Quantisation of Radiation:

Black-Body Radiation

(Thermal) Energy Transfer: i/. Conduction, ii/. Convection, iii/. Radiation.

1/. **Radiation**: (Example → Sun's Energy)

Hot objects emit energy in the form of electromagnetic radiation. (no medium needed)

2/. **Prévost's Theory**:

In thermal equilibrium with the surroundings, an object emits and absorbs the same amount of radiant energy per unit time.

Thermal Radiation

Ref: The Zeroth Law

(Equality of emission and absorption rates).

(P.T.O.)

3/. Any object at a temperature T, radiates energy over all wavelengths. When T is low, energy is mostly emitted at long wavelengths (and vice-versa).

4/. Absorption coefficient at a given wavelength, $a_\lambda \equiv$ fraction of radiant energy absorbed at that wavelength.

5/. For $\boxed{a_\lambda = 1}$, $\rightarrow$ Blackbody (and absorber) Emitter.

Blackbody

Also the most efficient emitter of electromagnetic energy.

[Gustav Kirchhoff].

Black surfaces absorb (and emit) radiation most efficiently. $\Rightarrow$ Blackbody Radiation.

(P.T.O.)

6/. Emissivity (equivalently known as emissive power or Spectral emittance), e_λ.

$\equiv$ Energy radiated per unit time over unit surface area, at a given wavelength, λ (Unit : $J\,m^{-2}\,s^{-1}\,m^{-1} \to SI$).

($J s^{-1} m^{-2} m^{-1}$)

7/. For a black body, $\boxed{a_\lambda = 1}$.

and $\boxed{e_\lambda \equiv \mathcal{E}_\lambda}$.

$$\boxed{\mathcal{E}_\lambda \equiv \mathcal{E}_\lambda(\lambda, T)}$$

(T → Absolute Temperature)

Spectral emissivity function of a black body

ONLY for a blackbody

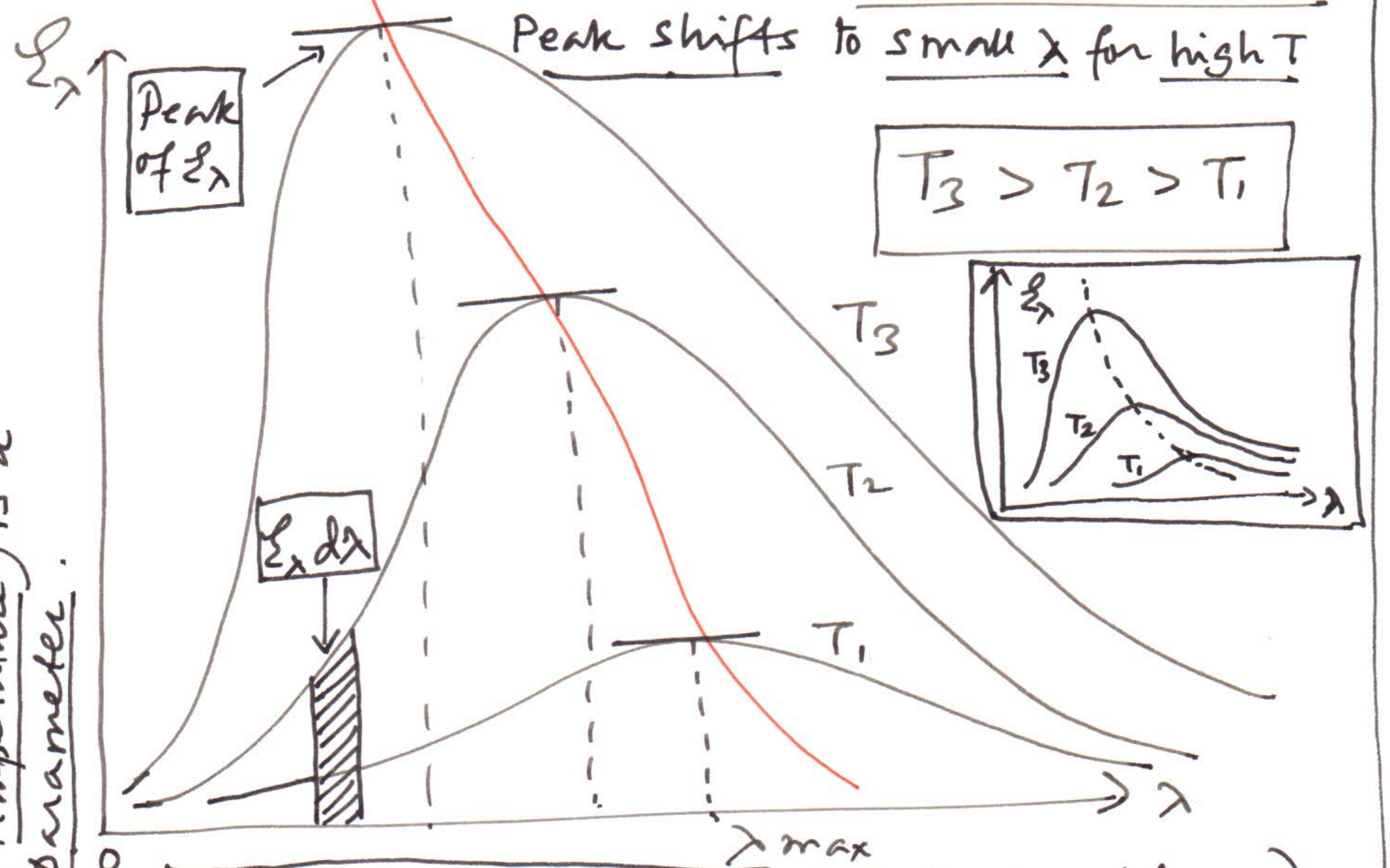
Independent of other factors like surface area, shape or material.

[Gustav Kirchhoff]

(from thermodynamics)

Experimental Data: Otto Lummer, Ernst Pringsheim

BLACKBODY RADIATION SPECTRUM



T (temperature) is a parameter.

$$\boxed{\mathcal{E}_\lambda \equiv \mathcal{E}_\lambda(\lambda, T)}$$ ($\mathcal{E}_\lambda$ peaks at λ_{max})

i/. Differentiation:

$$\boxed{\lambda_{max} \equiv \lambda_{max}(T)}$$

$$\frac{d\mathcal{E}_\lambda}{d\lambda} = 0 \quad \text{for} \quad \boxed{\lambda = \lambda_{max}}$$

ii/. Integration:

$$\boxed{\mathcal{E} \equiv \mathcal{E}(T)}$$

(integrate over all wavelengths $\Rightarrow$ free of λ.)

$$\int_0^\infty \mathcal{E}_\lambda \, d\lambda = \mathcal{E}_{radiated}$$

(Energy radiated per unit area per unit time).

Wien's DISPLACEMENT law:

Wilhelm Wien

$$\boxed{\frac{d\mathcal{E}_\lambda}{d\lambda} = 0}$$ leads to $$\boxed{\lambda_{max} T = b}$$

The maximum of $\mathcal{E}_\lambda$.

$$\boxed{b = 2.898 \times 10^{-3} \text{ m K}}$$ (Wien's constant)

$\lambda_{max} \simeq 5 \times 10^{-7}$ m

$$T_\odot \simeq \frac{b}{5 \times 10^{-7}\text{ m}} = \frac{2.898 \times 10^{4}}{5} \text{ K}$$

$$\Rightarrow \boxed{T_\odot \simeq 6000 \text{ K}}.$$

Solar surface temperature

Stefan - Boltzmann Law:

Josef Stefan & Ludwig Boltzmann

$$\mathcal{E} = \int_0^\infty \mathcal{E}_\lambda(\lambda, T)\, d\lambda \underbrace{= \sigma T^4}$$

$\rightarrow$ Area under the spectral curve.

$$\Rightarrow \boxed{\mathcal{E} = \sigma T^4}$$

$$\boxed{\sigma = 5.67 \times 10^{-8} \text{ W m}^{-2} \text{ K}^{-4}}$$

(Stefan's constant).

T $\rightarrow$ The absolute scale of temperature. The unit is Kelvin (K).

Find the formula of $\mathcal{E}_\lambda(\lambda T)$

Define an Energy Density function, $u(\lambda, T)$. [Energy per volume at a wavelength]

[Also known as the spectral distribution function].

$$\boxed{u(\lambda, T) = \frac{4}{c}\,\mathcal{E}_\lambda(\lambda, T).}$$

$\mathcal{E}_\lambda(\lambda, T)$

$$\frac{J s^{-1} m^{-2} m^{-1}}{m\, s^{-1}}$$

$u(\lambda, T) \to \equiv J m^{-3} m^{-1}$

Thermodynamical arguments of Wien, using the Stefan-Boltzmann law as a guide, (area under curve $= \sigma T^4$)

$$\boxed{u(\lambda, T) = \lambda^{-5} f(\lambda T)}$$

$f(\lambda T) \to$ Argument of f is a product of λT.

But what is $f(\lambda T)$?

Thermodynamical arguments do not provide the correct answer.

Wien's Distribution function:

$$\boxed{f(\lambda T) \equiv B\, e^{-a/\lambda T}}$$

a & B are constants

Gives the intensity at a given wavelength

$$\therefore \boxed{u(\lambda, T) = \frac{B}{\lambda^5} \cdot \frac{1}{e^{a/\lambda T}}}$$

Wien's DISTRIBUTION law.

(Different from Wien's DISPLACEMENT law).

Matches the data well for short wavelength, i.e. for small values of λ.

(a, B are determined from data

The formula is obtained by data analysis)

$$\int_0^\infty \frac{1}{\lambda^5} f(\lambda T)\, d\lambda = T^4 \int_0^\infty \frac{f(\lambda T)}{(\lambda T)^5}\, d(\lambda T) \sim T^4$$

Proportional to T^4

Considering standing electromagnetic waves

Rayleigh - Jeans Distribution function :

$$\boxed{u(\lambda, T) = \lambda^{-5} f(\lambda T)}$$

$$\boxed{f(\lambda T) = 8\pi k_B \lambda T}$$

$$\Rightarrow \boxed{u(\lambda, T) = \frac{8\pi k_B T}{\lambda^4}}$$

$k_B \rightarrow$ Boltzmann's constant

Lord Rayleigh + Sir James Jeans

From classical wave theory.

Satisfactory on long wavelengths.

On short wavelengths,

ULTRA VIOLET CATASTROPHE!

$$\int_0^\infty u(\lambda, T)\, d\lambda = \frac{4}{c}\int_0^\infty \mathcal{E}_\lambda(\lambda, T)\, d\lambda = 8\pi k_B T \int_\infty^0 \frac{d\lambda}{\lambda^4}$$

$$= 8\pi k_B T \left(-\frac{1}{\lambda^3}\right)_\infty^0$$

The integral goes to infinity for $\lambda \rightarrow 0$.

Mismatch of Wien's Distribution function on long wavelengths

$$\boxed{u(\lambda, T) = \frac{B}{\lambda^5}\,\frac{1}{e^{a/\lambda T}}} \rightarrow \text{Wien formula}$$

when $\boxed{\lambda \longrightarrow \infty} \Rightarrow \boxed{\dfrac{a}{\lambda T} \longrightarrow 0}$.

$\Rightarrow \boxed{e^{a/\lambda T} \longrightarrow 1} \rightarrow$ The exponential part converges faster to 1.

The λ^{-5} part dominates for large λ $\rightarrow$ $\therefore \boxed{u(\lambda, T) \simeq \dfrac{B}{\lambda^5}}$.

(Approximation of Wien's distribution law on large wavelengths) $\rightarrow$ [Not correct]

$$\boxed{\begin{array}{l}\text{The Rayleigh-Jeans law gives a better match with the data by the} \\ \text{formula } \boxed{u(\lambda, T) \propto \dfrac{1}{\lambda^4}} \text{ on long wavelengths.}\end{array}}$$

Planck's Distribution function :

Correctly matched the data at all wavelengths

$$\boxed{f(\lambda T) = \frac{B}{e^{a/\lambda T} - 1}}$$

Max Planck

$$\boxed{u(\lambda, T) = \frac{B}{\lambda^5} \cdot \frac{1}{e^{a/\lambda T} - 1}}$$

To arrive at this distribution function, Planck introduced the concept of the QUANTUM.

$$\boxed{B = 8\pi h c} \qquad \boxed{a = \frac{hc}{k_B}}$$

$h \rightarrow$ Planck's constant

$$\boxed{h \approx 6.62 \times 10^{-34} \text{ Js}}$$

The emission and absorption of the radiation from the blackbody surface was quantised.

Limiting cases of the Planck formula:

Long wavelength: Start with Planck's law.

$$\boxed{u(\lambda, T) = \frac{8\pi hc}{\lambda^5} \cdot \frac{1}{e^{hc/\lambda k_B T} - 1}}$$

When $\lambda \longrightarrow \infty$, $\frac{hc}{\lambda k_B T} \longrightarrow 0$ (i.e. $\frac{hc}{\lambda k_B T} \ll 1$)

$$\Rightarrow \boxed{e^{hc/\lambda k_B T} \simeq 1 + \frac{hc}{\lambda k_B T}}$$

$$\left(\text{Using } e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \right)$$

Binomial expansion

$$\Rightarrow u(\lambda, T) \simeq \frac{8\pi hc}{\lambda^5} \cdot \frac{1}{\cancel{1} + \frac{hc}{\lambda k_B T} - \cancel{1}}$$

Approximation

$$\Rightarrow \boxed{u(\lambda, T) \simeq \frac{8\pi hc}{hc} \cdot \frac{\lambda k_B T}{\lambda^5} \simeq \frac{8\pi k_B T}{\lambda^4}}$$

Rayleigh-Jeans law when $\lambda \to \infty$.

Without h $\Rightarrow$ classical formula

On large λ, the radiation is like a wave.

Short Wavelength : Start with Planck's law

$$\boxed{u(\lambda, T) = \frac{8\pi hc}{\lambda^5} \cdot \frac{1}{e^{hc/\lambda k_B T} - 1}}$$

when $\boxed{\lambda \to 0} \Rightarrow \boxed{\frac{hc}{\lambda k_B T} \longrightarrow \infty}$

$$\Rightarrow \boxed{e^{hc/\lambda k_B T} - 1 \simeq e^{hc/\lambda k_B T}}$$

Approximation

$$\Rightarrow \boxed{u(\lambda, T) \simeq \frac{8\pi hc}{\lambda^5} \cdot e^{-hc/\lambda k_B T}}$$

h is present on short λ

Comparing with the Wien's distribution law.

$$\boxed{u(\lambda, T) \simeq \frac{B}{\lambda^5} \cdot e^{-a/\lambda T}}$$

$\boxed{B = 8\pi hc}$, $\boxed{a = hc/k_B}$

Wien's Distribution law is obtained when $\lambda \longrightarrow 0$.

With h present, the quantum character holds.

Wien's Displacement Law from Planck's Distribution

$$\boxed{\frac{d\mathcal{E}_\lambda}{d\lambda} = 0} \qquad \boxed{u(\lambda, T) = \frac{4}{c}\,\mathcal{E}_\lambda(\lambda, T)}$$

$$\Rightarrow \boxed{\frac{du}{d\lambda} = 0}$$ At the peak of the emissivity curve.

$$\boxed{u(\lambda, T) = \frac{B}{\lambda^5} \cdot \frac{1}{e^{a/\lambda T} - 1}} \rightarrow \text{Planck formula}$$

where $B = 8\pi h c$, $a = hc/k_B$

$$\Rightarrow \boxed{u(\lambda, T) = B\,\lambda^{-5}\left(e^{a/\lambda T} - 1\right)^{-1}}$$

Taking the derivative

$$\Rightarrow \frac{du}{d\lambda} = B\left(e^{a/\lambda T} - 1\right)^{-1} \times \left(-\frac{5\lambda^{-5}}{\lambda}\right)$$

$$+ B\lambda^{-5}\frac{d}{d\lambda}\left(e^{a/\lambda T} - 1\right)^{-1}$$

$$\Rightarrow \frac{du}{d\lambda} = B\left(e^{a/\lambda T} - 1\right)^{-1} \times \left(\frac{-5\lambda^{-5}}{\lambda}\right)$$

$$+ \frac{B}{\lambda^5} \times \left[-\left(e^{a/\lambda T} - 1\right)^{-2} \times e^{a/\lambda T} \times \frac{a}{T} \times -\frac{1}{\lambda^2}\right]$$

(P. T. O.)

$$\frac{du}{d\lambda} = \frac{B\lambda^{-5}}{\lambda}\left(e^{a/\lambda T}-1\right)^{-1}\left[-5 + \frac{a}{T}\cdot\frac{1}{\lambda}\cdot\frac{e^{a/\lambda T}}{e^{a/\lambda T}-1}\right]$$

$$\Rightarrow \boxed{\frac{du}{d\lambda} = \frac{u(\lambda,T)}{\lambda}\left[\frac{a}{\lambda T}\cdot\frac{1}{1-e^{-a/\lambda T}} - 5\right]}$$

At the peak: $\boxed{\text{If } \frac{du}{d\lambda} = 0} \Rightarrow \boxed{\frac{a}{\lambda T}\cdot\frac{1}{1-e^{-a/\lambda T}} = 5}$

$$\Rightarrow \boxed{\lambda T = \frac{a}{5}\cdot\frac{1}{1-e^{-a/\lambda T}}}$$

Condition only at the maximum

Write, $\boxed{x = \lambda T}$ $[\lambda \equiv \lambda_{max}]$

$$\Rightarrow \boxed{x = \frac{a}{5}\cdot\frac{1}{1-e^{-a/x}}}$$

Transcendental Equation with no exact Solution.

Iterative Method: $\boxed{x \simeq \frac{a}{5}}$ Trial Solution

Start with this solution and check for consistency with the trial solution.

Approximate Approach

$$\boxed{x \simeq \frac{a}{5} \cdot \frac{1}{1 - e^{-a/a/5}}}$$

Apply the trial Solution $\boxed{x = a/5}$ only on the R.H.S.

$$\Rightarrow \quad x \simeq \frac{a}{5} \cdot \frac{1}{1 - e^{-5}} \simeq \frac{a}{5 \times 0.993}$$

$$\Rightarrow \quad \boxed{x \simeq \frac{a}{4.9657}}$$

Consistent

very close to $x = a/5$

$$\boxed{a = hc/k_B} \qquad \boxed{x = \lambda T \equiv \lambda_{max} T}$$

$$\therefore \quad \boxed{\lambda_{max} T = b = \frac{a}{4.9657}}$$

$$\Rightarrow \quad \boxed{\lambda_{max} T = \frac{hc}{4.9657 \times k_B}}$$

$$\boxed{\begin{aligned} h &= 6.62 \times 10^{-34} \ \mathrm{J\,s} \\ c &= 3 \times 10^{8} \ \mathrm{m\,s^{-1}} \\ k_B &= 1.38 \times 10^{-23} \ \mathrm{J\,K^{-1}} \end{aligned}}$$

S.I. values of the Constants

$$\therefore \quad \lambda_{max} T = \frac{1.439 \times 10^{-2}}{4.9657} \ \mathrm{m\,K}$$

$$\Rightarrow \quad \boxed{\lambda_{max} T = 2.898 \times 10^{-3} \ \mathrm{m\,K}}$$

Wien's Displacement Law.

Stefan-Boltzmann law from Planck's Distribution

$$\boxed{\mathcal{E} = \int_0^\infty \mathcal{E}_\lambda(\lambda, T)\, d\lambda = \sigma T^4}$$

(To prove)

and $\boxed{\mathcal{E}_\lambda = \frac{c}{4} u(\lambda, T)}$

where $\boxed{u(\lambda, T) = \frac{B}{\lambda^5} \cdot \frac{1}{e^{a/\lambda T} - 1}}$ $\boxed{B = 8\pi hc \quad a = \frac{hc}{k_B}}$

$$\boxed{\mathcal{E} = \frac{c}{4} B \int_0^\infty \frac{1}{\lambda^3} \frac{\lambda^{-2} d\lambda}{e^{a/\lambda T} - 1}}$$

$\boxed{a = hc/k_B}$ and $\boxed{\lambda \nu = c}$

$\Rightarrow$ $\boxed{\nu = c\lambda^{-1}}$ $\boxed{d\nu = -c\lambda^{-2} d\lambda}$

$\boxed{\frac{|d\nu|}{c} = \lambda^{-2} d\lambda}$ Take only the absolute value

The integral limits are reversed.

$$\boxed{\frac{a}{\lambda T} = \frac{hc}{k_B \lambda T} = \frac{h\nu}{k_B T}}$$

and $\boxed{\frac{1}{\lambda^3} = \frac{\nu^3}{c^3}}$.

With ν the lower limit of the integral is ∞, the upper is 0.

(P.T.O.)

$$\boxed{\mathcal{E} = \frac{c}{4} B \int_0^\infty \frac{\nu^3}{c^3} \frac{d\nu}{c\left[e^{h\nu/k_BT} - 1\right]}}$$

$$\Rightarrow \mathcal{E} = \frac{c}{4} \frac{B}{c^4} \int_0^\infty \frac{\nu^3 \, d\nu}{e^{h\nu/k_BT} - 1}$$

(All λ replaced by ν)

$$\boxed{B = 8\pi h c}$$

$$\mathcal{E} = \frac{c}{4} \cdot \frac{8\pi h c}{c^4} \cdot \left(\frac{k_B T}{h}\right)^4 \int_0^\infty \frac{(h\nu/k_BT)^3 \, d(h\nu/k_BT)}{e^{h\nu/k_BT} - 1}$$

$$\Rightarrow \boxed{\mathcal{E} = \frac{c}{4} \cdot \frac{8\pi h c}{c^4} \cdot \frac{k_B^4 T^4}{h^4} \int_0^\infty \frac{x^3 \, dx}{e^x - 1}} \rightarrow \boxed{\mathcal{E} \propto T^4}$$

where $\boxed{x = h\nu/k_B T}$

$$\boxed{\int_0^\infty \frac{x^3 \, dx}{e^x - 1} = \frac{\pi^4}{15}. \quad \left(6\,\zeta(4) = 6 \times \frac{\pi^4}{90} = \frac{\pi^4}{15}\right)}$$

[The integral above is solved by the Riemann Zeta function. $\boxed{6\,\zeta(4) = \frac{6 \times \pi^4}{90} = \frac{\pi^4}{15}}$]

(P.T.O.)

$$\boxed{\mathcal{E} = \frac{c}{4} \cdot \frac{8\pi hc}{c^4} \cdot \frac{k_B{}^4\, T^4}{h^4} \cdot \frac{\pi^4}{15}}$$

$$\Rightarrow \mathcal{E} = \frac{2}{15} \cdot \frac{\pi^5}{c^2} \cdot \frac{1}{h^3}\, k_B{}^4\, T^4$$

$$\Rightarrow \boxed{\mathcal{E} = \left(\frac{2}{15} \cdot \frac{\pi^5\, k_B{}^4}{c^2 h^3}\right) T^4}$$

$$\boxed{\mathcal{E} = \sigma T^4} \rightarrow$$ <u>Stefan-Boltzmann Law</u>.

$h = 6.626 \times 10^{-34}\, \mathrm{Js}$
$c = 3 \times 10^{8}\, \mathrm{m\, s^{-1}}$
$k_B = 1.38 \times 10^{-23}\, \mathrm{J\, K^{-1}}$

$$\boxed{\sigma = \frac{2}{15} \cdot \frac{\pi^5\, k_B{}^4}{c^2 h^3}} \rightarrow$$ Involves <u>3 fundamental constants, h, c, k_B</u>.

$$\sigma = 5.67 \times 10^{-8}\, \mathrm{W m^{-2} K^{-4}}$$

<u>(Stefan's constant)</u>

This gives a <u>theoretical estimate</u> of <u>Stefan's constant</u>.

Derivation of the Planck Distribution

1/. Emission and absorption of radiant energy is in discrete quanta of $\boxed{h\nu}$, from dipole oscillators on the black body surface (Planck).

2/. Radiation itself is quantised.

Quantum of Light → — PHOTONS (Einstein).

i.) Photoelectric Effect

$$\boxed{h\nu = W_0 + \frac{1}{2} m v^2}$$

ii.) Compton Effect. (uses $\boxed{E = h\nu}$ and special relativity)

3/. Photon gas, Bose-Einstein Statistics, Bosons, Bose-Einstein Condensation

(Satyendranath Bose).

↓

For short wavelengths, radiation becomes more particle-like ⇒ A gas of photons.