Probability Assignment-III

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I. Problem

Find the mean number of heads in three tosses of a fair coin.

II. Solution

Consider each trial results in success (i.e Heads) or failure (i.e Tails).

Let p and $q = (1-p)$ be the probability of success and failure respectively.

$$p = \frac{1}{2} \tag{1}$$

$$q = 1 - p = \frac{1}{2} \tag{2}$$

In n Bernoulli trials with k success and $(n-k)$ failures, the probablity of k success in n- Bernoulli trials can be given as

$$\Pr(X = k) = \begin{cases} {}^nC_k p^k q^{n-k} & 0 \le k \le n \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

where, $n = 3$

The generating function (or z-transform) of $\Pr(X = k)$ for non-negative integer values is defined as

$$M_k(z) = E[z^k] = \sum_{k=0}^{n} z^k \Pr(k) \tag{4}$$

$$E[z^k] = \sum_{k=0}^{n} z^k {}^nC_k p^k q^{n-k} \tag{5}$$

$$E[z^k] = \sum_{k=0}^{n} {}^nC_k q^{n-k} (pz)^k \tag{6}$$

Using binomial theorem eq(6) can be simplified as,

$$E[z^k] = (q + pz)^n \tag{7}$$

Now,
Mean is defined as the 1st moment of $E[z^k]$ i.e,

$$\text{Mean} = \frac{dE[z^k]}{dz}\Big|_{z=1} \tag{8}$$

$$= \frac{d(q+pz)^n}{dz}\Big|_{z=1} \tag{9}$$

$$= np(q+pz)^{n-1}\Big|_{z=1} \tag{10}$$

$$= np(q+p)^{n-1} \tag{11}$$

$$\therefore \text{Mean} = np \qquad \because (p+q=1) \tag{12}$$

$$= 3 \times \frac{1}{2} = 1.5 \tag{13}$$

The mean number of heads in three tosses of a fair coin is 1.5

The following code calculates the mean.

https://github.com/nikhilnair90/FWC/tree/main/Module–II/Probability/Prob3/Code/prob3.py