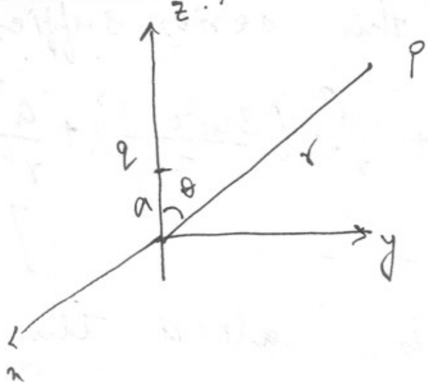
Multipole Expansion :



A point charge q is placed a distance a from the origin on the z-axis. We want to calculate the potential at $P(r,\theta)$ due to this charge q. This is easy. But we will treat it as an azimuthal symmetric problem. It is easy to find the potential on the z-axis. This is

$$V(r,0) = \frac{q}{4\pi\epsilon_0} \cdot \frac{1}{(r-a)} = \frac{q}{4\pi\epsilon_0 r} \frac{1}{\left(1-\frac{a}{r}\right)}$$

Here we will be interested in far away region i.e. $r >> a$. In this limit we will expand the above expression as a power series in $\left(\frac{a}{r}\right)$.

$$\therefore V(r,0) = \frac{q}{4\pi\epsilon_0 r} \sum_{l=0}^{\infty} \left(\frac{a}{r}\right)^l = \sum_{l=0}^{\infty} \frac{q}{r^{l+1}} \frac{a^l}{4\pi\epsilon_0}$$

The above expression is in the form

$$\sum_{l=0}^{\infty} A_l r^l + \frac{B_l}{r^{l+1}}$$

with $A_l = 0$ an $B_l = \frac{q a^l}{4\pi\epsilon_0}$.

From this it is easy to see that at P

$$V(r,\theta) = \sum_{l=0}^{\infty} \frac{q}{4\pi\epsilon_0} \frac{a^l}{r^{l+1}} P_l(\cos\theta)$$

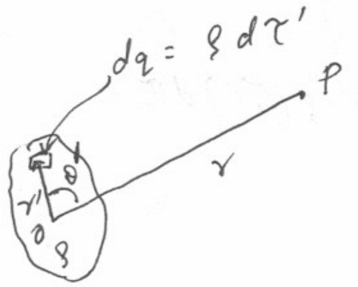
Since we are interested in region $r >> a$, generally only the first few term of this series suffies.

$$V(r,\theta) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{r} + \frac{a\cos\theta}{r^2} + \frac{a^2}{r^3}\left(\frac{3\cos^2\theta - 1}{2}\right) + \frac{a^3}{r^4}\left(\frac{5\cos^3\theta - 3\cos\theta}{2}\right) + \cdots\right]$$

The first term $\frac{q}{4\pi\epsilon_0 r}$ is called the monopole term. It is the most dominant term far away and this corresponds to the potential of a point charge q as if it is situated at the origin. However $a \neq 0$ and so we will have correction to this term.

The next term $\frac{qa\cos\theta}{4\pi\epsilon_0 r^2}$ is called the dipole term. This is equivalent to an electric dipole placed at the origin.

The other term are called quadrupole, ~~and~~ octupole and in general multipole terms. Each term corresponds to certain moments of charge distribution about the origin. The expression for the potential in this form is called the multipole expansion of the potential due to the charge distribution.



Now consider an arbitrary charge distribution $\rho(\vec{r}')$ near the origin of the co-ordinate system. We wish to find the potential due to this distribution at the point $P(\vec{r})$. This can be obtained by ~~the following~~ integrating the effect at P due to infinitesimal charges at positions $\vec{r}'$ that make an angle θ' with the position vector of P.

$$\therefore \quad V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_{l=0}^{\infty} \frac{1}{r^{(l+1)}} \int (r')^l P_l(\cos\theta') \rho(\vec{r}') d\tau'$$

Generally it is not easy to perform this integral since the θ' in the integrand is not measured from the z-axis but from the observation point. So it can be conveniently done only if we redefine the z-axis to be along $\vec{r}$, the position vector of P.

The monopole term i.e. the term $l=0$ is

$$V_{mon.} = \frac{1}{4\pi\epsilon_0} \frac{1}{r} \int \rho(\vec{r}') d\tau' = \frac{Q}{4\pi\epsilon_0 r}.$$

This is as if all the charge in the distribution is placed at the origin.

The dipole term is

$$V_{dip} = \frac{1}{4\pi\epsilon_0} \frac{1}{r^2} \int r' \cos\theta' \rho(\vec{r}') d\tau'$$

$$r' \cos\theta' = \vec{r}' \cdot \hat{r}$$

$$\therefore V_{dip} = \frac{1}{4\pi\epsilon_0} \frac{1}{r^2} \hat{r} \cdot \int \vec{r}' \rho(\vec{r}') d\tau'$$

The quantity

$$\vec{p} = \int \vec{r}' \rho(\vec{r}') d\tau'$$

is called the dipole moment of the charge distribution. So in term of $\vec{p}$

$$V_{dip} = \frac{1}{4\pi\epsilon_0} \frac{\vec{p} \cdot \hat{r}}{r^2}$$

Note that while the monopole potential varies as $\frac{1}{r}$, the dipole potential varies as $\frac{1}{r^2}$.

Now consider a charge distribution where a charge q is placed at $\vec{a}$ and $-q$ is placed at $-\vec{a}$. Since the total charge is 0; the monopole term of the potential at a point $P(\vec{r})$ is 0. So

$$V_{mon} = 0$$

The dipole moment of this distribution is

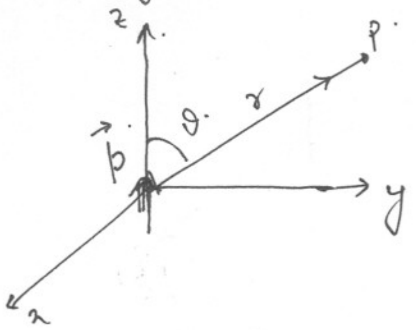
$$\vec{p} = q(\vec{a}) + (-q)(-\vec{a}) = 2q\vec{a}$$

The dipole term of the potential is

$$V_{dip} = \frac{1}{4\pi\epsilon_0} \frac{\vec{p} \cdot \hat{r}}{r^2} = \frac{1}{4\pi\epsilon_0} \cdot \frac{2q\vec{a} \cdot \hat{r}}{r^2}$$

Of course we will have the quadrupole and the other multipole term but in the absence of the monopole term, the dipole term is the most dominant far away i.e. for $r >> a$. An ideal dipole is made in the limit $a \to 0$, $q \to \infty$ such that $|\vec{p}| = 2qa$ is finite. For an ideal dipole, all the other multipole except the dipole potential is non-zero.

If an ideal dipole $\vec{p}$ is situated at the origin and along $\hat{z}$ axis then:



$$V_{dip}(r, \theta) = \frac{1}{4\pi\epsilon_0} \frac{\vec{p}\cdot\hat{r}}{r^2}$$

$$= \frac{1}{4\pi\epsilon_0} \frac{p\cos\theta}{r^2}$$

The electric field at the point P is

$$\vec{E}_{dip}(r, \theta) = -\vec{\nabla} V_{dip} = \frac{p}{4\pi\epsilon_0 r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)$$

In terms of only the dipole vector $\vec{p}$, and the position vector $\vec{r}$, this can be written as

$$\vec{E}_{dip}(\vec{r}) = \frac{1}{4\pi\epsilon_0 r^3}\left[3(\vec{p}\cdot\hat{r})\hat{r} - \vec{p}\right].$$