



Probability Assignment-III

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I. PROBLEM

Find the mean number of heads in three tosses of a fair coin.

II. SOLUTION

Consider each trial results in success (i.e Heads) or failure (i.e Tails).

Let p and $q = (1-p)$ be the probability of success and failure respectively.

$$p = \frac{1}{2} \quad (1)$$

$$q = 1 - p = \frac{1}{2} \quad (2)$$

In n Bernoulli trials with k success and $(n - k)$ failures, the probability of k success in n - Bernoulli trials can be given as

$$P(X = k) = {}^nC_k p^k q^{n-k} \quad (3)$$

Now the distribution of number of successes using (1), (2)&(3) can be given as,

k	0	1	2	3
P(X=k)	${}^3C_0(\frac{1}{2})^3$	${}^3C_1(\frac{1}{2})^3$	${}^3C_2(\frac{1}{2})^3$	${}^3C_3(\frac{1}{2})^3$

MGF of binomial distribution for discrete random variable is given by

$$M_k(t) = E[e^{tk}] = \sum_{k=0}^n e^{tk} P(k) \quad (4)$$

$$E[e^{tk}] = \sum_{k=0}^n e^{tkn} C_k p^k q^{n-k} \quad (5)$$

$$E[e^{tk}] = \sum_{k=0}^n {}^nC_k (pe^t)^k q^{n-k} \quad (6)$$

$$\begin{aligned} E[e^{tk}] &= {}^nC_0(pe^t)^0 q^n + {}^nC_1(pe^t)^1 q^{n-1} + {}^nC_2(pe^t)^2 q^{n-2} \\ &\quad + \dots + {}^nC_n(pe^t)^n q^{n-n} \\ &= q^n + {}^nC_1(pe^t)^1 q^{n-1} + {}^nC_2(pe^t)^2 q^{n-2} + \dots + (pe^t)^n \end{aligned} \quad (7)$$

Using binomial expansion (7) can be simplified as,

$$E[e^{tk}] = (q + pe^t)^n \quad (8)$$

Mean of the binomial distribution is given by,

$$\text{Mean} = \frac{dE[e^{tk}]}{dt} \Big|_{t=0} \quad (9)$$

$$= \frac{d(q + pe^t)^n}{dt} \Big|_{t=0} \quad (10)$$

$$= np e^t (q + pe^t)^{n-1} \Big|_{t=0} \quad (11)$$

$$= np(q + p)^{n-1} \quad (12)$$

$$\therefore \text{Mean} = np \quad \because (p + q = 1) \quad (13)$$

$$= 3 \times \frac{1}{2} = 1.5 \quad (14)$$

The mean number of heads in three tosses of a fair coin is 1.5

The following code calculates the mean.

<https://github.com/nikhilnair90/FWC/tree/main/Module-II/Probability/Prob3/Code>