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Probability Assignment-III

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I. Problem

Find the mean number of heads in three tosses of a fair coin.

II. Solution

Consider each trial results in success (i.e Heads) or failure (i.e Tails) represented by 1 and 0 respectively.

Let $X_i \in \{1,0\}, i = 1,2,3$ be the random variables representing the outcome for each coin toss.

p and $q = (1-p)$ are the probability of success and failure respectively.

$$p = P_{X_i}(1) = \frac{1}{2} \tag{1}$$

$$q = 1 - p = P_{X_i}(0) = \frac{1}{2} \tag{2}$$

In n Bernoulli trials with k success and $(n-k)$ failures, the probablity of k success in n- Bernoulli trials can be given as

$$\Pr(X_i = k) = \begin{cases} {}^nC_k p^k q^{n-k} & 0 \le k \le n \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

where, $n = 3$

The generating function (or z-transform) of $\Pr(X_i = k)$ is defined as

$$M_{X_i}(z) = E[z^{X_i}] = \sum_{k=-\infty}^{\infty} P_{X_i}(k) z^{-k} \tag{4}$$

For a single coin toss,

$$E[z^{X_i}] = \sum_{k=0}^{1} P_{X_i}(k) z^{-k} \tag{5}$$

$$= P_{X_i}(0) z^0 + P_{X_i}(1) z^{-1} \tag{6}$$

$$= q + pz^{-1} \tag{7}$$

$\therefore$n number of tosses can be represented as,

$$\begin{aligned} E[z^{X_1+X_2+\ldots+X_n}] &= E[z^{X_1} z^{X_2} \ldots . z^{X_n}] \\ &= E[z^{X_1}] E[z^{X_2}] \ldots . E[z^{X_n}] \end{aligned} \tag{8}$$

($\because X_1, X_2, \ldots, X_n$are independent and identically distributed)

$\Rightarrow M_{X_i}(z)$ for n number of coin tosses is

$$M_{X_i}(z) = (q + pz^{-1})^n \tag{9}$$

Now, mean is defined as the 1st moment of $M_{X_i}(z^{-1})$ at z=1 i.e,

$$\text{Mean} = \frac{dM_{X_i}(z^{-1})}{dz}\Big|_{z=1} \tag{10}$$

$$= \frac{d(q+pz)^n}{dz}\Big|_{z=1} \tag{11}$$

$$= np(q+pz)^{n-1}\Big|_{z=1} \tag{12}$$

$$= np(q+p)^{n-1} \tag{13}$$

$$\therefore \text{Mean} = np \qquad \because (p+q=1) \tag{14}$$

$$= 3 \times \frac{1}{2} = 1.5 \tag{15}$$

The mean number of heads in three tosses of a fair coin is 1.5

The following code simulates the mean.

https://github.com/nikhilnair90/FWC/tree/main/Module–II/Probability/Prob3/Code/prob3.py