Separation of Variables.

This is one method to convert a partial differential Equation like the Laplace's ~~Poisson's~~ Equation into ~~an~~ ~~ordin~~ set of ordinary differential Equations. We have to study this method for the various co-ordinate systems though the spirit of the method in every system is the same.

Cartesian System:

In the Cartesian System the Laplace's Equation is

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0.$$

where $V(x, y, z)$ is the potential in a chargeless region.

We try a solution to this equation of the form

$$V(x, y, z) = X(x)\, Y(y)\, Z(z)$$

where $X(x)$ is a function of only x, $Y(y)$ is a function of only y and $Z(z)$ is a function of only z.

Substituting this form in the Laplace's Eq^n gives

$$YZ \frac{\partial^2 X}{\partial x^2} + XZ \frac{\partial^2 Y}{\partial y^2} + XY \frac{\partial^2 Z}{\partial z^2} = 0.$$

Multiplying this Eq^n by $\frac{1}{V}$ gives

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} + \frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = 0$$

~~Ron~~. There are three terms in the above Eq^n.

The term $\frac{1}{X}\frac{\partial^2 X}{\partial x^2}$ is purely a function of x.

Likewise the other two terms are purely functions of y and ~~z~~ respectively. So we have

②

$$f(x) + g(y) + h(z) = 0$$

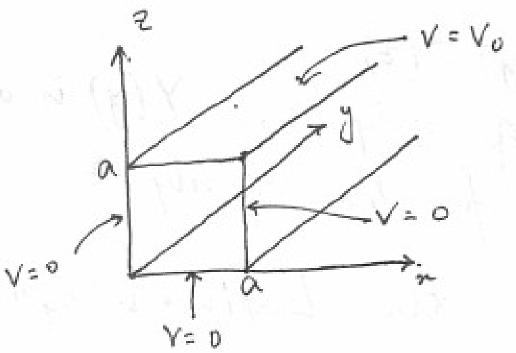
The only way we can satisfy this equation is when each of the functions $f(x)$, $g(y)$ and $h(z)$ is a constant i.e.

$$f(x) = C_1, \quad g(y) = C_2, \quad h(z) = C_3.$$

So $\frac{d^2X}{dx^2} = C_1 X$, $\frac{d^2Y}{dy^2} = C_2 Y$ and $\frac{d^2Z}{dz^2} = C_3 Z$

Each of above three differential Equans. are ordinary differential Equations and easy to solve.

The kind of values that C_1, C_2 and C_3 can take depend upon the type of boundary conditions in the problem. So now we consider an example



A metal tube with square cross-section has three sides grounded while the fourth surface at $z = a$ is maintained at a potential V_0. We have to find the potential at all the points inside the tube.

The potential will only vary with x and z. It is independent of y. After separation of variables in cartesian co-ordinates, the Laplace's. Eqn. becomes

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} = 0$$

Each term is a constant. Since the sum of these constants is 0, one of them is +ve while the other is -ve.

Let $\frac{1}{Z}\frac{d^2Z}{dz^2} = k^2$ and $\frac{1}{X}\frac{d^2X}{dx^2} = -k^2$

$\therefore$ $X = A \sin kx + B \cos kx$

$Z = C e^{kz} + D e^{-kz}$

At $x = 0$, $V = 0$ $\Rightarrow$ $B = 0$

At $z = 0$, $V = 0$ $\Rightarrow$ $C + D = 0$ $\Rightarrow$ $D = -C$

At $x = a$, $V = 0$

$\Rightarrow A \sin(ka) = 0 \Rightarrow k = \frac{n\pi}{a}$, $n = 1, 2, \ldots$

$$\therefore V = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{a}\right) \cdot C_n \left(e^{\frac{n\pi z}{a}} - e^{-\frac{n\pi z}{a}}\right)$$

$$= \sum_{n=1}^{\infty} K_n \sin\left(\frac{n\pi x}{a}\right) . 2 \sinh\left(\frac{n\pi z}{a}\right)$$ where $K_n = A_n C_n$.

At $z = a$ $V = V_0$

$$\therefore V_0 = \sum_{n=1}^{\infty} K_n \sin\left(\frac{n\pi x}{a}\right) \cdot 2 \sinh(n\pi) \quad \text{—— (1)}$$

To obtain K_n we use the following integral.

$$\int_0^a \sin\frac{n\pi x}{a} \sin\frac{n'\pi x}{a}\, dx = \frac{a}{2}\delta_{nn'}$$

$$\int_0^a \sin\frac{n\pi x}{a}\, dx = \frac{2a}{n\pi} \quad \text{for } n = 1, 3, 5, \ldots$$
$$= 0 \quad \text{for } n = 2, 4, 6 \ldots$$

Multiplying both sides of Eq (1) by $\sin\frac{n'\pi x}{a}$ and integrating form 0 to a we get

$$V_0 \cdot \frac{2a}{n'\pi} = K_{n'} \cdot a \cdot \sinh(n'\pi) \quad \text{for } n' = 1, 3, 5, \ldots$$

$$\therefore K_{n'} = \frac{2V_0}{n'\pi \sinh(n'\pi)}$$

$$K_{n'} = 0 \quad \text{for } n' = 2, 4, 6, \ldots$$

$$\therefore V(x, y, z) = \sum_{n=1,3,5\ldots} \frac{4V_0}{n\pi \sinh(n\pi)} \sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi z}{a}\right)$$

This is the appropriate potential for all points inside the tube satisfying the given boundary conditions.

Spherical Polar system :-

In the Spherical Polar co-ordinate system, the Laplace's Eqn. can be written as

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}=0$$

Generally in this system we encounter problems that have azimuthal symmetry i.e symmetry about the zenith axis. So the potential is independent of ϕ. So $V(r,\theta,\phi)$ can be written as $V(r,\theta)$. For variable separation technique we assume

$$V(r,\theta)=R(r)\,\Theta(\theta)$$

Substituting this into the Laplace's Eqn. we get

$$\frac{\Theta}{\cancel{r^2}}\frac{\partial}{\partial r}\left(r^2\frac{\partial R(r)}{\partial r}\right)+\frac{R}{\cancel{r^2}\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Theta}{\partial\theta}\right)=0$$

Dividing by $V(r,\theta)$ we get

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)=0$$

$$\therefore\quad f(r)+g(\theta)=0.$$

f is a function only of r and g is a function only of θ. So both must be constants. It proves convenient to take these constants as

$$f(r)=l(l+1)\qquad\text{and}\qquad g(\theta)=-l(l+1)$$

So we have the two separated ordinary differential Equations.

$$\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) = l(l+1) R. \qquad \text{and.}$$

$$\frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right) = -l(l+1)\sin\theta\, \Theta$$

The most general solution to the radial Eqn is

$$R(r) = A r^l + \frac{B}{r^{l+1}}$$

The acceptable solutions to the θ Eqn is

$$\Theta = C P_l(\cos\theta)$$

$P_l(\cos\theta)$ is a polynomial in $\cos\theta$ of degree. l. They are called Legendre polynomials.

$$P_0(\cos\theta) = 1$$

$$P_1(\cos\theta) = \cos\theta$$

$$P_2(\cos\theta) = \frac{1}{2}(3\cos^2\theta - 1)$$

$$P_3(\cos\theta) = \frac{1}{2}(5\cos^3\theta - 3\cos\theta) \qquad \text{etc.} \ldots$$

$P_l(\cos\theta)$ is even in $\cos\theta$ if l is even and odd in $\cos\theta$ if l is odd.

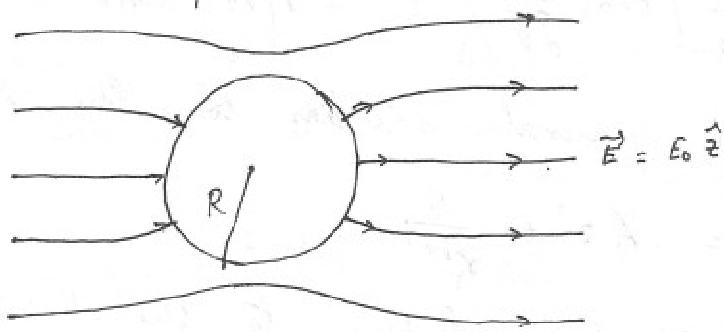
$$\therefore \quad V(r,\theta) = \left(A r^l + \frac{B}{r^{l+1}}\right) P_l(\cos\theta)$$

Note: The θ Eqn. has physically meaningful solutions only when l is an integer. So the most general solution to the Laplace's Eqn. with Azimuthal symmetry is

$$V(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right) P_l(\cos\theta).$$

(6)

Eg: A metal sphere of radius R is placed in ~~an~~ region having uniform electric field $\vec{E} = E_0 \hat{z}$. Find the potential at all points outside the sphere.



Far away from the sphere the ~~potential~~ electric field is uniform. $\vec{E} = E_0 \hat{z}$. The field distorts only near the sphere as shown. By symmetry along the z-axis it is convenient to keep the plane $z = 0$ at 0 potential. Hence the whole sphere will be at 0-potential.

Since the problem have azimuthal symmetry. we have.

$$V(r, \theta) = \sum_{l=0}^{\infty} \left(A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos\theta)$$

$$\vec{E} = -\vec{\nabla} V = -\left[\sum_{l=1}^{\infty} \left(l A_l r^{l-1} - (l+1) \frac{B_l}{r^{l+2}} \right) P_l(\cos\theta) - \frac{B_0}{r^2} \right] \hat{r}$$

$$+ \left[\sum_{l=1}^{\infty} \left(A_l r^{l-1} + \frac{B_l}{r^{l+2}} \right) \frac{d}{d\theta} \left(P_l(\cos\theta) \right) \right] \hat{\theta}$$

As $r \to \infty$ $\quad \vec{E} = E_0 \hat{z} = E_0 \cos\theta \, \hat{r} - E_0 \sin\theta \, \hat{\theta}$

Comparing the $\hat{r}$ component in this limit

$$-\sum_{l=1}^{\infty} l A_l r^{l-1} P_l(\cos\theta) = E_0 \cos\theta = E_0 P_1(\cos\theta) \quad \text{—— I}$$

The legendre's Polynomials $P_l(\cos\theta)$ are linearly independent of each other and they satisfy the following orthogonality conditions.

$$\int_0^{\pi} P_l(\cos\theta)\, P_{l'}(\cos\theta) \sin\theta \cdot d\theta = \frac{2}{2l+1}\delta_{ll'}$$

So multiplying Eqn. I. both sides by $P_{l'}(\cos\theta)$ and integrating from 0 to π we get.

$$-l' A_{l'} r^{(l'-1)} \cdot \frac{2}{2l'+1} = E_0 \cdot \frac{2}{2l'+1} \cdot \delta_{1l'}$$

$$\therefore -A_1 = E_0 \Rightarrow A_1 = -E_0$$

$$A_2 = A_3 \cdots = 0$$

The potential at $r = R$ is 0

$$\therefore \sum_{l=0}^{\infty}\left(A_l R^l + \frac{B_l}{R^{l+1}}\right) P_l(\cos\theta) = 0$$

Since $P_l(\cos\theta)$ are all linearly independent.

$$B_l = -A_l R^{2l+1} \quad ; \quad B_0 = -A_0 R$$

$$\therefore B_2 = B_3 = \cdots = 0$$

$$B_1 = -A_1 R^3 = E_0 R^3$$

$$\therefore V = A_0\left(1 - \frac{R}{r}\right) + \left(-E_0 r + \frac{E_0 R^3}{r^2}\right)\cos\theta$$

The electric field at $r = R$ is

$$\vec{E}(R) = -\frac{A_0}{R} + (E_0 + 2E_0)\cos\theta = \left(-\frac{A_0}{R} + 3E_0\cos\theta\right)\hat{r}$$

Since the sphere is chargeless.

$$\oint_{\text{sphere}} \vec{E}\cdot\hat{n}\, da = 0 \Rightarrow -\frac{A_0}{R} \times 4\pi R^2 = 0 \Rightarrow A_0 = 0.$$

$$\therefore V = -E_0\left(r - \frac{R^3}{r^2}\right)\cos\theta.$$