FFT: a fast way to implement DFT [Cooley-Tukey 1965]

- Direct matrix-vector multiplication requires $O(n^2)$ operations when using the Horner's method, i.e., $A(x) = a_0 + x(a_1 + x(a_2 + \ldots + xa_{n-1}))$.
- FFT: reduce $O(n^2)$ to $O(n \log_2 n)$ using divide-and-conqueror technique.
- How does FFT achieve this? Or what calculations are redundant in the direct matrix-vector multiplication approach?
- Note: The idea of FFT was proposed by Cooley and Tukey in 1965 when analyzing earth-quake data, but the idea can be dated back to F. Gauss.

Let's evaluate $A(x)$ at two special points first

- Consider evaluating a 7-degree polynomial $A(x) = a_0 + a_1 x + a_2 x^2 + ... + a_7 x^7$ at two special points $1, -1$.
- **Divide:** Break the polynomial into even and odd terms, i.e.,
  - $A_{even}(x) = a_0 + a_2 x + a_4 x^2 + a_6 x^3$
  - $A_{odd}(x) = a_1 + a_3 x + a_5 x^2 + a_7 x^3$

  Then we have the following equations:
  - $A(x) = A_{even}(x^2) + x A_{odd}(x^2)$
  - $A(-x) = A_{even}(x^2) - x A_{odd}(x^2)$
- **Combine:** For two special points $1, -1$, we have
  - $A(1) = A_{even}(1) + A_{odd}(1)$
  - $A(-1) = A_{even}(1) - A_{odd}(1)$
- In other words, the values of $A(x)$ at **2 points** $1, -1$ can be calculated based on the values of $A_{even}(x), A_{odd}(x)$ at only **1 point** .

Let's evaluate $A(x)$ at four special points further

- Consider evaluating a 7-degree polynomial $A(x) = a_0 + a_1 x + a_2 x^2 + ... + a_7 x^7$ at four special points $1, -i, -1, i$.
- **Divide:** Break the polynomial into even and odd terms, i.e.,
  - $A_{even}(x) = a_0 + a_2 x + a_4 x^2 + a_6 x^3$
  - $A_{odd}(x) = a_1 + a_3 x + a_5 x^2 + a_7 x^3$

  Then we have the following equations:
  - $A(x) = A_{even}(x^2) + x A_{odd}(x^2)$
  - $A(-x) = A_{even}(x^2) - x A_{odd}(x^2)$
- **Combine:** For 4 special points $1, -i, i, -1$, we have
  - $A(1) = A_{even}(1) + A_{odd}(1)$
  - $A(-i) = A_{even}(-1) - i A_{odd}(-1)$
  - $A(-1) = A_{even}(1) - A_{odd}(1)$
  - $A(i) = A_{even}(-1) + i A_{odd}(-1)$
- In other words, the values of $A(x)$ at **4 points** $1, -i, -1, i$ can be calculated based on the values of $A_{even}(x), A_{odd}(x)$ at **2 points** $1, -1$.

$\mathrm{FFT}(n, a_0, a_1, ..., a_{n-1})$

if $n == 1$ **then**

  return a_0 ;

end if

$(E_0, E_1, ..., E_{\frac{n}{2}-1}) = \mathrm{FFT}(\frac{n}{2}, a_0, a_2, ..., a_n)$;

$(O_0, O_1, ..., O_{\frac{n}{2}-1}) = \mathrm{FFT}(\frac{n}{2}, a_1, a_3, ..., a_{n-1})$;

for $k = 0$ to $\frac{n}{2} - 1$ **do**

  $\omega^k = e^{\frac{2\pi}{n}ki}$;

  $y_k = E_k + \omega^k O_k$;

  $y_{\frac{n}{2}+k} = E_k - \omega^k O_k$;

end for

return $(y_0, y_1, ..., y_{n-1})$;