Signals and Systems (CT 203)

Tutorial Sheet-05 DA-IICT, Gandhinagar.

Problem 1 Consider a linear time-invariant (LTI) system whose response to the signal $x_1(t)$ in Fig. 1a is the signal $y_1(t)$ in Fig. 1b. Determine and sketch carefully response of the system to the input $x_2(t)$ shown in Fig. 1c.

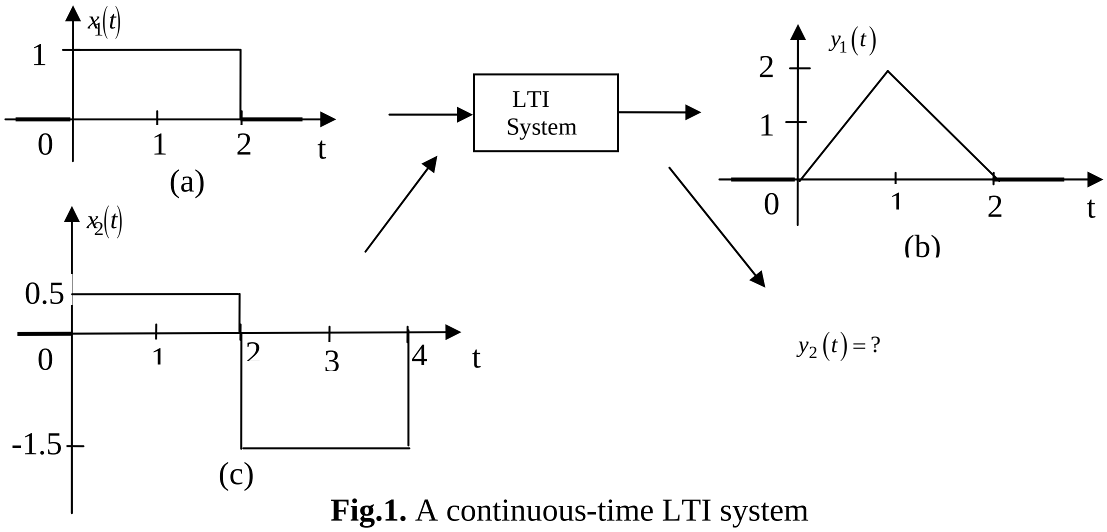


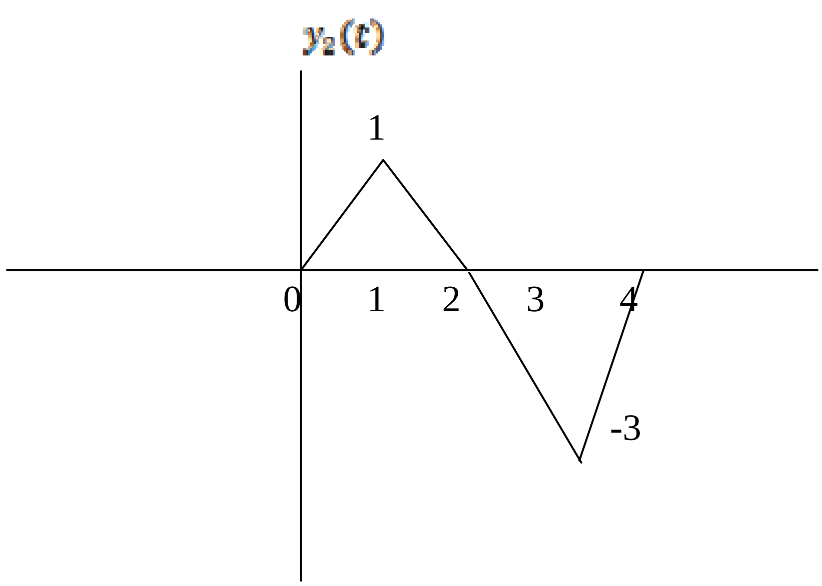
Fig.1. A continuous-time LTI system

Solution

$$x_2(t) = 0.5x_1(t) - 1.5x_1(t-2)$$

$$\therefore y_2(t) = 0.5y_1(t) - 1.5y_1(t-2)$$ (due to LTI system properties)

We can write above equation as the system is LTI.



Problem 2. Find the impulse response, $h(n)$ of following systems

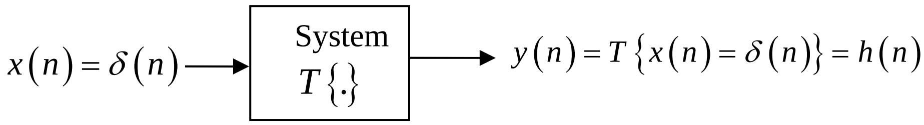


Fig.2a. Concept of Impulse Excitation

(a) Ideal Delay System, $y(n)=T\{x(n)\}=x(n-n_d)$

(b) Moving Average System, $y(n)=T\{x(n)\}=\dfrac{1}{N_1+N_2-1}\sum_{k=-N_1}^{N_2} x(n-k)$

(c) Accumulator System, $y(n)=T\{x(n)\}=\sum_{k=-\infty}^{n} x(k)$

(d) Forward Difference system, $y(n)=T\{x(n)\}=x(n+1)-x(n)$

(e) Backward Difference system, $y(n)=T\{x(n)\}=x(n)-x(n-1)$

(f) Linear interpolator system,

$$y(n)=T\{x(n)\}=x(n)+\frac{1}{2}\{x(n-1)-x(n+1)\}$$

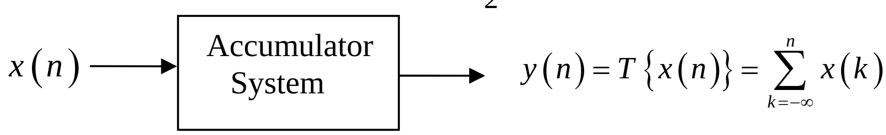


Fig.2b. Ideal Delay System

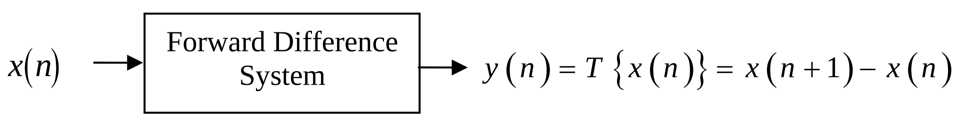


Fig.2c. Moving Average (MA) System

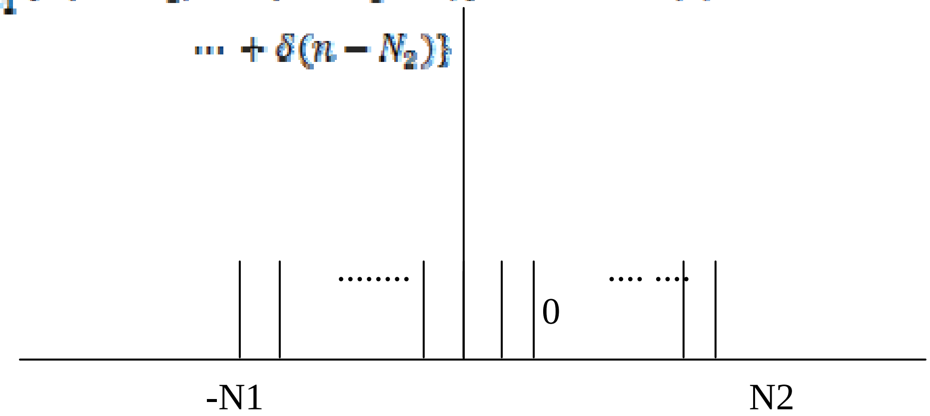
Solution:

For impulse response replace $x(n)=\delta(n)$ you will get $y(n)=h(n)$.

a) $h(n)=\delta(n-n_d)$

b) $h(n)=\frac{1}{N_1+N_2+1}\sum_{-N_1}^{N_2}\delta(n-k)$

$$= \frac{1}{N_1+N_2+1}\{\delta(n+N_1)+\delta(n+N_1+1)\}+\cdots\cdots+\delta(0)+\cdots+\delta(n-N_2)\}$$



(c)

$$h(n)=\sum_{k=-\infty}^{n}\delta(k)$$

$$h(-1)=\cdots\cdots\delta(-2)+\delta(-1)=0$$

$$h(0)=\cdots\cdots\delta(-1)+\delta(0)=1$$

$$h(1)=\cdots\cdots\delta(0)+\delta(1)=1$$

$$h(n)=\cdots\cdots\delta(n-1)+\delta(n)=1$$

$$\therefore h(n)=\sum_{k=0}^{n}\delta(k)=u(n)$$

(d) $h(n)=\delta(n+1)-\delta(n)$

(e) $h(n)=\delta(n)-\delta(n-1)$

(f) $h(n)=\delta(n)+\frac{1}{2}\{\delta(n-1)+\delta(n-1)\}$

Problem 3 Consider an input $x(n)$ and a unit impulse response $h(n)$ given by,

$$x(n)=\left(\frac{1}{2}\right)^{n-2}u(n-2)$$

$$h(n)=u(n+2)$$

Determine and plot output, $y(n)=x(n)*h(n)$.

Solution : -

$$y(n) = x(n) * h(n)$$
$$= \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$
$$= \sum_{k=-\infty}^{\infty} \left(\frac{1}{2}\right)^{k-2} u(k-2)u(n-k+2)$$

$$u(k-2) = \begin{cases} 1 \text{ for } k - 2 \geq 0 \Rightarrow k \geq 2 \\ 0 \text{ for } k < 2 \end{cases} \qquad \text{………(3.1)}$$

and

$$u(n-k-2) = \begin{cases} 1 \text{ for } n - k - 2 \geq 0 \Rightarrow k \leq n-2 \\ 0 \text{ for } k > n - 2 \end{cases} \qquad \text{……..(3.2)}$$

$$\therefore y(n) = \sum_{k=-\infty}^{1} \left(\frac{1}{2}\right)^{k-2} u(k-2)u(n-k+2) + \sum_{k=2}^{\infty} \left(\frac{1}{2}\right)^{k-2} u(k-2)u(n-k+2)$$

$$\therefore y(n) = \sum_{k=-\infty}^{1} \left(\frac{1}{2}\right)^{k-2} (0)u(n-k+2) + \sum_{k=2}^{\infty} \left(\frac{1}{2}\right)^{k-2} (1)u(n-k+2)$$ [because of (3.1)]

$$\therefore y(n) = \sum_{k=2}^{\infty} \left(\frac{1}{2}\right)^{k-2} u(n-k+2)$$
$$\therefore y(n) = \sum_{k=2}^{n-2} \left(\frac{1}{2}\right)^{k-2} u(n-k+2) + \sum_{k=n-1}^{\infty} \left(\frac{1}{2}\right)^{k-2} u(n-k+2)$$
$$\therefore y(n) = \sum_{k=2}^{n-2} \left(\frac{1}{2}\right)^{k-2} 1 + \sum_{k=n-1}^{\infty} \left(\frac{1}{2}\right)^{k-2} 0$$ [because of (3.2)]

$$\therefore y(n) = \sum_{k=2}^{n-2} \left(\frac{1}{2}\right)^{k-2}$$ [Note, here summation is nonzero for $2 \leq k \leq n+2$]

let, $m = k-2$, when $k = 2 \Rightarrow m = 0$ and when $k = n-2 \Rightarrow m = n$

$$\therefore y(n) = \sum_{m=0}^{n} \left(\frac{1}{2}\right)^{m} \qquad \text{….(3.3)}$$

This is defined for $n \geq 0$ **. For** $n < 0$ **,** $y(n)$ **is zero. This is because for** $n < 0$ **,** $2 \leq k \leq n+2$ **is not possible.**

G.P. with the first term $a = 1$ and ratio $r = \frac{1}{2}$

Sum of first n^{th} term $= \frac{a(1-r^{n+1})}{1-r}$

$$\boxed{\therefore y(n) = \frac{1(1-\{\frac{1}{2}\}^{n+1})}{1-\frac{1}{2}}u(n) = 2\left[1-\left(\tfrac{1}{2}\right)^{n+1}\right]u(n)}$$ **ANS**

Problem 4 Let $x(t) = u(t-3) - u(t-5)$ and $h(t) = e^{-3t}u(t)$, then compute following,

(a) $y(t) = x(t) * h(t)$

(b) $g(t) = \frac{d}{dt}[x(t)] * h(t)$

(c) How is $g(t)$ related to $y(t)$?

We will solve this problem using graphical methods to evaluate convolution.
Here, $x(t) = u(t-3) - u(t-5)$
and $h(t) = e^{-3t}u(t)$

$$u(t-3) = \begin{cases} 1 \text{ for } t\text{-}3 \geq 0 \Rightarrow t \geq 3 \\ 0 \text{ for } t < 3 \end{cases} \quad \text{and} \quad u(t-5) = \begin{cases} 1 \text{ for } t\text{-}5 \geq 0 \Rightarrow t \geq 5 \\ 0 \text{ for } t < 5 \end{cases}$$

$$x(t) = \begin{cases} 1 \text{ for } 3 \leq \text{t}<5 \\ 0 \text{ otherwise} \end{cases} \quad \text{and} \quad h(t) = \begin{cases} e^{-3t} & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$

(a) $y(t) = x(t) * h(t)$

$$\therefore y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

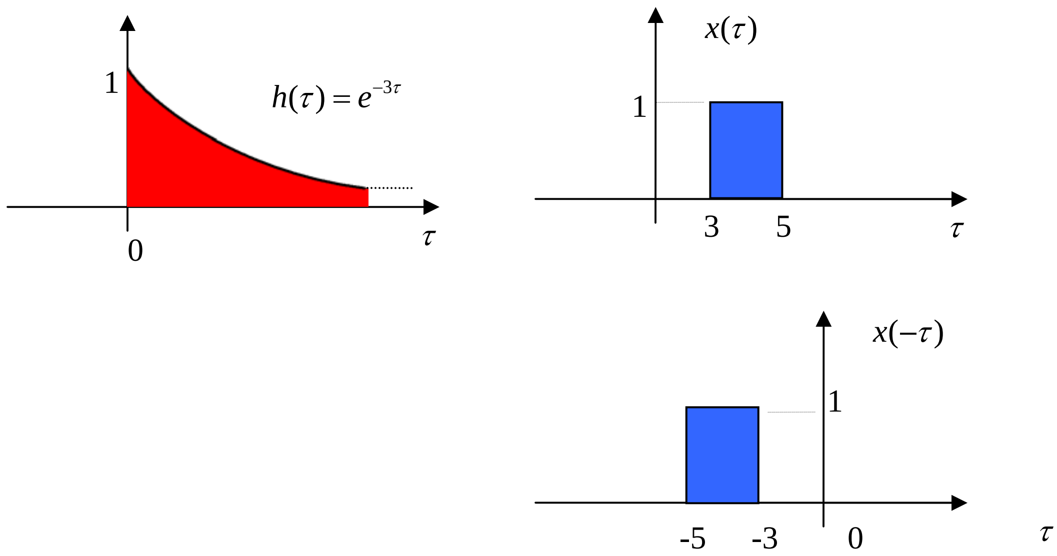


Figure 4.1. This figure shows given functions.

signal $x(t-\tau)$ is shifted version of $x(-\tau)$ by t units. (Be careful here, τ is independent variable not t, t is some arbitrary constant)
For different values of t, we can have 3 general cases.

Case 1:- There is no overlap between $x(t-\tau)$ and $h(\tau)$, i.e., $x(t-\tau)h(\tau)=0$
Case 2:- Partial overlap
Case 3:- Full overlap.

In this problem, **case 1:-** For $t<3$, there is no overlap. This can be seen in Figure 4.2

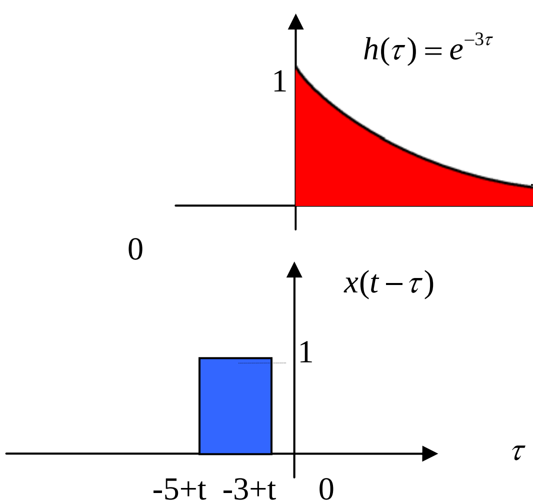


Figure 4.2. No overlap case.

$$x(t-\tau)h(\tau)=0 \Rightarrow y(t)=\int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau = 0 \quad \text{For } t<3$$

$\boxed{y(t)=0}$ For $t<3$

Case 2:-
for $t \geq 3$ curves start overlapping each other. $3 \leq t < 5$, there is partial overlapping of the curves. This can be seen in Figure 4.3.

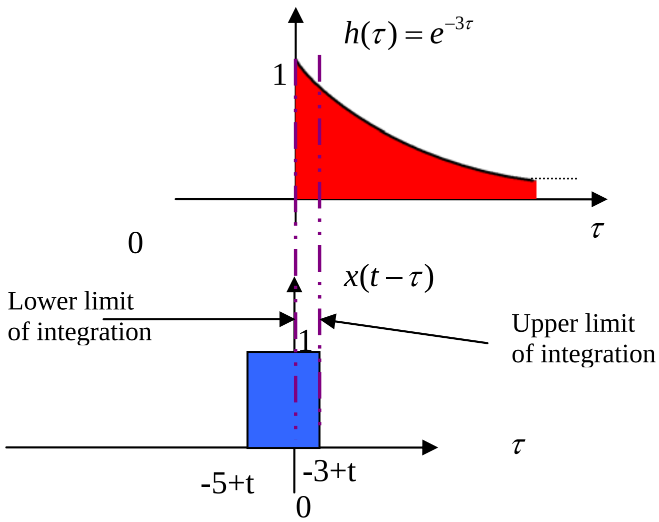


Figure 4.3. Partial overlap case.

$$y(t)=\int_{0}^{t-3} x(t-\tau)h(\tau)d\tau = \int_{0}^{t-3} 1e^{-3\tau}d\tau = \frac{e^{-3\tau}}{-3}\Bigg|_{\tau=0}^{\tau=t-3} = \frac{1}{3}\left(1-e^{3(3-t)}\right)$$

$$\boxed{\therefore y(t)=\frac{1}{3}\left(1-e^{3(3-t)}\right)} \quad \text{for } 3 \leq t < 5$$

Case 3:- for $t \geq 5$, the curve of $h(\tau)$ occupies the entire curve of $x(t-\tau)$, that means there is full overlap. This can be seen in Figure 4.4.

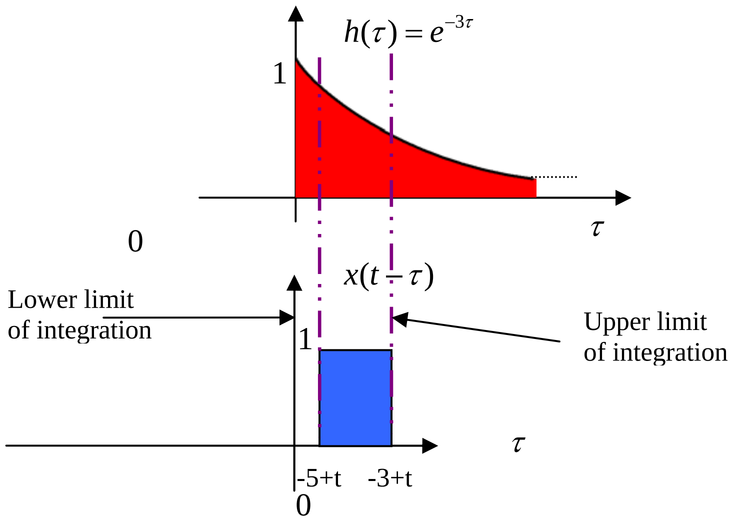


Figure 4.4. Full overlap case.

$$y(t)=\int_{t-5}^{t-3} x(t-\tau)h(\tau)d\tau=\int_{t-5}^{t-3} 1e^{-3\tau}d\tau=\left.\frac{e^{-3\tau}}{-3}\right|_{\tau=t-5}^{\tau=t-3}=\frac{1}{3}\left(e^{3(5-t)}-e^{3(3-t)}\right)$$

$$\boxed{\therefore y(t)=\frac{1}{3}\left(e^{3(5-t)}-e^{3(3-t)}\right)} \quad \text{for } t\geq 5$$

Combining all 3 cases …

$$\boxed{\therefore y(t)=\begin{cases} 0 & \text{for } t<3 \\ \frac{1}{3}\left(1-e^{3(3-t)}\right) & \text{for } 3\leq t<5 \\ \frac{1}{3}\left(e^{3(5-t)}-e^{3(3-t)}\right) & \text{for } t>5 \end{cases}}$$

ANS 4a

(b) $g(t)=\frac{d}{dt}[x(t)]*h(t)$ …..(b.1)

$x(t)=u(t-3)-u(t-5)$

$\therefore \frac{dx(t)}{dt}=\delta(t-3)-\delta(t-5)$ ..(b.2) $\quad [\because \frac{d}{dt}[u(t)]=\delta(t)]$

substitute (b.2) into eq. (b.1),

$$g(t) = \frac{d}{dt}[x(t)] * h(t)$$

$$\therefore g(t) = [\delta(t-3) - \delta(t-5)] * h(t) = [\delta(t-3) * h(t)] - [\delta(t-5) * h(t)]$$

(Distributive property for convolution operation)

$$\therefore g(t) = h(t-3) - h(t-5) \quad [\because \delta(t-t_0) * x(t) = x(t-t_0)]$$

$$\therefore g(t) = e^{-3(t-3)}u(t-3) - e^{-3(t-5)}u(t-5) = e^{3(3-t)}u(t-3) - e^{3(5-t)}u(t-5)$$

$$\boxed{\therefore g(t) = \begin{cases} 0 & \text{for } t < 3 \\ e^{3(3-t)} & \text{for } 3 \le t < 5 \\ e^{3(3-t)} - e^{3(5-t)} & \text{for } t > 5 \end{cases}}$$

ANS 4b

(c) Relation of $g(t)$ and $y(t)$?

Take derivative of $y(t)$ with respect to t.

$$\therefore \frac{dy(t)}{dt} = \left\{ \begin{array}{ll} 0 & \text{for } t < 3 \\ e^{3(3-t)} & \text{for } 3 \le t < 5 \\ e^{3(3-t)} - e^{3(5-t)} & \text{for } t > 5 \end{array} \right\} = g(t)$$

$$\boxed{\therefore g(t) = \frac{dy(t)}{dt}}$$

ANS 4c

Problem 5 Consider a causal LTI system whose input $x(n)$ and output $y(n)$ are related by the difference equation. $y(n) = \frac{1}{4}y(n-1) + x(n)$. Determine $y(n)$ if $x(n) = \delta(n-1)$.

SOLUTION:

We are given that, $y(n) = \frac{1}{4}y(n-1) + x(n)$

Now for causal system, $y(n) = 0$ for $n < 0$

Here applying the input as $x(n) = \delta(n-1)$, then output $y(n)$ is the following

$$y(n) = \frac{1}{4}y(n-1) + \delta(n-1)$$

$$y(0) = \frac{1}{4}y(-1) + \delta(-1) = 0$$

$$y(1) = \frac{1}{4}y(1-1) + \delta(1-1) = \frac{1}{4}y(0) + \delta(0) = 0 + 1 = 1$$

$$y(2) = \frac{1}{4}y(2-1) + \delta(2-1) = \frac{1}{4}y(1) + \delta(1) = \frac{1}{4} + 0 = \frac{1}{4}$$

$$y(3) = \frac{1}{4}y(3-1) + \delta(3-1) = \frac{1}{4}y(2) + \delta(2) = \frac{1}{4} \times \frac{1}{4} + 0 = \left(\frac{1}{4}\right)^2$$

$$y(4) = \left(\frac{1}{4}\right)^3$$

Therefore,

$$y(n) = \left(\frac{1}{4}\right)^{n-1} \cdot u(n-1)$$

Problem 6 Let $h(t)$ be the triangular pulse shown in Fig. 1(a) and let $x(t)$ be the impulse train shown in fig.(b)

Fig. 1a

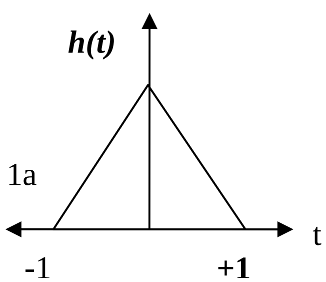
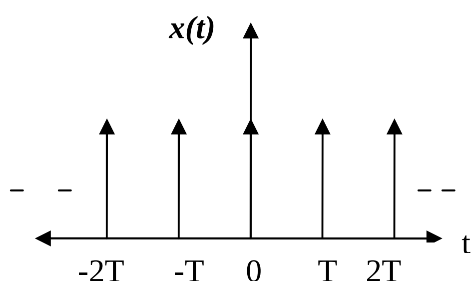


Fig.1b



Determine and sketch $y(t) = x(t) * h(t)$ for the following values of T.

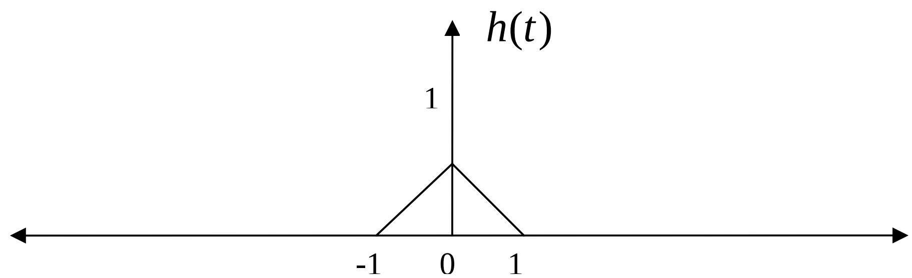
(a) T=4 **(b)** T=2 **(c)** T=3/2 **(d)** T=1

SOLUTION:

$$y(t) = x(t) * h(t) = \left[\sum_{k=-\infty}^{\infty} \delta(t-kT)\right] * h(t)$$

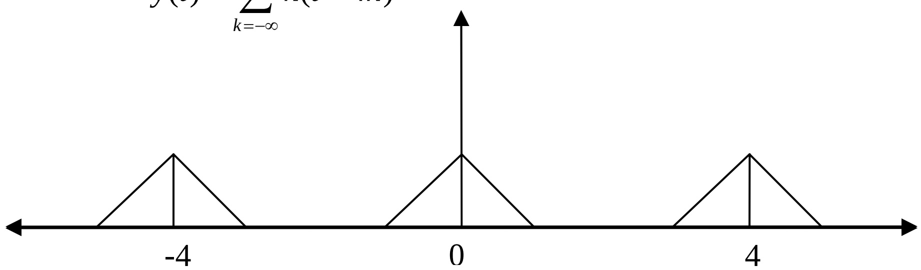
$$= \sum_{k=-\infty}^{\infty} \left[\delta(t-kT) * h(t)\right]$$

$$y(t) = \sum_{k=-\infty}^{\infty} h(t-kT)$$



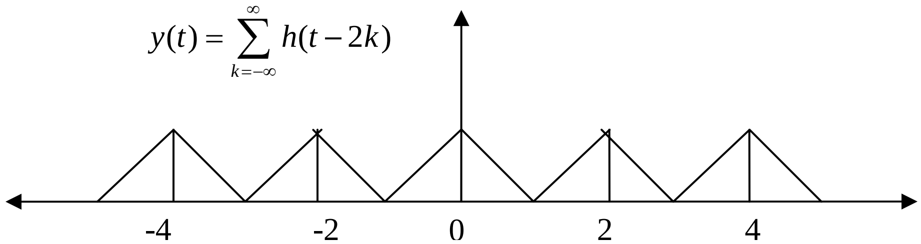
(a) T=4

$$y(t)=\sum_{k=-\infty}^{\infty} h(t-4k)$$



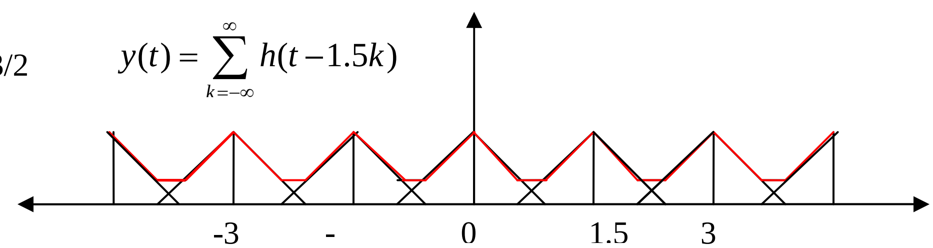
(b) T=2

$$y(t)=\sum_{k=-\infty}^{\infty} h(t-2k)$$



(c) T=3/2

$$y(t)=\sum_{k=-\infty}^{\infty} h(t-1.5k)$$



(d) T=1

$$y(t)=\sum_{k=-\infty}^{\infty} h(t-k)$$