Eigen values & vectors

Today's agenda

1) Intuition behind eigen values & vectors
2) Derivation
3) Properties of eigen values & vectors

Assumption :-

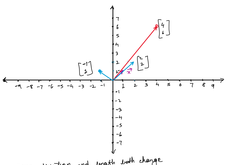
1) matrix multiplication can also be viewed as linear transformation.

When you multiply a vector with matrix, following transformation take place :-

i) change in length only
ii) change in direction only
iii) change in both length & direction
iv) no change

Ex:- Consider the matrix

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix} \quad \vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}_{2\times 1} \quad b = A\vec{x} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$



Have direction and length both change.

However there will be vectors where direction does not change. These vectors are unique to a matrix.

Consider another random point in the plane $\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$A\vec{x} = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

Thus every vector undergoes some transformation after matrix multiplication.

The vectors whose only length changes after matrix multiplication are called eigen vectors for that matrix.

Mathematically, we can write them as

$\boxed{A\vec{x} = \lambda\vec{x}}$; Here A is a matrix

$\vec{x} \rightarrow$ eigen vector

$\lambda \rightarrow$ scalar value

$\lambda \rightarrow$ It is the scalar value by which the original vector changes length also called eigen values

Solving the above equation to calculate the values of eigen values

$$A\vec{x} - \lambda\vec{x} = 0 \Rightarrow A\vec{x} - \lambda I\vec{x} = 0$$

$$\boxed{(A - \lambda I)\vec{x} = 0} \rightarrow (1)$$

To perform subtraction, $\boxed{A \text{ needs to be square matrix}}$ since λI will be square matrix.

$$\therefore (A - \lambda I)\vec{x} = 0$$

Consider $A - \lambda I = B$ $\therefore$ $\boxed{B\vec{x} = \vec{0}}$. (2)

Quick recap:

system of equations

$a_{11}x_1 + a_{12}x_2 = b_1$

$a_{21}x_1 + a_{22}x_2 = b_2$

In matrix form

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$\boxed{A \quad x = b}$$

system of equations

Here b is different from λ used above

Homogeneous S.o.E

When the constant term b is zero

$\therefore A\vec{x} = \vec{0}$

Trivial solution : $\vec{x} = 0$

$\therefore A\vec{x} = 0$

Non trivial solution $A\vec{x} = \vec{0}$

Non homogeneous S.o.E

When the constant term b is non-zero

$A\vec{x} = b$

b is a column vector with atleast one non-zero value

Coming back to $B\vec{x} = \vec{0}$

$$(A - \lambda I)\vec{x} = \vec{0}$$

For the above S.o.E.

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{bmatrix}$$

$$\therefore \begin{bmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\boxed{\begin{aligned} \therefore (a_{11} - \lambda)x_1 + a_{12}x_2 &= 0 \\ a_{21}x_1 + (a_{22} - \lambda)x_2 &= 0 \end{aligned}}$$

Now we know that $\vec{x} \neq \vec{0}$ $\therefore$

i) Either $x_1 = 0$ & $x_2 \neq 0$

ii) Either $x_1 \neq 0$ & $x_2 = 0$

iii) $x_1 \neq 0$ & $x_2 \neq 0$

i) $x_1 = 0$ & $x_2 \neq 0$

$\therefore$ $a_{12}x_2 = 0 \Rightarrow a_{12} = 0$

$(a_{22} - \lambda)x_2 = 0 \Rightarrow a_{22} = \lambda$

Now $a_{22} = \lambda$ $\therefore$ equation becomes

$a_{21}x_1 = 0 \Rightarrow a_{21} = 0$

$(a_{11} - \lambda)x_1 = 0 \Rightarrow a_{11} = \lambda$

$$\therefore A = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

ii) $x_1 \neq 0$ & $x_2 = 0$

$\therefore$ Equation becomes

$(a_{11} - \lambda)x_1 = 0 \qquad a_{11} = \lambda$

$a_{21}x_1 = 0 \qquad a_{21} = 0$

Substituting values back and getting a_{12}, a_{22}

$a_{12} = 0$; $a_{22} = \lambda$

$$\therefore A = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

iii) $x_1 \neq 0$ & $x_2 \neq 0$

$\therefore (a_{11} - \lambda)x_1 + a_{12}x_2 = 0$

$a_{21}x_1 + (a_{22} - \lambda)x_2 = 0$

$(a_{11} - \lambda)x_1 = -a_{12}x_2 \Rightarrow \dfrac{x_1}{x_2} = \dfrac{-a_{12}}{a_{11} - \lambda}$

$a_{21}x_1 = -(a_{22} - \lambda)x_2$

$\Rightarrow \dfrac{x_1}{x_2} = \dfrac{-(a_{22} - \lambda)}{a_{21}}$

$$\therefore \frac{x_1}{x_2} = \boxed{\frac{-a_{12}}{a_{11} - \lambda} = \frac{-(a_{22} - \lambda)}{a_{21}}}$$

$$\therefore -a_{12}a_{21} + (a_{22} - \lambda)(a_{11} - \lambda) = 0$$

$$(a_{11} - \lambda)(a_{22} - \lambda) - a_{12}a_{21} = 0 \Rightarrow \det(B) = 0$$

$\therefore B\vec{x} = \vec{0}$

When $B \neq 0$ & $\vec{x} \neq 0$ then for the S.o.E to have a non zero solution, $\det(B) = 0$

$\therefore$ Summary

i) You can calculate eigen values of only a square matrix & eigen vectors

ii) Eigen values can be 0 but eigen vectors by definition cannot be $\vec{0}$ vector

iii) Eigen values of a diagonal matrix are the entries of the diagonal.

Ex:- $A = \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}$ Eigen values $= 3, 5$

iv) Max number of eigen values a matrix can have is the dimension of that matrix

v) A matrix is fully diagonalizable if it has n linearly independent eigen vectors.

1. An **eigenvector** of A is a nonzero vector x in R n such that Ax=λx, for some scalar λ.
2. An **eigenvalue** of A is a scalar λ such that the equation Ax=λx has a nontrivialsolution. If Ax=λx for x≠0, we then say that λ is the **eigenvalue for** x, and that x is an **eigenvector for** λ.

$$\begin{bmatrix} -5 & 2 \\ -7 & 4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 2 \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$\begin{bmatrix} -5v_1 + 2v_2 \\ -7v_1 + 4v_2 \end{bmatrix} = \begin{bmatrix} 2v_1 \\ 2v_2 \end{bmatrix}$$

$-5v_1 + 2v_2 = 2v_1 \Rightarrow -7v_1 = -2v_2 \therefore 7v_1 = 2v_2$

$-7v_1 + 4v_2 = 2v_2 \Rightarrow -7v_1 = -2v_2 \therefore 7v_1 = 2v_2$

$\therefore 7v_1 = 2v_2$

If $v_2 = t$ then $v_1 = \frac{2}{7}t$

$$\therefore \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 2/7\, t \\ t \end{bmatrix} = t \begin{bmatrix} 2/7 \\ 1 \end{bmatrix}$$

For $\lambda = -3$

$$\begin{bmatrix} -5 & 2 \\ -7 & 4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = -3 \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$\begin{bmatrix} -5v_1 + 2v_2 \\ -7v_1 + 4v_2 \end{bmatrix} = \begin{matrix} -3v_1 \\ -3v_2 \end{matrix}$$

$-5v_1 + 2v_2 = -3v_1 \Rightarrow -2v_1 = -2v_2 \therefore v_1 = v_2$

$-7v_1 + 4v_2 = -3v_2 \Rightarrow -7v_1 = -7v_2 \therefore v_1 = v_2$

$\therefore v_1 = v_2$

Let $v_2 = t \therefore v_1 = t$

$$\therefore \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\therefore$ For $\lambda_1 = 2$, $v = t \begin{bmatrix} 2/7 \\ 1 \end{bmatrix}$

For $\lambda_2 = -3$, $v = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Properties of eigen values & vectors