Tutorial 11

SC-220 Groups and Linear algebra Autumn 2019
(Inner Products spaces)

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(1) Let V be an inner product space, then show that $\|x\| = \sqrt{\langle x \mid x\rangle}$ is a defines a norm on V.

(2) Parallelogram law
Let V be a real or complex vector space with an inner product. Show that the norm defined by the inner product satisfies the parallelogram law

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

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(3) In $M_2(\mathbb{R})$ we can define an inner product $\langle A \mid B\rangle = \mathrm{Tr}(AB^T)$. Verify that the set

$$\mathcal{B} = \left\{ \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \right\}$$

is an orthonormal basis. Compute the Fourier expansion of $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ with respect to $\mathcal{B}$

(4) Consider the space of real valued integrable functions on the interval $(-\pi, \pi)$ with inner product

$$\langle f \mid g\rangle = \int_{-\pi}^{\pi} f(t)g(t)dt$$

Verify that the set of trigonometric

$$\mathcal{B} = \left\{ \frac{1}{\sqrt{2\pi}}, \frac{\cos t}{\sqrt{\pi}}, \frac{\cos 2t}{\sqrt{\pi}}, \cdots, \frac{\sin t}{\sqrt{\pi}}, \frac{\sin 2t}{\sqrt{\pi}}, \cdots \right\}$$

is an orthonormal basis of this space. Find the expansion of the square wave function

$$f(t) = \begin{cases} -1 \text{ when } -\pi < t < 0 \\ 1 \text{ when } 0 < t < \pi \end{cases}$$

(5) Apply the Gram-Schmidt procedure for $\mathbb{C}^3$ to the vectors $\left\{ \begin{pmatrix} i \\ i \\ i \end{pmatrix}, \begin{pmatrix} 0 \\ i \\ i \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ i \end{pmatrix} \right\}$

(6) Consider the vector space P_3 of polynomials with rational coefficients that are of degree less than or equal to 3 on the domain $[-1, 1]$. Check that $\langle f \mid g\rangle = \int_{-1}^{1} f(x)g(x)dx$ is an inner product on P_3. The monomials $\{1, x, x^2, x^3\}$ form a basis of P_3. Use the Gram-Schmidt orthogonalization process to produce an ONB of P_3. Express the polynomial $x^3 + 3x^2 + 2x + 3$ as a linear combination of the ONB vectors.