Fuzzy Set Theory

Introduction

- The word “fuzzy” means “vaguness (ambiguity)”.
- Fuzziness occurs when the boundary of a piece of information is not clear-cut.
- Fuzzy sets - 1965 Lotfi Zadeh as an extension of classical notation set.
- Classical set theory allows the membership of the elements in the set in **binary terms**.
- Fuzzy set theory permits membership function valued in the interval [0,1].

Introduction

Example:

Words like young, tall, good or high are fuzzy.

- There is no single quantitative value which defines the term young.
- For some people, age 25 is young, and for others, age 35 is young.
- The concept young has no clean boundary.

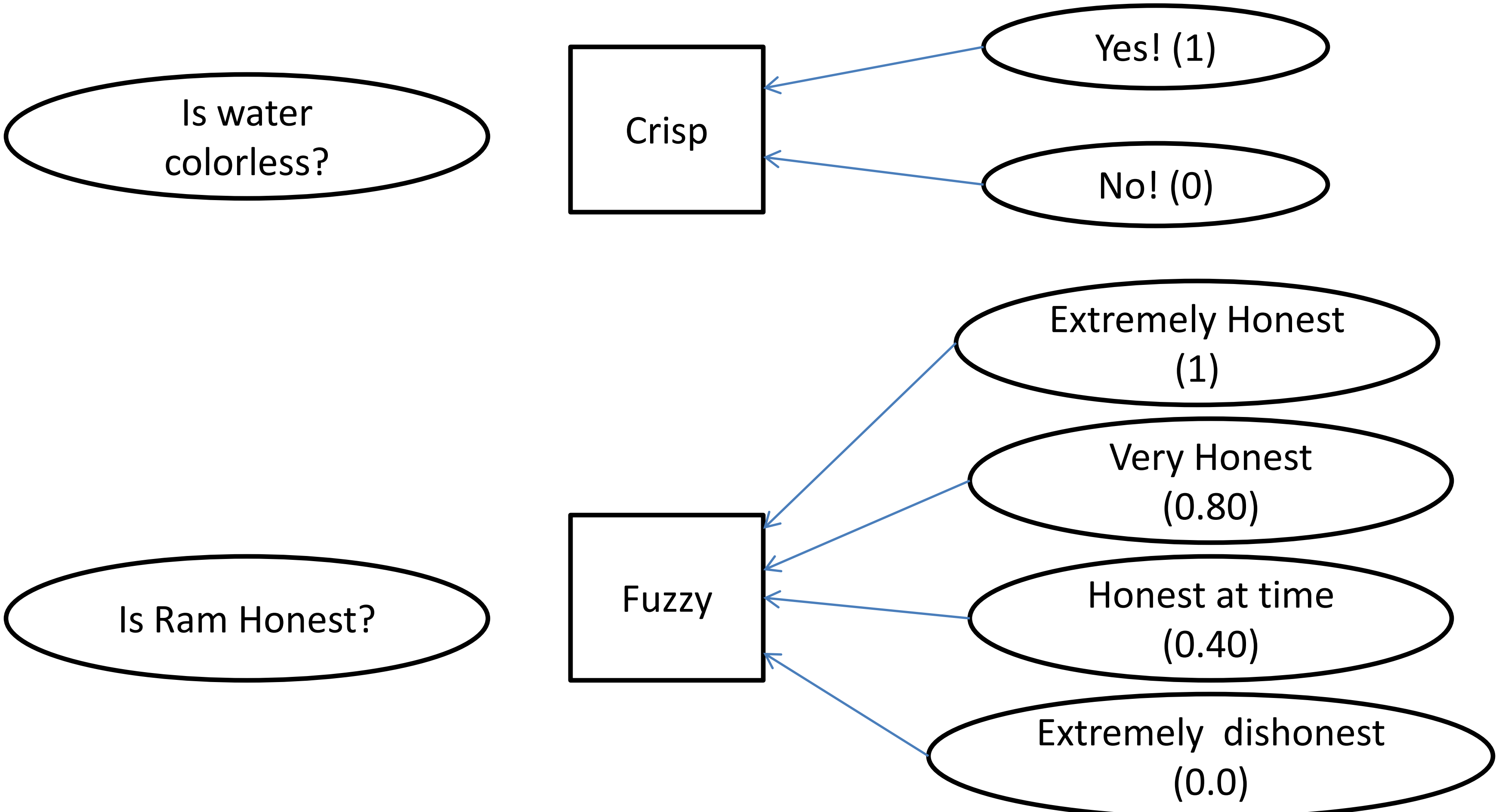
Fuzzy set theory is an extension of classical set theory where elements have degree of membership.

Introduction

Classical set theory	Fuzzy set theory
• Classes of objects with sharp boundaries.	• Classes of objects with un-sharp boundaries.
• A classical set is defined by crisp(exact) boundaries, i.e., there is no uncertainty about the location of the set boundaries.	• A fuzzy set is defined by its ambiguous boundaries, i.e., there exists uncertainty about the location of the set boundaries.
• Widely used in digital system design	• Used in fuzzy controllers.

Introduction (Continue)

Example



Fuzzy vs crips

Operations on classical set theory

Union: the union of two sets A and B is given as

$$A \cup B = \{ x \mid x \in A \text{ or } x \in B \}$$

Intersection: the intersection of two sets A and B is given as

$$A \cap B = \{ x \mid x \in A \text{ and } x \in B \}$$

Complement: It is denoted by Ã and is defined as

$$\tilde{A} = \{ x \mid x \text{ does not belongs } A \text{ and } x \in X \}$$

Fuzzy Sets

- Fuzzy sets theory is an extension of classical set theory.
- Elements have varying degree of membership. A logic based on two truth values,
- ***True*** and ***False*** is sometimes insufficient when describing human reasoning.
- Fuzzy Logic uses the whole interval between 0 (false) and 1 (true) to describe human reasoning.
- A Fuzzy Set is any set that allows its members to have different degree of membership, called **membership function**, having interval [0,1].

Fuzzy Sets

- **Fuzzy Logic** is derived from fuzzy set theory
- Many degree of membership (between 0 to 1) are allowed.
- Thus a membership function $\boldsymbol{\mu_A^{(x)}}$ is associated with a fuzzy sets Ã such that the function maps every element of universe of discourse X to the interval [0,1].
- The mapping is written as: $\mu_{\tilde{A}}(x): X \rightarrow [0,1]$.

- Fuzzy Logic is capable of handing inherently imprecise (vague or inexact or rough or inaccurate) concepts

Fuzzy Sets

- **Fuzzy set** is defined as follows:

- If X is an universe of discourse and x is a particular element of X, then a fuzzy set A defined on X and can be written as a collection of ordered pairs

$$A = \{(x, \mu_{\tilde{A}}(x)), x \in X \}$$

Fuzzy Sets (Continue)

Example

- Let $X = \{g_1, g_2, g_3, g_4, g_5\}$ be the reference set of students.
- Let Ã be the fuzzy set of "smart" students, where "smart" is fuzzy term.

$$\tilde{A} = \{(g_1,0.4)(g_2,0.5)(g_3,1)(g_4,0.9)(g_5,0.8)\}$$

Here Ã indicates that the smartness of g_1 is 0.4 and so on

Fuzzy Sets (Continue)

Membership Function

- The membership function fully defines the fuzzy set
- A membership function provides a measure of *the degree of similarity* of an element to a fuzzy set

Membership functions can

- either be chosen by the user arbitrarily, based on the user's experience (MF chosen by two users could be different depending upon their experiences, perspectives, etc.)
- Or be designed using machine learning methods (e.g., artificial neural networks, genetic algorithms, etc.)

Fuzzy Sets (Continue)

There are different shapes of membership functions;

- **Triangular,**
- **Trapezoidal,**
- **Gaussian, etc**

Fuzzy Set Operation

Given X to be the universe of discourse and Ã and Ḃ to be fuzzy sets with $\mu_A(x)$ and $\mu_B(x)$ are their respective membership function, the fuzzy set operations are as follows:

Union:

$$\mu_{A \cup B}(x) = \max(\mu_A(x), \mu_B(x))$$

Intersection:

$$\mu_{A \cap B}(x) = \min(\mu_A(x), \mu_B(x))$$

Complement:

$$\mu_A(x) = 1 - \mu_A(x)$$

Fuzzy Set Operation (Continue)

Example:

$A = \{(x_1,0.5),(x_2,0.7),(x_3,0)\}$ $B = \{(x_1,0.8),(x_2,0.2),(x_3,1)\}$

Union:

$$A \cup B = \{(x_1,0.8),(x_2,0.7),(x_3,1)\}$$

Because

$$\mu_{A \cup B}(x_1) = \max(\mu_A(x_1), \mu_B(x_1))$$
$$= \max(0.5,0.8)$$
$$= 0.8$$

$$\mu_{A \cup B}(x_2) = 0.7 \text{ and } \mu_{A \cup B}(x_3) = 1$$

Fuzzy Set Operation (Continue)

Example:

$A = \{(x_1,0.5),(x_2,0.7),(x_3,0)\}$ $B = \{(x_1,0.8),(x_2,0.2),(x_3,1)\}$

Intersection:

$$A \cap B = \{(x_1,0.5),(x_2,0.2),(x_3,0)\}$$

Because

$$\mu_{A \cap B}(x_1) = \min(\mu_A(x_1), \mu_B(x_1))$$
$$= \min(0.5,0.8)$$
$$= 0.5$$

$$\mu_{A \cap B}(x_2) = 0.2 \text{ and } \mu_{A \cap B}(x_3) = 0$$

Fuzzy Set Operation (Continue)

Example:

$A = \{(x_1,0.5),(x_2,0.7),(x_3,0)\}$

<u>Complement:</u>

$$A^c = \{(x_1,0.5),(x_2,0.3),(x_3,1)\}$$

Because

$$\mu_A(x_1) = 1 - \mu_A(x_1)$$
$$= 1 - 0.5$$
$$= 0.5$$

$$\mu_A(x_2) = 0.3 \text{ and } \mu_A(x_3) = 1$$

- **Support(A)** is set of all points x in X such that

$$\{(x|\ \mu_A(x) > 0\ \}$$

- **core(A)** is set of all points x in X such that

$$\{(x|\ \mu_A(x) = 1\ \}$$

- Fuzzy set whose support is a single point in X with $\mu_A(x) = 1$ is called fuzzy singleton