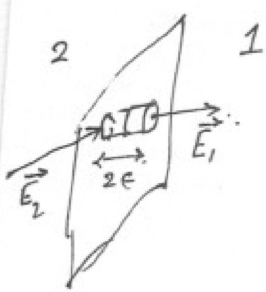
Boundary Conditions. : Once we solve the Poisson's Eq^n. we will get the potential due to a given charge distribution but ~~with~~ along with a number of arbitrary constants. The values of this constants have to be determined by a number of specified conditions on the fields in the region of interest. Generally these conditions are in the form of the value of potential or electric fields ~~at~~ over certain boundary surfaces of the region. The boundary conditions can also be certain specified surface charge density or linear charge densities.

Boundary condition on the Electric fields: A surface divides a region in two parts say 1 and 2.



Let $\vec{E}_1$ be the electric field near the surface in region 1 and $\vec{E}_2$ be the electric field in region 2. Consider a Gaussian surface of area da and thickness 2ϵ. where $\epsilon \to 0$.

The total flux of the electric field from this ~~sa~~ Gaussian surface is $\vec{E_1}\cdot\hat{n}\,da - \vec{E_2}\cdot\hat{n}\,da$. ———— (1).

It must be noted here that the flux through the curved cylindrical surface is not zero since the electric field is not neccesarily perpendicular to the surface. Secondly the '−' sign in Eq (1) above is because the normal $\hat{n}$ on the two sides of the surface are oppositely directed.

Now if the surface has a surface charge density σ then the total charge enclosed by the Gaussian surface is σda. Then by Gauss' law

$$\vec{E_1}\cdot\hat{n}\,da - \vec{E_2}\cdot\hat{n}\,da = \frac{\sigma da}{\epsilon_0}.$$

$$\therefore\ \vec{E_1}\cdot\hat{n} - \vec{E_2}\cdot\hat{n} = \frac{\sigma}{\epsilon_0}.$$

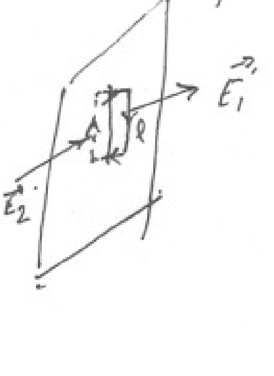
$$\therefore\ E_{1\perp} - E_{2\perp} = \frac{\sigma}{\epsilon_0} \qquad \text{———— (2)}$$

Eq 2 specifies the boundary condition on the perpendicular components of the electric field on the two sides. If ~~there~~ the surface charge density at a place is 0 then

$$E_{1\perp} = E_{2\perp}$$

Actually Eq (2) is a form of the differential form of Gauss' law.

The condition on the par-tangential component of Electric field comes from the property that $\vec{\nabla}\times\vec{E} = 0$.



Consider a rectangular loop as shown. The tangential elements are of length l and very near the surface. The parts of the loop that pierces the surface are negligibly small. So the integral becomes.

$$\oint \vec{E}\cdot d\vec{l} = \int (\vec{\nabla}\times\vec{E})\cdot\hat{n}\, da = 0$$

$$\therefore \int_{C_1} \vec{E_1}\cdot d\vec{l} + \int_{C_2} \vec{E_2}\cdot d\vec{l} = 0$$

The curves C_1 and C_2 are tangential to the surface and since they lie close to the surface we have

$$C_2 = -C_1$$

$$\therefore \int_{C_1} \vec{E_1}\cdot d\vec{l} + \int_{-C_1} \vec{E_2}\cdot d\vec{l} = 0$$

$$\therefore \int_{C_1} \vec{E_1}\cdot d\vec{l} - \int_{C_1} \vec{E_2}\cdot d\vec{l} = 0$$

$$\therefore \int_{C_1} \vec{E_1}\cdot d\vec{l} = \int_{C_1} \vec{E_2}\cdot d\vec{l}$$

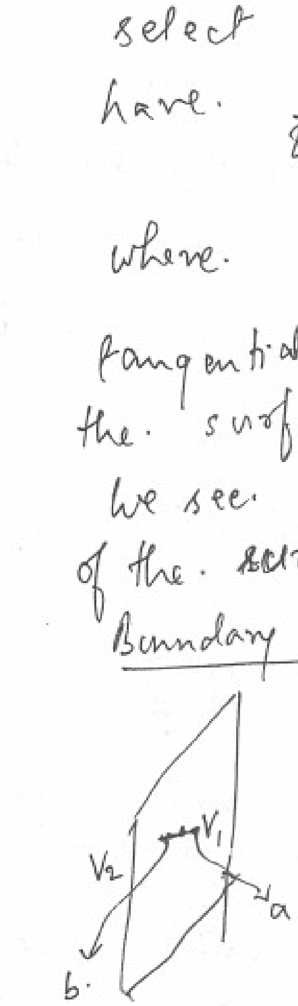
Since this is true for any arbitrary loop we select and hence any arbitrary curve C_1 we have

$$E_{1\parallel} = E_{2\parallel} \quad \text{———} \quad (3)$$

where $E_{1\parallel}$ and $E_{2\parallel}$ are the parallel or tangential components of the electric fields to the surface.

We see that the boundary condition (3) is independent of the surface charge density.

<u>Boundary condition on the Electric potential:</u>



$V_1 = V_2$ near the surface.

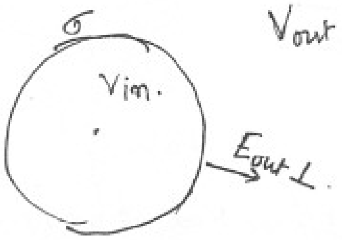
Let V_1 be the potential at a point very near to the surface in region 1 and V_2 in region 2.

$$V_2 - V_1 = \int_a^b \vec{E}\cdot d\vec{l}$$

Since $\vec{E}$ is finite in the two regions, though it may be discontinuous, the integral on R.H.S $\to 0$ as $a \to b$ as a and b approach the surface.

Therefore $V_2 \to V_1$

Eg: A sphere has a uniform charge density σ over its surface. Find the potential inside and outside the sphere.



Here $\rho = 0$ outside and $\rho = 0$ inside.

$$\nabla^2 V_{out} = 0$$

$$\therefore \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V_{out}}{\partial r}\right) = 0$$

$$\therefore V_{out} = -\frac{c_1}{r} + c_2$$

Similarly $\nabla^2 V_{in} = 0 \Rightarrow V_{in} = -\frac{d_1}{r} + d_2$

$$\vec{E}_{in} = -\vec{\nabla} V_{in} = -\frac{d_1}{r^2}\hat{r}$$

Applying Gauss's law over a spherical surface inside we get $-\frac{d_1}{\cancel{r^2}} \times 4\pi \cancel{r^2} = 0 \Rightarrow d_1 = 0$

If we demand that the potential at ∞ be 0 then $c_2 = 0$.

$$\therefore V_{out} = -\frac{c_1}{r}$$

$$\therefore \vec{E}_{out} = -\vec{\nabla} V_{out} = -\frac{c_1}{r^2}\hat{r}$$

At $r = R$ we have the boundary condition

$$E_{out\perp} - E_{in\perp} = \frac{\sigma}{\epsilon_0} \Rightarrow -\frac{c_1}{R^2} = \frac{\sigma}{\epsilon_0}$$

$$\therefore c_1 = -\frac{\sigma R^2}{\epsilon_0}$$

$$\therefore V_{out} = \frac{\sigma R^2}{r\epsilon_0}$$

$$V_{in} = d_2$$

At the surface of the sphere we have.

$$V_{out}\Big|_R = V_{in}\Big|_R.$$

$$\therefore \quad \frac{\sigma R^2}{R\epsilon_0} = d_2 \Rightarrow d_2 = \frac{\sigma R}{\epsilon_0}.$$

$$\therefore \quad V_{in} = \frac{\sigma R}{\epsilon_0}.$$

~~From this~~ So we get the potential everywhere. From this we can calculate the electric field.

$$E_{out} = -\vec{\nabla} V_{out} = \frac{\sigma R^2}{\epsilon_0}\frac{1}{r^2}\hat{r}$$

$$\vec{E}_{in} = 0$$

Verify that this is indeed the ~~ele~~ right electric field obtained by other methods.