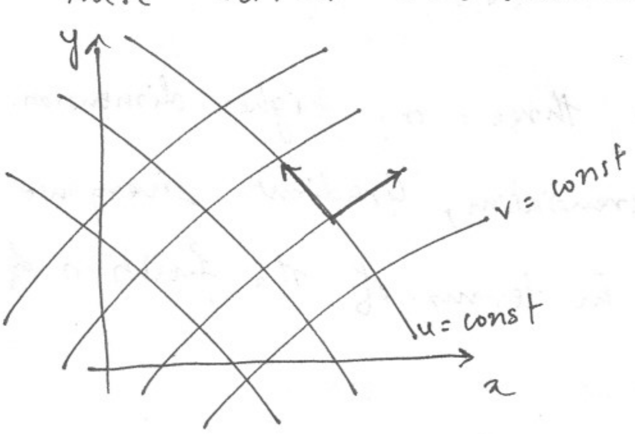
Curvilinear Co-ordinate System.

Cartesian co-ordinate system is simple to understand and convenient in many problems. However due to certain symmetries, some other co-ordinate system may be convenient to simplify computations. However we must be able to do calculus in these co-ordinate systems. In the x-y co-ordinate system, the co-ordinate axes are straight lines. This makes many aspects of calculus rather straight forward and simple to understand. For e.g. the unit vectors $\hat{i}$ and $\hat{j}$ are constant and hence immune to differentiation. If the co-ordinate axes are not straight lines, we call the co-ordinate system curvilinear. However we must have a one-one correspondence from any co-ordinate system to the cartesian system. Let (u, v) be a curvilinear co-ordinate system. We must be able to write x and y in terms of u and v. i.e. we have functions $x(u,v)$ and $y(u,v)$. Now if we keep v constant and only change u, we will trace a curve in the $x-y$ plane. These curves are denoted by the constant value of v



Likewise if we keep u - const and change v we trace another set of curves.

Now if we increase u by an ammount du ($v = \text{const}$), we will generate an infinetismal ~~vector~~ displacement in the x-y plane given by (see above figure)

$$\vec{dl_u} = \frac{\partial x}{\partial u} du\, \hat{i} + \frac{\partial y}{\partial u} du\, \hat{j}$$

$$= \left(\frac{\partial x}{\partial u}\hat{i} + \frac{\partial y}{\partial u}\hat{j}\right) du$$

Similarly if we increase v by an ammunt dv keeping u const, we will generate an infinitesimal displacement

$$\vec{dl_v} = \left(\frac{\partial x}{\partial v}\hat{i} + \frac{\partial y}{\partial v}\hat{j}\right) dv$$

We will concentrate on those co-ordinate system (u,v) which has $\vec{dl_u}$ and $\vec{dl_v}$ orthogonal.

Now $|\vec{dl_u}| = \sqrt{\left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2}\, du = h_u\, du$

and $|\vec{dl_v}| = \sqrt{\left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2}\, dv = h_v\, dv.$

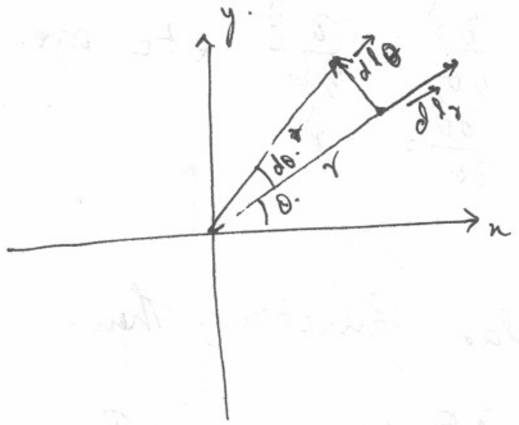
So the unit vectors along $\vec{dl_u}$ and $\vec{dl_v}$ are.

$$\hat{u} = \frac{\vec{dl_u}}{|\vec{dl_u}|} = \frac{1}{h_u}\left(\frac{\partial x}{\partial u}\hat{i} + \frac{\partial y}{\partial u}\hat{j}\right)$$

$$\hat{v} = \frac{\vec{dl_v}}{|\vec{dl_v}|} = \frac{1}{h_v}\left(\frac{\partial x}{\partial v}\hat{i} + \frac{\partial y}{\partial v}\hat{j}\right)$$

We can extend this to three or higher dimension. All the three types of differentiation, Gradient, divergence and curl can be expressed in terms of the functions ~~of the~~ $h_u, h_v, \ldots$

Eg: Polar Co-ordinates.



Here. $x = r\cos\theta$ $\quad 0 \le \theta < 2\pi$

$y = r\sin\theta$

$$\vec{dl_r} = \left(\frac{\partial x}{\partial r}\hat{i} + \frac{\partial y}{\partial r}\hat{j}\right)dr$$

$$\vec{dl_\theta} = \left(\frac{\partial x}{\partial \theta}\hat{i} + \frac{\partial y}{\partial \theta}\hat{j}\right)d\theta$$

$$\therefore \vec{dl_r} = \left(\cos\theta\,\hat{i} + \sin\theta\,\hat{j}\right)dr$$

$$\vec{dl_\theta} = \left(-r\sin\theta\,\hat{i} + r\cos\theta\,\hat{j}\right)d\theta$$

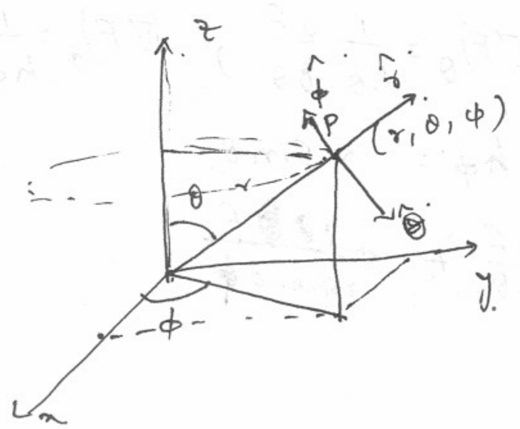
$$\therefore |\vec{dl_r}| = dr \Rightarrow h_r = 1.$$

$$|\vec{dl_\theta}| = r\,d\theta \Rightarrow h_\theta = r$$

$$\therefore \hat{r} = \frac{1}{h_r}\left(\cos\theta\,\hat{i} + \sin\theta\,\hat{j}\right) = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}.$$

$$\hat{\theta} = \frac{1}{h_\theta}\left(-r\sin\theta\,\hat{i} + r\cos\theta\,\hat{j}\right) = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}.$$

Spherical Polar co-ordinate.:



$x = r\sin\theta\cdot\cos\phi$

$y = r\sin\theta\,\sin\phi$

$z = r\cos\theta.$

Verify that

$\hat{r} = \sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k}$

$\hat{\theta} = \cos\theta\cos\phi\,\hat{i} + \cos\theta\sin\phi\,\hat{j} - \sin\theta\,\hat{k}$

$\hat{\phi} = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j}$

$0 \le \theta < \pi$ and $0 \le \phi < 2\pi$

Note that in both these co-ordinate systems the unit vectors $\hat{r}, \hat{\theta}, \hat{\phi}$ depends upon the co-ordinates. So differentiations like $\frac{\partial \hat{r}}{\partial \theta}, \frac{\partial \hat{\theta}}{\partial \theta}$ etc are non-zero. For e.g. $\frac{\partial \hat{r}}{\partial \theta} = \hat{\theta}$, $\frac{\partial \hat{\theta}}{\partial \theta} = -\hat{r}$

Gradient:

Let $F(r, \theta, \phi)$ be a scalar function. Then.

$$dF = \frac{\partial F}{\partial r} dr + \frac{\partial F}{\partial \theta} d\theta + \frac{\partial F}{\partial \phi} d\phi \qquad \text{--- I}$$

Now the increments $dr, d\theta, d\phi$ corresponds to an infinetisimal displacement vector.

$$\vec{dl} = h_r dr\, \hat{r} + h_\theta d\theta\, \hat{\theta} + h_\phi d\phi\, \hat{\phi}$$

Let the components of $\vec{\nabla} F$ along $\hat{r}, \hat{\theta}$ and $\hat{\phi}$ at the point (r, θ, ϕ) be $(\vec{\nabla} F)_r$, $(\vec{\nabla} F)_\theta$ and $(\vec{\nabla} F)_\phi$.

Then $dF = \vec{\nabla} F \cdot \vec{dl}$

$$= (\vec{\nabla} F)_r h_r dr + (\vec{\nabla} F)_\theta h_\theta d\theta + (\vec{\nabla} F)_\phi h_\phi d\phi \qquad \text{--- II}.$$

Comparing I and II we get

$$(\nabla F)_r = \frac{1}{h_r} \frac{\partial F}{\partial r}, \quad (\nabla F)_\theta = \frac{1}{h_\theta} \frac{\partial F}{\partial \theta}, \quad (\nabla F)_\phi = \frac{1}{h_\phi} \frac{\partial F}{\partial \phi}$$

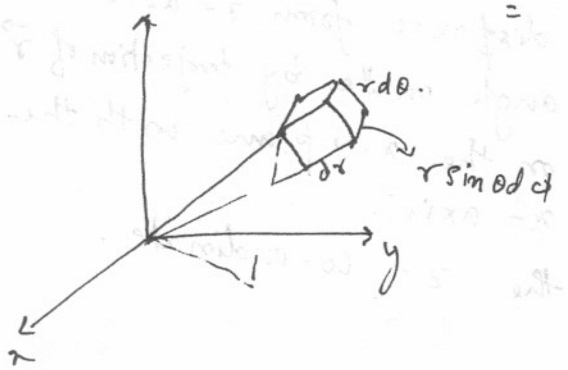
$$h_r = 1, \quad h_\theta = r, \quad h_\phi = r \sin\theta$$

$$\therefore \vec{\nabla} F = \hat{r} \frac{\partial F}{\partial r} + \hat{\theta} \frac{1}{r} \frac{\partial F}{\partial \theta} + \hat{\phi} \frac{1}{r \sin\theta} \frac{\partial F}{\partial \phi}$$

A volume element in spherical polar system is

$$dV = |\vec{dl}_r||\vec{dl}_\theta||\vec{dl}_\phi| = h_r h_\theta h_\phi \, dr\, d\theta\, d\phi$$

$$= r^2 \sin\theta \, d\theta\, d\phi\, dr$$



The three surface elements are

$h_\theta h_\phi \, d\theta\, d\phi \, \hat{r}$

$= r^2 \sin\theta \, d\theta \, d\phi \, \hat{r}$

on the surface $r =$ constant.

$h_r h_\theta \, dr\, d\theta = r\, dr\, d\theta$ on the surface $\phi =$ constant

$h_r h_\phi \, dr\, d\phi = r \sin\theta \, dr\, d\phi$

on the surface $\theta =$ constant

Using these we can evaluate the expression for divergence and the curl. These are

$$\vec{\nabla}\cdot\vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\, A_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial (A_\phi)}{\partial\phi}$$

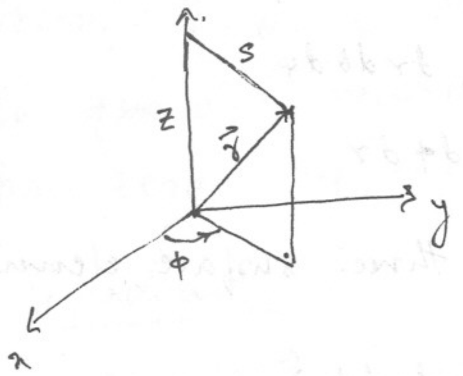
$$\vec{\nabla}\times\vec{A} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta\, A_\phi\right) - \frac{\partial A_\theta}{\partial\phi}\right]\hat{r} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial}{\partial r}(r A_\phi)\right]\hat{\theta}$$

$$+ \frac{1}{r}\left[\frac{\partial}{\partial r}(r A_\theta) - \frac{\partial A_r}{\partial\theta}\right]\hat{\phi}$$

And the Laplacian is

$$\nabla^2 F = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial F}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial F}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 F}{\partial\phi^2}$$

Cylindrical Co-ordinate System



This system specifies a point with three parameters (s, ϕ, z)

$s =$ distance from z-axis

$\phi =$ angle made by projection of $\vec{r}$ on the x-y plane with the x-axis.

$z =$ the z-co-ordinate.

$0 \leq \phi < 2\pi$.

$x = s\cos\phi$, $y = s\sin\phi$, $z = z$.

The infinitesimal length elements are.

$|\vec{dl}_s| = ds \Rightarrow h_s = 1$, $|\vec{dl}_z| = dz \Rightarrow h_z = 1$

$|\vec{dl}_\phi| = s\,d\phi \Rightarrow h_\phi = s$

The unit vectors are.

$\hat{s} = \cos\phi\,\hat{i} + \sin\phi\,\hat{j}$, $\hat{z} = \hat{k}$

$\hat{\phi} = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j}$,

The volume element is $dV = s\,ds\,d\phi\,dz$

Gradient: $\vec{\nabla} F = \hat{s}\frac{\partial F}{\partial s} + \hat{\phi}\frac{1}{s}\frac{\partial F}{\partial \phi} + \hat{z}\frac{\partial F}{\partial z}$

Divergence: $\vec{\nabla}\cdot\vec{A} = \frac{1}{s}\frac{\partial}{\partial s}(sA_s) + \frac{1}{s}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial V_z}{\partial z}$

Curl: $\vec{\nabla}\times\vec{A} = \left(\frac{1}{s}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right)\hat{s} + \left(\frac{\partial A_s}{\partial z} - \frac{\partial A_z}{\partial s}\right)\hat{\phi} + \frac{1}{s}\left(\frac{\partial}{\partial s}(sA_\phi) - \frac{\partial A_s}{\partial \phi}\right)\hat{z}$

Laplacian: $\nabla^2 F = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial F}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2 F}{\partial \phi^2} + \frac{\partial^2 F}{\partial z^2}$

Eg1) $F = \dfrac{z^2}{x^2+y^2+z^2}$. Find $\vec{\nabla} F$

In spherical polar co-ordinates

$x^2+y^2+z^2 = r^2$ and $z = r\cos\theta$

$\therefore F = \dfrac{r^2\cos^2\theta}{r^2} = \cos^2\theta$

$\therefore \vec{\nabla} F = \hat{\theta}\left[\dfrac{1}{r}\dfrac{\partial F}{\partial\theta}\right] = \hat{\theta}\,\dfrac{-2\cos\theta\sin\theta}{r} = \hat{\theta}\,\dfrac{\sin 2\theta}{r}$

Eg. 2) $\vec{A} = r^n\hat{r}$

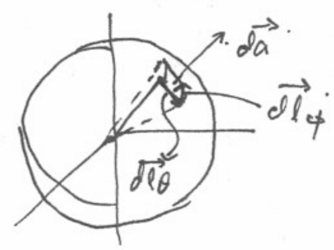
$\vec{\nabla}\cdot\vec{A} = \dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2 \cdot r^n\right) = (n+2)\,r^{n-1}$

Evaluate $\oint_S \vec{A}\cdot\vec{da}$ over the surface of a sphere of radius a.

On the surface of the sphere the ~~sa~~ surface elements are along $\hat{r}$ and $\vec{da}$ is given as

$\vec{da} = \vec{dl}_\theta \times \vec{dl}_\phi = a\,d\theta\,\hat{\theta} \times a\sin\theta\,d\phi\,\hat{\phi}$

$= a^2\sin\theta\,d\theta\,d\phi\,\hat{r}$



$\therefore \displaystyle\int_S \vec{A}\cdot\vec{da} = \int_0^{\pi}\int_0^{2\pi} a^{n+2}\sin\theta\,d\theta\,d\phi$

$= a^{n+2}\cdot 4\pi = 4\pi a^{n+2}$

Eg 3: Let $\vec{A} = y\hat{i} - x\hat{j}$. Find $\vec{\nabla}\times\vec{A}$

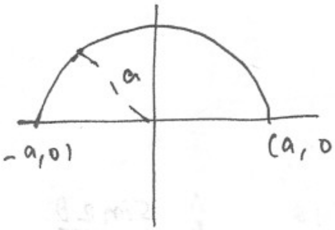
We will do this in cylindrical co-ordinates.

$y = s\sin\phi$, $x = s\cos\phi$

$\therefore \vec{A} = s(\cos\phi\,\hat{i} - \sin\phi\,\hat{j}) = -s\,\hat{\phi}$

$\vec{\nabla}\times\vec{A} = -\dfrac{\partial A_\phi}{\partial z}\hat{s} + \dfrac{1}{s}\dfrac{\partial}{\partial s}(sA_\phi)\,\hat{z} = \dfrac{1}{s}\dfrac{\partial}{\partial s}(-s^2)\,\hat{z} = -2\hat{z}$

Evaluate $\int_C \vec{A}\cdot d\vec{l}$ where C is the semicircle passing through $(-a,0)$, $(0,a)$ and $(a,0)$



$$\vec{dl} = a\, d\phi\, \hat{\phi}$$

$$\int_{+\pi}^{0} \vec{A}\cdot\vec{dl} = \int_{+\pi}^{0} (-a\,\hat{\phi})\cdot(a\, d\phi\, \hat{\phi})$$

$$= \int_{+\pi}^{0} -a^2\, d\phi = -a^2\left[0-(+\pi)\right] = \pi a^2$$