Nikhil Mehta, 190549.

Q-1 Concrete Terms :

$S_0 = \emptyset$

$S_{i+1} = \{true, false, 0\} \cup \{succ\ t_1, pred\ t_1, iszero\ t_1 \mid t_1 \in S_i\}$
$\cup \{if\ t_1\ then\ t_2\ else\ t_3 \mid t_1, t_2, t_3 \in S_i\}$

Let us try to ~~do~~ prove this by induction on value of i.

Base Case $i=0$, it is an empty set, so part of every set $S_0 \subseteq S_1$.

For $i>0$, let us assume $S_{i-1} \subseteq S_i$, we have to prove $S_i \subseteq S_{i+1}$ which means for every term $t \in S_i$, we have to show t also belongs to S_{i+1}.

~~t~~ for every t in S_i, we know that t belongs to ~~to~~ any one of the three sets. Let us consider them as:

Set 1 : $\{true, false, 0\}$

Set 2 : $\{succ\ t_1, pred\ t_1, iszero\ t_1 \mid t_1 \in S_{i-1}\}$

Set 3 : $\{if\ t_1\ then\ t_2\ else\ t_3 \mid t_1, t_2, t_3 \in S_{i-1}\}$

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For every t in S_i

(1) if $t \in$ Set 1, it definitely belongs to S_{i+1} by the definition.

(2) $t \in$ ~~Set 2~~ Set 2 i.e. $t \in \{succ\ t_1, pred\ t_1$, or $iszero(t_1)$, where $t_1 \in S_{i-1}\}$. By induction hypothesis $t_1 \in S_i$, & by definition of S_{i+1} i.e. it contains $succ\ t_1$, $pred\ t_1$, $iszero(t_1)$ which is t_2

(3) $t = \{$if t_1 then t_2 else $t_3 \mid t_1, t_2, t_3 \in S_{i-1}\}$

By induction hypothesis $t_1, t_2, t_3 \in S_i$ and by definition if t_1 then t_2 else t_3 belongs to S_{i+1}, so t is $t \in S_{i+1}$.

Hence Proved

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Q-2 Uniqueness of One Step Evaluation ($\rightarrow$)

To Prove if $t \rightarrow t'$ and $t \rightarrow t''$ then $t' = t''$

- Proof is based by induction using the evaluation properties.

i.e. At each step we will proceed by case analysis of smaller derivatives and have an induction on structure of t.

$\boxed{\text{3 Rules}}$ Rule 1) : if true then t_2 else $t_3 \rightarrow t_2$

Rule 2) : if false then t_2 else $t_3 \rightarrow t_3$

Rule 3) : $$\frac{t_1 \rightarrow t_1'}{\text{if } t_1 \text{ then } t_2 \text{ else } t_3 \rightarrow \text{if } t_1' \text{ then } t_2 \text{ else } t_3}$$

$\boxed{\text{Case - 1}}$: if last rule used in derivation of $t \rightarrow t'$ is Rule ①.

We now know t has the form if t_1 then t_2 else t_3 and $t_1 = $ true.

So for evaluation of $t \rightarrow t''$, Rule 2 can not be used as it will imply $t_1 = $ false which is contradictory.

Rule 3 can not be used as it demands t_1 to evaluate to something & true does not evaluate. So only $\boxed{\text{Rule 1}}$ is possible

So $t' = t''$

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$\boxed{\text{Case-2}}$: Reversing the arguments of $\boxed{\text{Case-1}}$ for $\boxed{\text{Rule-2}}$ gives similar expression.

- if Rule-2 is used in evaluation of $t \to t'$, Rule-1 & Rule-3 can be used for $t \to t''$ by similar arguments.

$\boxed{\text{Case-3}}$: if Rule-3 is applied in derivation of ~~t~~: $t \to t'$. It means t_1 evaluates to t_1'. So t_1 cannot be true/false.

So only Rule that can be applied in derivation of $t \to t''$ is Rule-3. So, $t_1 \to t_1''$.

By induction hypothesis, as t_1' & t_1'' are subderivatives of $t \to t'$ & $t \to t''$ which gives $t_1' = t_1''$ which in turn applies

$\boxed{t' = t''}$.

Q3 Types:

Q-4 Preservation:

- if $t : T$ and $t \to t'$ then $t' : T$

We will try to use a similar approach as in (Q-2).

Proof By Induction: At any step we assume that the properties holds for subderivatives & we will try to move forward case-wise

Let me re-write rules for convenience

- For Booleans $T ::= Bool$

1) T-True $\quad true : Bool$

2) T-False $\quad false : Bool$

3) T-if $\quad \dfrac{t_1 : Bool \quad t_2 : T \quad t_3 : T}{if\ t_1\ then\ t_2\ else\ t_3 : T}$

- For Numbers

$T ::= Nat$ $\qquad$ Nat denotes type of Natural Numbers

1) T-Zero $\quad 0 : Nat$

2) T-Succ $\quad \dfrac{t_1 : Nat}{succ\ t_1 : Nat}$

3) T-Pred $\quad \dfrac{t_1 : Nat}{iszero\ t_1 : Bool}$

P.T.O.

$\boxed{\text{Case - 1}}$ t is any constant true, false, 0

- t is already in normal form, so it does not make sense to evaluate to t'.

$\boxed{\text{Case - 2}}$ T - IF

t is of form if t_1 then t_2 else t_3

where t_1 : Bool , t_2, t_3 : T

Three subcases (E-IFTrue, E-IFFalse, E-If) similar

- E - IF True (t_1 = true)
  t' is t_2 which is of form T.
- E - IF False (t_1 = false)
  t' is t_3 which is form T.
- E - If $t_1 \rightarrow t_1'$ and t' = if t_1' then t_2 else t_3.

In this case we apply induction hypothesis & conclude t_1' is of form t_1 which is bool. Now t_2 : T & t_3 : T we can re-apply T-IF to conclude t' : T.

Case 3) T-Succ

$t = \text{succ } t_1 \qquad T = \text{Nat} \qquad t_1 : \text{Nat}$

By Rule : $\dfrac{t_1 \to t_1'}{\text{succ } t_1 \to \text{succ } t_1'}$

~~As t_1 belong to Nat, By induction hypothesis on subderivatives, we observe t_1' : Nat which implies succ t_1' : Nat, → succ (t_1) : Nat which ultimately says t' : T.~~

As t_1 belongs to Nat, by induction hypothesis on subderivatives, we observe t_1' : Nat, from which we say that succ t_1' : Nat, so succ(t_1) : Nat which says $t : T$.

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