Probability Assignment-III

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I. Problem

Find the mean number of heads in three tosses of a fair coin.

II. Solution

Consider each trial results in success (i.e Heads) or failure (i.e Tails).

Let p and $q = (1-p)$ be the probability of success and failure respectively.

$$p = \frac{1}{2} \tag{1}$$

$$q = 1 - p = \frac{1}{2} \tag{2}$$

In n Bernoulli trials with k success and $(n-k)$ failures, the probablity of k success in n- Bernoulli trials can be given as

$$\Pr\left(X = k\right) = \begin{cases} {}^{n}C_{k}p^{k}q^{n-k} & 0 \leq k \leq n \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

where, $n = 3$

The generating function (or z-transform) of $\Pr\left(X = k\right)$ is defined as

$$M_X(z) = E[z^{X=k}] = \sum_{k=-\infty}^{\infty} P_X(k) z^{-k} \tag{4}$$

$$M_X(z^{-1}) = \sum_{k=-\infty}^{\infty} P_X(k) z^{k} \tag{5}$$

Now, mean is defined as the 1st moment of $M_X(z^{-1})$ at z=1 i.e,

$$\text{Mean} = \frac{dM_X(z^{-1})}{dz}|_{z=1} \tag{6}$$

$$= \frac{d}{dz} \sum_{k=-\infty}^{\infty} P_X(k) z^{k}|_{z=1} \tag{7}$$

$$= \sum_{k=-\infty}^{\infty} k P_X(k) z^{k-1}|_{z=1} \tag{8}$$

$$= \sum_{k=-\infty}^{\infty} k P_X(k) \tag{9}$$

From eq(3) and eq(9)

$$\text{Mean} = \sum_{k=0}^{n=3} k\,{}^{n}C_{k}p^{k}q^{n-k} \tag{10}$$

$$= {}^{3}C_{1}\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^{2} + 2 \times {}^{3}C_{2}\left(\frac{1}{2}\right)^{2}\left(\frac{1}{2}\right) + 3 \times {}^{3}C_{3}\left(\frac{1}{2}\right)^{3} \tag{11}$$

$$\Rightarrow \text{Mean} = 1.5 \tag{12}$$

The mean number of heads in three tosses of a fair coin is 1.5

The following code simulates the mean.

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https://github.com/nikhilnair90/FWC/tree/main/
    Module−II/Probability/Prob3/Code/prob3.py
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