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Thurs 7pm-9pm

HW 0

1.
Favorite Movie Number : cid=542
Winter Travel Number : cid = 544

2.

(a) Just writing out the first few terms (for n=2 through 6) reveals a pattern:

$$\frac{3}{4}, \frac{4}{6}, \frac{5}{8}, \frac{6}{10}, \frac{7}{12}, \ldots$$

So, in general, the expression can be just simplified to $\frac{n+1}{2n}$

(b) $3^{1000} mod 7$

Looking at the first few exponents of $3^n mod 7$ we find a pattern:

$$3^0 mod 7 = 1$$
$$3^1 mod 7 = 3$$
$$3^2 \ mod 7 = 2$$
$$3^3 mod 7 = 6$$
$$3^4 mod 7 = 4$$
$$3^5 mod 7 = 5$$
$$3^6 mod 7 = 1$$
$$3^7 mod 7 = 3$$
$$\ldots$$

Following this pattern of 1,3,2,6,4,5,...
Now all we need is $1000 \ mod \ 6 = 4$, which corresponds to 4 on the list of the pattern(the first 6 are indexed as 0 through 5).

So $3^{1000} mod 7 = 4$

(c) $\sum_{r=1}^{\infty} \left(\frac{1}{2}\right)^r$

This is just a geometric series. The equation for the sum of an n-term geometric series is:

$$S_n = \frac{a_1(1 - r^n)}{1 - r}$$

Where a_1 is the first term, and r is the ratio being multiplied every time.
So in our case:

$$\lim_{n\to\infty} S_n = \frac{\frac{1}{2}\left(1-\frac{1}{2}^n\right)}{\frac{1}{2}}$$
$$\lim_{n\to\infty} S_n = (1-0) = 1$$

(d) $\frac{log_7 80}{log_7 9} = log_9 80$, using the log change of base formula in reverse.
(e) $log_2 4^{2n} = 2nlog_2 4 = 2(2n) = 4n$
(f) $\log_{17} 221 - \log_{17} 13 = \log_{17}\frac{221}{13} = \log_{17} 17 = 1$

3. I wrote out the first few terms:
2, 6, 24, 120
After writing out the first few sums, it was pretty obvious that the closed-form is
$$F(n) = (n+1)!$$
Base step:
$$F(1) = 1 + \sum_{i=1}^{1} i! * i = 1 + 1! * 1 = 2$$
$$F(1) = (1+1)! = 2! = 2$$
Inductive step:

Assume the closed-form is correct for all $n = x$, so it must work for any $n = x + 1$
So then, my formula would be correct only if:
$$1 + \sum_{j=1}^{x+1} j! * j = (x+1+1)!$$
$$1 + (x+1)(x+1)! + \sum_{j=1}^{x} j! * j = (x+2)!$$
$$(x+1)(x+1)! = (x+2)! - (\sum_{j=1}^{x} j! * j + 1)$$
$$(x+1)(x+1)! = (x+2)(x+1)! - (x+1)!$$
$$(x+1)(x+1)! = (x+2-1)(x+1)!$$
$$(x+1)(x+1)! = (x+1)(x+1)!$$
$$Q.E.D.$$

4.
(a) $f(n) = 4^{log_4 n}$, g(n)=2n+1
$$f(n) = 4^{\log_4 n} = n$$
We can prove $f(n)$ is $\theta(g(n))$ by showing $\lim_{n\to\infty}\frac{|f(n)|}{|g(n)|} = c, 0 < c < \infty$

$\lim_{n\to\infty} \frac{n}{2n+1} = \lim_{n\to\infty} \frac{\frac{n}{n}}{\frac{2n}{n}+\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{2+\frac{1}{n}}$
$= \frac{1}{2}$
$0 < \frac{1}{2} < \infty$, so f(n) is $\theta(g(n))$

(b) $f(n) = n^2, g(n) = \sqrt{2}^{\log_2 n}$
$f(n)\ is\ \Omega(g(n))\ iff\ \exists c \in \mathbb{R}: c > 0, k \geq 0: \forall n > k, |f(n)| \geq |cg(n)|$

$n^2 \geq c\sqrt{2}^{\log_2 n} \equiv \log_2 n^2 \geq \log_2\left(c\sqrt{2}^{\log_2 n}\right) \equiv \log_2 n^2 \geq \log_2 c + \log_2\left(\sqrt{2}^{\log_2 n}\right)$
$\equiv \log_2 n^2 \geq \log_2 c + \frac{1}{2}\log_2 n * \log_2 2 \equiv \log_2 n^2 \geq \log_2 c + \log_2 \sqrt{n}$
$\equiv \log_2 n^2 \geq \log_2 c\sqrt{n} \equiv n^2 \equiv n^2 \geq c\sqrt{n}$
$n^2 > \sqrt{n}$, for all n > 1
So we choose k = 1, and C = 1, to prove that
$f(n) \in \Omega(g(n))$

(c) $f(n) = \log_2 n!, g(n) = nlog_2 n$
$f(n) \in \Theta(g(n))$

(d)$f(n) = n^k, g(n) = c^n, c > 1$
$f(n) \in O(g(n))$

5.
(a). $T(n) = T\left(\frac{n}{2}\right) + 5, T(1) = 1, where\ n\ is\ a\ power\ of\ 2$
Since n is a power of two, we create a substitution
$S(k) = T(2^k), where\ k \geq 0$
$= T(2^{k-1}) + 5 = S(k-1) + 5, S(0) = 1$, which is just an arithmetic sequence given by:
$S_n = 1 + 5n$
$T(n) = S(\log_2 n) = 1 + 5(\log_2 n)$, assuming n is a power of 2.

(b). $T(n) = T(n-1) + \frac{1}{n}, T(0) = 0$
There is no solution for this recurrence. The function is essentially:
$$T(n) = \sum_{x=1}^{n} \frac{1}{x}, (n > 0)$$
which approximates ln(n),but there exists no closed form equation

(c). Prove that your answer to part (a) is correct using induction.

Once again, we transform the function, without loss of generality, back to:
$S(k) = T(2^k) = S(k-1) + 5, S(0) = 1$, which is solved by:
$S(n) = 1 + 5n$, and prove from there.

Base step: $S(0) = 1 + 5(0) = 1$

Inductive step:

Assume the closed-form is correct for all $n = x$, so it must work for any $n = x + 1$
So then, my formula would be correct only if:

$$S(x - 1 + 1) + 5 = 1 + 5(x + 1)$$
$$S(x) + 5 = 1 + 5x + 5$$
$$(1 + 5x) + 5 = 1 + 5x + 5$$
$$Q.E.D.$$

6.
(a.) Give a recurrence for worst case running time of the recursive Binary Search function in terms of n, the size of the search array. Assume n is a power of 2. Solve the recurrence.

Using the Master Theorem,

Recurrence: $T(n) = T\left(\frac{n}{2}\right) + O(1), n\ is\ a\ power\ of\ 2$
Base Case: $T(1) = O(1)$, since it's the only item and no comparison needed.
Recurrence solution:
$S(k) = T(2^k) = S(k-1) + O(1), S(0) = 0$
$S(k) = O(1) * k + O(1) = k[O(1) + 1] = k * O(1)$
$T(n) = S(\log_2 n) = O(1)\log_2 n \in O(\log n)$

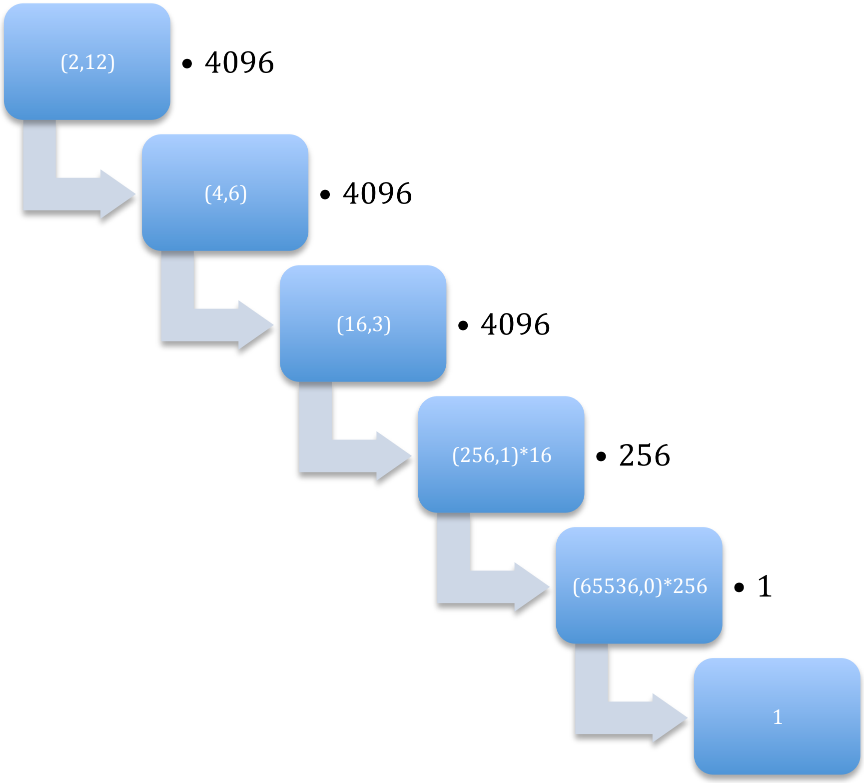
(b) Give a recurrence for worst case running time of the recursive Merge Sort function in terms of n, the size of the array being sorted. Solve the recurrence.

Again, using the Master Theorem,

Recurrence: $T(n) = 2T\left(\frac{n}{2}\right) + O(n)$
Base Case: $T(1) = O(1)$
Recurrence solution:
$S(k) = T(2^k) = S(k-1) + O(2^k), S(0) = O(1)$
$S(k) = O(2^k) * k + O(1)$
$T(n) = S(\log_2 n) = O(1) + O(n) * \log_2 n \in O(n * log\ n)$

7.

(a) What is the output when passed the following parameters: x = 2, n = 12. Show your work (activation diagram or similar).



derp(2,12)=4096

(b). Derp is a recursive implementation of the exponent function, where x is raised to n. ($2^{12} = 4096$)

(c). Once again, using the Master Theorem, we get a recurrence of:

$T(n) = 1 * T\left(\frac{n}{2}\right) + O(1), T(1) = O(1),$ which we've already solved to be:

(d). $O(\log n)$