Relativity Problems - 6 - Selected Solutions

Relativity - 2

Q2/. $\boxed{\mathcal{E} = mc^2}$ $\boxed{m = \gamma m_0}$ $\boxed{\beta = 0.97}$ $\boxed{V = 0.97c}$

Rest energy $= m_0c^2 = 9.11 \times 10^{-31} \times (3\times 10^{8})^2$
$\simeq 82 \times 10^{-15}$ joules.

Convert to eV units: $m_0c^2 = \dfrac{82 \times 10^{-15}}{1.6 \times 10^{-19}}$ eV

(Charge of an electron) $\longrightarrow 1.6 \times 10^{-19}$

$\Rightarrow$ Rest Energy of an electron $= \boxed{0.51\ \text{MeV}}$

$\therefore \gamma = \dfrac{\mathcal{E}}{m_0c^2} = \dfrac{2\ \text{MeV}}{0.51\ \text{MeV}} = 3.92.$ Answer

$\therefore \boxed{m = 3.92 \times 0.51 = 2\ \text{MeV}/c^2}$

Unit of energy/c^2 (energy in MeV)

Q4/. Check solved Example 1.6 in Arthur Beiser.

Q5/. $\boxed{m = \gamma m_0}$ $\boxed{\gamma = 11}$ $\boxed{\text{Rest mass} = 0.51\ \frac{\text{MeV}}{c^2}}$

Kinetic energy, $\boxed{T = (\gamma - 1)\, m_0c^2} = 10 \times \underbrace{(0.51\ \text{MeV})}_{\text{Rest energy}}$

$= \boxed{5.1\ \text{MeV}}$ ← Answer

Momentum, $\boxed{p = mv = \gamma m_0 \dfrac{c^2 v}{c^2}}$

$\Rightarrow \boxed{p = \gamma\beta \dfrac{(m_0c^2)}{c}}$ $\boxed{\beta = \sqrt{1 - \frac{1}{\gamma^2}}}$ (P.T.O.)

Q5/. $\boxed{\beta = \sqrt{1-\frac{1}{\gamma^2}} = \sqrt{1-\frac{1}{11^2}} \simeq 0.996}$ Answer

$\therefore p = 0.996 \times 11 \times 0.51 \frac{\text{MeV}}{c} = \boxed{5.6 \frac{\text{MeV}}{c}}$

Q8/. $\boxed{p = mv}$, $\boxed{p = \gamma m_0 c \frac{v}{c} = \gamma\beta m_0 c}$

$\therefore \boxed{\frac{p_2}{p_1} = \frac{\gamma_2\beta_2}{\gamma_1\beta_1}}$ $\boxed{\beta_1 = 0.2}$ $\boxed{\beta_2 = 0.4}$ $\boxed{\gamma = \frac{1}{\sqrt{1-\beta^2}}}$ $\boxed{\begin{array}{l}\gamma_1 = 1.021 \\ \gamma_2 = 1.091\end{array}}$

$\Rightarrow \boxed{\frac{p_2}{p_1} = \frac{1.091 \times 0.4}{1.021 \times 0.2} = 2.14}$ ← Answer

In the second case, $\boxed{\beta_1 = 0.4}$, $\boxed{\beta_2 = 0.8}$

$\boxed{\gamma_1 = 1.091}$, $\boxed{\gamma_2 = 1.667}$ $\boxed{\frac{p_2}{p_1} = \frac{1.667 \times 0.8}{1.091 \times 0.4} = 3.06}$

Answer

Although the velocity increases by a factor of 2, momentum increases by a factor that is greater than 2.

Q10/. $\boxed{p_1 = \frac{E}{c}}$ for a PHOTON ← No ~~[illegible]~~ rest mass

$\boxed{p_2 = \gamma m_0 v}$ for a PROTON.

$\boxed{p_1 = p_2}$ $\Rightarrow$ Kinetic Energy, $\boxed{T = (\gamma - 1) m_0 c^2}$

(P.T.O.) $\boxed{m_0 c^2}$ → Rest energy of proton.

Q10/. $\Rightarrow \boxed{\gamma = 1 + \frac{T}{M_0 C^2}}$ $\Rightarrow$ ~~[illegible]~~ T = 10 MeV

The mass of a proton is 1836 times greater than the mass of an electron. The rest energy of an electron is 0.51 MeV. The rest energy of a proton (mass of a proton) = 1836 × 0.51 MeV ≈ 938 MeV.

OR $m_0 c^2 = \frac{1.67 \times 10^{-27} \times 9 \times 10^{16}}{1.6 \times 10^{-19}} = 938\text{ MeV}$

Charge of an electron/proton (in eV unit)

$\therefore \boxed{\gamma = 1 + \frac{10}{938} = 1.011}$

$\therefore \boxed{\beta = \sqrt{1 - \frac{1}{\gamma^2}} = \sqrt{1 - \frac{1}{1.011^2}} = 0.15}$.

Since $\boxed{p_1 = p_2}$, $\Rightarrow \frac{\mathcal{E}}{c} = \gamma m_0 V \Rightarrow \boxed{\mathcal{E} = \gamma m_0 V c}$

$\Rightarrow \mathcal{E} = \gamma \beta m_0 c^2 = 1.011 \times 0.15 \times 938\text{ MeV}$

$\Rightarrow \boxed{\mathcal{E} = 139\text{ MeV}}$ (photon energy) ← Answer

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Q11/. $\boxed{p = \gamma m_0 V}$ Momentum, $\boxed{T = (\gamma - 1) m_0 c^2}$ → Kinetic energy

$\boxed{p = \gamma \beta m_0 c}$ $\Rightarrow p = \frac{\gamma \beta}{c} (m_0 c^2)$ → Rest energy

$\Rightarrow \boxed{p = \frac{\gamma \beta}{c} \cdot \frac{T}{\gamma - 1}}$ ——— ① (P.T.O.)

Q11/. $\boxed{\gamma = \frac{1}{\sqrt{1-\beta^2}}} \Rightarrow 1-\beta^2 = \frac{1}{\gamma^2} \Rightarrow \boxed{\beta^2 = 1-\frac{1}{\gamma^2}}$

$\Rightarrow \gamma^2\beta^2 = \gamma^2 - 1 \Rightarrow \boxed{\gamma\beta = \sqrt{\gamma^2-1}}$.

Hence, $\boxed{\frac{pc}{T} = \frac{\gamma\beta}{\gamma-1} = \frac{\sqrt{\gamma^2-1}}{\gamma-1}}$ from ~~eq~~ eq(1) in last page

$\Rightarrow \frac{(pc)^2}{T^2} = \frac{\gamma^2-1}{(\gamma-1)^2} = \frac{\gamma+1}{\gamma-1}$

$p = 335 \frac{MeV}{c} \Rightarrow \boxed{pc = 335 \text{ MeV}}$

$\boxed{T = 62 \text{ MeV}} \Rightarrow \frac{pc}{T} = \frac{335}{62} = 5.4$

$\Rightarrow \frac{\gamma+1}{\gamma-1} = (5.4)^2 \Rightarrow \boxed{\gamma+1 = (\gamma-1)\times 29.2}$

$\Rightarrow 29.2\gamma - \gamma = 29.2+1 \Rightarrow \boxed{\gamma = \frac{30.2}{28.2} = 1.071}$

Hence, $\boxed{m_0 = \frac{T}{(\gamma-1)c^2} = \frac{62 \text{ MeV}}{0.071\, c^2} = 873 \frac{\text{MeV}}{c^2}}$

$\beta = \sqrt{1-\frac{1}{\gamma^2}} \Rightarrow \boxed{\beta = \sqrt{1-\frac{1}{1.071^2}} = 0.36} \Rightarrow \boxed{v = 0.36c}$ ↑Answer

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Q12/. Initial total momentum $= \boxed{\gamma_1 m_0 v_1}$

Final total momentum $= \boxed{\gamma_2 M_0 v_2}$

($M_0 \rightarrow$ Composite final mass)

By momentum conservation $\boxed{\gamma_1 m_0 v_1 = \gamma_2 M_0 v_2}$

$\boxed{v_1 = \beta_1 c}$; $\boxed{v_2 = \beta_2 c}$

(P.T.O.)

Q12/. $\Rightarrow \cancel{\text{}}\ \gamma_1 \beta_1 m_0 \cancel{c} = \gamma_2 M_0 \beta_2 \cancel{c}$

$$\Rightarrow \boxed{\gamma_1 \beta_1 m_0 = \gamma_2 \beta_2 M_0} \quad \text{—— (1)}$$

[M_0 is the final composite mass] ↑ Rest mass

Initial total energy $= \boxed{3 m_0 c^2 + \gamma_1 m_0 c^2}$

Final total energy $= \boxed{\gamma_2 M_0 c^2}$

By energy conservation $\boxed{\gamma_2 M_0 c^2 = 3 m_0 c^2 + \gamma_1 m_0 c^2}$

$$\Rightarrow \boxed{\gamma_2 M_0 = (3 + \gamma_1) m_0} \quad \text{—— (2)}$$

~~Combining~~ From Eq. (1) $\gamma_2 M_0 = \dfrac{\gamma_1 \beta_1 m_0}{\beta_2}$

In Eq. (2) ↓

$$\therefore \frac{\gamma_1 \beta_1 \cancel{m_0}}{\beta_2} = (3 + \gamma_1) \cancel{m_0} \Rightarrow \boxed{\beta_2 = \frac{\gamma_1 \beta_1}{3 + \gamma_1}}$$

$$\boxed{\beta_1 = 0.8} \Rightarrow \boxed{\gamma_1 = \frac{1}{\sqrt{1 - \beta_1^2}} = 10/6 = 1.667}$$

$$\therefore \boxed{\beta_2 = \frac{1.667 \times 0.8}{3 + 1.667} \simeq 0.29}, \boxed{\gamma_2 = \frac{1}{\sqrt{1 - \beta_2^2}} = 1.045}.$$

Hence, from Eq. (2), $\boxed{M_0 = \left(\dfrac{3 + \gamma_1}{\gamma_2}\right) m_0}$

Answer ↓

$$\therefore M_0 = \left(\frac{3 + 1.667}{1.045}\right) m_0 \Rightarrow M_0 = \frac{4.667}{1.045} m_0 = \boxed{4.47 m_0}$$

Q13/. $\boxed{\dfrac{\nu_{obs}}{\nu_s} = \dfrac{\lambda_s}{\lambda_{obs}} = \sqrt{\dfrac{1-\beta}{1+\beta}}}$ $\quad \boxed{\beta = \dfrac{15000 \times 10^3}{3 \times 10^8} = \dfrac{v}{c} \Rightarrow \beta = 5 \times 10^{-2}}$

$$\therefore \boxed{\lambda_{obs} = \lambda_s \sqrt{\frac{1+\beta}{1-\beta}}} \Rightarrow \lambda_{obs} = 550 \sqrt{\frac{1+0.05}{1-0.05}} = \boxed{578\,\text{nm}}$$

↑ Answer

Q15/. $\boxed{\text{Approaching car} \Rightarrow \text{observed frequency increases}}$

Frequency of waves incident on the car is (Doppler Effect)

$\boxed{\nu_1 = \nu_0 \sqrt{\frac{1+\beta}{1-\beta}}}$. Frequency of waves reflected from the car

is $\boxed{\nu_2 = \nu_1 \sqrt{\frac{1+\beta}{1-\beta}}} \Rightarrow \boxed{\nu_2 = \nu_0 \left(\frac{1+\beta}{1-\beta}\right)}$

For a car, $\beta \ll 1$. $\Rightarrow \nu_2 = \nu_0 (1+\beta)(1-\beta)^{-1}$

$\Rightarrow \boxed{\nu_2 \simeq \nu_0 (1+\beta)^2}$

$\Rightarrow \boxed{\nu_2 \simeq \nu_0 (1+2\beta)}$ → Binomial approximation done twice.

$\Rightarrow \nu_2 \simeq \nu_0 + 2\beta\nu_0 \Rightarrow \boxed{\nu_2 - \nu_0 \simeq 2\nu_0 \frac{v}{c}}$

Beat frequency, $\boxed{\nu_2 - \nu_0 = \Delta\nu \simeq 2\nu_0 \frac{v}{c}}$.

$\therefore \boxed{v = \frac{\Delta\nu}{2\nu_0} c}$ $\boxed{\Delta\nu = 1600\,Hz}$, $\boxed{\nu_0 = 8 \times 10^9\,Hz}$

$\Rightarrow v = \frac{1600 \times 3 \times 10^8}{2 \times 8 \times 10^9}\,ms^{-1} = \boxed{30\,ms^{-1}}$.

$\therefore v = \frac{30 \times 10^{-3}}{(3600)^{-1}} = \boxed{108\,km\,hr^{-1}}$ ← Answer

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Q17/. $\boxed{\frac{\Delta\lambda}{\lambda} = \frac{GM}{Rc^2}}$ → Gravitational Redshift.

$\boxed{M = M_\odot, \quad R = R_\odot}$ (Solar data).

$\Rightarrow \frac{\Delta\lambda}{\lambda} = \frac{6.67 \times 10^{-11} \times 2 \times 10^{30}}{7 \times 10^8 \times 9 \times 10^{16}} \simeq \boxed{2 \times 10^{-6}}$ ← Answer

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