Probability Assignment-III

Nikhil Nair

I. Problem

Find the mean number of heads in three tosses of a fair coin.

II. Solution

Consider each trial results in success (i.e Heads) or failure (i.e Tails).

Let p and $q = (1-p)$ be the probability of success and failure respectively.

$$p = \frac{1}{2} \tag{1}$$

$$q = 1 - p = \frac{1}{2} \tag{2}$$

In n Bernoulli trials with k success and $(n-k)$ failures, the probablity of k success in n- Bernoulli trials can be given as

$$P(X=k) = {}^nC_k p^k q^{n-k} \tag{3}$$

Now the distribution of number of successes using $(1), (2)\&(3)$ can be given as,

k	0	1	2	3
P(X=k)	${}^3C_0(\frac{1}{2})^3$	${}^3C_1(\frac{1}{2})^3$	${}^3C_2(\frac{1}{2})^3$	${}^3C_3(\frac{1}{2})^3$

MGF of binomial distribution for discrete random variable is given by

$$M_k(t) = \sum_{k=0}^{n} e^{tk} P(k) \tag{4}$$

$$M_k(t) = \sum_{k=0}^{n} e^{tkn} C_k p^k q^{n-k} \tag{5}$$

$$M_k(t) = \sum_{k=0}^{n} {}^nC_k (pe^t)^k q^{n-k} \tag{6}$$

$$\begin{aligned} M_k(t) = {}^nC_0(pe^t)^0 q^n + {}^nC_1(pe^t)^1 q^{n-1} + {}^nC_2(pe^t)^2 q^{n-2} \\ +..... + {}^nC_n(pe^t)^n q^{n-n} \\ = q^n + {}^nC_1(pe^t)^1 q^{n-1} + {}^nC_2(pe^t)^2 q^{n-2} + ... + (pe^t)^n \end{aligned} \tag{7}$$

Using binomial expansion (7) can be simplified as,

$$M_k(t) = (q + pe^t)^n \tag{8}$$

Mean of the binomial distribution is given by,

$$\text{Mean} = \frac{dM_k(t)}{dt}\Big|_{t=0} \tag{9}$$

$$= \frac{d(q+pe^t)^n}{dt}\Big|_{t=0} \tag{10}$$

$$= npe^t(q+pe^t)^{n-1}\Big|_{t=0} \tag{11}$$

$$= np(q+p)^{n-1} \tag{12}$$

$$\therefore \text{Mean} = np \qquad \because (p+q=1) \tag{13}$$

$$= 3 \times \frac{1}{2} = 1.5 \tag{14}$$

The mean number of heads in three tosses of a fair coin is 1.5