Covariance matrix

Video proof quickly understand the definition of variance and covariance.

Standard formula for

Variance $(X) = \dfrac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{n-1}$;

Covariance $(X,Y) = \dfrac{\sum_{i=1}^{n}(x_i-\bar{x})(y_i-\bar{y})}{n-1}$

X	Y
x_1	y_1
x_2	y_2
x_3	y_3
$\vdots$	$\vdots$
x_n	y_n

Consider the dataset

$$D = \begin{array}{c} \begin{array}{cccc} \text{Var_1} & \text{Var_2} & \text{Var_3} \cdots & \text{Var_p} \end{array} \\ \begin{bmatrix} x_{11} & x_{12} & x_{13} & x_{1p} \\ x_{21} & x_{22} & x_{23} & x_{2p} \\ x_{31} & x_{32} & x_{33} & x_{3p} \\ \vdots & \vdots & \vdots & \vdots \\ x_{n1} & x_{n2} & x_{n3} & x_{np} \end{bmatrix}_{n\times p} \end{array}$$

Matrix formula to calculate variance & covariance of the above matrix

$$\boxed{\Sigma = \frac{1}{n-1}(X-\bar{X})^T(X-\bar{X})}$$

Here $X \in \mathbb{R}_{n\times p}$

Let us solve for $n=3$ & $p=2$ $\therefore$

$$\therefore X = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \\ x_{31} & x_{32} \end{bmatrix}_{3\times 2}$$

$$\bar{X} = \begin{bmatrix} \frac{x_{11}+x_{21}+x_{31}}{3} & \frac{x_{12}+x_{22}+x_{32}}{3} \end{bmatrix} = \begin{bmatrix} \bar{x}_1 & \bar{x}_2 \end{bmatrix}_{1\times 2}$$

$$X - \bar{X} = \begin{bmatrix} x_{11}-\bar{x}_1 & x_{12}-\bar{x}_2 \\ x_{21}-\bar{x}_1 & x_{22}-\bar{x}_2 \\ x_{31}-\bar{x}_1 & x_{32}-\bar{x}_2 \end{bmatrix}_{3\times 2}$$

$$(X-\bar{X})^T = \begin{bmatrix} x_{11}-\bar{x}_1 & x_{21}-\bar{x}_1 & x_{31}-\bar{x}_1 \\ x_{12}-\bar{x}_2 & x_{22}-\bar{x}_2 & x_{32}-\bar{x}_2 \end{bmatrix}_{2\times 3}$$

$$\therefore \frac{(X-\bar{X})^T(X-\bar{X})}{n-1} =$$

$$\begin{bmatrix} x_{11}-\bar{x}_1 & x_{21}-\bar{x}_1 & x_{31}-\bar{x}_1 \\ x_{12}-\bar{x}_2 & x_{22}-\bar{x}_2 & x_{32}-\bar{x}_2 \end{bmatrix}_{2\times 3} \begin{bmatrix} x_{11}-\bar{x}_1 & x_{12}-\bar{x}_2 \\ x_{21}-\bar{x}_1 & x_{22}-\bar{x}_2 \\ x_{31}-\bar{x}_1 & x_{32}-\bar{x}_2 \end{bmatrix}_{3\times 2}$$

$$= \begin{bmatrix} \frac{(x_{11}-\bar{x}_1)^2+(x_{21}-\bar{x}_1)^2+(x_{31}-\bar{x}_1)^2}{n-1} & \frac{(x_{11}-\bar{x}_1)(x_{12}-\bar{x}_2)+(x_{21}-\bar{x}_1)(x_{22}-\bar{x}_2)+(x_{31}-\bar{x}_1)(x_{32}-\bar{x}_2)}{n-1} \\ \frac{(x_{12}-\bar{x}_2)(x_{11}-\bar{x}_1)+(x_{22}-\bar{x}_2)(x_{21}-\bar{x}_1)+(x_{32}-\bar{x}_2)(x_{31}-\bar{x}_1)}{n-1} & \frac{(x_{12}-\bar{x}_2)^2+(x_{22}-\bar{x}_2)^2+(x_{32}-\bar{x}_2)^2}{n-1} \end{bmatrix}_{2\times 2}$$

$$= \begin{bmatrix} \text{Var}(X_1) & \text{Cov}(X_1,X_2) \\ \text{Cov}(X_1,X_2) & \text{Var}(X_2) \end{bmatrix}$$

* Keep in mind about the dimensions of the matrix

If $X \in \mathbb{R}_{p\times n}$ then $\text{Cov}(X) = \Sigma = \frac{1}{n-1}(X-\bar{X})(X-\bar{X})^T$

$\text{Cov}(X) \in \mathbb{R}_{p\times p}$

* Most of the times, you will see the definition of

$\text{Cov}(X) = XX^T$ or X^TX $\Rightarrow$ this means X is already centered.

* Another way to center the data i.e. matrix is

$H = I - n^{-1}ee^T$ $\Rightarrow$ centering matrix ; $H \in \mathbb{R}^{n\times n}$

If $n=3, p=2$ $\quad H \in \mathbb{R}^{3\times 3}$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{1}{n}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{1}{n}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1/n & 1/n & 1/n \\ 1/n & 1/n & 1/n \\ 1/n & 1/n & 1/n \end{bmatrix}$$

$$= \begin{bmatrix} 1-1/n & -1/n & -1/n \\ -1/n & 1-1/n & -1/n \\ -1/n & -1/n & 1-1/n \end{bmatrix}_{3\times 3} \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \\ x_{31} & x_{32} \end{bmatrix}$$

$$= \begin{bmatrix} x_{11}-\frac{x_{11}}{n}-\frac{x_{21}}{n}-\frac{x_{31}}{n} & x_{12}-\frac{x_{12}}{n}-\frac{x_{22}}{n}-\frac{x_{32}}{n} \\ -\frac{x_{11}}{n}+x_{21}-\frac{x_{21}}{n}-\frac{x_{31}}{n} & -\frac{x_{12}}{n}+x_{22}-\frac{x_{22}}{n}-\frac{x_{32}}{n} \\ -\frac{x_{11}}{n}-\frac{x_{21}}{n}+x_{31}-\frac{x_{31}}{n} & -\frac{x_{12}}{n}-\frac{x_{22}}{n}+x_{32}-\frac{x_{32}}{n} \end{bmatrix}$$

$$= \begin{bmatrix} x_{11}-\bar{x}_1 & x_{12}-\bar{x}_2 \\ x_{21}-\bar{x}_1 & x_{22}-\bar{x}_2 \\ x_{31}-\bar{x}_1 & x_{32}-\bar{x}_2 \end{bmatrix}$$

* Properties of Covariance matrix

i) Always a square matrix (determinants, eigenvalues, powers, inverses)

ii) Symmetric matrix i.e. $\text{Cov}(X) = (\text{Cov}(X))^T$

iii) Diagonal elements represents variance of each variable

Off-diagonal element represents covariance between two variables

$x_{ij} \rightarrow$ covariance between i^{th} and j^{th} variable

iv) Non-negative definite:- variances cannot be negative

$$\text{Cov}(X) = \frac{1}{n-1}(X-\bar{X})^T(X-\bar{X}) \qquad X \in \mathbb{R}_{n\times p}$$

$$\text{Cov}(X) = \frac{1}{n-1}(X-\bar{X})(X-\bar{X})^T \qquad X \in \mathbb{R}_{p\times n}$$