Fundamental theorem of Calculus:

Let $F(x)$ be a ~~scalar~~ function of x and let

$$f(x) = \frac{dF}{dx}$$

A small change in F due to a small change in x is given as

$$dF = \frac{dF}{dx}.dx = f(x)\,dx.$$

When we add up these increments from $x=a$ to $x=b$, we get

$$\int_a^b dF = F(b) - F(a) = \int_a^b f(x)\,dx = \int_a^b \frac{dF}{dx}.dx.$$

This is the fundamental theorem of Calculus.

Now if F is a function of more than one variable then we will have directional derivative of F given by the gradient.

Along an infinitesimal vector $\vec{dl}$ we will have

$$dF = \vec{\nabla}F.\vec{dl}$$

When we add up these increments along a curve C from point a to point b, we will have

$$\int_a^b dF = F(b) - F(a) = \int_{a\,(\text{along } C)}^{b} \vec{\nabla}F.\vec{dl}$$

Since the result only depends upon the end points a and b of the curve C, it doesn't matter whichever curve C we take. So we can write

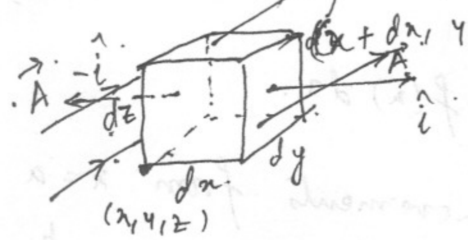
$$\int_a^b \vec{\nabla}F.\vec{dl} = F(b) - F(a)$$

If this integral is done over a closed loop then

$$\oint \vec{F} \cdot \vec{dl} = 0$$

Divergence Theorem :

Consider an infinitesimal volume $dV = dx\,dy\,dz$.



Consider a vector field $\vec{A}$ crossing this volume element. We will calculate the net flux of $\vec{A}$ through this volume element. We will find the net flux through all the six bounding surfaces. Through the element $dy\,dz$ parallel to the x-axis we have

$$\left(\vec{A}(x+dx, y, z)\cdot\hat{i}\right)dy\,dz - \left(\vec{A}(x, y, z)\cdot\hat{i}\right)dy\,dz$$

$$= \left[A_x(x+dx, y, z) - A_x(x, y, z)\right] dy\,dz$$

$$= \frac{\partial A_x}{\partial x}\,dx\,dy\,dz$$

Similarly the flux through the other surfaces will be

$$\frac{\partial A_y}{\partial y}\,dx\,dy\,dz \quad \text{and} \quad \frac{\partial A_z}{\partial z}\,dx\,dy\,dz$$

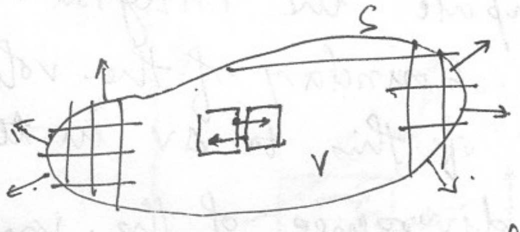
So the net flux through this volume element is

$$\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right) dx\,dy\,dz$$

$\therefore$ Net flux per unit volume $= \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right) = \vec{\nabla}\cdot\vec{A}$

This is the definition of divergence.

Now if we have a finite volume V enclosed by a surface S then we can cut this volume into small infinitesimal ~~element~~ volume elements like the one above.



We add up the flux through all these infinitesimal volume elements. The flux from adjoining cells cancel eachother since the normal on the common wall are oppositely directed for the two cells. In other words the flux which goes out (+ve) from once cell enters (−ve) the adjoining cell. Finally what remains, is the flux through the boundary surface S. So we have

$\sum$ flux through each cell = Flux through surface S.

$$\text{i.e.} \quad \int_V \vec{\nabla}\cdot\vec{A}\, dV = \oint_S \vec{A}\cdot d\vec{a}$$

<u>This is the divergence theorem.</u>

The divergance theorem gives us a nice way to understand divergence ~~of~~ itself which is just like the way we understand derivatives.
Consider an infinitesimal volume ΔV enclosed by a closed surface S. Then we can say from divergance theorem that

$$(\vec{\nabla}\cdot\vec{A})\,\Delta V = \oint_S \vec{A}\cdot\vec{da}$$

$$\therefore \vec{\nabla}\cdot\vec{A} = \lim_{\Delta V\to 0} \frac{1}{\Delta V}\oint_S \vec{A}\cdot\vec{da}$$

On the R.H.S. we compute the integral over a surface S which is the boundary of the volume element ΔV. The ratio of this to ΔV in the limit $\Delta V \to 0$ is the divergence of the vector function $\vec{A}$ at the point.

Compare this with the definition of derivative.

$$\frac{dF}{dx} = \lim_{\Delta x\to 0} \frac{1}{\Delta x}\left[F(x+\Delta x) - F(x)\right]$$

$F(x)$ and $F(x+\Delta x)$ are the values of the function F at the boundaries of the interval Δx.

Eg: Consider $\vec{A} = x\hat{i} + y\hat{j} + z\hat{k}$. Consider a volume V bounded by the surfaces $x=1$, $x=-1$, $y=1$, $y=-1$, $z=1$ and $z=-1$.

We have seen earlier that the integral

$\oint_S \vec{A}\cdot\vec{da} = 24$ where S is the surface of the cube formed.

Now $\vec{\nabla}\cdot\vec{A} = 3$ everywhere inside the cube

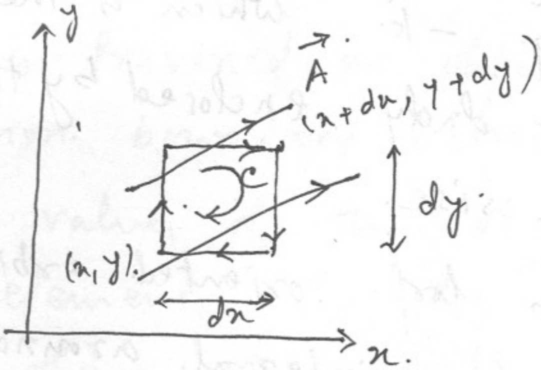
$$\therefore \int_V \vec{\nabla}\cdot\vec{A}\, dV = 3\times \text{volume of the cube}$$

$$= 3\times(2\times 2\times 2) = 24.$$

$$\therefore \oint_S \vec{A}\cdot\vec{da} = \int_V \vec{\nabla}\cdot\vec{A}\, dV$$

Stokes' Theorem:

Consider a vector field $\vec{A}$ and an infinitesimal surface element ~~aa~~ $dx\,dy$ on the $x-y$ plane.



Let us calculate $\oint_C \vec{A}\cdot\vec{dl}$ along the boundary loop enclosing this area $dx\,dy$.

The contribution from the upper horizontal element $\hat{i}dx$ is

$$\vec{A}(x, y+dy)\cdot\hat{i}\,dx$$

The contribution from the lower horizontal element $(-\hat{i}\,dx)$ is

$$\vec{A}(x, y)\cdot(-\hat{i}\,dx)$$

Similarly we will have contribution from the vertical elements. Finally we will have

$$\oint_C \vec{A}\cdot\vec{dl} = \left[\vec{A}(x, y+dy)\cdot\hat{i} - \vec{A}(x,y)\cdot\hat{i}\right]dx$$

$$+\left[\vec{A}(x+dx, y)\cdot(-\hat{j}) + \vec{A}(x,y)\cdot\hat{j}\right]dy$$

$$= \left[A_x(x, y+dy) - A_x(x,y)\right]dx + \left[-A_y(x+dx, y) + A_y(x,y)\right]dy$$

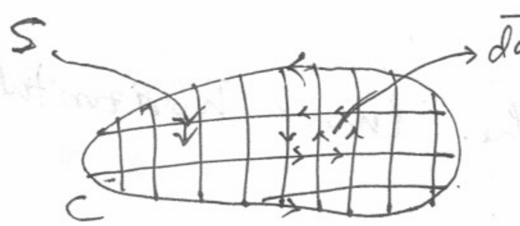
$$= \frac{\partial A_x}{\partial y}\,dy\,dx - \frac{\partial A_y}{\partial x}\,dx\,dy = -\left[\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right]dx\,dy.$$

$$\therefore \oint_C \vec{A}\cdot d\vec{l} = -(\vec{\nabla}\times\vec{A})\cdot\hat{k}\,dx\,dy$$
$$= (\vec{\nabla}\times\vec{A})\cdot(-\hat{k})\,dx\,dy.$$

So this integral gives the component of the curl of $\vec{A}$ along $-\hat{k}$ which is the normal to the surface element $dx\,dy$ enclosed by the ~~surface~~ loop traversed clockwise.

So if we have this loop oriented arbitrarily in space, then a line integral around this loop will give the curl of $\vec{A}$ along ~~the~~ normal to the surface enclosed by the loop multiplied by the area of the infinitesimal loop.

Now let us consider a finite loop enclosing a surface S.



This finite loop can be thought of made up of large number of infinitesimal loops enclosing infinitesimal surfaces da. The sense of traversing the infinitesimal loops is same as the sense of traversing the finite loop C.

Now we add up $\oint \vec{A}\cdot d\vec{l}$ from each infinitesimal ~~to~~ loops. We can see that the contribution to this sum from the common boundary between two loops cancell as shown in the figure

This cancellation is because the line element $\vec{dl}$ is traversed in opposite direction on the common boundary between the two loops, while the value of the vector field is same on this element. So the sum of the contribution from these two loops is equal to the loop integral over the outer boundary only.

Like wise when we sum over all these loops, all the contribution that lies inside the surface S cancell, while the contribution from the boundary C enclosing the surface S remains. So this sum gives

$$\oint_C \vec{A}\cdot dl.$$

Since the integral over individual infinitesimal loops is $\vec{\nabla}\times\vec{A}\cdot\hat{n}\,da$ where $\hat{n}$ is the local normal to the surface of the infinitesimal loop, we have

$$\oint_C \vec{A}\cdot\vec{dl} = \int_S (\vec{\nabla}\times\vec{A})\cdot\hat{n}\,da$$

This is the statement of the Stokes' Theorem.

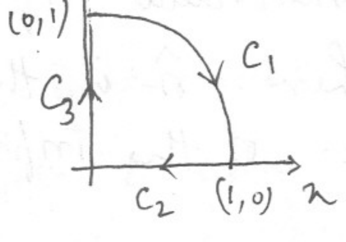
Stokes theorem gives us an alternative definition of the curl.

If we want the component of $\vec{\nabla}\times\vec{A}$ along a direction $\hat{n}$, then we consider a small convenient loop of area Δa whose normal is along $\hat{n}$. Then we do the integral $\oint_C \vec{A}\cdot d\vec{l}$ along a curve C enclosing da. The curl of $\vec{A}$ is then defined as

$$(\vec{\nabla}\times\vec{A})\cdot\hat{n} = \lim_{\Delta a\to 0}\frac{1}{\Delta a}\oint_C \vec{A}\cdot d\vec{l}$$

Here again we see that curl is a derivative which is given by the value of the field $\vec{A}$ along a boundary divided by the area enclosed by the boundary in the limit as the area $(\Delta a)\to 0$.

Eg: Let $A = y\hat{i} - x\hat{j}$. Let us verify stokes' theorem along the curve C shown. C_1 is a part of the circle $x^2+y^2=1$, C_2 and C_3 are along x and y axes



$$\oint_C \vec{A}\cdot d\vec{l} = \int_{C_1}\vec{A}\cdot d\vec{l} + \int_{C_2}\vec{A}\cdot d\vec{l} + \int_{C_3}\vec{A}\cdot d\vec{l}$$

Along C_1 $\quad x^2+y^2=1$

$$\therefore\ 2x\,dx + 2y\,dy = 0 \Rightarrow dy = -\left(\frac{x}{y}\right)dx$$

$$\vec{A}\cdot d\vec{l} = y\,dx - x\,dy = \left(y+\frac{x^2}{y}\right)dx = \frac{y^2+x^2}{y}\,dx$$

$$= \frac{1}{\sqrt{1-x^2}}\,dx$$

$$\therefore \int_{C_1} A \cdot dl. = \int_0^1 \frac{1}{\sqrt{1-x^2}}\, dx = \frac{\pi}{2}$$

Along C_2, $y = 0$

$\therefore dy = 0 \Rightarrow \vec{dl} = dx\,\hat{i}$

$\vec{A} \cdot \vec{dl} = y\,dx = 0$ since $y = 0$

$$\therefore \int_{C_2} \vec{A} \cdot \vec{dl} = 0$$

Along C_3, $x = 0$

$\therefore dx = 0 \Rightarrow \vec{dl} = dy\,\hat{j}$

$\vec{A} \cdot \vec{dl} = -x\,dy = 0$ since $x = 0$

$$\therefore \int_{C_3} \vec{A} \cdot \vec{dl}. = 0$$

$$\therefore \oint_C \vec{A} \cdot \vec{dl}. = \int_{C_1} \vec{A} \cdot dl. + \int_{C_2} \vec{A} \cdot dl. + \int_{C_3} \vec{A} \cdot \vec{dl}$$

$$= \frac{\pi}{2}$$

Now let us evaluate $\int_S \vec{\nabla} \times \vec{A} \cdot \hat{n} da$ over the surface S enclosed by the closed loop C. We consider the flat ~~plane~~ surface on the $x-y$ plane. The normal $\hat{n}$ is taken along $-\hat{k}$; This is because the loop is traversed clock wise. (according to right hand screw rule.)

$\vec{\nabla} \times \vec{A} = -2\hat{k}$ everywhere. So.

$$\int_S (\vec{\nabla} \times \vec{A}) \cdot \hat{n} da. = (-2\hat{k}) \cdot (-\hat{k}) \times \frac{\pi}{4} = \frac{\pi}{2}$$