190549 . CS340 Assignment-4 ,
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Q-1

a) $L_1 = \{ \langle M, N \rangle \mid M, N$ are two TM's & M takes fewer steps than N on input $\epsilon \}$

$\boxed{\text{• Undecidable}}$

- We will try to prove L_1 is undecidable using reduction.
- let $G_{\epsilon TM} = \{ \langle N \rangle \mid N$ is a TM accepting $\epsilon \}$
- $G_{\epsilon TM}$ is undecidable (Class Lecture & Notes).

Claim : $G_\epsilon \le_m L_1$

We will justify our claim by constructing a function f which takes $\langle N \rangle$ as input & produces output $\langle N_1, N_2 \rangle$ s.t. ~~$\epsilon \in G(M)$~~ $\epsilon \in G(N) \iff N_1$ takes fewer steps than N_2 on ϵ.

f :

(i) TM N_1 is constructed such that on every input.
- simulate N on ϵ
- ~~Accept if~~
- Accept if N accepts & if N reject ϵ go into infinite loop.

(ii) Construct another TM N_2 which goes into infinite loop on all input.

$\therefore$ N_2 takes ∞ steps on ε.

~~if $\varepsilon \in G(M)$~~

~~if $\varepsilon \in G(M) \Rightarrow N_1$ takes ∞ steps~~

if $\varepsilon \in G(N) \Rightarrow N_1$ takes finite & N_2 ∞ steps

if $\varepsilon \notin G(N) \Rightarrow N_1$ & N_2 both takes ∞ steps.

$$\therefore G_{\varepsilon} \leq_m L_1$$

L_1 is unpredictable.

(b) $L_2 = \{ \langle M \rangle \mid M$ takes at most 2^{340} steps on some input $\}$

• Decidable

Observation:
Only 2^{340} bits of input are relevant because it will be accepted within at-most 2^{340} steps according to definition.

TM

Algorithm for TM: for input $\langle M \rangle$.

- M is run for at most 2^{340} steps for all input length of atmost 2^{340}.
- If input is accepted by M within these steps, accept it otherwise reject it

(c) $L_3 = \{ \langle M \rangle \mid$ there are infinitely many TMs equivalent to M $\}$.

$\boxed{\text{• Decidable}}$

• Fact: Every TM has infinitally many equivalent TMs.

It is infinite as unreachable states can be added to any TM not changing the accepting Language.

So Accept input if it is a TM otherwise reject.

(d) $L_4 = \{ \langle M, N \rangle \mid L(M) \cap L(N)$ is infinite $\}$.

$\boxed{\text{• Undecidable}}$

• We try to design a computable function f s.t. it takes $\langle M, w \rangle$ as input giving $\langle M_1, M_2 \rangle$ as output :-

$$w \in L(M) \Leftrightarrow L(M_1) \cap L(M_2) \text{ is infinite}$$

We claim that $\boxed{A_{TM} \leq L_4}$.

f: on input $\langle M, w \rangle$ outputs $\langle M_1, M_2 \rangle$.

• M_1 is designed s.t. $\forall$ input

(a) simulate M on w.

(b) Accept if M accepts w, reject if M reject w.

• Also design $M_1 = M_2$

P.T.O

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Proof

(i) $\omega \in L(M)$:-

(a) M_1 accepts every string $\Rightarrow L(M_2)$ is infinite

(b) $L(M_1) \cap L(M_2)$ is infinite as $M_1 = M_2$.

(ii) $\omega \notin L(M)$:-

$\rightarrow M_1$ is finite as it reject every string

$\Rightarrow L(M_1) \cap L(M_2)$ is ϕ

~~Thus~~ A_{TM}

$\therefore$ $\omega \in L(M) \Rightarrow L(M_1) \cap L(M_2)$ is infinite.

Thus L_4 is undecidable

Question - 2

$L' = \{0w \mid w \in L\} \cup \{1w \mid w \notin L\}$.

Given L is TR & $\bar{L}$ is non-TR.

- We will try to prove that $\boxed{\bar{L} \leq_m L'}$ $\Rightarrow$ L' is unpredictable
- We will ~~compute~~ construct a computable function $f \mid x \in \bar{L} \Leftrightarrow f(x) \in L'$.
- For computational function f & input string x:
  $\Rightarrow$ Add 1 in front of x if $x \notin L$.

  Output: $1x$

Proof:

- $x \in \bar{L} \Rightarrow x \notin L$ s.t. $x \in L' \Rightarrow f(x) \in L'$.
- $x \notin \bar{L} \Rightarrow x \in L$ s.t. $x \notin L' \Rightarrow f(x) \notin L'$.

~~$\Rightarrow$ Thus, $x \in \bar{L}$ $x \in \bar{L}$ is equivalent $f(x) \in$~~

$\Rightarrow$ Thus, $x \in \bar{L} \Leftrightarrow f(x) \in L'$ $\forall$ input x.

$\bar{L} \leq_m L'$

$\boxed{L' \text{ is not-TR}}$

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Question-3

(a) $Infinite_{TM} = \{\langle M\rangle \mid M$ is a TM & $L(M)$ is an infinite language$\}$

- We will try to prove that $A_{TM} \leq_m Infinite_{TM}$.

Make a computational function f that takes input $\langle M, w\rangle$ producing output $\langle M_1\rangle \mid w \in L(M) \iff L(M_1)$ is infinite.

Let $\langle M, w\rangle$ be input for computational function f producing output $\langle M_1\rangle$ as:

- TM M_1 is constructed with following things for every input string y.
  (i) Simulate M on w.
  (ii) Accept w if it is accepted by M. ~~otherwise reject~~. & reject if rejected by M.

Proof:

$w \in L(M) \Rightarrow$ Every string is accepted $M_1 \Rightarrow L(M_1)$ is infinite.
$w \notin L(M) \Rightarrow M_1$ do not accept any string $\Rightarrow L(M_1)$ is finite
i.e. ϕ

$\therefore A_{TM} \leq_m Infinite$.

$Infinite_{TM}$ is $\boxed{\text{undecidable}}$

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(b) $ALL_{TM} = \{ \langle M \rangle \mid M \text{ is a TM } \& \ L(M) = \Sigma^* \}$

- We will try to prove that $A_{TM} \leq_m ALL_{TM}$.

~~Make a computable function f that takes input~~
$\langle M, w \rangle$

Let $\langle M, w \rangle$ be input & construct TM M' with following on input string y:

(i) Simulate M on w.

(ii) Accept it if accepted by M.

- $L(M') = \Sigma^*$ if M accept w.
- $L(M') = \phi$ if M does not accept w.

Proof

$w \in L(M) \Rightarrow M'$ accept all string $\in \Sigma^*$ $\Rightarrow L(M')$ is Σ

$w \notin L(M) \Rightarrow M'$ do not accept string $\Rightarrow L(M')$ not Σ^*

~~$A_{TM} \leq_m$ Infinite~~. $A_{TM} \leq_m ALL_{TM}$

$\therefore$ ALL_{TM} is unpredictable