POPL Assignment 2

Diamond Property: Let R be a binary relation. We say that R has diamond property if, whenever aRb and aRc, then $\exists d$ such that bRd and cRd.

β-reduction as a binary relation: We can define $\xrightarrow{\beta}$ as a binary relation between two λ terms t_1 and t_2:

$$t_1 \xrightarrow{\beta} t_2 \text{ iff } t_2 \text{ can be obtained from } t_1 \text{ using 1 } \beta \text{ reduction step.}$$

1. Show that $\xrightarrow{\beta}$ does not have the Diamond property. That is, give an example λ-term t_1 such that

    1. $t_1 \xrightarrow{\beta} t_2$
    2. $t_1 \xrightarrow{\beta} t_3$
    3. $\nexists\, t_4$ s.t. $t_2 \xrightarrow{\beta} t_4$ and $t_3 \xrightarrow{\beta} t_4$

    (5 Marks)

2. **Non-associativity of Substitutions:** Let M, N, and P be λ-terms. Assume $x \neq y$. Show that the order of substitution matters, i.e., in general

    $$M[x := N][y := P] \not\equiv M[y := P][x := N]$$

    (5 Marks)

3. **Constrained-associativity of Substitutions:** Let M, N, and P be λ-terms. Assume $x \neq y$ and $x \notin FV(P)$. Show that

    $$M[x := N][y := P] \equiv M[y := P][x := N'] \text{ where } N' = N[y := P]$$

    (10 Marks)

4. **Church Numerals:** Explain with suitable examples, what simple arithmetic function does the following λ-term represents:

    $$\lambda n.\ n\ (\lambda p\ z.\ z\ (succ\ (p\ true))(p\ true))\ (\lambda z.\ z\ zero\ zero)\ false$$

    Here, *succ*, *true*, *zero*, *false* represent the λ-terms defined in the lectures. (5 Marks)