# DYNAMIC PROGRAMMING IN CRICKET: PROTECTING THE WEAKER BATSMAN

Stephen R. CLARKE
*School of Mathematical Sciences*
*Swinburne University of Technology*
*Hawthorn 3122, Australia*

John M. NORMAN
*Management School*
*Sheffield University, U.K.*

A simple Dynamic Programming model of cricket is presented. The state is the facing batsmen and the number of runs on offer. The decision is whether to run or not, with the objective to maximise the chance the better batsman is on strike at the start of the next over. The model is solved analytically to find the optimal policy and the value of the objective function. The simple initial model is extended to a more realistic one requiring no further calculations and a numerical example is given. An alternative optimality criterion is investigated and we demonstrate that trying to put the better batsman on strike at the start of the over does not necessarily maximise the expected duration of the partnership. This alternative objective function is investigated numerically, and it is shown that the better batsman should generally run if possible off the second last or last ball of the over.

**Keywords**: sports, cricket, Dynamic Programming, Markov processes

## 1. Introduction

There are several examples where a Dynamic Programming (DP) formulation has the potential to assist the sports person with decision making. Norman (1995) in giving one example of an application of DP in each sport lists 10 papers. There have been few applications of DP in cricket, which is surprising as the ball by ball nature of the game should lend itself to this structure. Clarke (1988) uses a DP formulation to advise on optimal run rates in both the first and second innings, and Johnston (1992), Johnston et al. (1992, 1993) use the first innings formulation to provide measures of a batsman's performance. There may of course be several possible optimality criteria depending on the different stages of a cricket match. In the first innings batsmen are generally trying to maximise the expected number of runs, but they may prefer to maximise the probability of achieving a certain number of runs. The team batting last is usually trying to maximise the probability of achieving the opponent's score. However in test cricket, teams may often be just trying to avoid a loss. In this case they may wish to avoid finishing in a certain state (team dismissed) with runs being immaterial. In other situations teams may wish to bat for as long as possible. For example, in the fourth test between Australia and the West Indies played April 29 to May 3 1995, in Australia's first innings on the third day, Steve Waugh was the last batsman dismissed for 200. His partners, after the last specialist batsman was dismissed, made 6, 8, 23, 0 and 3 not out. While runs were still important, some commentators made the point that it was also important to occupy the crease for as long as possible, to give the pitch time to break up and so assist the Australian spinner.

The above situation, where a top order batsman is paired with a batsman of lesser ability, often arises towards the end of an innings. In cricket, the facing batsman changes whenever an odd number of runs is scored, and also at the completion of each 6 ball over. The bowling team wish to bowl to the weaker batsman and will often set deep fields to concede a single to the good batsman early in the over. The good batsman in turn will sometimes decline to take the single, in the hope of protecting the weaker batsman for a few balls and taking a run nearer the end of the over. The desired result is to take a single off the last ball, so the better bat is again on strike at the beginning of the next over. In most such situations runs are still important, but in other cases runs are immaterial except in that they allow batsmen to change ends. Describing the last session of play in the famous drawn test between Australia and the West Indies in 1961, Lunn (1993) says

> But with half an hour to go Mackay was no longer the only person who thought he could do it. Taking just a single off the last ball of almost every over (eight ball overs in those days) he faced almost every ball instead of Kline. No attempt to score other than this. ... The last ball of the second last over: Mackay scores a single to face the last over against the fastest man in the world, giant Wes Hall.

In this case Mackay would clearly bat out the last over without taking a run. But in the second last over, if he wants to maximise the chance that he will be on strike for the final Wes Hall over, should he wait until the last ball before taking a single? We look here at a simple DP model to analyse this end play strategy - i.e. maximise the probability that the weaker batsman finishes the over on strike, so that after the change of ends he will be protected from the strike for the beginning of the new over. We leave until later discussion on whether this is a sensible objective function.

## 2. The Model

Suppose you have two batsman whom we will refer to as **G** and **B** for **G**ood and **B**ad, although they can be any two batsmen of different abilities such as **G**reg Chappel and **B**ruce Reid. We assume each batsman can score zero, one (more correctly hit a stroke for which he has the opportunity to run a single,) or be dismissed. (While this initial model is obviously not realistic it is presented for simplicity. We will see that it can be extended with no further calculations to something more realistic). For the good batsman these occur with probability $p_0$, $p_1$ and $p_d$, and for the bad batsman $q_0$, $q_1$ and $q_d$ . Let $p = 1-p_d = p_0 + p_1$, $q = 1- q_d = q_0 + q_1$ and $p>q$ since the poor batsman has a greater chance of dismissal. There are nine wickets down, so once a batsman is dismissed the innings is **E**nded (E).

Since the batsmen have to decide whether to take the runs on offer or not, consider the the situation **after** a ball is bowled but before a run is taken. The stage $n$ is the number of balls still to be bowled in the over. There are five possible states $S_n$ ε {G0, G1, B0, B1, E } representing the facing batsman and the runs on offer. Note that because the epoch in which the state is defined and decision is made is after the ball is bowled, stage 0 refers to the last ball of the over and stage 5 to the first ball in a six ball over. The only decisions occur at G1 and B1 and are whether the batsmen take the run on offer YES (Y) or not NO (N).

The transition diagram is shown in Figure 1.

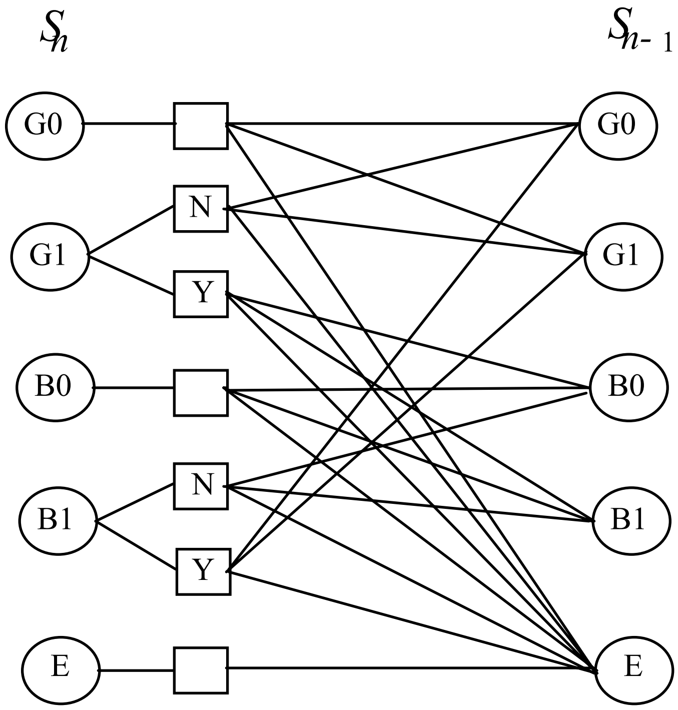


Figure 1. Transition diagram

The transition probabilities are easily calculated provided we ignore complicating factors such as runouts. Thus for example at G1, if the good batsman refuses the run, he will be on strike for the next ball and so after it is bowled there will be a probability $p_0$ of no run on offer (so with probability $p_0$ the next state is G0). Similarly with probability $p_1$ the next state is G1 and with probability $p_d$ the next state is E. Exactly the same transition probabilities arise if the state is B1 and the batsmen take the run. Thus in the above diagram all lines entering G0 have probability $p_0$, all entering G1 have probability $p_1$, all entering $B_0$ are $q_0$, all entering $B_1$ are $q_1$. Those lines entering E from G0, G1(N) and B1(Y) have probability $p_d$, those from B0, B1(N) and G1(Y) have probability $q_d$.

We wish to maximise the chance that at the end of the over the bad batsman will be on strike. Note there is no need to consider in any special way the change of ends at the end of the over. The batsmen wish to maximise for each over the chance that the bad batsman finishes the over on strike. Whether they succeed or not, they have exactly the same problem in the following over.

Let $f_n(S_n)$ be the probability under an optimal policy of ending the over with the bad batsman on strike when in state $S_n$ with $n$ balls to go.

**Initial conditions**

For the last ball of the over, the bad bat will not run, and the good bat will definitely run if possible, so any of the states B0, B1, and G1 will put the bad bat on strike at the end of the over. Thus

$$f_0(\mathrm{G0}) = f_0(\mathrm{E}) = 0$$
$$f_0(\mathrm{B0}) = f_0(\mathrm{B1}) = f_0(\mathrm{G1}) = 1 \qquad (1)$$

**Functional Equations**

In general

$$f_n(S_n) = \underset{\substack{\text{admissible}\\ \text{decisions}}}{Max} \sum_{S_{n-1}\,\mathcal{E}\{G0,G1,B0,B1,E\}} \text{prob}(S_n \to S_{n-1}) f_{n-1}(S_{n-1}) \text{ for } n = 1, 2, \ldots$$

where admissible decisions are defined in Figure 1.

In the particular cases this becomes

State E: $f_n(\text{E}) = f_{n-1}(\text{E}) \Rightarrow f_n(\text{E}) = f_0(\text{E}) = 0$ (2)

State G0: $f_n(\text{G0}) = p_0 f_{n-1}(\text{G0}) + p_1 f_{n-1}(\text{G1}) + p_d f_{n-1}(\text{E})$

$$= p_0 f_{n-1}(\text{G0}) + p_1 f_{n-1}(\text{G1}) \qquad (3)$$

State B0: $f_n(\text{B0}) = q_0 f_{n-1}(\text{B0}) + q_1 f_{n-1}(\text{B1})$ (4)

State G1:
$$f_n(\text{G1}) = \text{MAX}\begin{cases}\text{YES} : q_0 f_{n-1}(\text{B0}) + q_1 f_{n-1}(\text{B1})\\ \text{NO} : p_0 f_{n-1}(\text{G0}) + p_1 f_{n-1}(\text{G1})\end{cases}$$

$$= \text{MAX}\begin{cases}\text{YES} : f_n(\text{B0})\\ \text{NO} : f_n(\text{G0})\end{cases} \quad \text{from (3), (4)} \qquad (5)$$

State B1:
$$f_n(\text{B1}) = \text{MAX}\begin{cases}\text{YES} : p_0 f_{n-1}(\text{G0}) + p_1 f_{n-1}(\text{G1})\\ \text{NO} : q_0 f_{n-1}(\text{B0}) + q_1 p_{n-1}(\text{B1})\end{cases}$$

$$= \text{MAX}\begin{cases}\text{YES} : f_n(\text{G0})\\ \text{NO} : f_n(\text{B0})\end{cases} \qquad (6)$$

So from (5) & (6), $f_n(\text{G1}) = f_n(\text{B1})$ for $n = 1, 2\ldots$
and if optimal decision is YES at G1 then it is NO at B1 and vice versa. (7)

This is a sensible result. Since runs are not important, clearly if it is optimal for the good bat to change ends at some stage, it is optimal for the bad bat not to change ends at the same stage. From now we will refer to the optimal policy as simply YES or NO, meaning the decision at G1 only, as B1 is implied to be the opposite.

This model is completely solvable analytically. The appendix shows that the values of the objective functions can be calculated in a closed form for all stages and states.

**Case 1:** If $q \le p_1$ the optimal decision is NO for all $n > 0$. The better batsman should obtain and remain on strike for the whole over and only run off the last ball. In this case

$$f_n(\text{G1}) = f_n(\text{G0}) = f_n(\text{B1}) = p^{n-1} p_1$$

$$f_n(\text{B0}) = q_0{}^{n-1} q + \frac{q_1 p_1}{q_0 - p}\left(q_0{}^{n-1} - p^{n-1}\right)$$

**Case 2:** If $q > p_1$ there is a stage $N \geq 1$, below which the decision at G1 is YES, and for $n \geq N$ the optimal decision is NO. Thus there will always be a number of balls to go in the over (which may be greater than 6 or 8), before which the optimal strategy is to protect the weaker batsman, and after which the optimal strategy is to put the weaker batsman on strike. $N$ is given by the smallest integer satisfying the inequality

$$n > \frac{\ln\left(\frac{p-q}{p_1}\right)}{\ln\left(\frac{p_0}{q}\right)}.$$

For $n = 1,2,3,...N\text{-}1$ $\quad f_n(\text{G0}) = \frac{p_1}{p_0 - q}\left(p_0^{\,n} - q^n\right)$

and $\quad f_n(\text{G1}) = f_n(\text{B0}) = f_n(\text{B1}) = q^n.$

For $n = N, N+1, N+2,$ $\quad f_n(\text{G0}) = f_n(\text{G1}) = f_n(\text{B1}) = \frac{p^{n-N} p_1}{p_0 - q}\left(p_0^{\,N} - q^N\right)$

and $\quad f_n(\text{B0}) = q_0^{\,n-N} q^N + \frac{q_1}{q_0 - p}\left(q_0^{\,n-N} - p^{n-N}\right) f_N(\text{G0}).$

**Prior Probabilities**

It could be argued that probabilities before a given ball is bowled would be more useful. These are easily obtained by weighting the above state probabilities with the chances of them arising. Thus if we let $F_n(\text{G})$ and $F_n(\text{B})$ be the probabilities of finishing in the required state with G and B facing BEFORE the $n$th last ball of the over is bowled we have

$F_n(\text{B}) = q_0 f_{n\text{-}1}(\text{B0}) + q_1 f_{n\text{-}1}(\text{B1}) = f_n(\text{B0})$ from equation (4)

Similarly $\quad F_n(\text{G}) = f_n(\text{G0}).$

Thus although in the above formulation the stage $n$ only goes to 5, it is useful to calculate $f_6(\text{G0})$ and $f_6(\text{B0})$ as they give the probabilities before the beginning of the over.

## 3. Extension

With no further calculations this model can be extended by considering the state G0 to be the Good bat facing and a score of zero or a boundary (four or six) is made, so no decision on running can be made; G1 that there are 1, 2, 3 runs or a 4 all run on offer so that a decision on running is possible; and the decision to be made is NO to 'run' an even number (quotes because it includes not taking any runs when 1 is on offer) or YES run an odd number of runs. These decisions could be rephrased as NO don't change ends and YES change ends. Thus $p_0$ becomes the probability of a score of 0, 4 (boundary) or 6, $p_1$ becomes the probability of a score of 1, 2, 3 or 4 all run, and $p_d$ remains the probability of dismissal. Similar states and probabilities apply for the bad batsman. The state E can be thought of as the partnership ending rather than the innings ending. The above model and results are then directly applicable.

## 4. Discussion

The probabilities $p_0$ etc should be estimable from scoresheets when batsmen are taking all available runs. The optimal strategy can then be calculated. This is done in the example following. However some of the ramifications of the above seem quite interesting and can be used to make general statements irrespective of the scoring profiles of the batsmen.

For example, equation 7 says that if at any stage, one batsman takes an odd number of runs, at the same stage the other batsman should take an even number of runs. Thus the policy of taking all the runs is NEVER optimal. (Except of course when $p = q$ ).

If $q > p_1$ then the model says that there is always some value $N$ balls to go (admittedly $N$ may be greater than 5) where the optimal strategy changes from NO in one ball to YES for the next. Now if $N$ is greater than 5 this implies the weak batsman will never be protected from the strike (in fact he should be given it). At any stage in the over the good bat will put the weak on strike if possible, and for the rest of the over the weaker bat will stay on strike. Note this only occurs when $p_1$ is so low that it is highly unlikely that the good bat will ever be able to put the bad bat back on strike. This seems unlikely to occur in practice. (It also suggests that we should perhaps look at other objective functions, such as minimise the proportion of times we are in a given state, or maximise the chance of not finishing in State E).

However if $N$ is any number less than 6, the results confirm what is done in practice. It implies that at some stage in the over the optimal decision changes from NO (protect the weak batsman) to YES (put the weak batsman on strike). This is the common strategy used but has an interesting implication when the bad bat is on strike at the beginning of the over. It implies the bad batsman if on strike should change ends if possible, only to again change ends the next ball if possible. Table 1 gives the optimal policy for the good and bad bat when $N$=3. If the bad bat is on strike, he should change ends if possible with 3 balls to go. In this case the good bat will be on strike and should now also immediately change ends if possible with 2 balls to go. This would probably be criticised by commentators.

Table 1. Optimal strategies for the case $N$=3.

| Ball of over | Balls to go $n$ | Good Bat | Bad bat |
|---|---|---|---|
| 1 | 5 | No | Yes |
| 2 | 4 | No | Yes |
| 3 | 3 | No | Yes |
| 4 | 2 | Yes | No |
| 5 | 1 | Yes | No |
| 6 | 0 | Yes | No |

**Example.**

In the World cup match between SA and Australia on 26th Feb 1992, for two batsmen we had from the official score sheets:

Steve Waugh - 1311121114121111112, 27 runs from 51 balls

Bruce Reid - 1211 - 5 runs from 10 balls.

Although Bruce Reid was not out, we might give him an honorary dismissal and from these figures estimate the probabilities as follows.

Steve Waugh: $p_d = 1/51 = 0.02$, $p_1$ = probability of 1, 2 or 3 = 18/51 = 0.35, $p_0$ = probability of 0, 4 or 6 = 32/51 = 0.63

Bruce Reid: $q_d = 1/10 = 0.1$, $q_1 = 4/10 = .4$, $q_0 = 5/10 = 0.5$

Now $q = 0.9 > p_1 = 0.35$, $N >$ ln(0.08/0.35)/ln(0.63/0.9)= 4.14, so $N = 5$. Thus if on strike at the beginning of the over, Steve Waugh should only take an even number of runs for the first ball, but an odd number thereafter. If Reid is on strike he should take an odd number of runs for the first ball, but an even number thereafter. If Waugh is on strike at the start of the over then the chance of Reid finishing the over on strike is given by

$$F_6(\mathrm{G}) = f_6(\mathrm{G0}) = pf_5(\mathrm{G0}) = p\frac{p_1}{p_0 - q}(p_0^5 - q^5) = 0.624$$

If Reid starts the over on strike his chance of finishing on strike is

$$F_6(\mathrm{B}) = f_6(\mathrm{B0}) = q_0 q^5 + q_1 f_5(\mathrm{G0}) = 0.550$$

The optimal policy to put Reid on strike very early is here caused by the high probability of Waugh scoring zero. If we alter $p_0$ to 0.35 and $p_1$ to 0.63, we get $N = 3$, and the optimal strategy would revert to that given in Table 1. The new values of Reid finishing on strike become 0.740 and 0.626. Note how the probability of success alters depending on the good batsman's distribution of probability. Two batsmen could have the same average and run rate, but the one that has a higher probability of scoring runs rather than boundaries is the more flexible and may be a better batsman with lower order players. Maybe more extensive statistics on cricketers should be published. This has also been suggested in the context of measuring consistency by Clarke (1991, 1994).

## 5. An alternative criterion of optimality

By providing a counter example we show that attempting to put the better batsman on strike at the start of the over does not necessarily maximise the expected duration of the partnership.

Consider a Markov chain where the states are B, G and E being the facing batsman when a ball is bowled and the partnership Ended. If the policy is NO (for the good bat, implying YES for the bad bat), then we have using the probabilities as before of a good and bad bat scoring runs, the following transition matrix.

$$\mathrm{P_N} = \begin{array}{c} B \\ G \\ E \end{array}\begin{pmatrix} q_0 & q_1 & q_d \\ 0 & p & p_d \\ 0 & 0 & 1 \end{pmatrix}$$

For example, for the good bat, the probability of not being dismissed is $p_0 + p_1 = p$, which as he does not run is the probability of the good bat being on strike next ball.

If the policy is YES we get the following transition matrix.

$$P_{\mathrm{Y}} = \begin{array}{c} B \\ G \\ E \end{array}\begin{pmatrix} q & 0 & q_d \\ p_1 & p_0 & p_d \\ 0 & 0 & 1 \end{pmatrix}$$

Then using the example given above with Waugh and Reid, we have

$$P_{\mathrm{N}} = \begin{pmatrix} 0.5 & 0.4 & 0.1 \\ 0 & 0.98 & 0.02 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad P_{\mathrm{Y}} = \begin{pmatrix} 0.9 & 0 & 0.1 \\ 0.35 & 0.63 & 0.02 \\ 0 & 0 & 1 \end{pmatrix}$$

This gives, for the optimal policy of NO for the first ball and YES for the last 5 balls a transition matrix for the over of $P_N^1 P_Y^5 = \begin{pmatrix} 0.550 & 0.040 & 0.410 \\ 0.624 & 0.097 & 0.279 \\ 0 & 0 & 1 \end{pmatrix}$

Note the answers 0.550 and 0.624 agree with that given above for the bad bat ending the over on strike.

Alternatively, if we look at a strategy of the good bat running if possible only off the last ball, ie NO from the first 5 balls and YES from the last ball, we get a transition matrix for the over of $P_N^5 P_Y^1 = \begin{pmatrix} 0.283 & 0.458 & 0.259 \\ 0.316 & 0.569 & 0.114 \\ 0 & 0 & 1 \end{pmatrix}$

This certainly gives much lower probabilities of the bad bat ending the over on strike. It also (not surprisingly) gives lower probabilities of the partnership ending by the end of the over. Our hope was these would be cancelled out by the higher chance of the partnership ending earlier because the bad bat has a higher probability of facing up to the next over.

However, by swapping the first and second columns to allow for the change of ends at the completion of the over, we now have a transition matrix where the stage is an over and the states are the batsmen on strike at the beginning of the over. This gives for the optimal policy

$\begin{pmatrix} 0.040 & 0.550 & 0.410 \\ 0.097 & 0.624 & 0.279 \\ 0 & 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0.458 & 0.283 & 0.259 \\ 0.569 & 0.316 & 0.114 \\ 0 & 0 & 1 \end{pmatrix}$ for the other policy.

The expected number of times in each state can be calculated by applying normal methods for absorbing Markov chains. The fundamental matrix gives for the two cases $\begin{pmatrix} 1.222 & 1.788 \\ 0.316 & 3.123 \end{pmatrix}$ and $\begin{pmatrix} 3.264 & 1.349 \\ 2.719 & 2.587 \end{pmatrix}$. Again, the smaller numbers in the first column show the optimal strategy clearly produces a much lower expected number of times the bad batsman is on strike at the beginning of the new over. However, the expected number of completed overs the partnership will last is given by summing the rows to give $\begin{pmatrix} 3.011 \\ 3.439 \end{pmatrix}$ and $\begin{pmatrix} 4.613 \\ 5.306 \end{pmatrix}$. Thus the second case gives longer expected partnership times, whoever is facing the first over, than the previous optimal policy. These calculations can be repeated for each of the seven possible policies and are shown in Table 2. This shows in fact to optimise the length of the partnership the good bat should run on the fifth ball, although there is little difference in the two options of running on the fifth or sixth ball.

Table 2. Expected number of completed overs partnership lasts if good batsman runs on ball $n$

| Facing Batsman at beginning of over | $n$ | | | | | | |
|---|---|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | 5 | 6 | Never |
| Bad | 2.573 | 3.011 | 3.545 | 4.145 | 4.629 | 4.613 | 3.892 |
| Good | 2.960 | 3.439 | 4.050 | 4.753 | 5.325 | 5.306 | 4.447 |

The best policy under the criteria of maximising the expected number of completed overs was investigated numerically in this manner for a range of values. Using SAS/IML the optimal ball on which the good batsman should run was calculated for each value of $p_d$, $q_d$ = 0.01 to 0.10 in steps of 0.01, $p_0$, $q_0$ = 0.1 to 0.9 in steps of 0.1, $p_d < q_d$. In 65% of these cases the optimal strategy was to run off the last ball, 32% the second last ball with the remaining 3% of cases giving the 4th ball. Thus the simple strategy of the good batsman getting off the strike if possible on the second last or last ball of the over, and the bad batsman doing the same off the first four balls, generally optimises the expected number of completed overs the partnership will last.

## 6. Conclusion

A simple DP model was set up to solve a specific and very limited problem. The model can be solved completely analytically, and the solution used to suggest a suitable strategy for every over except the last. In practice, the model is probably deficient in a couple of respects. The fielding side often set widespread fields to the good batsman early in the over to encourage a single, and bring the field in later in the over to prevent the batsmen taking a single. In this case the values of $p_0$ etc would depend on the stage $n$ as well as the state. The model would still possibly be soluble analytically, and certainly numerically. However the model can also suggest reasons why some cricketers' scoring profile could make them more suited than others to certain situations such as playing with tail-enders. Commentators often comment on the ability (or lack of it) of a player at rotating the strike, but commonly kept statistics do not measure this.

It is clear that minimising the exposure of the weaker batsman to the first ball of the new over is not necessarily the appropriate objective function. It does not necessarily minimise his exposure to the strike, the chance of the partnership ending within a certain number of balls, or maximise the number of balls until the partnership is broken. The optimal strategy to maximise the number of completed overs for the partnership can be found numerically, and usually requires the better batsman running an odd number of runs if possible on the second last or last ball of the over.

The ball by ball nature of cricket makes it particularly suitable for a DP approach. Clarke and Norman (1996) have constructed several other models using alternative objective functions such as maximising the expected number of runs in the remainder of the innings, which take into account the number of runs, run rate, the number of wickets down and the change of ends between overs. Such models could be used to assist with end play strategies in both the first and second innings.

Note: An earlier version of this paper was presented at the 13th ASOR conference in Canberra, 1995.

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## Appendix

The functional equations (1) to (6) can be solved to obtain analytical solutions.

**Theorem 1.** *If there is some stage n for which the optimal decision is NO, then it is also NO for the previous stage n + 1.*

**Proof**: If decision is NO at $n$ then from (5) $f_n(\text{G0}) > f_n(\text{B0})$ (8)

and $f_n(\text{G1}) = f_n(\text{G0}) = f_n(\text{B1})$ (9)

Then $f_{n+1}(\text{G0}) - f_{n+1}(\text{B0})$

$= p_0 f_n(\text{G0}) + p_1 f_n(\text{G1}) - q_0 f_n(\text{B0}) - q_1 f_n(\text{B1})$ from (3) and (4)

$\geq p_0 f_n(\text{G0}) + p_1 f_n(\text{G0}) - q_0 f_n(\text{G0}) - q_1 f_n(\text{G0})$ from (8) and (9)

$= (p_0 + p_1 - q_0 - q_1) f_n(\text{G0})$

$= (p - q) f_n(\text{G0})$

$> 0$ since $p > q$

ie. $f_{n+1}(\text{G0}) > f_{n+1}(\text{B0})$ and by (5) the optimal decision at $n$+1 is NO. ■

Consider the stage $n$= 1

$f_1(\text{B0}) = q_0 f_0(\text{B0}) + q_1 f_0(\text{B1})$ from (4)

$= q_0 + q_1$ from (1)

$= q$ (10)

$f_1(\text{G0}) = p_0 f_0(\text{G0}) + p_1 f_0(\text{G1})$ from (3)

$= p_1$ from (1) (11)

So using (5), if $q > p_1$ then decision is YES and $f_1(\text{G1}) = q$

and if $q \leq p_1$ then decision is NO and $f_1(\text{G1}) = p_1$. (12)

We thus have two cases.

**Case 1: $q \leq p_1$**

From (12) and Theorem 1, decision is always NO and so for $n \geq 1$

$f_n(\text{G1}) = f_n(\text{G0})$ from (5)

$= p_0 f_{n-1}(\text{G0}) + p_1 f_{n-1}(\text{G1})$ from (3)

$= (p_0 + p_1) f_{n-1}(\text{G0})$ from above

$= p f_{n-1}(\text{G0})$

$= p^{n-1} f_1(\text{G0})$

$= p^{n-1} p_1$ from (11)

$f_n(\text{B1}) = f_n(\text{G0}) = p^{n-1} p_1$ from (6)

$f_n(\text{B0}) = q_0 f_{n-1}(\text{B0}) + q_1 f_{n-1}(\text{B1})$ from (4)

$= q_0 f_{n-1}(\text{B0}) + q_1 p^{n-2} p_1$ from above

$$= q_0^{n-1} q + \frac{q_1 p_1}{q_0 - p}(q_0^{n-1} - p^{n-1})$$

**Case 2: $q > p_1$**

From (10) decision at $n$=1 is YES and $f_1(\mathrm{G1}) = q$. Let $n = N$ be the first time the decision is NO, then for $n$ = 1,2,... $N$–1 the decision is YES and

$$f_n(\mathrm{G1}) = f_n(\mathrm{B0}) = f_n(\mathrm{B1}) \text{ for } n = 1,2,...N\text{-}1 \text{ from (5)} \tag{13}$$

So for n = 1 to $N$

$$\begin{aligned} f_n(\mathrm{B0}) &= q_0 f_{n-1}(\mathrm{B0}) + q_1 f_{n\text{-}1}(\mathrm{B1}) \quad \text{from (4)} \\ &= q_0 f_{n-1}(\mathrm{B0}) + q_1 f_{n\text{-}1}(\mathrm{B0}) \\ &= q\, f_{n-1}(\mathrm{B0}) \\ &= q^2 f_{n-2}(\mathrm{B0})....= q^n f_0(\mathrm{B0}) \\ &= q^n \end{aligned} \tag{14}$$

So by (13), for $n$ = 1, 2.....$N$-1, $f_n(\mathrm{G1}) = f_n(\mathrm{B0}) = f_n(\mathrm{B1}) = q^n$ (15)

Now $f_1(\mathrm{G0}) = p_1$ by (11)

So $f_n(\mathrm{G0}) = p_0 f_{n\text{-}1}(\mathrm{G0}) + p_1 f_{n\text{-}1}(\mathrm{G1})$

$$\begin{aligned} &= p_0 f_{n\text{-}1}(\mathrm{G0}) + p_1 q^{n\text{-}1} \text{ by (15)} \\ &= \frac{p_1}{p_0 - q}(p_0{}^n - q^n) \quad \text{for } n = 1, 2, 3...., N \text{ by induction} \end{aligned} \tag{16}$$

Now, since N is smallest n for which decision at G1 is NO, we have from (5)

$$f_N(\mathrm{G0}) > f_N(\mathrm{B0})$$

$$\frac{p_1}{p_0 - q}[p_0{}^N - q^N] > q^N \qquad \text{from (14), (16)}$$

$$p_1 p_0{}^N - p_1 q^N > p_0 q^N - q^{N+1} \text{ where sign reverses if } p_0 < q$$

 $p_1 p_0{}^N > (p_0 + p_1) q^N - q^{N+1}$

$$= (p - q) q^N$$

 $\left(\frac{p_0}{q}\right)^N > \frac{p-q}{p_1}$

$$N > \frac{\ln\frac{p-q}{p_1}}{\ln\frac{p_0}{q}} \text{ where sign reverses back if } p_0 < q$$

$$= \frac{\ln\frac{q_\mathrm{d} - p_\mathrm{d}}{p_1}}{\ln\frac{p_0}{1 - q_\mathrm{d}}} \tag{17}$$

Now for $n = N, N + 1,...$ ($n > N$). Decision is NO

So $f_n(\mathrm{G0}) = f_n(\mathrm{B1}) = f_n(\mathrm{G1})$ by (5), (6) (18)

But

$$\begin{aligned} f_n(\mathrm{G0}) &= p_0 f_{n-1}(\mathrm{G0}) + p_1 f_{n-1}(\mathrm{G1}) \text{ by (3)} \\ &= p_0 f_{n-1}(\mathrm{G0}) + p_1 f_{n-1}(\mathrm{G0}) \text{ by (18) if } n-1 \geq N \\ &= (p_0 + p_1) f_{n-1}(\mathrm{G0}) \qquad \text{if } n \geq N + 1 \\ &= p^{n-N} f_N(\mathrm{G0}) \end{aligned} \tag{19}$$

OR $f_{m+N}(\mathrm{G0}) = p^m f_N(\mathrm{G0})$

so $f_n(G0) = f_n(B1) = f_n(G1) = p^{n-N} \dfrac{p_1}{p_0 - q} (p_0{}^N - q^N)$ for $n > N$(20)

OR $f_{m+N}(G0) = f_{m+N}(B1) = f_{m+N}(G1) = p^m \dfrac{p_1}{p_0\ q} (p_0{}^N - q^N)$ for $m > 0$(21)

$$
\begin{aligned}
f_n(B0) &= q_0 f_{n-1}(B0) + q_1 f_{n-1}(B1) \text{ from (4)} \\
&= q_0 f_{n-1}(B0) + q_1 p^{n-N-1} f_N(G0) \\
&= q_0{}^{n-N} q^N + \frac{q_1}{q_0 - p} (q_0{}^{n-N} - p^{n-N})\ f_N(G0) \\
&\text{for } n = N+1, N+2, \ldots
\end{aligned}
$$

(22)

OR $f_{m+N}(B0) = q_0{}^m\ q^N + \dfrac{q_1}{q_0 - p} (q_0{}^m - p^m)\ f_N(G0)$ for $m > 0$.

So we have an analytic solution for $f_n(G0), f_n(G1), f_n(B0)$ and $f_n(B1)$ for all $n$.

**Author details:**

Stephen R Clarke is an Associate Professor in the School of Mathematical Sciences at Swinburne University. He holds a BSc(Hons) Dip Ed from Melbourne University and an MA in Operational Research from Lancaster University. He is a member of ASOR, ORS and INFORMS. His major research interest is in OR applications in sport, mainly racquet sports, golf, football and cricket. He has published in international journals such as JORS, APJOR, The Statistician and Journal of Applied Statistics. He is a winner of 1988 ORS President's medal for a paper in JORS on Dynamic Programming in cricket.

John Norman has a BA in Philosophy, Politics and Economics and an MSc and PhD in Operational Research. His main research interest is in Dynamic Programming applications, especially in sport and he is currently working on the effectiveness of university-based management education. He has published papers in JORS, Operations Research, EJOR, the IMA Journal of Mathematics applied to Business and Industry, the Statistician and other journals. He has held visiting professorships in India, the USA and Japan. He worked for McVitie & Price, Dunlop, and the Centre for Business Research before entering academic life as a lecturer at the University of Lancaster. Since 1978 he has been Professor of Management Studies at the University of Sheffield.