4.20 Narrowband Band-pass Noise

Band-pass filtering of signals arises in many situations, the basic arrangement being shown in Fig. 4.20.1. The filter has an equivalent noise bandwidth B_N (see Section 4.2) and a center frequency f_c. A narrowband system is one in which the center frequency is much greater than the bandwidth, which is the situation to be considered here.

The signal source is shown as a voltage generator of internal resistance R_S. System noise is referred to the input as a thermal noise source at a noise temperature T_s. The available power spectral density is, from Eq. (4.2.9),

$$G_a(f) = kT_s \tag{4.20.1}$$

For the ideal band-pass system shown, the spectral density is not altered by transmission through the filter, but the filter bandwidth determines the available noise power as kT_sB_N. So far, this is a result that has already been encountered in general. An alternative description of the output noise, however, turns out to be very useful, especially in connection with the modulation systems described in later chapters. The waveforms of input and output noise voltages are shown in Fig. 4.20.2.

The output waveform has the form of a modulated wave and can be expressed mathematically as

$$n(t) = A_n(t)\cos(\omega_c t + \phi_n(t)) \tag{4.20.2}$$

This represents the noise in terms of a randomly varying voltage envelope $A_n(t)$ and a random phase angle $\phi_n(t)$. These components are readily identified as part of the waveform, as shown in Fig. 4.20.2, but an equivalent although not so apparent expression can be obtained by trigonometric expansion of the output waveform as

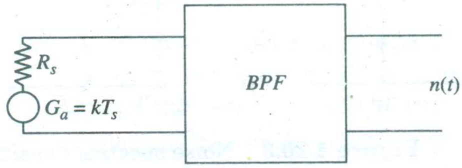

Figure 4.20.1 Noise in a band-pass system.

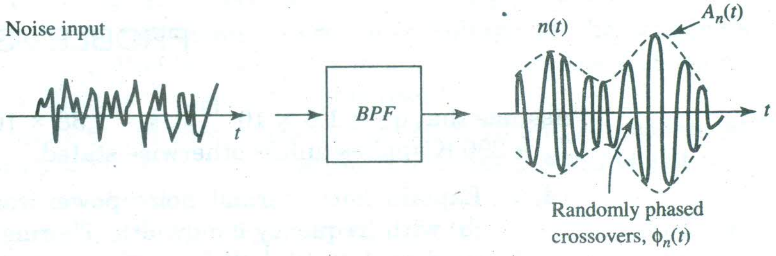

Figure 4.20.2 Input and output noise waveforms for a band-pass system.

$$n(t) = n_I(t) \cos \omega_c t - n_Q(t) \sin \omega_c t \tag{4.20.3}$$

Here, $n_I(t)$ is a random noise voltage termed the *in-phase component* because it multiplies a cosine term used as a reference phasor, and $n_Q(t)$ is a similar random voltage termed the *quadrature component* because it multiplies a sine term, which is therefore 90° out of phase, or in quadrature with, the reference phasor. The reason for using this form of equation is that, when dealing with modulated signals, the output noise voltage is determined by these two components (this is described in detail in later chapters on modulation). The two noise voltages $n_I(t)$ and $n_Q(t)$ appear to modulate a carrier at frequency f_c and are known as the *low-pass equivalent noise voltages.* The carrier f_c may be chosen anywhere within the passband, but the analysis is simplified by placing it at the center as shown. This is illustrated in Fig. 4.20.3

A number of important relationships exist between $n_I(t)$ and $n_Q(t)$ and $n(t)$, some of which will be stated here without proof. All three have similar noise characteristics and $n_I(t)$ and $n_Q(t)$ are uncorrelated. Of particular importance in later work on modulation is that where the power spectral density of $n(t)$ is $G_a(f) = kT_s$ the power spectral densities for $n_I(t)$ and $n_Q(t)$ are

$$G_I(f) = G_Q(f) = 2kT_s \tag{4.20.4}$$

This important result, which is illustrated in Fig. 4.20.3, will be encountered again in relation to modulated signals.

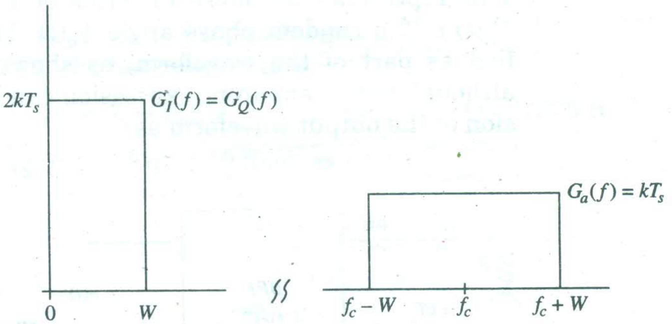

Figure 4.20.3 Noise spectral densities.

PROBLEMS

Assume that $q_e = 1.6 \times 10^{-19}$ C, $k = 1.38 \times 10^{-23}$ J/K and room temperature $T_o = 290$ K applies unless otherwise stated.

4.1. Explain how thermal noise power varies (a) with temperature and (b) with frequency bandwidth. Thermal noise from a resistor is measured as 4×10^{-17} W for a given bandwidth and at a temperature of 20°C. What will the noise power be when the temperature is changed to (c) 50°C; (d) 70 K?

8.14 Noise in AM Systems

For AM systems that carry analog-type message signals, the signal-to-noise ratio is the most commonly used measure of performance. As discussed in Chapter 4, all the noise generated within the receiver can be referred to the receiver input, which makes it easy to compare receiver noise, antenna noise, and received signal. The antenna and receiver noise powers can be added, so the receiving system can be modeled as shown in Fig. 8.14.1.

From the point of view of determining the signal-to-noise ratio, the additional information needed is the bandwidth of the system. From Fig. 8.14.1, it is seen that between the antenna and the detector stage there is the bandwidth of the RF stages, followed by the bandwidth of the IF stages. Normally, the bandwidth of the IF stages is very much smaller than the RF bandwidth, and this will be the bandwidth that determines the noise reaching the detector. Following the detector, the bandwidth is that of the *baseband,* which is that required by the modulating signal. To distinguish these clearly, the baseband bandwidth will be denoted by W and the IF bandwidth by B_{IF}. Also, for AM systems it may be assumed that $B_{IF} \cong 2\,W$. In Section 4.2, the concept of equivalent noise bandwidth was explained, and it will be further assumed that B_{IF} and W refer to the equivalent noise bandwidths.

Equation (4.20.3), which gives the noise output from a bandpass system, is rewritten here as

$$n_{IF}(t) = n_I(t)\cos 2\pi f_{IF}t - n_Q(t)\sin \omega_{IF}t \tag{8.14.1}$$

When the noise waveform is passed through the detector, the resulting noise output is very much dependent on whether or not a carrier is present and on the size of the carrier. When evaluating the signal-to-noise ratio, a carrier must be present. Let the modulated carrier be represented by

$$\begin{aligned} e(t) &= E_{c\,\max}(1 + m\cos 2\pi f_m t)\cos 2\pi f_{IF}t \\ &= A_c(t)\cos 2\pi f_{IF}t \end{aligned} \tag{8.14.2}$$

The input to the detector is therefore

$$\begin{aligned} e_{\text{det}}(t) &= e(t) + n_{IF}(t) \\ &= (A_c(t) + n_I(t))\cos 2\pi f_{IF}t - n_Q(t)\sin 2\pi f_{IF}t \end{aligned} \tag{8.14.3}$$

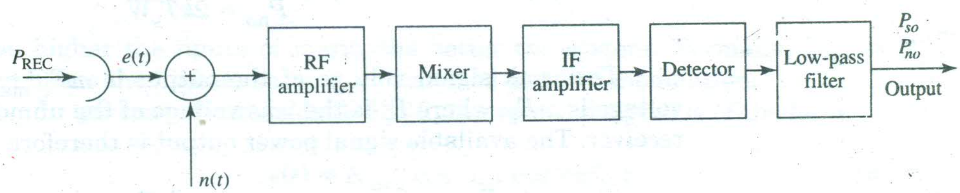

Figure 8.14.1 Model of an AM receiver showing noise added.

The AM envelope detector recovers the envelope of this waveform, and therefore the waveform needs to be expressed as a cosine carrier wave of the form $R(t)\cos(\omega_{IF}t + \psi(t))$. Derivation of the amplitude and phase angle terms is left as an exercise for the student (see Problem 8.51). For the present application, the phase angle $\psi(t)$ can be ignored, and the amplitude term is given by

$$R(t) = \sqrt{[(A_c(t) + n_I(t)]^2 + [n_Q(t)]^2} \tag{8.14.4}$$

Although complete, this expression for $R(t)$ needs to be simplified to get a clearer picture of how the noise adds to the output. For many situations, it can be assumed that the AM carrier is much greater than the noise voltage for most of the time (remembering that the noise is random and that occasional large spikes will be encountered that are greater than the carrier). With this assumption, $R(t)$ simplifies to

$$\begin{aligned} R(t) &= \sqrt{[A_c(t)]^2 + 2A_c(t)n_I(t) + [n_I(t)]^2 + [n_Q(t)]^2} \\ &\cong \sqrt{[A_c(t)]^2 + 2A_c(t)n_I(t)} \end{aligned} \tag{8.14.5}$$

A further expansion and simplification of the square-root term can be made using the binomial theorem.

$$\begin{aligned} R(t) &= \sqrt{[A_c(t)]^2 + 2A_c(t)n_I(t)} \\ &= A_c(t)\left(1 + 2\frac{n_I(t)}{A_c(t)}\right)^{1/2} \\ &\cong A_c(t) + n_I(t) \end{aligned} \tag{8.14.6}$$

For sinusoidal modulation, this becomes

$$R(t) = E_{c\,\max} + mE_{c\,\max}\cos 2\pi f_m t + n_I(t) \tag{8.14.7}$$

The envelope is seen to consist of a dc term, the modulating signal voltage, and the noise voltage $n_I(t)$. As shown in Section 8.11, the dc output from the detector is blocked so that only the ac components contribute to the final output. For the noise, the available power spectral density is given by Eq. (4.20.4) as $2kT_s$, and hence the available noise power output is

$$P_{no} = 2kT_sW \tag{8.14.8}$$

The peak signal voltage at the output is $mE_{c\,\max}$, and hence the rms voltage is mE_c, where E_c is the rms voltage of the unmodulated carrier at the receiver. The available signal power output is therefore

$$P_{so} = \frac{m^2E_c^2}{4R_{out}} \tag{8.14.9}$$

The output signal-to-noise ratio is

$$\left(\frac{S}{N}\right)_o = \frac{P_{so}}{P_{no}}$$

$$= \frac{m^2E_c^2}{8R_{out}kT_sW} \tag{8.14.10}$$

Standard practice is to compare the output signal-to-noise ratio to a reference ratio, which is the signal-to-noise ratio at the detector input *but with the noise calculated for the baseband bandwidth* (W in this case). The noise power spectral density at the detector input is kT_s, and so the reference noise power

$$P_{n\,\mathrm{REF}} = kT_sW \tag{8.14.11}$$

The available signal power from a source with internal resistance R_s is [see Eq. (8.6.3)]

$$P_R = \frac{E_c^2}{4R_s}\left(1 + \frac{m^2}{2}\right) \tag{8.14.12}$$

Hence the reference signal-to-noise ratio is

$$\left(\frac{S}{N}\right)_{\mathrm{REF}} = \frac{P_R}{P_{n\,\mathrm{REF}}}$$

$$= \frac{E_c^2(1 + m^2/2)}{4R_SkT_sW} \tag{8.14.13}$$

A figure of merit that is used is the ratio of these two signal-to-noise ratios. Denoting this by R_{AM}, then

$$R_{AM} = \frac{(S/N)_o}{(S/N)_{\mathrm{REF}}}$$

$$= \frac{m^2}{(2 + m^2)}\frac{R_s}{R_{out}} \tag{8.14.14}$$

The higher the figure of merit, the better the system. Normally, $R_{out} \cong R_s$ and hence the highest value is $\frac{1}{3}$, achieved at 100% modulation.

In the case of sinusoidal DSBSC, the received signal is of the form

$$e(t) = E_{max}\cos\omega_m t\cos 2\pi f_{IF}t \tag{8.14.15}$$

where E_{max} is the peak value of the received signal. The input to the detector is therefore

$$
\begin{aligned}
e_{\text{det}}(t) &= e(t) + n(t) \\
&= (E_{\max}\cos\omega_m t + n_I(t))\cos 2\pi f_{IF}t - n_Q(t)\sin\omega_{IF}t \\
&= A(t)\cos\omega_{IF}t - n_Q(t)\sin\omega_{IF}t
\end{aligned} \tag{8.14.16}
$$

where $A(t) = (E_{\max}\cos\omega_m t + n_I(t))$. With DSBSC a different type of detection is used, known as *coherent detection*. Demodulation of the DSBSC signal utilizes a balanced mixer (or similar circuit) as described in Section 5.10. A locally generated carrier is required that is exactly locked onto the incoming carrier $\cos\omega_{IF}t$, and the two signals are fed into the balanced mixer. One complication with DSBSC detection (which also applies to SSB detection as described in Chapter 9) is generating the local carrier. However, methods are available for achieving this, and as a result the output of the balanced demodulator is

$$e_{\text{out}}(t) = ke_{\text{det}}(t)\cos\omega_{IF}t \tag{8.14.17}$$

where k is a constant of the multiplier circuit. Multiplying this out in full, which is left as an exercise for the student, results in

$$e_{\text{out}}(t) = \frac{k}{2}(E_{\max}\cos\omega_m t + n_I(t) \;+\; \text{high frequency terms} \tag{8.14.18}$$

Low-pass filtering following the balanced demodulator removes the high-frequency terms, leaving, as the baseband output,

$$e_{BB}(t) = \frac{k}{2}(E_{\max}\cos\omega_m t + n_I(t)) \tag{8.14.19}$$

Thus, apart from the multiplying constant $k/2$ and the absence of a dc term, this output is the same as that given in Eq. (8.14.7) for the standard AM case. The $k/2$ factor is common to signal and noise and can be ignored. It should be noted, however, that, whereas the standard AM result requires that the carrier be much greater than the noise, this approximation is not required for the DSBSC case. Equation (8.14.10) applies for the output signal-to-noise ratio, but with $E_{\max}$ replacing $mE_{c\max}$, as seen by comparing eqs. (8.14.19) and (8.14.7). The result is

$$
\begin{aligned}
\left(\frac{S}{N}\right)_o &= \frac{(E_{\max}/\sqrt{2})^2}{8R_{\text{out}}kT_sW} \\
&= \frac{E^2_{\max}}{16R_{\text{out}}kT_sW}
\end{aligned} \tag{8.14.20}
$$

For the reference signal-to-noise ratio, the reference noise is $P_{n\,\text{REF}} = kT_sW$ as given by Eq. (8.14.11). The rms voltage of the received (DSBSC) signal $E_{\max}\cos\omega_m t\cos\omega_{IF}t$ is $E_{\max}/2$ (note *not* $E_{\max}/\sqrt{2}$,) as is readily checked from ac circuit theory. The available signal power at the input is therefore

$$P_R = \frac{(E_{\max}/2)^2}{4R_s}$$

$$= \frac{E^2_{\max}}{16\,R_s} \qquad (8.14.21)$$

The reference signal-to-noise is therefore

$$\left(\frac{\mathrm{S}}{\mathrm{N}}\right)_{\mathrm{REF}} = \frac{E^2_{\max}}{16R_S kT_s W} \qquad (8.14.22)$$

The figure of merit is therefore

$$R_{\mathrm{DSBSC}} = \frac{(\mathrm{S/N})_o}{(\mathrm{S/N})_{\mathrm{REF}}}$$

$$= \frac{R_s}{R_{\mathrm{out}}} \qquad (8.14.23)$$

For $R_{\mathrm{out}} \cong R_s$, the figure of merit is unity, which is three times better than the best that can be achieved for R_{AM}. It will be shown in Chapter 9 that single-sideband (SSB) transmission also has a unity figure of merit, and since this requires half the bandwidth of a DSBSC signal, it is the preferred method of AM carrier transmission (apart from AM broadcast applications, where simplicity of receiver design has established standard AM as the preferred method).

PROBLEMS

8.1 A sinusoidal carrier is amplitude modulated by a square wave that has zero dc component and a peak-to-peak value of 2 V. The periodic time of the square wave is 0.5 ms. The carrier amplitude is 2.5 V, and its frequency is 10 kHz. Write out the equations for the modulating signal, the carrier, and the modulated wave, and plot these functions over a time base equal to twice the periodic time of the square wave.

8.2. A sinusoidal carrier is amplitude modulated by a triangular wave. The triangular wave has zero mean value and is an even function, the first quarter-cycle being described by $-1 + 8t$ volts, with t in milliseconds. The periodic time of the triangular wave is 0.5 ms. The carrier amplitude is 2.5 V, and its frequency is 100 kHz. Write out the equations for the modulating signal, the carrier, and the modulated wave, and plot these functions over a time base equal to twice the periodic time of the triangular wave.

8.3. Calculate the modulation index for each of the modulated waves in Problems 8.1 and 8.2.

It was stated that most types of FM detectors require amplitude limiting in order to function properly. The ratio detector (see Section 10.14) is an exception to this, because it has a degree of inherent limiting built into it. In critical applications it is still necessary to provide additional limiting, but the ratio detector performs well enough otherwise.

10.17 Noise in FM Systems

Noise in an FM receiver can be referred to the input as shown in Fig. 10.17.1(a). With FM receivers the limiter stages described in Section 10.16 help to reduce impulse-type noise, so FM has this advantage over AM.

Another advantage arises from the nature of the noise modulation process. The noise at the receiver input cannot directly frequency modulate the incoming carrier since its frequency is fixed at a distant transmitter, which may in fact be crystal controlled. The noise phase modulates the carrier at the receiver and, as will be shown, this leads to a reduction in output compared to the AM situation.

An important advantage with FM reception is that an improvement in signal-to-noise ratio can be achieved by increasing the frequency deviation. This requires an increased bandwidth, but at least the option is there for an exchange of bandwidth for signal-to-noise ratio. This aspect of FM reception will be explained in detail in this section.

Figure 10.17.1(b) shows the signal and noise voltages at the FM detector. The noise, having passed through a band-pass filter, can be represented by narrow-band noise as described in Section 4.20. This will be analyzed shortly. At this point it need only be noted that the power spectral density as given by Eq. (4.20.1) is kT_s, and hence the available noise power at the detector input for a bandwidth W is $P_{n\mathrm{REF}} = kT_sW$. The rms signal voltage is $E_c = E_{c\,\max}/\sqrt{2}$, and therefore the signal power at the detector input is $P_R = E^2_{c\,\max}/(8R_s)$. The reference signal-to-noise ratio, (introduced in Eq. [8.14.13]), is for the FM case

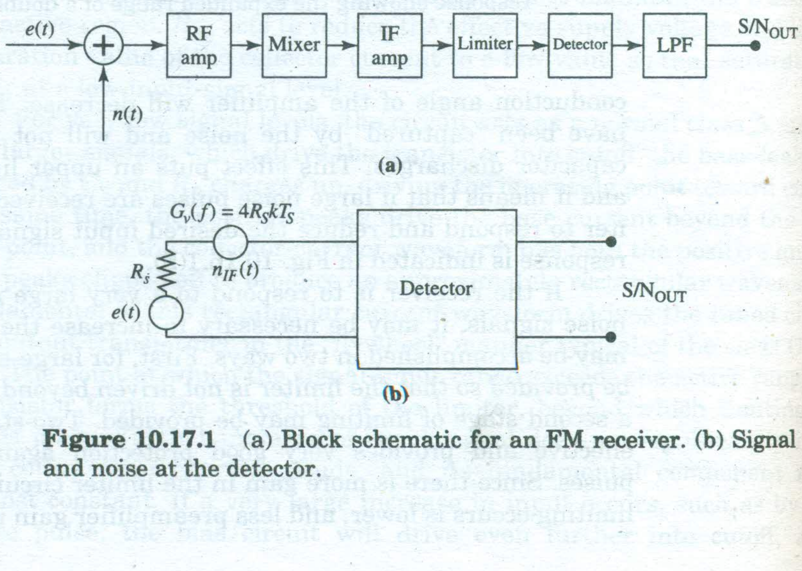

Figure 10.17.1 (a) Block schematic for an FM receiver. (b) Signal and noise at the detector.

$$\left(\frac{\mathrm{S}}{\mathrm{N}}\right)_{\mathrm{REF}} = \frac{P_R}{P_{n\mathrm{REF}}}$$

$$= \frac{E_{c\,\max}^2}{8R_S k T_s W} \tag{10.17.1}$$

Here again it is emphasized that, although the reference signal-to-noise ratio is determined at the input side of the detector, the bandwidth W is that determined by the LPF at the output side.

Consider now the signal output from the band-pass filter in the absence of noise, when a sinusoidally modulated carrier is received. The instantaneous frequency is

$$f_i = f_{IF} + \Delta f \cos \omega_m t \tag{10.17.2}$$

The FM detector converts this to a signal output given by

$$v_m(t) = C\Delta f \cos \omega_m t \tag{10.17.3}$$

where C, a constant, is the frequency-to-voltage coefficient of the detector. The rms voltage output is $E_m = C\Delta f/\sqrt{2}$, and hence the available power output is

$$P_{so} = \frac{(C\Delta f)^2}{8R_{\mathrm{out}}} \tag{10.17.4}$$

The next part of the analysis requires finding the noise at the output of the detector. The noise voltage output from a band-pass system is given by Eq. (4.20.3), which is rewritten here as

$$n_{IF}(t) = n_1(t) \cos \omega_{IF} t - n_Q(t) \sin \omega_{IF} t \tag{10.17.5}$$

The incoming carrier in this case is frequency modulated and can be written as

$$e(t) = E_{c\,\max} \cos (\omega_{IF} t + \phi_m(t)) \tag{10.17.6}$$

where $\phi_m(t)$ is the equivalent phase modulation produced by the signal (see Section 10.9). An analysis of the detector output when noise and modulating signal are present *simultaneously* is very complicated, but fortunately the additional noise terms in the output, resulting from the interaction of $\phi_m(t)$ and the noise modulation, lie outside the baseband. As a result, the noise analysis can be made assuming that an unmodulated carrier is received. In this case, the input to the detector is

$$\begin{aligned} e_{\mathrm{det}}(t) &= e_c(t) + n_{IF}(t) \\ &= (E_{c\,\max} + n_I(t)) \cos \omega_{IF} t - n_Q(t) \sin \omega_{IF} t \end{aligned} \tag{10.17.7}$$

This should be compared with Eq. (8.14.3) for the AM case. As with the AM case, the waveform can be expressed in an equivalent form: $e_{det}(t) = R(t)\cos(\omega_{IF}t + \psi(t))$; but in this situation it is the phase angle $\psi(t)$, rather than the envelope $R(t)$, that is of interest. A trigonometric analysis of the equation gives

$$\psi(t) = \tan^{-1}\frac{n_Q(t)}{E_{c\,\max} + n_I(t)} \tag{10.17.8}$$

Also, as in the AM case, the analysis will be limited to the situation where the carrier amplitude is much greater than the noise voltage for most of the time. Under these circumstances the noise phase angle becomes

$$\psi(t) \cong \tan^{-1}\frac{n_Q(t)}{E_{c\,\max}} \cong \frac{n_Q(t)}{E_{c\,\max}} \tag{10.17.9}$$

The total carrier angle is $\theta(t) = \omega_{IF}t + \psi(t)$, and from Eq. (10.9.2) the equivalent frequency modulation is

$$f_{ieq}(t) = f_{IF} + \frac{1}{2\pi}\frac{d\psi(t)}{dt} = f_{IF} + \frac{\dot{n}_Q(t)}{2\pi E_{c\,\max}} \tag{10.17.10}$$

where the dot notation is used for the differential coefficient. The noise voltage output is therefore

$$v_n(t) = \frac{C\dot{n}_Q(t)}{2\pi E_{c\,\max}} \tag{10.17.11}$$

To find the noise power output, it is best to work in the frequency domain. A result of Fourier analysis shows that, if a voltage waveform $v(t)$ has a power spectral density $G(f)$, then the power spectral density for $\dot{v}(t)$ is $\omega^2 G(f)$. Equation (4.20.4) gives the spectral density for $n_Q(t)$ as $2kT_s$, and hence the spectral density for $\dot{n}_Q(t)$ is $\omega^2 2kT_s$. Combining this with Eq. (10.17.11) and simplifying gives, for the power spectral density of $v_n(t)$,

$$G_v(f) = \frac{C^2 f^2 2kT_s}{E_{c\,\max}^2} \tag{10.17.12}$$

From the definition of power spectral density (see Section 2.17), the noise power output is

$$P_{no} = \int_0^W G_v(f)df$$

$$= \frac{W^3 C^2 2kT_s}{3E^2_{c\,\max}} \qquad (10.17.13)$$

Combining this with Eq. (10.17.4) and simplifying gives, for the output signal-to-noise ratio,

$$\left(\frac{\text{S}}{\text{N}}\right)_o = \frac{P_{so}}{P_{no}}$$

$$= \frac{3\Delta f^2 E^2_{c\,\max}}{16R_{\text{out}}kT_s W^3} \qquad (10.17.14)$$

The signal-to-noise figure of merit introduced in Section 8.14 is, for FM,

$$R_{\text{FM}} = \frac{(\text{S/N})_o}{(\text{S/N})_{\text{REF}}}$$

$$= 1.5 \cdot \left(\frac{\Delta f}{W}\right)^2 \frac{R_s}{R_{\text{out}}}$$

$$= 1.5\beta^2_W \frac{R_s}{R_{\text{out}}} \qquad (10.17.15)$$

Here β_W is the modulation index calculated for the highest baseband frequency W. Note that this is not the same in general as the modulation index for the signal, which is $\beta = \Delta f / f_m$.

The information on FM signal-to-noise ratio is often presented in another way, using the carrier-to-noise ratio as the input parameter at the detector. The carrier-to-noise ratio is similar to the $(\text{S/N})_{\text{REF}}$ ratio except that the total IF bandwidth is used rather than W. Denoting the carrier-to-noise ratio by (C/N), Eq. (10.17.1) is modified to

$$\left(\frac{\text{C}}{\text{N}}\right) = \frac{E^2_{c\,\max}}{8R_S kT_s B_{IF}} \qquad (10.17.16)$$

Also, applying Carson's rule, $B_{IF} = 2(\beta_W + 1)W$ gives

$$\left(\frac{\text{C}}{\text{N}}\right) = \frac{E^2_{c\,\max}}{16R_S kT_s(\beta_W + 1)W} \qquad (10.17.17)$$

The ratio of output signal-to-noise ratio to carrier-to-noise ratio is known as the receiver (or detector) processing gain and is