Nature of Mass

Inertial Mass : $\boxed{F = ma}$

For a fixed force, $\boxed{m \propto \frac{1}{a}}$.

1/. m is a measure of inertia, i.e., the ~~[illegible]~~ tendency to resist a change in the state of rest or a constant velocity.

Gravitational Mass : $\boxed{W = \frac{GM_e}{r_e^2} m'}$

No inertia.

Write $\boxed{g = \frac{GM_e}{r_e^2}}$ $\therefore$ $\boxed{W = m'g}$

1/. g has a fixed value of $9.8\ ms^{-2}$.

2/. m′ is a measure of the gravitational force (on the surface of the earth this attractive force is weight).

Experimental facts :

1/. m (inertial mass) = m′ (gravitational mass)

2/. Gravity is EVERYWHERE.

3/. No frame is really inertial.

Non-inertial frames

$$\boxed{\begin{array}{l}\text{1/. } S \text{ is an } \underline{\text{inertial}} \text{ frame.} \\ \text{2/. } S' \text{ is a } \underline{\text{non-inertial}} \text{ frame, moving with } \underline{\text{acceleration}}\ \vec{A}. \\ \qquad\qquad\qquad (\underline{\text{with respect to } S}) \end{array}}$$

By <u>Galilean transformation</u>, i.e.,

$$\boxed{\vec{u} = \vec{u}' + \vec{v}} \Rightarrow \boxed{\vec{u}' = \vec{u} - \vec{v}},$$

we get

$$\boxed{\frac{d\vec{u}'}{dt} = \frac{d\vec{u}}{dt} - \frac{d\vec{v}}{dt}}$$

If ~~...~~ a <u>particle</u> has mass m,

then, $m\frac{d\vec{u}'}{dt} = m\frac{d\vec{u}}{dt} - m\frac{d\vec{v}}{dt}$

$$\boxed{\begin{array}{l}\vec{a}' = \frac{d\vec{u}'}{dt} \\ \vec{a} = \frac{d\vec{u}}{dt} \\ \vec{A} = \frac{d\vec{v}}{dt}\end{array}}$$

$$\Rightarrow \boxed{m\vec{a}' = m\vec{a} - m\vec{A}}$$

Write $\boxed{\vec{F} = m\vec{a}}$ and $\boxed{\vec{F}' = m\vec{a}'}$

$$\therefore \boxed{\vec{F}' = \vec{F} - m\vec{A}}$$

Write $\underline{-m\vec{A} = \vec{F}_{in}}$

In a <u>non-inertial frame</u> a particle is subject <u>to an additional</u> <u>"inertial force"</u>, $\boxed{\vec{F}_{in} = -m\vec{A}}$ (Property of inertia).

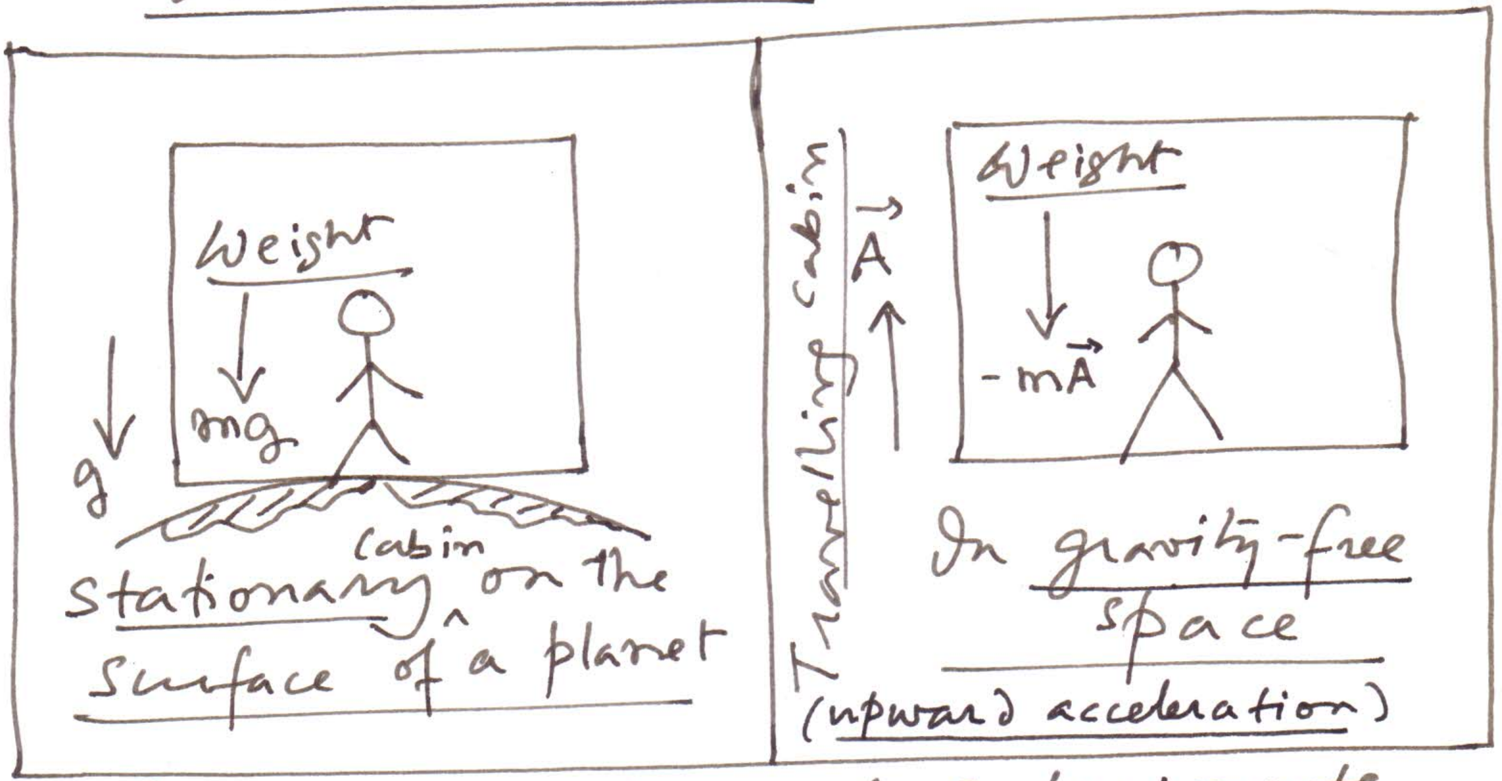
Einstein's Equivalence Principle

1/. Inertial force: $\boxed{\vec{F}_{in} = -m\vec{A}}$

2/. Gravitational force: $\boxed{\vec{W} = \vec{F}_{gr} = m\vec{g}}$

Both forces are proportional to m.

Consider an enclosed cabin with an observer inside, in two scenarios.

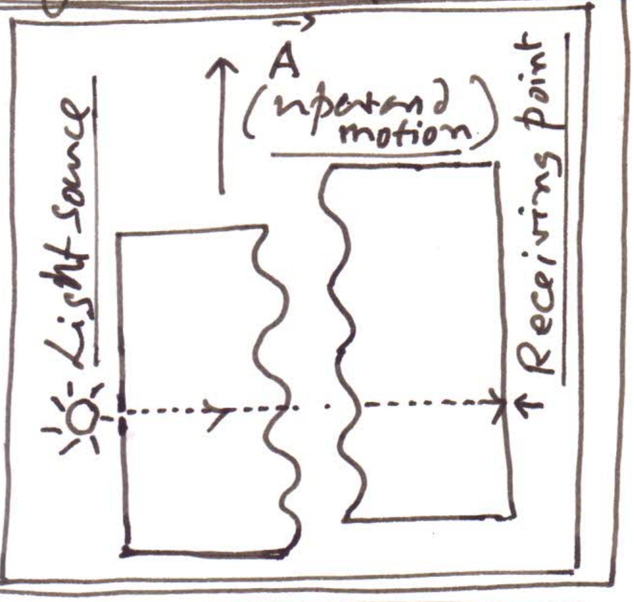


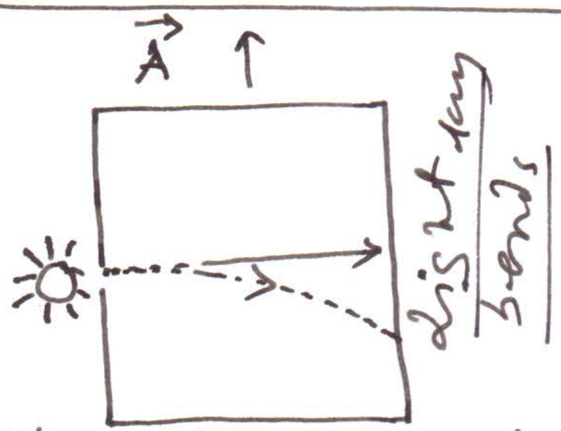
Both are mechanical experiments.

> No experiment, mechanical or otherwise, can distinguish between a uniform gravitational field ($\vec{g}$) and an equivalent uniform acceleration ($\vec{A} = -\vec{g}$)
>
> – Albert Einstein.

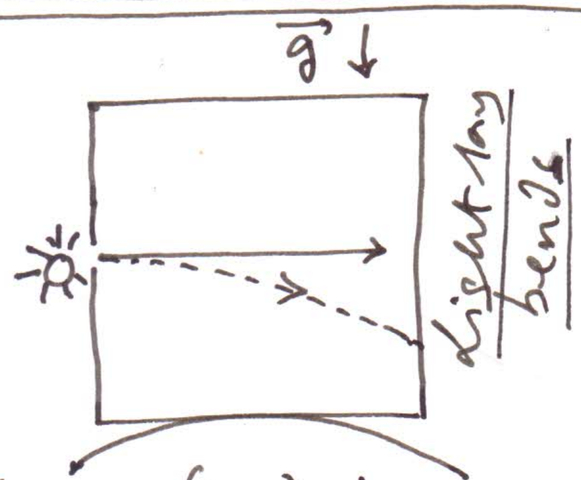
Bending of Light by Gravity

Consider an enclosed cabin accelerating upwards, with a beam of light travelling across it.





Upwardly accelerating frame in gravity-free space.



Frame fixed in a gravitational field.

$\boxed{\frac{d^2x}{dt^2} = 0} \Rightarrow \frac{dx}{dt} = v_0 \Rightarrow \boxed{x = x_0 + v_0 t}$

$\boxed{\frac{d^2z}{dt^2} = -g} \Rightarrow \frac{dz}{dt} = -gt + u_0$ When $t = 0$, $\frac{dz}{dt} = 0 \Rightarrow u_0 = 0$

$\Rightarrow \boxed{z = -\frac{1}{2}gt^2 + z_0} \Rightarrow z - z_0 = -\frac{1}{2}gt^2$

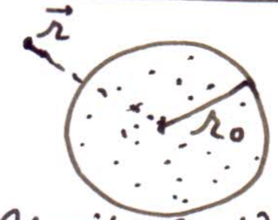
& $x - x_0 = v_0 t$.

$\therefore \boxed{z - z_0 = -\frac{1}{2}g\left(\frac{x - x_0}{v_0}\right)^2}$ If $\boxed{X = x - x_0}$ & $\boxed{Z = z - z_0}$

Parabola $\boxed{Z = -\frac{1}{2}g\frac{X^2}{v_0^2}}$ Projectile motion

Escape Velocity and Black Holes

Gravity: $\boxed{m v \frac{dv}{dr} = -\frac{GMm}{r^2}}$



Gravity field

Travel to infinity

$$\Rightarrow \int_{v_{esc}}^{0} v\,dv = -GM \int_{r_0}^{\infty} r^{-2}\,dr$$

$$\Rightarrow \left.\frac{v^2}{2}\right|_{v_{esc}}^{0} = -\left.\frac{GM}{-1} r^{-1}\right|_{r_0}^{\infty}$$

$$\Rightarrow \boxed{\frac{v_{esc}^2}{2} = \frac{GM}{r_0}} \Rightarrow \boxed{v_{esc}^2 = 2G\left(\frac{M}{r_0}\right)}$$

$\boxed{v_{esc} \rightarrow c}$ but $\left(M/r_0\right)$ can grow as large as possible (arbitrarily)

When $\boxed{v_{esc} = c}$, write $\boxed{r_0 = R_s}$

(massive)

R_s is Schwarzschild radius (Karl Schwarzschild) of a spherical BLACK HOLE.

i.) Large mass
ii.) Small radius (compact)

$$\therefore \boxed{\frac{M}{R_s} = \frac{c^2}{2G}} \quad \left(\text{OR } \frac{M}{r_0} \geq \frac{c^2}{2G}\right)$$

$$\Rightarrow \boxed{R_s = \frac{2GM}{c^2}} \quad \left(\text{OR } r_0 \leq \frac{2GM}{c^2}\right)$$

$r_0 = R_s$ for equality

$\hookrightarrow$ Radius of Event Horizon.

For Earth, $M = 6 \times 10^{24}$ kg, $R_s \simeq 9$ mm.

For the Sun, $M_\odot = 2 \times 10^{30}$ kg, $R_s = 3$ km.

Astrophysical Objects (Physical Context for Black Holes)

1/. Massive astrophysical objects such as stars, ~~[illegible]~~ where gravity dominates.

- i.) White dwarfs: $\leq 1.4 M_\odot$ (Chandrasekhar limit)
  (gravity against electron degeneracy)
- ii.) Neutron stars: $\lesssim 5 M_\odot$.
  (gravity against neutron degeneracy)
- iii.) Black holes: $\gtrsim 5 M_\odot - 10 M_\odot$
  (Stellar black holes).

Super-massive black holes $\sim 10^5 - 10^{10} \times M_\odot$

2/. Gravity is MOST effective on large length scales, with charge neutral matter.

Eg. For 2 electrons:

$$\left|\frac{F_{elec}}{F_{grav}}\right| = \underbrace{\left(\frac{e^2}{4\pi\epsilon_0 r^2}\right)}_{\text{Electrical}} \times \underbrace{\left(\frac{r^2}{G m_e^2}\right)}_{\text{Gravitation}} \sim 4 \times 10^{42}.$$

Electrical forces are strong on small scales.

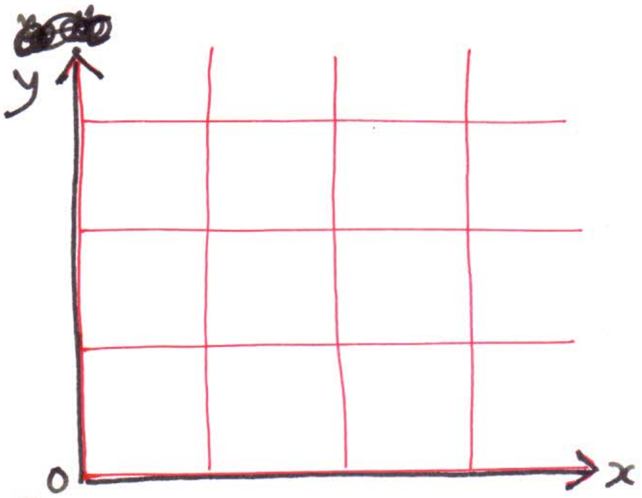
3/. Stars have a spherical configuration.

- i.) Stellar black holes have an event horizon whose shape is spherical.
- ii.) Fluid origin with associated fluid properties like surface tension.

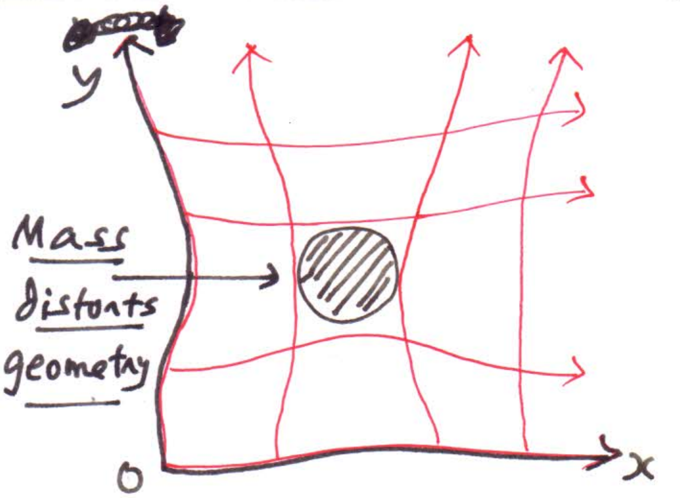
Gravity as Geometry

Newton - Field

Einstein - Geometry



Geometry in gravity-free conditions (two-dimensional plane).

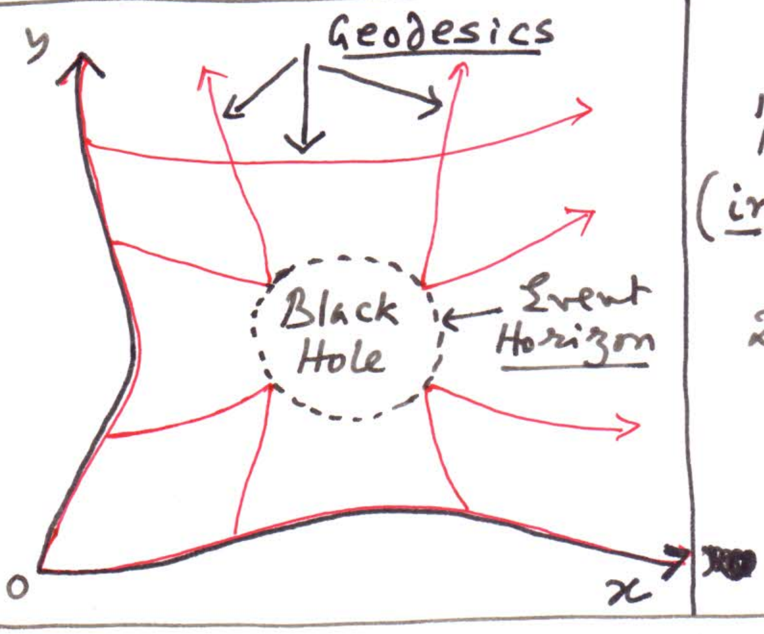


Warping of geometry due to gravity. Both space and time are warped.

1/. Geometry is physically influenced by the presence of matter.

2/. Light follows paths known as "GEODESICS" in the geometric grid. | Eg. longitudes

3/. Near massive objects light is deviated.

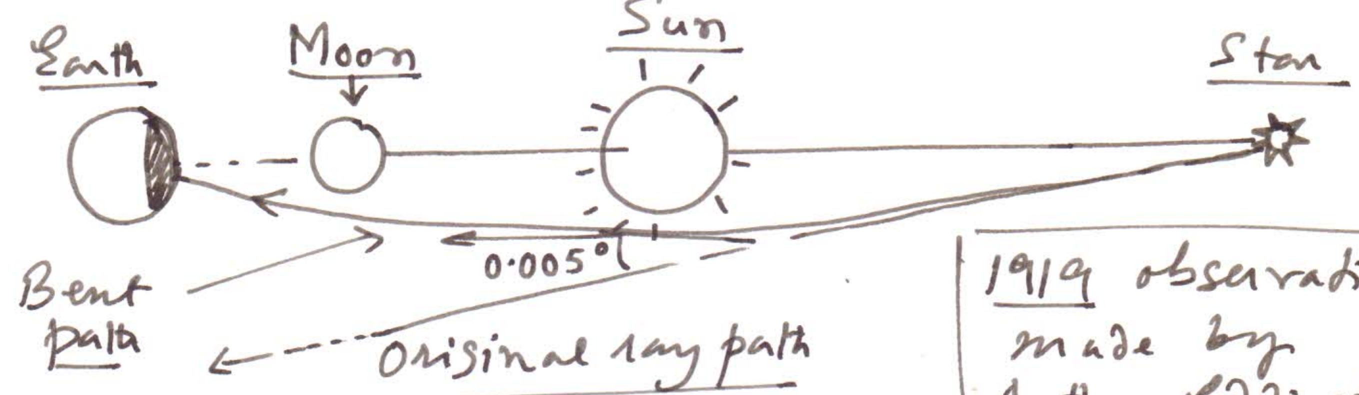


(are ruptures)

1/. Black holes rupture (in) the geometric fabric.

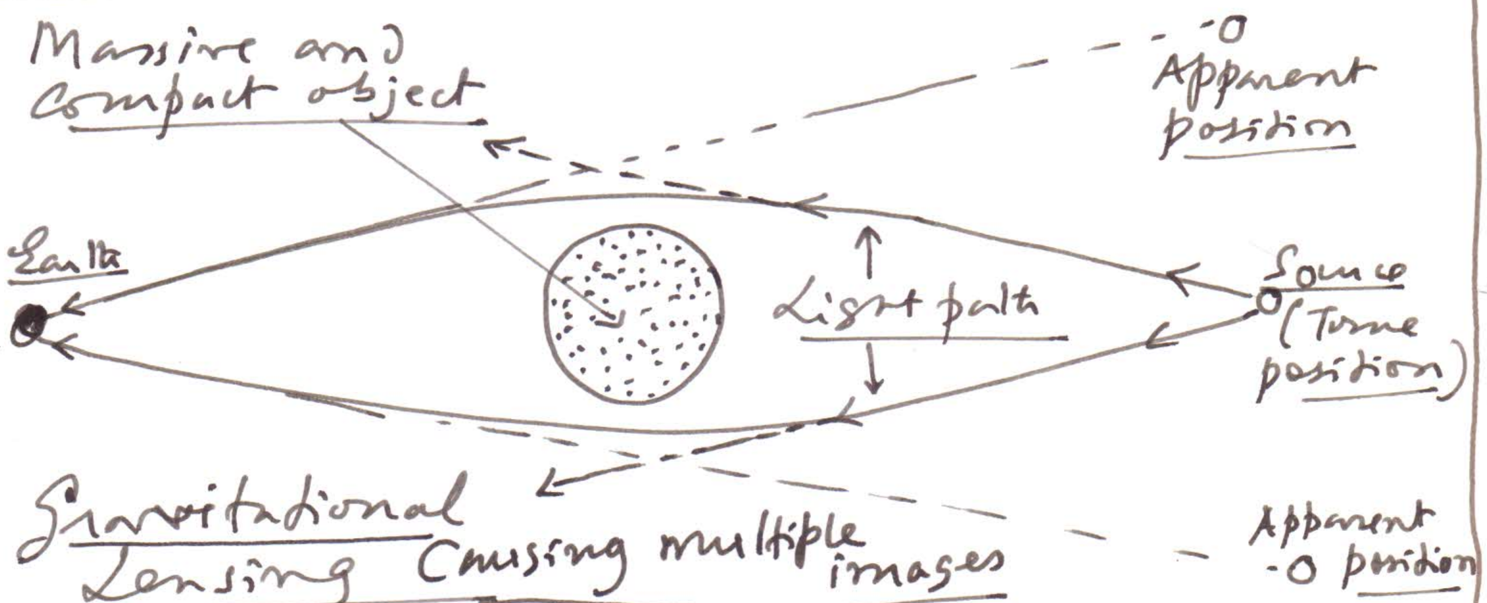
2/. Light does not have a continuous passage through the rupture.

Astrophysical Contexts of Bending of Light due to Gravity.



1919 observation made by Arthur Eddington.

(Observed during a solar eclipse).



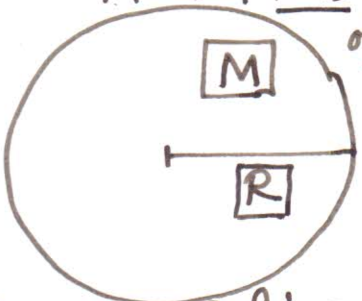
Gravitational Lensing Causing multiple images

A massive and compact astrophysical object (Such as a QUASAR - quasi-Stellar object), Can act as a strong gravitational lens for light coming from distant galaxies.

Gravitational Redshift

Consider a compact and massive astrophysical object, radiating light.

M → Mass of the object



R → Radius at the surface.

1/. Gravitational Potential energy at the surface is $\boxed{-\frac{GMm}{R}}$ for a particle of mass m.

2/ For light an "effective mass" is

$\boxed{m = \frac{\mathcal{E}}{c^2}}$, where $\boxed{\mathcal{E} = h\nu}$ → Planck formula

$\therefore$ $\boxed{m = \frac{h\nu}{c^2}}$ → "Effective" mass of a photon (particle)

3/ Total energy of light is

$$\boxed{h\nu - \frac{GM}{R}\left(\frac{h\nu}{c^2}\right)}$$

Approximately Correct (for $GM/Rc^2 \ll 1$)

The second term is gravitational.

Light has a gravitational behaviour. → (mass)

4/. Travelling far from the compact object, the gravitational influence becomes weak, but energy is conserved.

This energy is $h\nu'$. (P.T.O.)

Hence, $\boxed{h\nu' = h\nu - \frac{GM}{R}\frac{h\nu}{c^2}}$

Redshift → shifted to larger λ.

$\Rightarrow$ $\boxed{\frac{\nu'}{\nu} = 1 - \frac{GM}{Rc^2}}$ h cancels out (NOT a quantum effect)

Now $\frac{\nu'}{\nu} - 1 = \frac{\nu' - \nu}{\nu} = \frac{\Delta\nu}{\nu}$

Also, $\boxed{\nu = c\lambda^{-1}}$ $\Rightarrow$ $\boxed{\Delta\nu = -\frac{c}{\lambda^2}\Delta\lambda.}$

$\Rightarrow$ $\boxed{\frac{\Delta\nu}{\nu} = -\frac{GM}{Rc^2}}$ Final frequency is lower. Why?

Energy is used up to work against gravity.

and $\boxed{-\frac{c}{\nu\lambda}\frac{\Delta\lambda}{\lambda} = -\frac{GM}{Rc^2}}$

Gravitational Red Shift → $\boxed{\frac{\Delta\lambda}{\lambda} = \frac{GM}{Rc^2}}$ $\left(\because \lambda\nu = c\right)$

Wavelength increases.

i.) If M/R is large (for a massive and compact object), then $\frac{\Delta\lambda}{\lambda}$ is also appreciably large.

(Different from DOPPLER EFFECT)

ii.) Light is redshifted due to gravity. (NOT a quantum effect)

iii.) Effect is appreciably observed in white dwarfs.

iv.) Applies for any electromagnetic wave.

Application of Relativity: Global Positioning System

1/. Determines location on the Earth (latitude, longitude, elevation) within 1 m.

2/. Uses 24 satellites in circular orbits at height of 20,200 km above the Earth. Satellites travel at 6 $km\,s^{-1}$.

3/. At least 4 satellites cover any spot on earth at any time. (Why?)

4/. There are 3 spatial coordinates (x, y, z) and 1 time coordinate (t). So the number of unknowns is $\boxed{4}$. $\vec{r} \equiv \vec{r}(\vec{x}, \vec{y}, \vec{z})$

$$\boxed{|\vec{r} - \vec{r}_j| = c\,|t - t_j|} \quad (j = 1, 2, 3, 4)$$

5/. Satellites send radio signals (electromagnetic waves) to the location.

6/. Corrections are ~~require~~ made for:

i.) Index of refraction of the atmosphere, which slows down the radio waves.

ii.) Time dilation of the atomic clocks on the moving satellites.

iii.) Gravitational redshift through 20,200 km.

iv.) Relativistic velocity addition