

Fuzzy Relations

Crisp relations

To understand the fuzzy relations, it is better to discuss first **crisp relation**.

Suppose, A and B are two (crisp) sets. Then Cartesian product denoted as $A \times B$ is a collection of order pairs, such that

$$A \times B = \{(a, b) | a \in A \text{ and } b \in B\}$$

Note :

$$(1) A \times B \neq B \times A$$

$$(2) |A \times B| = |A| \times |B|$$

(3) $A \times B$ provides a mapping from $a \in A$ to $b \in B$.

The mapping so mentioned is called a **relation**.

Crisp relations

Example 1:

Consider the two crisp sets A and B as given below. $A = \{1, 2, 3, 4\}$
 $B = \{3, 5, 7\}$.

Then, $A \times B = \{(1, 3), (1, 5), (1, 7), (2, 3), (2, 5), (2, 7), (3, 3), (3, 5), (3, 7), (4, 3), (4, 5), (4, 7)\}$

Let us define a relation R as $R = \{(a, b) | b = a + 1, (a, b) \in A \times B\}$

Then, $R = \{(2, 3), (4, 5)\}$ in this case.

We can represent the relation R in a matrix form as follows.

$$R = \begin{matrix} & \begin{matrix} 3 & 5 & 7 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \end{matrix}$$

Operations on crisp relations

Suppose, $R(x, y)$ and $S(x, y)$ are the two relations define over two crisp sets $x \in A$ and $y \in B$

Union:

$$R(x, y) \cup S(x, y) = \max(R(x, y), S(x, y));$$

Intersection:

$$R(x, y) \cap S(x, y) = \min(R(x, y), S(x, y));$$

Complement:

$$\overline{R(x, y)} = 1 - R(x, y)$$

Example: Operations on crisp relations

Example:

Suppose, $R(x, y)$ and $S(x, y)$ are the two relations define over two crisp sets $x \in A$ and $y \in B$

$$R = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } S = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} ;$$

Find the following:

- 1 $R \cup S$
- 2 $R \cap S$
- 3 \overline{R}

Composition of two crisp relations

Given R is a relation on X, Y and S is another relation on Y, Z . Then $R \circ S$ is called a composition of relation on X and Z which is defined as follows.

$$R \circ S = \{(x, z) | (x, y) \in R \text{ and } (y, z) \in S \text{ and } \forall y \in Y\}$$

Max-Min Composition

Given the two relation matrices R and S , the **max-min composition** is defined as $T = R \circ S$;

$$T(x, z) = \max\{\min\{R(x, y), S(y, z) \text{ and } \forall y \in Y\}\}$$

Composition: Composition

Example:

Given

$$X = \{1, 3, 5\}; Y = \{1, 3, 5\}; R = \{(x, y) | y = x + 2\}; S = \{(x, y) | x < y\}$$

Here, R and S is on $X \times Y$.

Thus, we have

$$R = \{(1, 3), (3, 5)\}$$

$$S = \{(1, 3), (1, 5), (3, 5)\}$$

$$R = \begin{matrix} & \begin{matrix} 1 & 3 & 5 \end{matrix} \\ \begin{matrix} 1 \\ 3 \\ 5 \end{matrix} & \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{matrix} \text{ and } S =$$

$$\begin{matrix} & \begin{matrix} 1 & 3 & 5 \end{matrix} \\ \begin{matrix} 1 \\ 3 \\ 5 \end{matrix} & \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{matrix}$$

Using max-min composition $R \circ S =$

$$\begin{matrix} & \begin{matrix} 1 & 3 & 5 \end{matrix} \\ \begin{matrix} 1 \\ 3 \\ 5 \end{matrix} & \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{matrix}$$

Fuzzy relations

- Fuzzy relation is a fuzzy set defined on the Cartesian product of crisp set X_1, X_2, \dots, X_n
- Here, n-tuples (x_1, x_2, \dots, x_n) may have varying degree of memberships within the relationship.
- The membership values indicate the strength of the relation between the tuples.

Example:

$X = \{ \text{typhoid, viral, cold} \}$ and $Y = \{ \text{running nose, high temp, shivering} \}$

The fuzzy relation R is defined as

	<i>runningnose</i>	<i>hightemperature</i>	<i>shivering</i>
<i>typhoid</i>	0.1	0.9	0.8
<i>viral</i>	0.2	0.9	0.7
<i>cold</i>	0.9	0.4	0.6

Fuzzy Cartesian product

Suppose

A is a fuzzy set on the universe of discourse X with $\mu_A(x) | x \in X$

B is a fuzzy set on the universe of discourse Y with $\mu_B(y) | y \in Y$

Then $R = A \times B \subset X \times Y$; where R has its membership function given by $\mu_R(x, y) = \mu_{A \times B}(x, y) = \min\{\mu_A(x), \mu_B(y)\}$

Example :

$A = \{(a_1, 0.2), (a_2, 0.7), (a_3, 0.4)\}$ and $B = \{(b_1, 0.5), (b_2, 0.6)\}$

$$R = A \times B = \begin{array}{cc} & \begin{array}{cc} b_1 & b_2 \end{array} \\ \begin{array}{c} a_1 \\ a_2 \\ a_3 \end{array} & \left[\begin{array}{cc} 0.2 & 0.2 \\ 0.5 & 0.6 \\ 0.4 & 0.4 \end{array} \right] \end{array}$$

Operations on Fuzzy relations

Let R and S be two fuzzy relations on $A \times B$.

Union:

$$\mu_{R \cup S}(a, b) = \max\{\mu_R(a, b), \mu_S(a, b)\}$$

Intersection:

$$\mu_{R \cap S}(a, b) = \min\{\mu_R(a, b), \mu_S(a, b)\}$$

Complement:

$$\mu_{\overline{R}}(a, b) = 1 - \mu_R(a, b)$$

Composition

$$T = R \circ S$$
$$\mu_{R \circ S} = \max_{y \in Y} \{ \min(\mu_R(x, y), \mu_S(y, z)) \}$$

Operations on Fuzzy relations: Examples

Example:

$$X = (x_1, x_2, x_3); Y = (y_1, y_2); Z = (z_1, z_2, z_3);$$

$$R = \begin{matrix} & \begin{matrix} y_1 & y_2 \end{matrix} \\ \begin{matrix} x_1 \\ x_2 \\ x_3 \end{matrix} & \begin{bmatrix} 0.5 & 0.1 \\ 0.2 & 0.9 \\ 0.8 & 0.6 \end{bmatrix} \end{matrix}$$

$$S = \begin{matrix} & \begin{matrix} z_1 & z_2 & z_3 \end{matrix} \\ \begin{matrix} y_1 \\ y_2 \end{matrix} & \begin{bmatrix} 0.6 & 0.4 & 0.7 \\ 0.5 & 0.8 & 0.9 \end{bmatrix} \end{matrix}$$

$$R \circ S = \begin{matrix} & \begin{matrix} z_1 & z_2 & z_3 \end{matrix} \\ \begin{matrix} x_1 \\ x_2 \\ x_3 \end{matrix} & \begin{bmatrix} 0.5 & 0.4 & 0.5 \\ 0.5 & 0.8 & 0.9 \\ 0.6 & 0.6 & 0.7 \end{bmatrix} \end{matrix}$$

$$\begin{aligned} \mu_{R \circ S}(x_1, y_1) &= \max\{\min(x_1, y_1), \min(y_1, z_1), \min(x_1, y_2), \min(y_2, z_1)\} \\ &= \max\{\min(0.5, 0.6), \min(0.1, 0.5)\} = \max\{0.5, 0.1\} = 0.5 \text{ and so on.} \end{aligned}$$