Second-order autonomous systems

$\boxed{\frac{d^2x}{dt^2} = \ddot{x} = f(x)}$ → Ordinary, autonomous differential equation of the second-order. Define $\boxed{f(x) = -\varphi'(x)}$ where the prime denotes a derivative in x.

with a negative sign

$$\Rightarrow \boxed{\ddot{x} = -\varphi'(x)} \Rightarrow \boxed{\frac{2}{2}\, \dot{x}\,\ddot{x} = -\dot{x}\,\frac{d\varphi}{dx}}$$

$$\Rightarrow \frac{d}{dt}\left(\frac{1}{2}\dot{x}^2\right) = -\frac{dx}{dt}\frac{d\varphi}{dx} = -\frac{d\varphi}{dt}$$

$$\Rightarrow \frac{d}{dt}\left(\frac{1}{2}\dot{x}^2 + \varphi\right) = 0 \Rightarrow \boxed{\frac{1}{2}\dot{x}^2 + \varphi = c}$$

Where c is a constant. This gives a conservation equation $\boxed{\frac{1}{2}\dot{x}^2 + \varphi(x) = \text{constant}}$

Hence $\boxed{\ddot{x} = f(x)}$ is a conservative system.

$\boxed{\frac{d^2x}{dt^2} = f(x)}$ is invariant when $\boxed{t \rightarrow -t}$.

$\therefore$ $\boxed{\ddot{x} = f(x)}$ also observes time reversal symmetry.

We decompose $\boxed{\ddot{x} = f(x)}$ into a coupled set of first-order autonomous equations.

Define $\boxed{\frac{dx}{dt} = \dot{x} = y} \Rightarrow \boxed{\frac{dy}{dt} = \dot{y} = f(x)}$

Both the equations are coupled to each other.

When $\boxed{t \to -t}$, $\boxed{\frac{dx}{dt} \to -\frac{dx}{dt}} \Rightarrow \boxed{y \to -y}$.

$\Rightarrow \boxed{\frac{dx}{dt} = y} \longrightarrow \boxed{\frac{dx}{d(-t)} = -y}$ and $\boxed{\frac{d(-y)}{d(-t)} = \frac{dy}{dt} = f(x)}$

$\therefore$ $\boxed{\dot{x} = y}$ and $\boxed{\dot{y} = f(x)}$ are a reversible system.

A general coupled second-order system is $\boxed{\dot{x} = f(x,y)}$ and $\boxed{\dot{y} = g(x,y)}$. Invariance under $\boxed{t \to -t}$ and $\boxed{y \to -y}$ implies

i.) $\boxed{f(x,-y) = -f(x,y)} \Rightarrow$ f is odd in y.

ii) $\boxed{g(x,-y) = g(x,y)} \Rightarrow$ g is even in y.

In that case the system is reversible in time.

Examples: 1/. $\boxed{\ddot{x} + \omega^2 x = 0} \Rightarrow \boxed{\ddot{x} = -\omega^2 x}$ Undamped oscillator.

Decompose $\boxed{\dot{x} = v}$ and $\boxed{\dot{v} = -\omega^2 x}$. When $\boxed{t \to -t,\ v \to -v}$ but $\boxed{\dot{v} \text{ is invariant}}$ under this time reversal. (Conservative system is reversible).

2/. $\boxed{\ddot{x} + 2b\dot{x} + \omega^2 x = 0} \Rightarrow \boxed{\ddot{x} = -2b\dot{x} - \omega^2 x}$ Damped oscillator.

$\Rightarrow \boxed{\dot{x} = v}$ and $\boxed{\dot{v} = -2bv - \omega^2 x}$ Not reversible.

But $\boxed{\ddot{x} = -2b\dot{x}^2 - \omega^2 x}$ is reversible. All conservative systems are reversible, but the opposite is not true.

Curl of the Gradient Operator

For a scalar function $\boxed{\psi \equiv \psi(x, y, z)}$

$$\boxed{\vec{\nabla}\psi = \hat{x}\frac{\partial \psi}{\partial x} + \hat{y}\frac{\partial \psi}{\partial y} + \hat{z}\frac{\partial \psi}{\partial z}} \rightarrow \text{Vector quantity}$$

The curl of the above is $\boxed{\vec{\nabla}\times(\vec{\nabla}\psi)}$.

$$\therefore \vec{\nabla}\times(\vec{\nabla}\psi) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \psi}{\partial x} & \frac{\partial \psi}{\partial y} & \frac{\partial \psi}{\partial z} \end{vmatrix}$$

$$= \hat{x}\left[\frac{\partial^2 \psi}{\partial y \partial z} - \frac{\partial^2 \psi}{\partial z \partial y}\right] - \hat{y}\left[\frac{\partial^2 \psi}{\partial x \partial z} - \frac{\partial^2 \psi}{\partial z \partial x}\right]$$

$$+ \hat{z}\left[\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial y \partial x}\right]$$

$$\Rightarrow \boxed{\vec{\nabla}\times(\vec{\nabla}\psi) = 0\hat{x} + 0\hat{y} + 0\hat{z} = \vec{0}} \text{ (vector)}$$

(A general mathematical result)

$\vec{\nabla}\psi$ is an irrotational vector. It shows the direction along which the quickest increase (change) of ψ occurs. For a vector $\vec{F}$, if $\boxed{\vec{\nabla}\times\vec{F} = \vec{0}}$ then $\boxed{\vec{F} = \vec{\nabla}\psi}$ (gradient of a scalar).