Vector Potential.

In electrostatics since $\vec{\nabla} \times \vec{E} = 0$, we can express $\vec{E} = -\vec{\nabla}\phi$ where $\phi$ is the electrostatic potential. Once $\phi$ is found we can find the electric field at all points. The equivalent equation in magnetostatics is

$$\vec{\nabla} \cdot \vec{B} = 0.$$

Since divergence of a curl of a vector field is zero, we claim that $\vec{B}$ can be expressed as the curl of a vector field. i.e. —— I.

$$\vec{B} = \vec{\nabla} \times \vec{A}.$$

Indeed we can always find a vector field $\vec{A}$ that satisfies Eq I. $\vec{A}$ is called a magnetic vector potential for the magnetic field $\vec{B}$. Just like the electrostatic scalar potential $\phi$, the magnetic vector potential is not unique. We can always add a constant to the scalar potential to $\phi$ and it still gives the same electric field. For the case of magnetic vector potential $\vec{A}$ the quantity we can add is more complicated. We can add the gradient of a scalar i.e if

$$\vec{A}' = \vec{A} + \vec{\nabla} F$$

where $F$ is a scalar field. then

$$\vec{\nabla} \times \vec{A}' = \vec{\nabla} \times \vec{A} + \vec{\nabla} \times (\vec{\nabla} F) = \vec{\nabla} \times \vec{A} = \vec{B}.$$

So $\vec{A}'$ is also a valid magnetic vector potential.

To specify $\vec{A}$ more precisely we must also know $\vec{\nabla}\cdot\vec{A}$. So far we don't have any condition on $\vec{\nabla}\cdot\vec{A}$. If we specify some value ~~of~~ for $\vec{\nabla}\cdot\vec{A}$ say $\vec{\nabla}\cdot\vec{A} = 0$ then $\vec{A}$ can be uniquely determined. But this is purely a matter of choice as fixing a ~~constant~~ reference point to the scalar potential. So let us have

$$\vec{\nabla}\times\vec{A} = \vec{B} \quad \text{and} \quad \vec{\nabla}\cdot\vec{A} = 0.$$

These Equation look identical to the following

$$\vec{\nabla}\times\vec{B} = \mu_0\vec{J} \quad \text{and} \quad \vec{\nabla}\cdot\vec{B} = 0.$$

We know that given $\vec{J}$,

$$\vec{B} = \frac{\mu_0}{4\pi}\int \frac{\vec{J}\times\hat{\imath}}{\imath^2}\,d\tau$$

So for a given magnetic field $\vec{B}$, the vector potential $\vec{A}$ can be written in analogy as

$$\vec{A} = \frac{1}{4\pi}\int \frac{\vec{B}\times\hat{\imath}}{\imath^2}\,d\tau.$$

The source of a magnetic field is the current density $\vec{J}$ in the region. So we would like to evaluate the ~~ve~~ vector potential for a given source. This can be obtained as follows.

$$\vec{\nabla}\times\vec{B} = \vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \nabla^2(\vec{A}) = \mu_0\vec{J}$$

$$\therefore \quad \nabla^2(\vec{A}) = -\mu_0\vec{J}$$

This is a set of three equations along the three components of the vector $x, y$ and $z$ i.e.

$$\nabla^2 A_x = -\mu_0 J_x, \quad \nabla^2 A_y = -\mu_0 J_y, \quad \nabla^2 A_z = -\mu_0 J_z$$

We know the solution to these equations from the Poisson's Equan in electrostatics. $\nabla^2\phi = -\frac{\rho}{\epsilon_0}$.

Reading off the solution from there we have

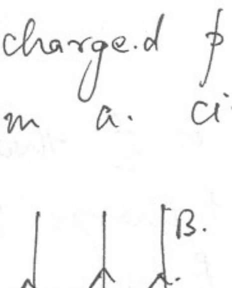
$$A_x = \frac{\mu_0}{4\pi}\int \frac{J_x}{r}\, d\tau$$

Likewise for $A_y$ and $A_z$. Combining all the components we have

$$\vec{A} = \frac{\mu_0}{4\pi}\int \frac{\vec{J}}{r}\, d\tau$$

The vector potential $\vec{A}$ doesn't have a straightforward interpretation as the scalar potential in electrostatics. ~~Howev~~ The scalar potential is the potential energy or the work done by a ~~posit~~ unit charged particle in moving from infinity to a point. Dimensionally the vector potential ~~has~~ is that of momentum per unit charge. In certain cases we can indeed relate it to the momentum of a charged particle in a magnetic field. Let us consider a simple example.

Consider a uniform magnetic field $\vec{B}$ along $\hat{z}$. Any charged particle moving perpendicular to $\vec{B}$ will perform a circular motion. The radius of this circular orbit is given by




$$qvB = \frac{mv^2}{r}$$

$$\therefore mv = qBr$$

$mv$ is the momentum of the particle. Considering the vectors we have $\vec{p} = q(\vec{B} \times \vec{r})$.

Let us now ~~calcu~~ find a vector potential for this magnetic field.

Since $\vec{B}$ is along $\hat{z}$, if we want to find $\vec{A}$ at each point on the orbit of the particle we can use the stokes' theorem.

$$\oint_C \vec{A}\cdot d\vec{l} = \int_S \vec{\nabla}\times\vec{A}\cdot\hat{n}\,da = \int_S \vec{B}\cdot\hat{n}\,da$$

where $S$ is the surface perpendicular to the $z$ axis. Enclosed by the circuit $C$.

$$\therefore \oint_C \vec{A}\cdot d\vec{l} = B\times\pi r^2.$$

Along the chosen curve $\vec{dl} = r\,d\phi\,\hat{\phi}$. By symmetry of the problem $A_\phi$ is constant over $\phi$. So

$$\oint_C \vec{A}\cdot d\vec{l} = \int_0^{2\pi} A_\phi\, r\, d\phi = B\pi r^2.$$

$$\therefore A_\phi \cdot 2\pi r = B\cdot\pi r^2.$$

$$\therefore A_\phi = \frac{1}{2}Br$$

$$\therefore \vec{A} = \frac{1}{2}(\vec{B}\times\vec{r}) = \frac{\vec{p}}{2q}$$

So our choice of vector potential is $\frac{1}{2}$ that of the momentum of ~~the particle for~~ a unit charge performing a circular motion in the magnetic field.

The interpretation of the vector potential as momentum is very interesting because in relativity the four components $(t, x, y, z)$ has $t$ and the three space components. They become dimensionally eqvt if we consider $(ct, x, y, z)$. The electric & magnetic fields $(\vec{E}, \vec{B})$ also become dimensionally equivalent if we consider $\left(\frac{\vec{E}}{c}, \vec{B}\right)$

Now if we consider the four quantities $(\phi, A_x, A_y, A_z)$, they indeed form a four dimensional vector in relativistic formulation. Here $\phi$ is the scalar potential of electrostatics and $\vec{A}$ is the magnetic vector potential. Infact $\frac{1}{c}\phi$ has the same dimension as $\vec{A}$.

In relativity the space-time gets mixed i.e. transform into each other for moving observers. Likewise the four components of the potential $(\phi, A_x, A_y, A_z)$ gets mixed for a moving observer producing the appropriate electro-magnetic field for the observer. In fact the energy and the momentum $(E, p_x, p_y, p_z)$ transform in the same way.

Eg 1): What current density would produce the vector potential $\vec{A} = k\hat{\phi}$ (in cylindrical co-ordinates).

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

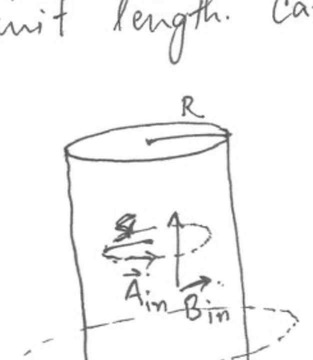
$$= \left(\frac{1}{s}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right)\hat{s} + \left(\frac{\partial A_s}{\partial z} - \frac{\partial A_z}{\partial s}\right)\hat{\phi} + \frac{1}{s}\left(\frac{\partial}{\partial s}(sA_\phi) - \frac{\partial A_s}{\partial \phi}\right)\hat{z}$$

$$= \frac{1}{s}\frac{\partial}{\partial s}(s \cdot k)\,\hat{z} = \frac{k}{s}\hat{z}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$

$$\therefore \vec{J} = \frac{1}{\mu_0}\left(\vec{\nabla} \times \vec{B}\right) = \frac{1}{\mu_0}\left[-\frac{\partial}{\partial s}\left(\frac{k}{s}\right)\right]\hat{\phi} = \frac{1}{\mu_0}\frac{k}{s^2}\hat{\phi}$$

Eg 2: Let us find a vector potential for the case of an infinite solenoid of radius R. The solenoid has n turns per unit length carrying current I.




We have found the magnetic field inside the solenoid.

$$\vec{B}_{in} = \mu_0 n I \hat{z}$$

Consider a loop of radius s inside the solenoid as shown.

$$\oint_C \vec{A}_{in} \cdot d\vec{l} = \iint_S (\vec{\nabla} \times A_{in}) \cdot \hat{n}\, da = \int_S \vec{B}_{in} \cdot \hat{n}\, da$$

$$\therefore A_{in} \times 2\pi s = \mu_0 n I \times \pi s^2$$

$$\therefore A_{in} = \frac{\mu_0 n I s}{2} \Rightarrow \vec{A}_{in} = \frac{\mu_0 n I s}{2}\hat{\phi}$$

For outside.

$$A_{out} \times 2\pi s = \int_S \vec{B} \cdot \hat{n}\, da = \mu_0 n I \times \pi R^2$$

($\vec{B}$ outside the solenoid is 0)

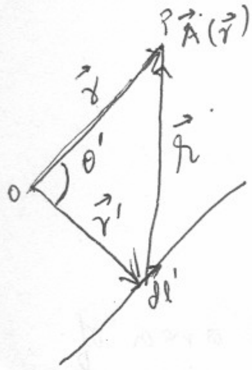
$$\therefore A_{out} = \frac{\mu_0 n I R^2}{2s} \Rightarrow \vec{A}_{out} = \frac{\mu_0 n I R^2}{2s}\hat{\phi}$$

So there exist a vector potential outside though $\vec{B}_{out} = 0$

## Multipole Expansion of the Vector potential:

Just like expanding the potential at a point in terms of monopole, dipole, quadropole etc / potential, we can expand the vector potential $\vec{A}$; for a wire carrying a current $I$.

$$\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi}\int \frac{d\vec{l}'}{\mathscr{r}}$$




$$\frac{1}{\mathscr{r}} = \frac{1}{\sqrt{r^2 + r'^2 - 2rr'\cos\theta'}} = \frac{1}{r\sqrt{1+\left(\frac{r'}{r}\right)^2 - 2\frac{r}{r'}\cos\theta'}}$$

$$= \frac{1}{r}\sum_{l=0}^{\infty}\left(\frac{r'}{r}\right)^l P_l(\cos\theta')$$

$$\therefore \vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi}\int \frac{1}{r}\sum_{l=0}^{\infty}\left(\frac{r'}{r}\right)^l P_l(\cos\theta')\, d\vec{l}'$$

In this we are assuming that $r > r'$ all along the wire. This will certainly not be true for a very long wire stretching to infinity. So this expression is valid only for confined currents. This is the case for current loops which are confined in some region and we are observing $\vec{A}(\vec{r})$ at a large distance. So under these condition the integral becomes a closed loop integral.

$$\vec{A}(\vec{r}) = \frac{\mu_0 I}{4\pi}\sum_{l=0}^{\infty}\frac{1}{r^{l+1}}\oint (r')^l P_l(\cos\theta')\, d\vec{l}'$$

$$= \frac{\mu_0 I}{4\pi}\left[\frac{1}{r}\oint d\vec{l}' + \frac{1}{r^2}\oint r'\cos\theta'\, d\vec{l}' + \cdots\right]$$

For a closed loop $\oint d\vec{l}' = 0$. So the monopole term corresponding to $l=0$ does not contribute to the magnetic vector potential. This bolsters the fact that the Biot-Savart laws only gives magnetic fields for dipoles and other higher multipoles. There is not magnetic monopoles so far in nature.

So unlike electrostatics the most dominant term of magnetic field is always the dipole term and not the monopole term. The dipole term is written as

$$\vec{A}_{dip} = \frac{\mu_0 I}{4\pi r^2} \oint r' \cos\theta' \, d\vec{l}'$$

$$= \frac{\mu_0 I}{4\pi r^2} \oint (\hat{r} \cdot \vec{r}') \, d\vec{l}'$$

$$= \frac{\mu_0 I}{4\pi r^2} \left( \oint \frac{1}{2} (\vec{r}' \times d\vec{l}') \right) \times \hat{r}$$ (see next page). *

$$= \frac{\mu_0 I}{4\pi r^2} \vec{a} \times \hat{r}$$

where $\vec{a} = \oint \frac{1}{2} (\vec{r}' \times d\vec{l}')$ is the vector area of the loop. The quantity

$\vec{m} = I\vec{a}$ is called the dipole moment of the loop.

$$\therefore \vec{A}_{dip} = \frac{\mu_0}{4\pi r^2} (\vec{m} \times \hat{r})$$

$$\vec{B}_{dip} = \vec{\nabla} \times \vec{A}_{dip} = \frac{\mu_0}{4\pi} \left[ \vec{\nabla} \times \left( \vec{m} \times \frac{\hat{r}}{r^2} \right) \right]$$

$$= \frac{\mu_0}{4\pi} \left[ -(\vec{m} \cdot \vec{\nabla}) \frac{\hat{r}}{r^2} + \vec{m} \left( \vec{\nabla} \cdot \frac{\hat{r}}{r^2} \right) \right]$$

$\vec{\nabla} \cdot \frac{\hat{r}}{r^2} = 4\pi \, \delta^{(3)}(\vec{r})$. So for $r \neq 0$

$$\vec{B}_{dip} = \frac{\mu_0}{4\pi} \left[ -(\vec{m} \cdot \vec{\nabla}) \frac{\hat{r}}{r^2} \right].$$

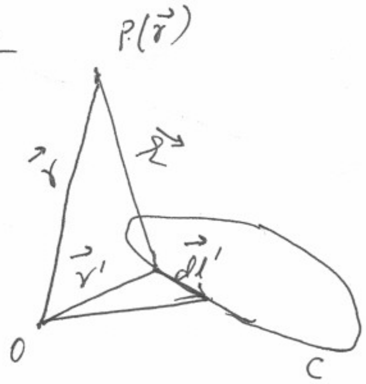
Consider $\vec{m} = m\hat{z}$ i.e. a dipole along $\hat{z}$. Then

$$(\vec{m} \cdot \vec{\nabla}) \frac{\hat{r}}{r^2} = m \frac{\partial}{\partial z} \left( \frac{\vec{r}}{r^3} \right) = -\vec{r} \left( \frac{-3mz}{r^5} \right) + \frac{\vec{m}}{r^3}$$

$$= \left( -3 (\vec{m} \cdot \hat{r}) \hat{r} + \vec{m} \right) \frac{1}{r^3}$$

$$\therefore \vec{B}_{dip} = \frac{\mu_0}{4\pi r^3} \left[ 3 (\vec{m} \cdot \hat{r}) \hat{r} - \vec{m} \right] \quad \text{for } r > 0.$$

* $P(\vec{r})$




We want to evaluate

$$\oint_C (\hat{r}\cdot\vec{r}')\,\vec{dl}'$$

As we move along the curve $\vec{r}'$ changes. We write $\vec{dl}'$ as $\vec{dr}'$. So

$$\oint(\hat{r}\cdot\vec{r}')\,\vec{dl}' = \oint(\hat{r}\cdot\vec{r}')\,\vec{dr}'.$$

Let the curve C be described by a parameter ~~t~~ ~~so~~ that we traverse the curve as t goes from 0 to 1. Then $\vec{r}'$ is a function of t and

$$\vec{dr}' = \cancel{\frac{d\vec{r}'}{dt}dt}\cdot \frac{d}{dt}(\vec{r}')\,dt.$$

$$\therefore \oint(\hat{r}\cdot\vec{r}')\,\vec{dl}' = \int_0^1 (\hat{r}\cdot\vec{r}')\frac{d(\vec{r}')}{dt}\,dt \quad \text{--- I.}$$

$$(\hat{r}\cdot\vec{r}')\frac{d\vec{r}'}{dt} = \frac{d}{dt}\left[(\hat{r}\cdot\vec{r}')\,\vec{r}'\right] - \frac{d}{dt}(\hat{r}\cdot\vec{r}')\,\vec{r}' \quad \text{--- II.}$$

$$= \frac{d}{dt}\left[(\hat{r}\cdot\vec{r}')\,\vec{r}'\right] - \left(\hat{r}\cdot\frac{d\vec{r}'}{dt}\right)\vec{r}'$$

$$\hat{r}\times\left(\frac{d\vec{r}'}{dt}\times\vec{r}'\right) = \frac{d\vec{r}'}{dt}(\hat{r}\cdot\vec{r}') - \vec{r}'\left(\hat{r}\cdot\frac{d\vec{r}'}{dt}\right).$$

$\therefore$ II becomes

$$(\hat{r}\cdot\vec{r}')\frac{d\vec{r}'}{dt} = \frac{d}{dt}\left[(\hat{r}\cdot\vec{r}')\,\vec{r}'\right] + \hat{r}\times\left(\frac{d\vec{r}'}{dt}\times\vec{r}'\right) - \frac{d\vec{r}'}{dt}(\hat{r}\cdot\vec{r}')$$

$$\therefore (\hat{r}\cdot\vec{r}')\frac{d\vec{r}'}{dt} = \frac{1}{2}\frac{d}{dt}\left[(\hat{r}\cdot\vec{r}')\,\vec{r}'\right] + \frac{1}{2}\left\{\hat{r}\times\left(\frac{d\vec{r}'}{dt}\times\vec{r}'\right)\right\}.$$

So integral I becomes

$$\oint(\hat{r}\cdot\vec{r}')\,\vec{dl}' = \frac{1}{2}\int_0^1 \frac{d}{dt}\left[(\hat{r}\cdot\vec{r}')\,\vec{r}'\right]dt + \frac{1}{2}\int_0^1 \hat{r}\times\left(\frac{d\vec{r}'}{dt}\times\vec{r}'\right)dt$$

$$= \frac{1}{2}\left[(\hat{r}\cdot\vec{r}')\,\vec{r}'\right]_0^1 + \frac{1}{2}\oint \hat{r}\times(\vec{dl}'\times\vec{r}')$$

$$= 0 + \frac{1}{2}\oint(\vec{r}'\times\vec{dl}')\times\hat{r}$$