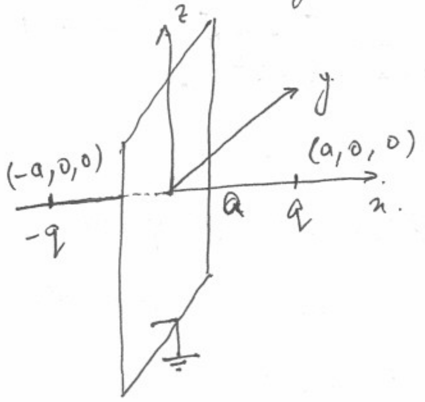
Method of images :

The electrostatic problem is to find the electrostatic potential in
a region with a specified charge distribution, bounded by
certain surfaces. Often the charge distribution may not be
very complex but to account for the boundary conditions on the
surfaces may be very complicated or too detailed. It may
however be possible to find certain additional charge
distribution in regions, not of our interest, but that
can cause the same potential on the surfaces
as specified by the boundary conditions. These
additional charge distribution will then produce the
same potential in the region of interest along with
the given charge distribution. This is illustrated by
the following simple example.




Consider an infinite plane conductor
maintained at potential $V=0$.
We put a point charge $q$ in front
of the conductor at a distance a
from it. We can consider the
plane to be along the $y-z$ plane and the charge $q$ at
$x=a$. The problem is to find the potential at all points
in the region $x>0$

Now, if ~~we~~ don't maintain the plane $x=0$ at $V=0$
then ~~this plane will be at~~ the potential of this plane will
be a function of $(y,z)$ due to the charge $q$ at $x=a$.
If we want to cancell this potential everywhere
on the plane then we can place an equal and

opposite charge symmetrically on the other side of the plane, i.e, at $x = -a$. Once we have the potential $V = 0$ at $x=0$, then the potential at all points $x > 0$ will be exactly as required by the problem. It is easy to calculate the potential everywhere with this two charges $q$ and $-q$ at $x = a$ and $x = -a$. Of course we have altered the charge distribution in the region $x < 0$ and hence this method doesn't give the correct potential in this region. But this region is not of our interest. Moreover, the problem is trivial in this region. The potential everywhere is 0 for $x < a$.

For a general point $(x, y, z)$ the potential is given by

$$V(x,y,z) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{(x-a)^2+y^2+z^2}} - \frac{1}{\sqrt{(x+a)^2+y^2+z^2}}\right] \quad \text{— I}$$

It is easy to see that at $x = 0$, $V = 0$.

Eq$^n$ I gives the required potential at all points in the region $x > 0$. For $x < 0$ it doesn't give the correct potential, but that doesn't matter.

$$\nabla^2 V = -\frac{q}{\epsilon_0}\delta^{(3)}(\vec{r} - a\hat{i}) + \frac{q}{\epsilon_0}\delta^{(3)}(\vec{r} + a\hat{i})$$

So this gives the correct charge density for $x > 0$ and hence it satisfies the required Poisson's Eq$^n$ in this region. Also $V = 0$ at $x = 0$. So this satisfies all the boundary condition.

The electric field can be obtained by computing $-\vec{\nabla} V$. At $x = 0$

$$\vec{E} = -\vec{\nabla} V = \frac{-qa}{2\pi\epsilon_0 (a^2+y^2+z^2)^{3/2}}\hat{i}$$

So the electric field on the plane is perpendicular to the plane and pointed into the plane from the region $x > 0$. If $\sigma(y, z)$ is the charge density over the surface at $(y, z)$ then

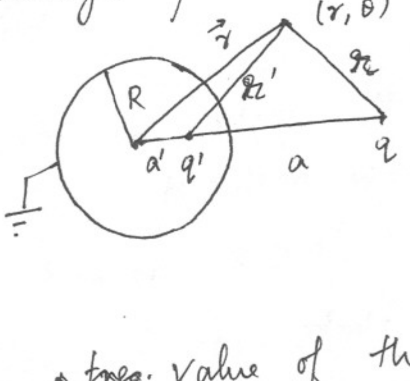
$$E = \frac{\sigma(y,z)}{\epsilon_0}$$

$$\therefore \sigma(y,z) = -\frac{q a}{2\pi (a^2 + y^2 + z^2)^{3/2}}$$

The total charge over the plane is $-q$ which can be obtained by $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sigma(y,z)\, dy\, dz$.

This is expected since we got this electric field by considering the charge $-q$ in the region $x < 0$. The plane $x = 0$ can be considered to be a Gaussian surface that encloses this charge and hence $\oint \vec{E} \cdot \hat{n}$ over this plane gives $-\frac{q}{\epsilon_0}$.

Eg2: A grounded conducting sphere of radius $R$ is placed near a point charge $q$. The distance of the point charge from the centre of the sphere is $a$.




Here the image charge can be found inside the grounded sphere at a distance $a' = \frac{R^2}{a}$.

The ~~stree~~ value of the image charge is

$$q' = -\frac{R}{a} q.$$

These choices makes the potential of the sphere 0. So this configuration will give the required potential everywhere outside the sphere. Obviously this doesn't give the right potential for $r < 0$, where $V = 0$ everywhere as we know.

Once we know the image charge, we can calculate all the required electrostatic quantities like the electric field, the surface charge density, the energy of the system.

Sometimes the boundary conditions are such that one image charge won't work. A combination of two or more image charge can give the required boundary condition. It may always be possible to find image charges which are not point charges but continuous distribution. But such an exercise ~~be~~ comes extremely difficult and it may not be easier than solving the Poisson's Equation.

In more than one dimensions, Poisson's Equation is a ~~partial~~ 2nd order partial differential Equation. We would like to convert this partial differential equation into a number of ordinary differential equations, which can be solved. One method to do this is called the separation of variables.