Dynamic Modeling and System Identification

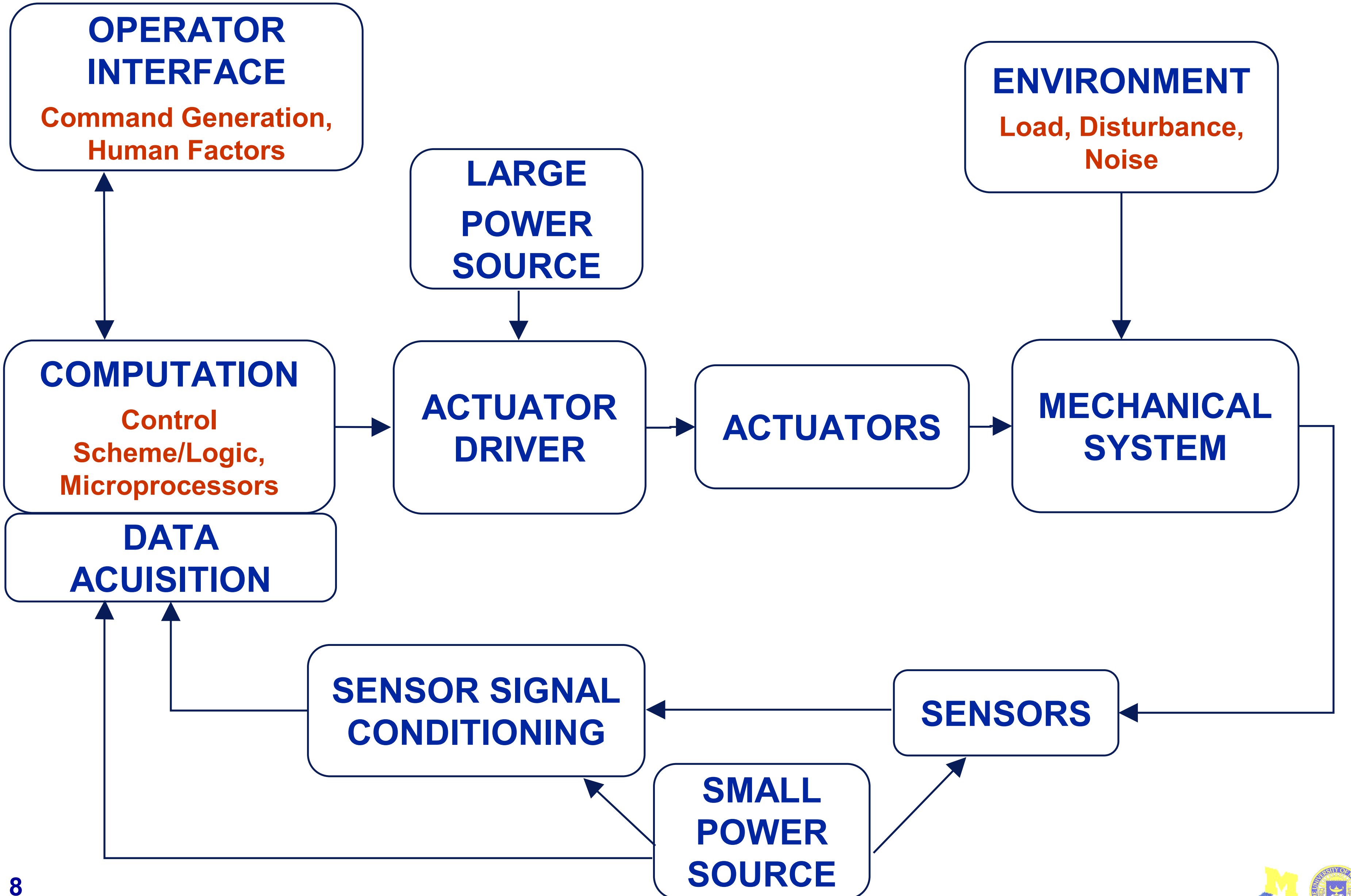
Mechatronics
Rotary Inverted Pendulum System

Physical System Components

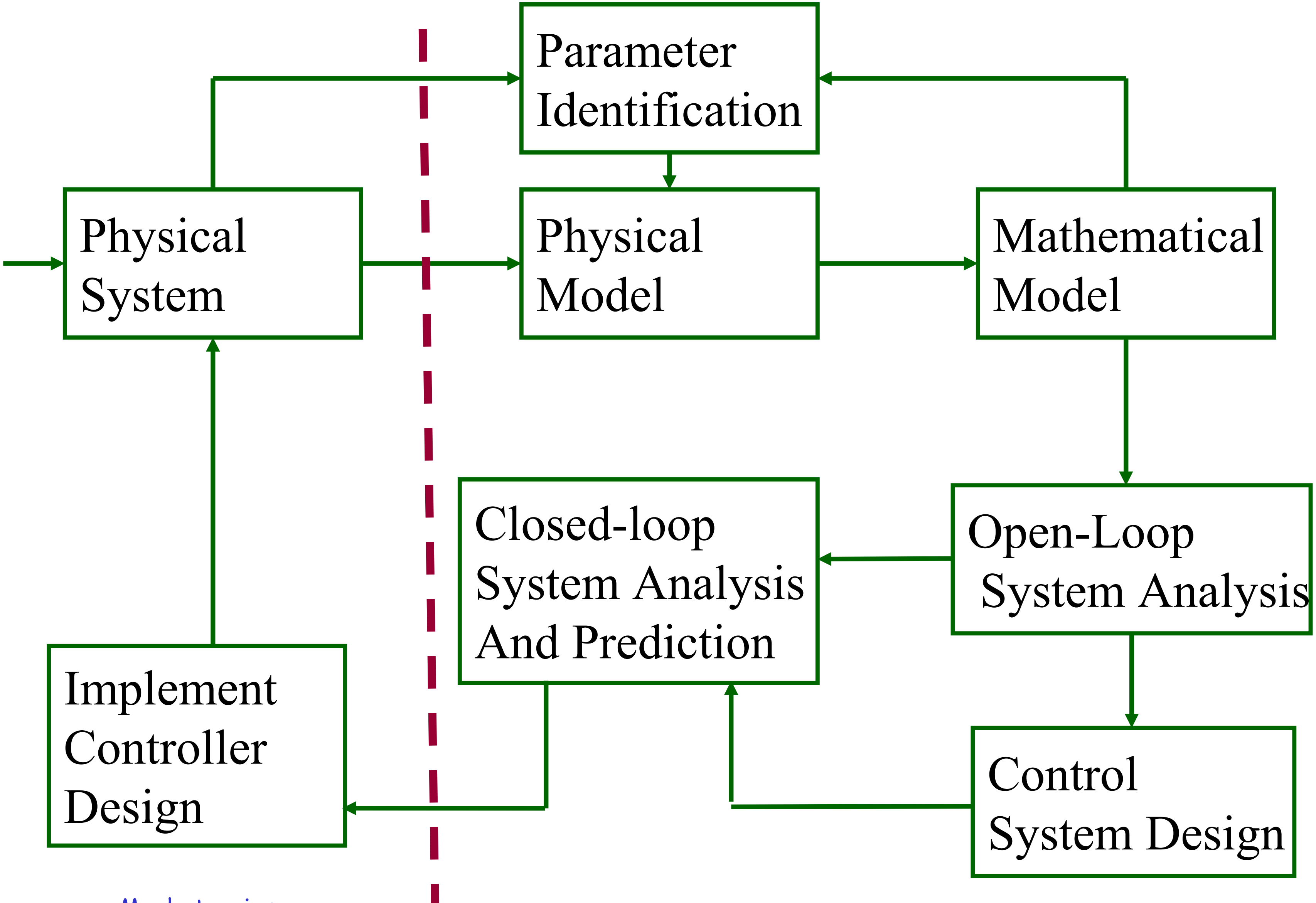
- Two Links
  - Motor-driven horizontal link
  - Un-actuated vertical pendulum link
- Permanent-magnet, brushed DC motor actuator
- 24-volt, 2-amp, DC power supply
- Pulse-width-modulated servo-amplifier (AMC)
- Two rotary incremental optical encoders

  - One encoder measures pendulum angle
  - One encoder measures horizontal link angle
  - Velocity data is derived for each link from the encoder data using simple derivative followed by a filter.
- Counter-weight on the horizontal arm
- The command to the amplifier is provided by a feedback controller implemented in NI LabView

Mechatronic Systems



Mechatronic System Investigation Process



Physical Model: Simplifying Assumptions

- Links are rigid
- Two-degree-of-freedom system; generalized coordinates are:
  - θ - horizontal link angle
  - ϕ - pendulum arm angle
- Assume both Coulomb and viscous friction in the motor, and at the pendulum revolute joint and perform tests to identify those parameters
- Dynamic response of the encoders is sufficiently fast that it can be considered instantaneous
- Dynamic response of the servo-amplifier is sufficiently fast that it can be considered instantaneous

- Motor operates in the torque mode with $V_{in}K_A = i$ and $T = K_T i$ where:
  - K_T is the motor torque constant (N-m/A)
  - K_A is the amplifier constant (A/V)
  - V_{in} is the command voltage (V)
  - T is the electromagnetic torque applied to the motor rotor (N-m)
  - i is the motor current (A)

- Motor is modeled assuming lumped parameter where:
  - J is the rotor inertia (N-m-s^2/rad = kg-m^2)
  - B_f is the viscous friction coefficient (N-m-s/rad)
  - T_f is the Coulomb friction torque (N-m)

Physical & Mathematical Modeling

<u>Reference Frames</u>:

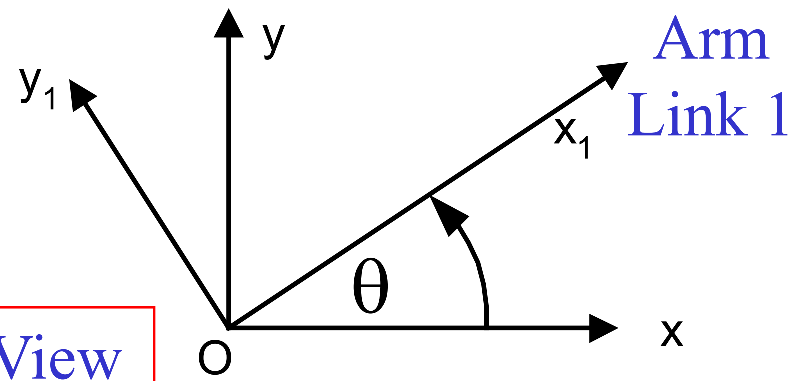
R: ground xyz

R_1: arm $x_1y_1z_1$

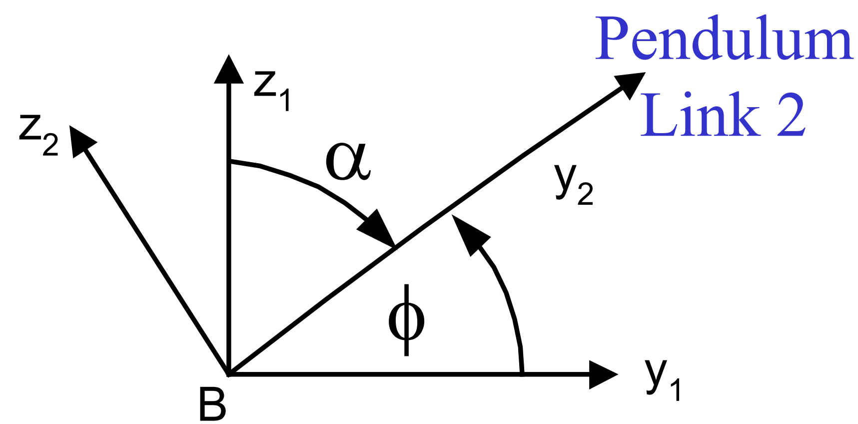
R_2: pendulum $x_2y_2z_2$

$$\begin{bmatrix} \hat{i}_1 \\ \hat{j}_1 \\ \hat{k}_1 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix}$$

$$\begin{bmatrix} \hat{i}_2 \\ \hat{j}_2 \\ \hat{k}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \hat{i}_1 \\ \hat{j}_1 \\ \hat{k}_1 \end{bmatrix}$$



Top View



Front View

- Angular Velocities of Links

$$ {}^{R}\vec{\omega}^{R_1} = \dot{\theta}\hat{k} = \dot{\theta}\hat{k}_1 $$

$$ \begin{aligned} {}^{R}\vec{\omega}^{R_2} &= \dot{\phi}\cos\theta\hat{i} + \dot{\phi}\sin\theta\hat{j} + \dot{\theta}\hat{k} \\ &= \dot{\phi}\hat{i}_1 + \dot{\theta}\hat{k}_1 \\ &= \dot{\phi}\hat{i}_2 + \dot{\theta}\sin\phi\hat{j}_2 + \dot{\theta}\cos\phi\hat{k}_2 \end{aligned} $$

- Velocities of CG's of Links
  - Point A is CG of Link 1 (Arm)
  - Point C is CG of Link 2 (Pendulum)

$$ {}^{R}\vec{v}^{A} = \left(-\ell_{11}\dot{\theta}\sin\theta\right)\hat{i} + \left(\ell_{11}\dot{\theta}\cos\theta\right)\hat{j} $$

$$ \begin{aligned} {}^{R}\vec{v}^{C} &= \left(-\dot{\theta}\ell_1\sin\theta - \dot{\theta}\ell_{21}\cos\phi\cos\theta + \dot{\phi}\ell_{21}\sin\phi\sin\theta\right)\hat{i} \\ &+ \left(\dot{\theta}\ell_1\cos\theta - \dot{\theta}\ell_{21}\cos\phi\sin\theta - \dot{\phi}\ell_{21}\sin\phi\cos\theta\right)\hat{j} \\ &+ \left(\dot{\phi}\ell_{21}\cos\phi\right)\hat{k} \end{aligned} $$

$$\left({}^{R}v^{A}\right)^{2} = \ell_{11}^{2}\dot{\theta}^{2}$$

$$\left({}^{R}v^{C}\right)^{2} = \ell_{21}^{2}\dot{\phi}^{2} + \ell_{1}^{2}\dot{\theta}^{2} - 2\dot{\theta}\dot{\phi}\ell_{1}\ell_{21}\sin\phi + \dot{\theta}^{2}\ell_{21}^{2}\cos^{2}\phi$$

- <u>Definitions</u>:

$\ell_{1} =$ length of link 1

$\ell_{11} =$ distance from pivot O to CG of link 1

$\ell_{2} =$ length of link 2 $= \ell_{21} + \ell_{22}$

$\ell_{21} =$ distance from pivot B to CG of link 2

$\ell_{22} =$ distance from CG of link 2 to end of link 2

Lagrange's Equations

- Lagrange's Equations

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right)-\frac{\partial T}{\partial q_i}+\frac{\partial V}{\partial q_i}=Q_i$$

- Generalized Coordinates

$$q_1=\theta$$

$$q_2=\phi$$

Kinetic and Potential Energy

- Kinetic Energy T of System

$$T = \frac{1}{2} m_1 \left({}^R v^A \right)^2 + \frac{1}{2} \bar{I}_{1_{z_1}} \dot{\theta}^2 + \frac{1}{2} m_2 \left({}^R v^C \right)^2 +$$

$$\frac{1}{2} \left[\bar{I}_{2_{x_2}} \dot{\phi}^2 + \bar{I}_{2_{y_2}} \left(\sin^2 \phi \right) \dot{\theta}^2 + \bar{I}_{2_{z_2}} \left(\cos^2 \phi \right) \dot{\theta}^2 \right] + \bar{I}_{2_{x_2 y_2}} \dot{\phi} \dot{\theta} \sin \phi$$

This assumes a pendulum geometry with

- Symmetry about the X_2-Y_2 plane => $I_{2y2z2} = I_{2x2z2} = 0$
- Slight asymmetry about the Y_2 - Z_2 plane => $I_{2x2y2} \neq 0$

- Potential Energy V of the System

$$V = -m_2 g \ell_{21} \left(1 - \sin \phi \right)$$

- ## Generalized Forces

$$Q_\theta = T - B_\theta\dot{\theta} - T_{f\theta}\,\mathrm{sgn}\left(\dot{\theta}\right)$$

$$Q_\phi = -B_\phi\dot{\phi} - T_{f\phi}\,\mathrm{sgn}\left(\dot{\phi}\right)$$

$T =$ motor torque

$B_\theta =$ viscous damping constant θ joint

$T_{f\theta} =$ Coulomb friction constant θ joint

$B_\phi =$ viscous damping constant ϕ joint

$T_{f\phi} =$ Coulomb friction constant ϕ joint

- ## Equations of Motion

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{\theta}} - \frac{\partial T}{\partial \theta} + \frac{\partial V}{\partial \theta} = Q_\theta$$

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{\phi}} - \frac{\partial T}{\partial \phi} + \frac{\partial V}{\partial \phi} = Q_\phi$$

Nonlinear Equations of Motion

$$\left[m_1\ell_{11}^2+\bar{I}_{1_{z_1}}+m_2\ell_1^2+m_2\ell_{21}^2\cos^2\phi+\bar{I}_{2_{z_2}}\cos^2\phi+\bar{I}_{2_{y_2}}\sin^2\phi\right]\ddot{\theta}+$$
$$\left[\bar{I}_{2_{x_2y_2}}-m_2\ell_1\ell_{21}\right]\sin\phi\ddot{\phi}+\left[\bar{I}_{2_{x_2y_2}}-m_2\ell_1\ell_{21}\right]\cos\phi\dot{\phi}^2+$$
$$\left[\bar{I}_{2_{y_2}}-m_2\ell_{21}^2-\bar{I}_{2_{z_2}}\right](2\cos\phi\sin\phi)\dot{\phi}\dot{\theta}=T-\left[B_\theta\dot{\theta}+T_{f\theta}\,\mathrm{sgn}\left(\dot{\theta}\right)\right]$$

[1]

$$\left[m_2\ell_{21}^2+\bar{I}_{2_{x_2}}\right]\ddot{\phi}+\left[\bar{I}_{2_{x_2y_2}}-m_2\ell_1\ell_{21}\right]\sin\phi\ddot{\theta}+$$
$$\left[m_2\ell_{21}^2-\bar{I}_{2_{y_2}}+\bar{I}_{2_{z_2}}\right](\cos\phi\sin\phi)\dot{\theta}^2+m_2g\ell_{21}\cos\phi=-\left[B_\phi\dot{\phi}+T_{f\phi}\,\mathrm{sgn}\left(\dot{\phi}\right)\right]$$

[2]

<u>What are the equilibrium conditions?</u>

$\boxed{\text{Define:}}$ $\alpha = \dfrac{\pi}{2} - \phi$

$$\left[m_1\ell_{11}^2 + \bar{I}_{1_{z_1}} + m_2\ell_1^2 + m_2\ell_{21}^2\sin^2\alpha + \bar{I}_{2_{z_2}}\sin^2\alpha + \bar{I}_{2_{y_2}}\cos^2\alpha\right]\ddot{\theta} -$$

$$\left[\bar{I}_{2_{x_2y_2}} - m_2\ell_1\ell_{21}\right]\cos\alpha\ddot{\alpha} + \left[\bar{I}_{2_{x_2y_2}} - m_2\ell_1\ell_{21}\right]\sin\alpha\dot{\alpha}^2 +$$

$$\left[+\bar{I}_{2_{z_2}} + m_2\ell_{21}^2 - \bar{I}_{2_{y_2}}\right]\left(2\cos\alpha\sin\alpha\right)\dot{\alpha}\dot{\theta} = T - \left[B_\theta\dot{\theta} + T_{f\theta}\,\mathrm{sgn}\left(\dot{\theta}\right)\right]$$

[1A]

$$-\left[m_2\ell_{21}^2 + \bar{I}_{2_{x_2}}\right]\ddot{\alpha} + \left[\bar{I}_{2_{x_2y_2}} - m_2\ell_1\ell_{21}\right]\cos\alpha\ddot{\theta} +$$

$$\left[m_2\ell_{21}^2 - \bar{I}_{2_{y_2}} + \bar{I}_{2_{z_2}}\right]\left(\cos\alpha\sin\alpha\right)\dot{\theta}^2$$

$$+m_2 g\ell_{21}\sin\alpha = \left[B_\alpha\dot{\alpha} + T_{f\alpha}\,\mathrm{sgn}\left(\dot{\alpha}\right)\right]$$

[2A]

Linearization:

$$\left.\begin{aligned}\theta &= 0\\ \alpha &= 0\end{aligned}\right\}\ \text{Operating Point}$$

$$\left[m_1\ell_{11}^2+\bar{I}_{1_{z_1}}+m_2\ell_1^2+\bar{I}_{2_{y_2}}\right]\ddot{\theta}-\left[\bar{I}_{2_{x_2y_2}}-m_2\ell_1\ell_{21}\right]\ddot{\alpha}=T-B_\theta\dot{\theta} \qquad [3]$$

$$-\left[m_2\ell_{21}^2+\bar{I}_{2_{x_2}}\right]\ddot{\alpha}+\left[\bar{I}_{2_{x_2y_2}}-m_2\ell_1\ell_{21}\right]\ddot{\theta}+m_2g\ell_{21}\alpha=B_\alpha\dot{\alpha} \qquad [4]$$

Definitions:

$$C_1\ddot{\theta}+C_2\ddot{\alpha}=T-B_\theta\dot{\theta} \qquad [5]$$

$$C_3\ddot{\alpha}+C_2\ddot{\theta}-C_4\alpha=-B_\alpha\dot{\alpha} \qquad [6]$$

$$\left\{\begin{aligned}C_1 &= m_1\ell_{11}^2+\bar{I}_{1_{z1}}+m_2\ell_1^2+\bar{I}_{2_{y_2}}\\ C_2 &= m_2\ell_1\ell_{21}-\bar{I}_{2_{x_2y_2}}\\ C_3 &= m_2\ell_{21}^2+\bar{I}_{2_{x_2}}\\ C_4 &= m_2g\ell_{21}\end{aligned}\right.$$

Transfer Functions (with damping):

$$\frac{\theta}{T}=\frac{C_3 s^2+B_\alpha s-C_4}{\left[\left(C_1C_3-C_2^2\right)s^4+\left(B_\alpha C_1+B_\theta C_3\right)s^3+\left(B_\alpha B_\theta-C_1C_4\right)s^2-B_\theta C_4 s\right]}$$

$$\frac{\alpha}{T}=\frac{-C_2 s^2}{\left[\left(C_1C_3-C_2^2\right)s^4+\left(B_\alpha C_1+B_\theta C_3\right)s^3+\left(B_\alpha B_\theta-C_1C_4\right)s^2-B_\theta C_4 s\right]}$$

State-Space Equations (with damping):

$$x_1=\theta \quad x_2=\dot{\theta} \quad x_3=\alpha \quad x_4=\dot{\alpha}$$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix}=\begin{bmatrix}0 & 1 & 0 & 0\\ 0 & \dfrac{-B_\theta C_3}{C_1C_3-C_2^2} & \dfrac{-C_2C_4}{C_1C_3-C_2^2} & \dfrac{B_\alpha C_2}{C_1C_3-C_2^2}\\ 0 & 0 & 0 & 1\\ 0 & \dfrac{B_\theta C_2}{C_1C_3-C_2^2} & \dfrac{C_1C_4}{C_1C_3-C_2^2} & \dfrac{-B_\alpha C_1}{C_1C_3-C_2^2}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}+\begin{bmatrix}0\\ \dfrac{C_3}{C_1C_3-C_2^2}\\ 0\\ \dfrac{-C_2}{C_1C_3-C_2^2}\end{bmatrix}[T]$$

Transfer Functions (neglect damping terms):

$$\frac{\theta}{T} = \frac{C_3 s^2 - C_4}{s^2\left[\left(C_1C_3 - C_2^2\right)s^2 - C_1C_4\right]}$$

$$\frac{\alpha}{T} = \frac{-C_2 s^2}{s^2\left[\left(C_1C_3 - C_2^2\right)s^2 - C_1C_4\right]}$$

State-Space Equations (neglect damping terms):

$$x_1 = \theta$$
$$x_2 = \dot{\theta}$$
$$x_3 = \alpha$$
$$x_4 = \dot{\alpha}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \dfrac{-C_2C_4}{C_1C_3 - C_2^2} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \dfrac{C_1C_4}{C_1C_3 - C_2^2} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{C_3}{C_1C_3 - C_2^2} \\ 0 \\ \dfrac{-C_2}{C_1C_3 - C_2^2} \end{bmatrix} [T]$$