Hawking Radiation

1/. Consider a black hole to be a spherical cavity in the continuous space-time geometry (x, y, z and t).

2/. A Newtonian perspective (simplified):

$$\boxed{m\frac{dv}{dt} = mv\frac{dv}{dr} = -\frac{GMm}{r^2}} \Rightarrow \boxed{v\frac{dv}{dr} = -\frac{GM}{r^2}}$$

$$\Rightarrow \int v\,dv = -GM\int\frac{dr}{r^2} \Rightarrow \boxed{\frac{v^2}{2} = \frac{GM}{r} + \text{constant}}$$

When $r \to \infty$, $v \to 0$ $\Rightarrow$ $\boxed{\text{Constant} = 0}$.

$\therefore$ $\boxed{v^2 = \frac{2GM}{r}}$. At the event horizon of a spherical black hole,

$$\boxed{r = R_s = \frac{2GM}{c^2}} \quad \therefore \quad v^2 = \frac{\cancel{2GM}}{\cancel{2GM}}c^2 \Rightarrow \boxed{v = c}$$

($v = \pm c$)

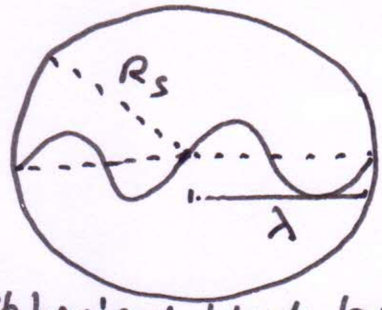
Hence, at the event horizon, a particle falling into a black hole attains the speed c. Since the speed c is attained only by particles without rest mass, at ~~on~~ the ~~surface~~ horizon of a black hole particles are "photonized" (like photons).

(P.T.O.)

(Wave-particle duality)

3/. The photons have a wavelength comparable to the radius of the black hole – the Schwarzschild radius.

i.) The wave can be seen as a standing wave inside a spherical cavity. If the wavelength were to be much longer, then the wave would "spill out" of the black hole.



Spherical black hole

ii.) A shorter wavelength would require the photons to gain more energy. $\boxed{E = \frac{hc}{\lambda}}$.

Blue shift

This energy would have to be supplied to the photons in the form of work (by gravity).

But work is $\boxed{\text{force} \times \text{displacement}}$.

Since there is no space inside a black hole, there can be no displacement. $\Rightarrow$ (No work)

Uncertainty

iii.) Hence, the wavelength of the photons is of the order of R_s. $\Rightarrow \boxed{\lambda \sim R_s}$

$\hookrightarrow$ Nothing much more, nothing much less $\Rightarrow \boxed{\lambda \sim \frac{2GM}{c^2}}$

(same order-of-magnitude)

(P.T.O.)

(Energy-mass equivalence)

4/. For a black hole of mass M, the total energy contained is $\boxed{Mc^2}$.

5/. Average energy $\boxed{\langle \text{energy} \rangle \sim k_B T}$.

$$\boxed{\langle \text{energy} \rangle = \frac{Mc^2 \text{ (total energy)}}{\text{No. of particles (photon number)}}}$$

$$\boxed{\text{Photon number} = \frac{\text{Total energy } (Mc^2)}{\text{Energy of a photon } \left(\frac{hc}{\lambda}\right)}}$$

$$\therefore \boxed{\langle \text{energy} \rangle = \frac{Mc^2}{Mc^2/(hc/\lambda)} = \frac{hc}{\lambda}}$$

All photons have (nearly) the same λ.

(good averaging)

$$\Rightarrow \boxed{\frac{hc}{\lambda} \sim k_B T} \quad \text{But} \quad \boxed{\lambda \sim R_s = \frac{2GM}{c^2}}.$$

$$\Rightarrow T \sim \frac{hc^3}{k_B (2GM)} \Rightarrow \boxed{T_H \sim \left(\frac{hc^3}{2\,G k_B}\right)\frac{1}{M}}$$

$T_H \rightarrow$ Hawking temperature (Stephen Hawking)

6/. $$\boxed{T_H = \frac{\alpha_H}{2}\left(\frac{hc^3}{G k_B}\right)\frac{1}{M}}, \quad \boxed{\alpha_H = \frac{1}{8\pi^2}}$$

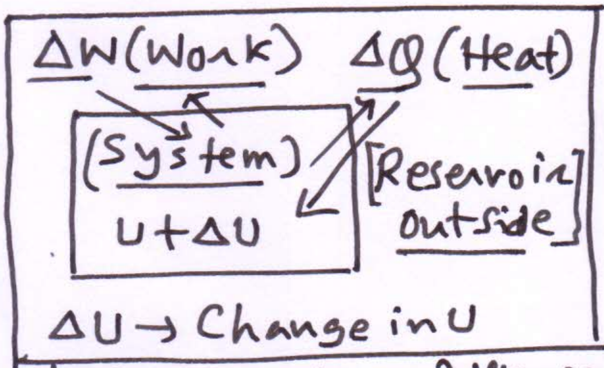
(dimensionless factor)

Black holes of smaller mass (less energy) are warmer.

Black Hole Thermodynamics

1/. The First Law of Thermodynamics:
(energy conservation)

i.) $\boxed{\Delta U = \Delta W + \Delta Q}$



ii.) $\boxed{\Delta W}$ → Work done (by the system or on it).

iii.) $\boxed{\Delta Q}$ → Heat exchanged with the reservoir (due to temperature difference).

iv) $\boxed{U}$ → Internal energy of the system. It is a function of the thermodynamic variables. It is a state function (not a path function).

v.) ΔQ is the heat exchanged with the reservoir due to temperature difference. [Zeroth law]

vi.) A reservoir is a large external system that absorbs or rejects heat in unlimited amount, without undergoing any change in its own thermodynamic variables.

2/. At the event horizon of a black hole, no (more) work is done. Hence, $\boxed{\Delta W = 0}$.

3/. The Second Law of Thermodynamics:

$\boxed{\Delta Q = T\Delta S}$. $\boxed{S \text{ is the Entropy}}$ T → Temperature.

(P.T.O.)

4/. Since $\boxed{\Delta W = 0}$, $\Rightarrow \boxed{\Delta U = \Delta Q}$.

Hence, $\boxed{\Delta U = \Delta Q = T\Delta S}$. But the internal energy of a black hole is $\boxed{U = Mc^2}$. $\Rightarrow \boxed{\Delta U = (\Delta M)c^2}$.

$$\Rightarrow \boxed{\Delta S = \frac{\Delta U}{T} = \frac{(\Delta M)c^2}{T_H} = \frac{(c^2 \Delta M)\, M}{(\alpha_H/2)(hc^3/Gk_B)}}$$

$$\Rightarrow \boxed{\Delta S = \frac{2}{\alpha_H}\left(\frac{Gk_B}{hc}\right) M \Delta M}$$

Considering infinitesimals $\boxed{\Delta S \to dS}$, $\boxed{\Delta M \to dM}$

Integral

$$\Rightarrow \boxed{S = \frac{\cancel{2}}{\alpha_H}\,\frac{Gk_B}{hc}\,\frac{M^2}{\cancel{2}} = \frac{1}{\alpha_H}\left(\frac{Gk_B}{hc}\right)M^2}$$

Black Hole Entropy

5/. Area of a black hole $\boxed{A = 4\pi R_s^2}$ $\boxed{R_s = \frac{2GM}{c^2}}$

$$\Rightarrow A = 4\pi\left(\frac{2GM}{c^2}\right)^2 = 4\pi\,\frac{4G^2}{c^4}\,M^2$$

$$\Rightarrow \boxed{M^2 = \frac{Ac^4}{16\pi G^2}} \Rightarrow \boxed{S = \frac{1}{\alpha_H}\,\frac{Gk_B}{hc}\,\frac{Ac^4}{16\pi G^2}}$$

$$\Rightarrow \boxed{S = \frac{8\pi^2}{16\pi}\,\frac{k_B c^3}{Gh}\,A = \frac{2\pi}{4}\,\frac{k_B c^3}{Gh}\,A}$$

Now $\boxed{\hbar = h/2\pi}$

Planck length $\rightarrow \boxed{l_{Pl} = \sqrt{\frac{G\hbar}{c^3}}} \Rightarrow \boxed{S = \frac{k_B}{4}\,\frac{A}{l_{Pl}^2}}$

$\therefore \boxed{S = S_{BH} = \frac{k_B}{4}\,\frac{A}{l_{Pl}^2}} \rightarrow$ Bekenstein-Hawking Entropy

 (Jacob Bekenstein)

Black Hole Luminosity

1/. $\boxed{T_H = \left[\frac{\alpha_H}{2}\left(\frac{hc^3}{Gk_B}\right)\frac{1}{M_\odot}\right]\frac{M_\odot}{M}}$ $\boxed{\begin{array}{l} M_\odot = 2\times10^{30}\,\text{kg} \\ M_\odot \to \text{Solar Mass}\end{array}}$

$\boxed{\begin{array}{l} h = 6.626\times10^{-34}\,\text{Js},\ c = 3\times10^{8}\,\text{ms}^{-1},\ \alpha_H = \frac{1}{8\pi^2} \\ G = 6.67\times10^{-11}\,\text{Kg}^{-1}\text{m}^3\text{s}^{-2},\ k_B = 1.38\times10^{-23}\,\text{JK}^{-1}\end{array}}$

$\Rightarrow \boxed{T_H \simeq 6.2\times10^{-8}\,(M_\odot/M)\ \text{kelvin}} \rightarrow$ very cold.

In comparison, the Cosmic Microwave Background Radiation (a remnant of the Big Bang) is $\boxed{2.7\,K}$. Hence, it is nearly impossible to detect Hawking radiation coming out of astrophysical black holes, against the background temperature of 2.7 K.

2/. The feeble radiation from a black hole can be approximated as a black-body radiation, for which the intensity of radiation is $\boxed{\mathcal{E} = \sigma T_H^4}$ (energy per unit time per unit area).
(Stefan-Boltzmann)

The total luminosity is $\boxed{\mathcal{W} = \sigma T_H^4 A}$ (power emitted over the total area), $\boxed{A = 4\pi R_S^2}$, in which $\boxed{R_S = 2GM/c^2}$ ($\sigma \to$ Stefan's constant).

(P.T.O.)

3/. $$\boxed{W = \underbrace{\left(\frac{2\pi^5 k_B^4}{15 c^2 h^3}\right)}_{\text{Stefan's Constant} \rightarrow \boxed{\sigma}} 4\pi \underbrace{\left(\frac{2GM}{c^2}\right)^2}_{\text{Schwarzschild Radius} \rightarrow \boxed{R_S}} \underbrace{\left[\frac{\alpha_H}{2}\left(\frac{hc^3}{Gk_B}\right)\frac{1}{M}\right]^4}_{\text{Hawking Temperature} \rightarrow \boxed{T_H}}}$$

$$\Rightarrow \boxed{W = \frac{1}{60 \times 8^3 \pi^2} \cdot \left(\frac{hc^6}{G^2}\right)\frac{1}{M^2} = \frac{1}{15360\pi}\left(\frac{\hbar c^6}{G^2}\right)\frac{1}{M^2}}$$

i.) Bekenstein-Hawking Luminosity – Energy radiated per unit time from the entire surface of the black hole (total power).

ii.) Since $\boxed{W \propto M^{-2}}$, as the black hole radiates away energy (and loses mass), it becomes more luminous (γ-ray bursts).

4/. Now, energy contained in a black hole is $\boxed{Mc^2}$. Since energy is lost, $\boxed{dM/dt < 0}$. Hence, $\boxed{W = -c^2 dM/dt}$.

$$\therefore \boxed{-c^2 \frac{dM}{dt} = \frac{1}{15360\pi} \cdot \left(\frac{\hbar c^6}{G^2}\right) \cdot \frac{1}{M^2}} \text{ (Power radiated)}$$

$$\Rightarrow -\int_{M_0}^{0} M^2 dM = \int_0^{M_0} M^2 dM = \frac{\hbar c^4}{15360\pi G^2}\int_0^{\tau} dt$$

$$\Rightarrow \boxed{\frac{M_0^3}{3} = \frac{\hbar c^4 \tau}{15360\pi G^2}}$$ $M_0 \rightarrow$ initial mass
$\tau \rightarrow$ life time of a black hole.

(P.T.O.)

$(M_0 \equiv M)$

$$\boxed{\tau = \left(\frac{5120\pi G^2}{\hbar c^4}\right) M^3}$$ → Time taken for a black hole to dissipate fully.

Now $\tau = \left(\frac{5120 \times 2\pi^2 G^2 M_\odot^3}{h c^4}\right)\left(\frac{M}{M_\odot}\right)^3$ → In terms of the solar mass.

$$\Rightarrow \boxed{\tau \simeq 2.1 \times 10^{67} (M/M_\odot)^3 \text{ years}}$$

(Much longer than any practical time). Age of the Universe is 14×10^9 years

5/. By Wien's Displacement Law, $\boxed{\lambda_{max} T = b}$, the peak emission ~~[illegible]~~ of the black hole radiation has the wavelength $\boxed{\lambda_{max} = b/T_H}$.

$\boxed{b \simeq \frac{hc}{5k_B}}$ and $\boxed{T_H \sim \left(\frac{hc^3}{Gk_B}\right)\frac{1}{M}}$. Hence, we get

$$\boxed{\lambda_{max} \sim \frac{hc}{\cancel{k_B}} \cdot \frac{G\cancel{k_B} \cdot M}{hc^3} \sim \frac{GM}{c^2}}$$

, a result that is consistent with our original argument that the photons in a black hole have wavelengths comparable to the radius of the black hole, $\boxed{R_s = 2GM/c^2}$.

I.) The Hawking temperature and the Bekenstein-Hawking entropy bring together h, c, G, k_B (fundamental constants).

II.) Point to a quantum theory of gravity.

Additional Remarks (Hawking Radiation)

Radiation from Black Holes

1/. The average measure of the energy of a large aggregate of particles on the event horizon of a black hole is given by the Hawking temperature T_H (in kelvin).

2/. Any ~~~~ physical system characterised by temperature is a thermodynamical system.

3/. An object radiates heat at a rate that depends on its surface temperature. Hence, a black hole, with its Hawking temperature, can be treated as a black body radiator (Hawking radiation).

4/. Since the Hawking temperature is small, the Hawking radiation is feeble.

Space-time at the Event Horizon

1/. Time dilation: $\boxed{t = t_0/\sqrt{1-(v/c)^2}}$.

2/. Length contraction: $\boxed{l = l_0\sqrt{1-(v/c)^2}}$.

3/. At the event horizon $\boxed{v = c}$. Hence $t \to \infty$, and $\boxed{l \to 0}$ $\Rightarrow$ At the speed of light space collapses to a point, time is infinitely dilated.

No space is left.

 (Penzias & Wilson)

Cosmic Microwave Background Radiation

1/. The Universe started from a very small region of space (of the order of Planck length), and following an explosion (the Big Bang), expanded outwards ever since.

2/. The expansion was of the geometric fabric of the space time continuum.

3/. The expanding Universe was filled with thermal radiation (blackbody radiation).

4/. As the Universe expanded, it cooled. By the first law of thermodynamics, we write $\boxed{\Delta U = \Delta W + \Delta Q}$. Since the Universe is a closed adiabatic system (no heat is exchanged with the surroundings), $\boxed{\Delta Q = 0}$ and $\boxed{\Delta U = \Delta W}$. Hence, the work done to expand the Universe, was supplied by the internal energy of the system.

5/. λ_{max} of the radiation of the Cooled Universe is $\sim \boxed{10^{-3}\,m}$.

Wien displacement law

$\therefore\ T \simeq b/\lambda_{max} \simeq 3\times10^{-3}/10^{-3} \simeq 3K \rightarrow$ The temperature of the Cosmic Microwave Back Ground Radiation.

(actually 2.7K)