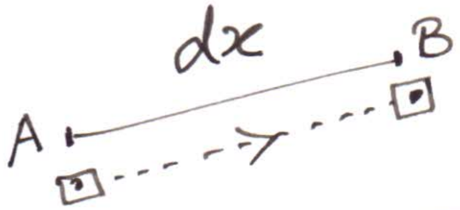
Momentum, Force and Energy

Rest mass, $\boxed{m_0}$ → Mass of an object measured in a frame with respect to which the object is at rest.

Motion from A to B



Momentum,

$$p = mv = m\frac{dx}{dt}$$

In the body frame $\boxed{m \longrightarrow m_0}$, $\boxed{t \longrightarrow t_0}$ rest mass proper time

Satisfies Momentum Conservation Rule

DEFINITION : $\boxed{p = m_0 \dfrac{dx}{dt_0}}$

But $\boxed{dt = \gamma\, dt_0}$ (Dilated time) , $\boxed{\gamma = \dfrac{1}{\sqrt{1-(V/c)^2}}}$

$$\Rightarrow \boxed{p = m_0 \gamma \frac{dx}{dt} = \gamma m_0 V}$$

Momentum modified under Special Relativity

Momentum

$$\Rightarrow \boxed{p = m\frac{dx}{dt} = \gamma m_0 v}$$

As $v \to c$, inertia increases.

where MASS (in special Relativity)

$$\boxed{m = \gamma m_0 = \frac{m_0}{\sqrt{1-(v/c)^2}}}$$

No material object can reach the speed of light.

$$\boxed{\text{Force},\ F = \frac{dp}{dt} = \frac{d(\gamma m_0 v)}{dt}}$$

$$\Rightarrow \boxed{F = m_0 \frac{d(\gamma v)}{dt}}$$

$\boxed{p = \gamma m_0 v}$ ↳ In special relativity

$$\boxed{\text{Work done} = \int F\,dx}$$

(one-dimensional motion)

Work done ≡ Kinetic Energy (starting from rest).
(in the absence of other forces)

[Follows energy conservation] ≡ Change in kinetic Energy
(more general, if starting from non-zero velocity)

Work done = Kinetic Energy

Kinetic Energy,

$$\boxed{T = \int F\,dx = \int \frac{dp}{dt}\,dx}$$

$$\left[\frac{dp}{dt}\,dx \equiv dp\left(\frac{dx}{dt}\right)\right]$$

$$\Rightarrow T = \int v\,dp$$

$$\boxed{\text{But, } v\,dp = d(vp) - p\,dv}$$

$$\Rightarrow T = \int d(vp) - \int p\,dv$$

$p \rightarrow$ momentum

$\boxed{p = \gamma m_0 v}$

$v \rightarrow$ velocity of the particle in it's own body frame.

$\boxed{v = \frac{dx}{dt}}$

$$\Rightarrow T = vp - \int p\,dv$$

$$\boxed{\text{But } p = \gamma m_0 v = \frac{m_0 v}{\sqrt{1-(v/c)^2}}}$$

$$\Rightarrow T = \frac{m_0 v^2}{\sqrt{1-(v/c)^2}} - m_0 \int \frac{v\,dv}{\sqrt{1-(v/c)^2}}$$

$$\Rightarrow T = \frac{m_0 v^2}{\sqrt{1-(v/c)^2}} - m_0 c^2 \int \frac{(v/c)\,d(v/c)}{\sqrt{1-(v/c)^2}}$$

$$\boxed{\Rightarrow T = \frac{m_0 v^2}{\sqrt{1-(v/c)^2}} - m_0 c^2 \int \frac{\beta\,d\beta}{\sqrt{1-\beta^2}}} \qquad \boxed{\beta = v/c}$$

$\boxed{\int \frac{\beta \, d\beta}{\sqrt{1-\beta^2}}}$ can be solved by substituting $\boxed{\omega^2 = 1-\beta^2}$.

$$\boxed{\begin{aligned} &\Rightarrow \quad 2\omega \, d\omega = -2\beta \, d\beta \\ &\Rightarrow \quad \beta \, d\beta = -\omega \, d\omega \end{aligned}}$$

The integral becomes, $\int \frac{\beta \, d\beta}{\sqrt{1-\beta^2}}$

$$= -\int \frac{\omega \, d\omega}{\sqrt{\omega^2}} = -\int d\omega = -\omega$$

$$= -\sqrt{1-\beta^2}$$

$$\therefore \quad \boxed{-\int \frac{\beta \, d\beta}{\sqrt{1-\beta^2}} = +\sqrt{1-\beta^2}}$$

$$\Rightarrow \quad \boxed{T = \frac{m_0 v^2}{\sqrt{1-(v/c)^2}} + m_0 c^2 \sqrt{1-(v/c)^2} + A}$$

where A is a constant of integration.

$$\Rightarrow T = \frac{m_0 v^2}{\sqrt{1-(v/c)^2}} + \frac{m_0 c^2 [1-(v/c)^2]}{\sqrt{1-(v/c)^2}} + A$$

$$\Rightarrow T = \frac{\cancel{m_0 v^2} + m_0 c^2 - \cancel{m_0 v^2}}{\sqrt{1-(v/c)^2}} + A$$

$$\Rightarrow \boxed{T = \frac{m_0 c^2}{\sqrt{1-(v/c)^2}} + A}$$

when $\boxed{v = 0, \quad T = 0}$ → No velocity ⇒ No kinetic energy

(Kinetic energy is energy due to motion).

$$\Rightarrow \boxed{A = -m_0 c^2}$$

$$\Rightarrow \boxed{T = \frac{m_0 c^2}{\sqrt{1-(v/c)^2}} - m_0 c^2}$$

$$\Rightarrow \boxed{\mathcal{E} = \frac{m_0 c^2}{\sqrt{1-(v/c)^2}} = T + m_0 c^2}$$

$$\boxed{m = \frac{m_0}{\sqrt{1-(v/c)^2}} = \gamma m_0}$$ → Mass in relativity

Relevant only in relativity

$\mathcal{E} = \gamma m_0 c^2$ → Total Energy

T → Kinetic Energy.

(only relevant in relativity)

$m_0 c^2$ → Rest-mass Energy (Rest Energy)

Rest energy (m_0c^2) → Relevant only in relativity

$$\boxed{\mathcal{E} = T + m_0 c^2}$$

Since, $\boxed{m = \gamma m_0}$

Kinetic Energy in Special Relativity

Total Energy → $\boxed{\mathcal{E} = m c^2}$

$\Rightarrow \boxed{T = (m - m_0)c^2}$

$\Rightarrow \boxed{T = m_0 c^2 (\gamma - 1)}$

For $\boxed{v \ll c}$.

CHECKING LIMIT

$$T = m_0 c^2 \left[\left\{ 1 - \left(\frac{v}{c}\right)^2 \right\}^{-1/2} - 1 \right]$$

$$T \simeq m_0 c^2 \left[\cancel{1} + \frac{1}{2}\left(\frac{v}{c}\right)^2 - \cancel{1} \right]$$

Binomial Theorem : $\boxed{(1+x)^n \simeq 1 + nx}$ when $x \ll 1$

Derive the Newtonian expression of kinetic energy $\frac{1}{2} m_0 v^2$

Newtonian formula of kinetic energy

$$\boxed{T \simeq m_0 c^2 \cdot \frac{1}{2}\frac{v^2}{c^2} \approx \frac{1}{2} m_0 v^2}$$

(Standard classical result)

ENERGY AND MOMENTUM

$$\boxed{\mathcal{E} = \gamma m_0 c^2 \text{ and } p = \gamma m_0 v}$$

$$\boxed{\mathcal{E}^2 = \gamma^2 m_0^2 c^4} \text{ and } \boxed{p^2 c^2 = \gamma^2 m_0^2 v^2 c^2}$$

$$\Rightarrow \mathcal{E}^2 - p^2c^2 = \gamma^2 m_0^2 c^2 (c^2 - v^2)$$

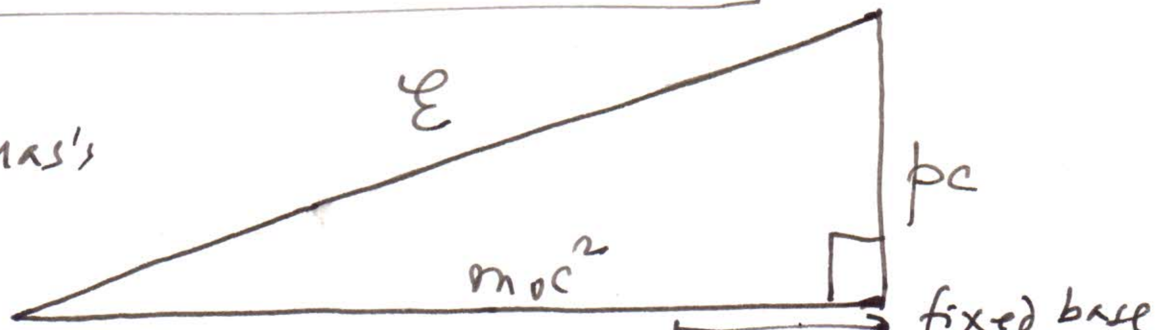
$$\Rightarrow \mathcal{E}^2 - p^2c^2 = \frac{m_0^2 c^2}{1 - v^2/c^2}(c^2 - v^2)$$

$$\Rightarrow \mathcal{E}^2 - p^2c^2 = \frac{m_0^2 c^4}{\cancel{c^2 - v^2}} \cdot \cancel{(c^2 - v^2)}$$

$$\Rightarrow \boxed{\mathcal{E}^2 = (pc)^2 + (m_0 c^2)^2}$$

Only p, and $\mathcal{E}$ can change

Pythagoras's Relation



MASSLESS PARTICLES WITH MOMENTUM

$$\boxed{\mathcal{E} = \gamma m_0 c^2} \text{ and } \boxed{p = \gamma m_0 v}$$

$$\Rightarrow \boxed{\frac{p}{\mathcal{E}} = \frac{v}{c^2}} \Rightarrow \boxed{\beta = \frac{v}{c} = \frac{pc}{\mathcal{E}}}$$

But $\boxed{\mathcal{E}^2 = (pc)^2 + (m_0 c^2)^2}$

when $\boxed{m_0 = 0}$, $\boxed{\mathcal{E} = pc}$ Zero rest mass

(A Massless particle, e.g. photon).

$$\therefore \boxed{\frac{pc}{\mathcal{E}} = 1} \Rightarrow \boxed{\beta = \frac{v}{c} = 1}$$

$\Rightarrow \boxed{v = c}$. i.e. Massless particles travel at the speed of light

No rest

space? time?

(small λ $\Rightarrow$ More particle-like)

Photon Momentum: $\boxed{p = \frac{\mathcal{E}}{c}}$ $\rightarrow$ for $\boxed{m_0 = 0}$

$$\boxed{\mathcal{E} = h\nu = \frac{hc}{\lambda}} \Rightarrow \boxed{p = \frac{h\nu}{c} = \frac{h}{\lambda}}$$

(Planck formula) Large p for small λ

Special Relativity : Formulae

$\boxed{t = \gamma t_0}$, $\boxed{\gamma = \frac{1}{\sqrt{1-\beta^2}}}$, $\boxed{\beta = v/c}$

$\boxed{l = \frac{l_0}{\gamma}}$, $\boxed{m = \gamma m_0}$, $\boxed{p = \gamma m_0 v}$

$\boxed{\mathcal{E} = \gamma m_0 c^2 = m c^2}$

$T = (m - m_0)c^2 = m_0 c^2 (\gamma - 1)$

$\boxed{\mathcal{E} = T + m_0 c^2}$ ($T \rightarrow$ Kinetic Energy)

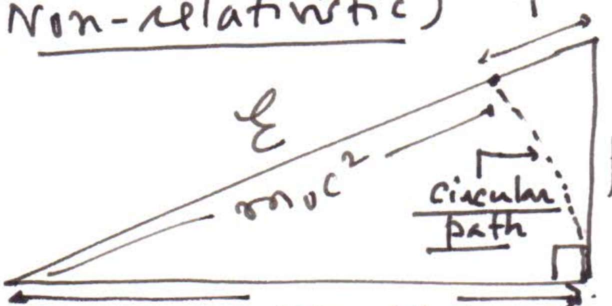
$\boxed{\mathcal{E}^2 = (pc)^2 + (m_0 c^2)^2}$ & $\beta = \frac{pc}{\mathcal{E}}$

($m_0 = 0$)

When $\boxed{pc \ll m_0 c^2} \Rightarrow \boxed{p \ll m_0 c}$

i.) $\mathcal{E}$ is mostly $m_0 c^2$

(Non-relativistic)

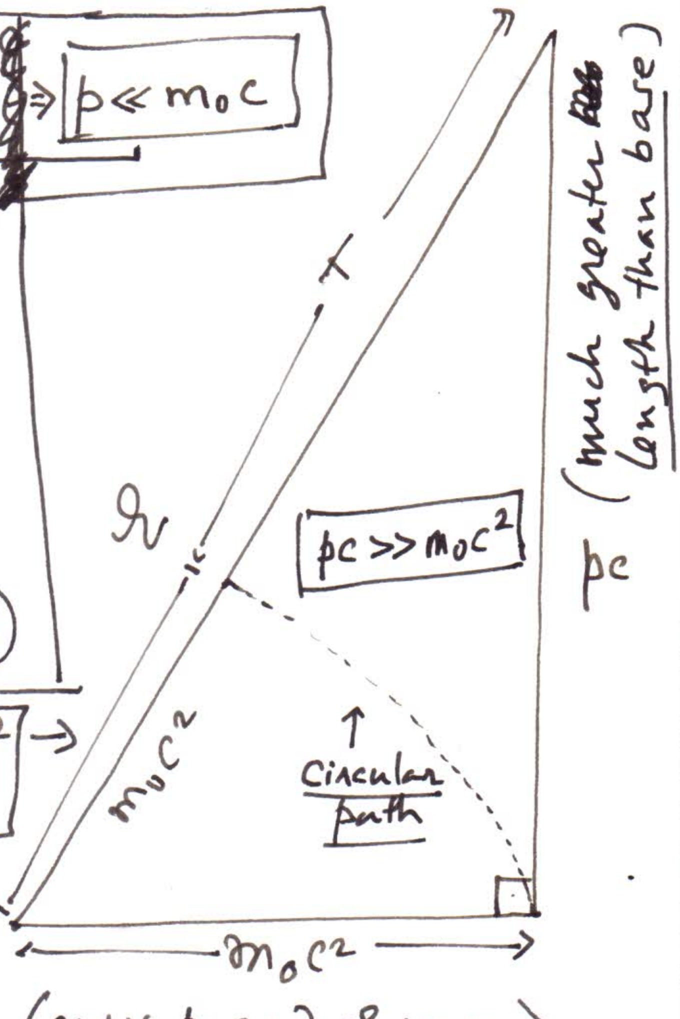


$\boxed{\mathcal{E} \simeq m_0 c^2}$ (fixed rest energy)

ii.) When $\boxed{pc \gg m_0 c^2 \Rightarrow p \gg m_0 c}$ →

$\mathcal{E}$ is mostly T.

(Highly-relativistic)



(ONLY p and $\mathcal{E}$ vary)