: $\boxed{m\dfrac{d^2x}{dt^2} = -kx - D\dfrac{dx}{dt}}$ $\boxed{\begin{matrix} m>0 \\ k>0 \\ D>0 \end{matrix}}$

For small oscillations and velocities, damping is taken as proportional to the velocity. $D \to$ Damping coefficient.

$\Rightarrow \boxed{\ddot{x} + 2b\dot{x} + \omega^2 x = 0}$, $\boxed{2b = \dfrac{D}{m}}$, $\boxed{\omega^2 = \dfrac{k}{m}}$.

Since there are both first and second order terms in the damped equation, it is convenient to apply a trial solution going as $\boxed{x \sim e^{\lambda t}}$. Periodic solutions are captured by complex λ.

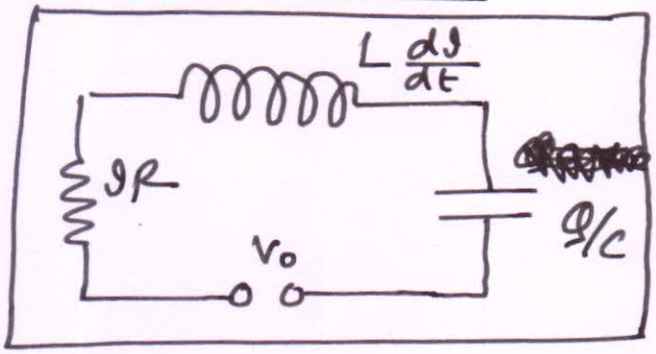
P.T.O.

This follows the Euler formula,

$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$ relating the exponential function to periodic behaviour.
(periodic ⇒ sinusoidal)

Example: The L-R-C Circuit

$$\boxed{V_0 = IR + \frac{Q}{C} + L\frac{dI}{dt}}$$



Since V_0 is constant taking a derivative gives $\boxed{L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{I}{C} = 0}$ $\left(\because I = \frac{dQ}{dt}\right)$

$\Rightarrow \boxed{\frac{d^2I}{dt^2} + \frac{R}{L}\frac{dI}{dt} + \frac{I}{LC} = 0}$. Establishing an equivalence with $\boxed{\ddot{x} + 2b\dot{x} + \omega^2 = 0}$, we see $\boxed{\omega^2 = \frac{1}{LR}}$ and $\boxed{2b = \frac{R}{L}}$ → damping is related to the resistance R. The frequency of oscillations is $\boxed{f = \frac{1}{2\pi}\cdot\frac{1}{\sqrt{LR}}}$.

An electrical analogue of an oscillator, given by a second-order differential equation.

With the solution $\boxed{x \sim e^{\lambda t}}$, we get $\boxed{\dot{x} = \lambda x}$ and $\boxed{\ddot{x} = \lambda^2 x}$. From the damped oscillator equation $\boxed{\ddot{x} + 2b\dot{x} + \omega^2 x = 0}$ we get

(P.T.O.)

the eigenvalue equation, ($b, \omega^2 > 0$)

$$\boxed{\left(\frac{d^2}{dt^2} + 2b\frac{d}{dt} + \omega^2\right)x = \left(\lambda^2 + 2b\lambda + \omega^2\right)x = 0}.$$

This requires solving for the eigenvalues, in $\boxed{\lambda^2 + 2b\lambda + \omega^2 = 0}$. (a quadratic equation).

$$\Rightarrow \boxed{\lambda_{1,2} = -\frac{2b \pm \sqrt{4b^2 - 4\omega^2}}{2} = -b \pm \sqrt{b^2 - \omega^2}}$$

Case 1: Overdamped — $\boxed{b^2 > \omega^2}$

In this case both eigenvalues, $\boxed{\lambda_1, \lambda_2 < 0}$, and they are real numbers.

Each will give a solution $\boxed{x_1 = Ae^{\lambda_1 t}}$ and $\boxed{x_2 = Be^{\lambda_2 t}}$. where A and B are constants (amplitude). Since both $\lambda_1, \lambda_2 < 0$, there will be a decay.

The general solution is a superposition of the two solutions x_1 and x_2. $\Rightarrow$ $\boxed{x = Ae^{\lambda_1 t} + Be^{\lambda_2 t}}$. The decay will be slowed by the eigenvalue of smaller absolute magnitude.

Case II: Underdamped — $\boxed{b^2 < \omega^2}$.

In this case λ_1, λ_2 will be complex, given by $\boxed{\lambda_{1,2} = -b \pm i\sqrt{\omega^2 - b^2}}$. The solutions will be $\boxed{x_1 = \tilde{A}\, e^{\lambda_1 t}}$ and $\boxed{x_2 = \tilde{B}\, e^{\lambda_2 t}}$, where $\tilde{A}$ and $\tilde{B}$ are in general complex constants. The linear superposition of ~~the~~ x_1 and x_2 gives the general solution,

$$\boxed{x = \tilde{A}\, e^{\lambda_1 t} + \tilde{B}\, e^{\lambda_2 t}}. \quad \boxed{\begin{aligned}\lambda_1 &= -b + i\sqrt{\omega^2 - b^2} \\ \lambda_2 &= -b - i\sqrt{\omega^2 - b^2}\end{aligned}}$$

$$\Rightarrow \boxed{x = e^{-bt}\left[\tilde{A}\, e^{i\sqrt{\omega^2 - b^2}\, t} + \tilde{B}\, e^{-i\sqrt{\omega^2 - b^2}\, t}\right]}$$

Now we use the Euler formula $\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$ and extract only the real part.

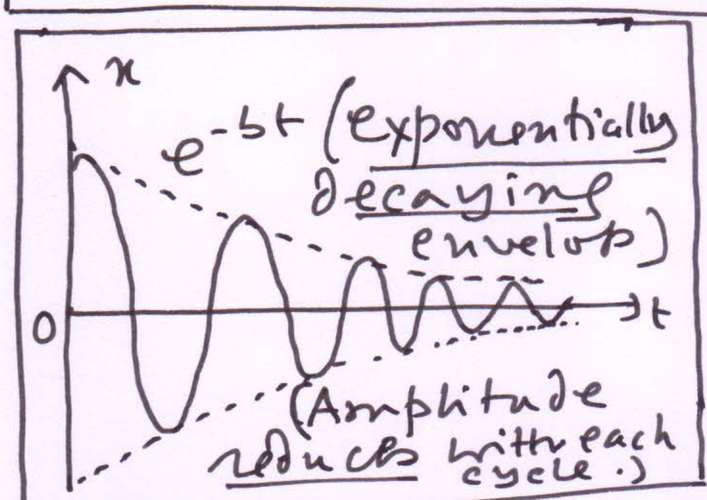
Writing $\boxed{\tilde{A} = \alpha_1 + i\alpha_2}$ and $\boxed{\tilde{B} = \beta_1 + i\beta_2}$, ($\alpha_1, \alpha_2, \beta_1, \beta_2$ are real)

we get

$$x = e^{-bt}\Big[(\alpha_1 + i\alpha_2)\cos\left(\sqrt{\omega^2 - b^2}\, t\right) + i(\alpha_1 + i\alpha_2) \times \sin\left(\sqrt{\omega^2 - b^2}\right)t + (\beta_1 + i\beta_2)\cos\left(\sqrt{\omega^2 - b^2}\, t\right) - i(\beta_1 + i\beta_2)\sin\left(\sqrt{\omega^2 - b^2}\, t\right)\Big].$$

Retaining only the physically meaningful real part, we get

$$\boxed{x = e^{-bt}\left[(\alpha_1+\beta_1)\cos\left(\sqrt{\omega^2-b^2}\,t\right) - (\alpha_2-\beta_2)\times \sin\left(\sqrt{\omega^2-b^2}\,t\right)\right]}$$



Writing $\boxed{C = \alpha_1+\beta_1}$ and $\boxed{D = -(\alpha_2-\beta_2)}$, we get a completely real solution;

$$\boxed{x = e^{-bt}\left[C\cos\left(\sqrt{\omega^2-b^2}\,t\right) + D\sin\left(\sqrt{\omega^2-b^2}\,t\right)\right]}$$

The solution will be oscillatory with a decaying amplitude $\boxed{e^{-bt}}$ (reduces exponentially)

Special Case: $\boxed{b=0} \rightarrow$ No damping.

$\Rightarrow \boxed{x = C\cos(\omega t) + D\sin(\omega t)}$ Undamped oscillator

Case III: Critically damped - $\boxed{b^2=\omega^2}$

$\because \boxed{\lambda_{1,2} = -b \pm \sqrt{b^2-\omega^2}}$ when $b^2=\omega^2$, $\boxed{\lambda_1 = \lambda_2 = -b}$

Both eigenvalues are equal and negative.

One solution is $\boxed{x_1 = A_1 e^{\lambda_1 t}}$. The other

(P.T.O.)

Solution cannot just be $\boxed{x_2 = A_2 e^{\lambda_2 t}}$, because with $\lambda_1 = \lambda_2$, the general solution will be reduced to $\boxed{x_1 + x_2 = (A_1 + A_2) e^{\lambda_1 t}}$.

$\therefore$ We write $\boxed{x_2 = u(t)\, e^{\lambda_2 t}}$ in a more general form.

$$\Rightarrow \frac{dx_2}{dt} = \frac{du}{dt} e^{\lambda_2 t} + \lambda_2 u e^{\lambda_2 t} = e^{\lambda_2 t}\left(\frac{du}{dt} + \lambda_2 u\right)$$

$$\Rightarrow \frac{d^2 x_2}{dt^2} = \lambda_2 e^{\lambda_2 t}\left(\frac{du}{dt} + \lambda_2 u\right) + e^{\lambda_2 t}\left(\frac{d^2 u}{dt^2} + \lambda_2 \frac{du}{dt}\right)$$

Using these derivatives in $\boxed{\ddot{x}_2 + 2b\dot{x}_2 + \omega^2 x_2 = 0}$,

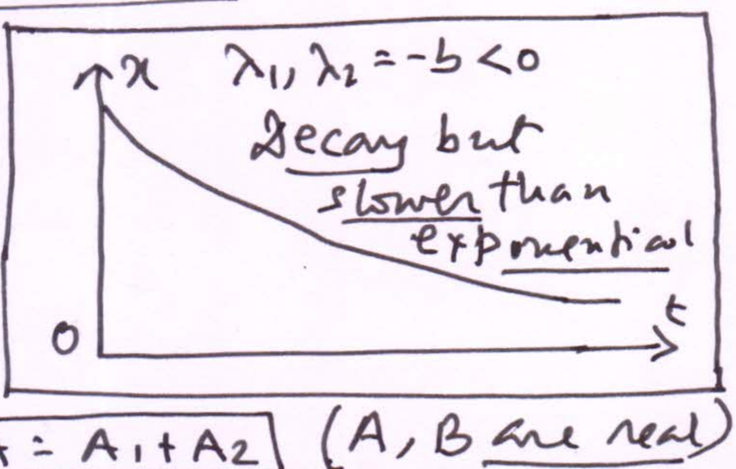
$$\left(\frac{du}{dt} + \lambda_2 u\right)\lambda_2 \cancel{e^{\lambda_2 t}} + \cancel{e^{\lambda_2 t}}\left(\frac{d^2u}{dt^2} + \lambda_2 \frac{du}{dt}\right)$$
$$+ 2b\cancel{e^{\lambda_2 t}}\left(\frac{du}{dt} + \lambda_2 u\right) + \omega^2 u \cancel{e^{\lambda_2 t}} = 0$$

$$\Rightarrow \frac{d^2u}{dt^2} + \frac{du}{dt}\left(\underbrace{\lambda_2 + \lambda_2 + 2b}_{\to 0}\right) + \left(\underbrace{\lambda_2^2 + \omega^2 + 2b\lambda_2}_{0 \leftarrow}\right)u = 0$$

Since $\boxed{\lambda_2^2 + 2b\lambda_2 + \omega^2 = 0}$ and $\boxed{\lambda_2 = -b}$,

we get finally $\boxed{\frac{d^2u}{dt^2} = 0}$. Solving this equation we get $\boxed{u(t) = A_2 + Bt}$

The general solution is $x = x_1 + x_2 = (A_1 + A_2 + Bt) \times e^{-bt}$



$$\Rightarrow \boxed{x = (A + Bt)e^{-bt}} \longrightarrow \boxed{A = A_1 + A_2} \quad (A, B \text{ are real})$$

Resonance (Large divergence of amplitude)

With a periodic driving force $\boxed{F_0 \cos(\Omega t)}$

We write $\boxed{m\frac{d^2x}{dt^2} + kx = F_0 \cos(\Omega t)}$.

$\Rightarrow \boxed{\frac{d^2x}{dt^2} + \omega^2 x = f_0 \cos(\Omega t)}$, $\omega^2 = k/m$, $f_0 = F_0/m$.

When $f_0 = 0$, the homogeneous equation $\boxed{\ddot{x} + \omega^2 x = 0}$

gives the solution $\boxed{x_h = A\cos(\omega t) + B\sin(\omega t)}$.

If the driving force is $\boxed{f_0 \cos(\Omega t)}$, the particular solution is $\boxed{x_p = C_1 \cos(\Omega t) + C_2 \sin(\Omega t)}$

$$\boxed{\begin{aligned} \frac{dx_p}{dt} &= -C_1 \Omega \sin(\Omega t) + C_2 \Omega \cos(\Omega t) \\ \frac{d^2x_p}{dt^2} &= -C_1 \Omega^2 \cos(\Omega t) - C_2 \Omega^2 \sin(\Omega t) \end{aligned}}$$

Using the foregoing derivatives in $\boxed{\ddot{x} + \omega^2 x = f_0 \cos(\Omega t)}$, we get,

$$\boxed{\begin{aligned} -\Omega^2 C_1 \cos(\Omega t) - C_2 \Omega^2 \sin(\Omega t) + \omega^2 C_1 \cos(\Omega t) \\ + \omega^2 C_2 \sin(\Omega t) = f_0 \cos(\Omega t) \end{aligned}}$$

Comparing the $\sin(\Omega t)$ and $\cos(\Omega t)$ on both sides, we get $\boxed{-\Omega^2 C_2 + \omega^2 C_2 = 0}$

and $\boxed{-\Omega^2 C_1 + \omega^2 C_1 = f_0} \Rightarrow \boxed{C_2 = 0}$, $\boxed{C_1 = \frac{f_0}{\omega^2 - \Omega^2}}$

The general solution is $\boxed{x = x_h + x_p}$. Hence,

$$\boxed{x = A\cos(\omega t) + B\sin(\omega t) + \frac{f_0}{\omega^2 - \Omega^2}\cos(\Omega t)}$$

from $\boxed{x = A\cos(\omega t) + B\sin(\omega t) + \frac{f_0}{\omega^2 - \Omega^2}\cos(\Omega t)}$

clearly when $\boxed{\omega \to \Omega}$, the <u>amplitude of the last term will be large.</u> <u>(Resonance)</u>

i.) $\boxed{\omega = \sqrt{\frac{k}{m}}}$ is the <u>natural frequency</u> of the <u>oscillator</u> <u>(physical property)</u>.

ii.) <u>Ω is the frequency</u> of the <u>periodic Driving force</u> <u>(depends on external factors)</u>.

iii) The <u>oscillator</u> vibrates <u>strongly when</u> $\boxed{\omega = \Omega}$. This is <u>resonance</u>.

Numerical Integration by Euler's Method

The <u>damped oscillator</u> is given by $\boxed{\ddot{x} + 2b\dot{x} + \omega^2 x = 0}$ <u>(linear equation)</u>

Write $\boxed{\frac{dx}{dt} = v}$ and $\boxed{\frac{dv}{dt} = -2bv - \omega^2 x}$.

$\therefore \boxed{x_{i+1} - x_i = v_i \, \Delta t} \Rightarrow \boxed{x_{i+1} = x_i + v_i \Delta t}$

$\boxed{v_{i+1} - v_i = \left(-2bv_i - \omega^2 x_i\right)\Delta t} \Rightarrow \boxed{v_{i+1} = v_i - \left(2bv_i + \omega^2 x_i\right)\Delta t}$

i.) The <u>choice of Δt</u> is guided by the <u>natural time period</u> $\boxed{T = 2\pi/\omega}$ $\boxed{\Delta t \ll T}$.

ii.) <u>Not considering higher-order terms</u> will <u>introduce numerical errors</u>, even in the <u>conservative case</u> of <u>$b = 0$</u>.