Indian Institute of Technology Hyderabad

Probability Assignment-III

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I. Problem

Find the mean number of heads in three tosses of a fair coin.

II. Solution

Consider each trial results in success (i.e Heads) or failure (i.e Tails).

Let p and $q = (1-p)$ be the probability of success and failure respectively.

$$p = \frac{1}{2} \tag{1}$$

$$q = 1 - p = \frac{1}{2} \tag{2}$$

In n Bernoulli trials with k success and $(n-k)$ failures, the probablity of k success in n- Bernoulli trials can be given as

$$P(X = k) = {}^{n}C_{k} p^{k} q^{n-k} \tag{3}$$

Now the distribution of number of successes using $(1), (2) \& (3)$ can be given as,

k	0	1	2	3
P(X=k)	${}^{3}C_{0}(\frac{1}{2})^{3}$	${}^{3}C_{1}(\frac{1}{2})^{3}$	${}^{3}C_{2}(\frac{1}{2})^{3}$	${}^{3}C_{3}(\frac{1}{2})^{3}$

MGF of binomial distribution for discrete random variable is given by

$$M_k(t) = \sum_{k=0}^{n} e^{tk} P(k) \tag{4}$$

$$M_k(t) = \sum_{k=0}^{n} e^{tk\,n} C_k p^k q^{n-k} \tag{5}$$

$$M_k(t) = \sum_{k=0}^{n} {}^{n}C_k (pe^t)^k q^{n-k} \tag{6}$$

$$\begin{aligned} M_k(t) &= {}^{n}C_0 (pe^t)^0 q^n + {}^{n}C_1 (pe^t)^1 q^{n-1} + {}^{n}C_2 (pe^t)^2 q^{n-2} \\ &\quad + \ldots.. + {}^{n}C_n (pe^t)^n q^{n-n} \\ &= q^n + {}^{n}C_1 (pe^t)^1 q^{n-1} + {}^{n}C_2 (pe^t)^2 q^{n-2} + \ldots.. + (pe^t)^n \end{aligned} \tag{7}$$

Using binomial expansion (7) can be simplified as,

$$M_k(t) = (q + pe^t)^n \tag{8}$$

Mean of the binomial distribution is given by,

$$\begin{aligned} \text{Mean} &= \frac{dM_k(t)}{dt}\Big|_{t=0} \quad (9) \\ &= \frac{d(q+pe^t)^n}{dt}\Big|_{t=0} \quad (10) \\ &= npe^t (q+pe^t)^{n-1}\Big|_{t=0} \quad (11) \\ &= np(q+p)^{n-1} \quad (12) \\ \therefore \text{Mean} &= np \qquad \because (p+q=1) \quad (13) \\ &= 3 \times \frac{1}{2} = 1.5 \quad (14) \end{aligned}$$

The mean number of heads in three tosses of a fair coin is 1.5

The following code calculates the mean.

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https://github.com/nikhilnair90/FWC/tree/main/
    Module−II/Probability/Prob3/Code
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