Probability Assignment-III

Nikhil Nair

I. Problem

Find the mean number of heads in three tosses of a fair coin.

II. Solution

Consider each trial results in success (i.e Heads) or failure (i.e Tails).

Let p and $q = (1-p)$ be the probability of success and failure respectively.

$$p = \frac{1}{2} \tag{1}$$

$$q = 1 - p = \frac{1}{2} \tag{2}$$

In n Bernoulli trials with k success and $(n-k)$ failures, the probablity of k success in n- Bernoulli trials can be given as

$$\Pr(X = k) = \begin{cases} {}^nC_k p^k q^{n-k} & 0 \le k \le n \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

where, $n = 3$

The generating function (or z-transform) of $\Pr(X = k)$ for non-negative integer values is defined as

$$M_X(z) = E[z^{X=k}] = \sum_{k=0}^{n} P_X(k) z^{-k} \tag{4}$$

$$M_X(z^{-1}) = E[z^{-X}] = \sum_{k=0}^{n} P_X(k) z^{k} \tag{5}$$

Now, mean is defined as the 1st moment of $E[z^{-X}]$ at z=1 i.e,

$$\text{Mean} = \frac{dE[z^{-X}]}{dz}|_{z=1} \tag{6}$$

$$= \frac{d}{dz} \sum_{k=0}^{n} P_X(k) z^k|_{z=1} \tag{7}$$

$$= \sum_{k=0}^{n} k P_X(k) z^{k-1}|_{z=1} \tag{8}$$

$$= \sum_{k=0}^{n} k P_X(k) \tag{9}$$

$$\therefore \text{Mean} = \sum_{k=0}^{n=3} k\, {}^nC_k p^k q^{n-k} \tag{10}$$

$$= {}^3C_1 \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^2 + 2 \times {}^3C_2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right) + 3 \times {}^3C_3 \left(\frac{1}{2}\right)^3 \tag{11}$$

$$\Rightarrow \text{Mean} = 1.5 \tag{12}$$

The mean number of heads in three tosses of a fair coin is 1.5

The following code simulates the mean.

https://github.com/nikhilnair90/FWC/tree/main/Module–II/Probability/Prob3/Code/prob3.py