Tutorial 1

Getting Started with MATLAB

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The purpose of this tutorial is to present basics of MATLAB. We do not assume any prior knowledge of this package. This tutorial is intended for users running a professional version of MATLAB 5.3, Release 11 under Windows 95. Topics discussed in this tutorial include the **Command Window**, numbers and arithmetic operations, saving and reloading a work, using help, MATLAB demos, interrupting a running program, long command lines, and MATLAB resources on the Internet.

1.1 The Command Window

You can start MATLAB by double clicking on the MATLAB icon that should be on the desktop of your computer. This brings up the window called the **Command Window**. This window allows a user to enter simple commands. To clear the **Command Window** type **clc** and next press the **Enter** or **Return** key. To perform a simple computations type a command and next press the **Enter** or **Return** key. For instance,

```
s = 1 + 2

s =
     3

fun = sin(pi/4)

fun =
    0.7071
```

In the second example the trigonometric function sine and the constant π are used. In MATLAB they are named **sin** and **pi**, respectively.

Note that the results of these computations are saved in *variables* whose names are chosen by the user. If they will be needed during your current MATLAB session, then you can obtain their values typing their names and pressing the **Enter** or **Return** key. For instance,

```
s
```

```
s =
     3
```

Variable name begins with a letter, followed by letters, numbers or underscores. MATLAB recognizes only the first 31 characters of a variable name.

To change a format of numbers displayed in the **Command Window** you can use one of the several formats that are available in MATLAB. The default format is called **short** (four digits after the decimal point.) In order to display more digits click on **File**, select **Preferences…**, and next select a format you wish to use. They are listed below the **Numeric Format**. Next click on **Apply** and **OK** and close the current window. You can also select a new format from within the **Command Window**. For instance, the following command

```
format long
```

changes a current format to the format **long.** To display more digits of the variable **fun** type

```
fun
```

```
fun =
   0.70710678118655
```

To change a current format to the default one type

```
format short
```

```
fun
```

```
fun =
    0.7071
```

To close MATLAB type **exit** in the **Command Window** and next press **Enter** or **Return** key. A second way to close your current MATLAB session is to select **File** in the MATLAB's toolbar and next click on **Exit MATLAB** option. All unsaved information residing in the MATLAB **Workspace** will be lost.

1.2 Numbers and arithmetic operations in MATLAB

There are three kinds of numbers used in MATLAB:

- integers
- real numbers
- complex numbers

Integers are enterd without the decimal point

```
xi = 10

xi =
    10
```

However, the following number

```
xr = 10.01

xr =
   10.0100
```

is saved as the real number. It is not our intention to discuss here machine representation of numbers. This topic is usually included in the numerical analysis courses.
Variables **realmin** and **realmax** denote the smallest and the largest positive real numbers in MATLAB. For instance,

```
realmin

ans =
  2.2251e-308
```

Complex numbers in MATLAB are represented in rectangular form. The imaginary unit $\sqrt{-1}$ is denoted either by **i** or **j**

```
i

ans =
        0 + 1.0000i
```

In addition to classes of numbers mentioned above, MATLAB has three variables representing the nonnumbers:

- **-Inf**
- **Inf**
- **NaN**

The **–Inf** and **Inf** are the IEEE representations for the negative and positive infinity, respectively. Infinity is generated by overflow or by the operation of dividing by zero. The **NaN** stands for the *not-a-number* and is obtained as a result of the mathematically undefined operations such as 0.0/0.0 or $\infty - \infty$.

List of basic arithmetic operations in MATLAB include six operations

Operation	Symbol
addition	+
subtraction	-
multiplication	*
division	/ or \
exponentiation	^

MATLAB has two division operators / - the right division and \ - the left division. They do not produce the same results

```
rd = 47/3

rd =
   15.6667


ld = 47\3

ld =
    0.0638
```

1.3 Saving and reloading your work

All variables used in the current MATLAB session are saved in the **Workspace**. You can view the content of the Workspace by clicking on **File** in the toolbar and next selecting **Show Workspace** from the pull-down menu. You can also check contents of the Workspace typing **whos** in the **Command Window**. For instance,

```
whos

  Name      Size          Bytes  Class

  ans       1x1              16  double array (complex)
  fun       1x1               8  double array
  ld        1x1               8  double array
  rd        1x1               8  double array
  s         1x1               8  double array
  xi        1x1               8  double array
  xr        1x1               8  double array

Grand total is 7 elements using 64 bytes
```

shows all variables used in current session. You can also use command **who** to generate a list of variables used in current session

```
who
```

```
Your variables are:

ans       ld        s         xr
fun       rd        xi
```

To save your current workspace select **Save Workspace as…** from the **File** menu. Chose a name for your file, e.g. **filename.mat** and next click on **Save**. Remember that the file you just created must be located in MATLAB's search path. Another way of saving your workspace is to type **save filename** in the **Command Window**. The following command **save filename s** saves only the variable **s**.

Another way to save your workspace is to type the command **diary filename** in the **Command Window**. All commands and variables created from now will be saved in your file. The following command: **diary off** will close the file and save it as the text file. You can open this file in a text editor, by double clicking on the name of your file, and modify its contents if you wish to do so.

To load contents of the file named **filename** into MATLAB's workspace type **load filename** in the **Command Window**.

More advanced computations often require execution of several lines of computer code. Rather than typing those commands in the **Command Window** you should create a file. Each time you will need to repeat computations just invoke your file. Another advantage of using files is the ease to modify its contents. To learn more about files, see [1], pp. 67-75 and also Section 2.2 of Tutorial 2.

1.4 Help

One of the nice features of MATLAB is its help system. To learn more about a function you are to use, say **rref**, type in the **Command Window**

```
help svd
```

```
 SVD    Singular value decomposition.
    [U,S,V] = SVD(X) produces a diagonal matrix S, of the same
    dimension as X and with nonnegative diagonal elements in
    decreasing order, and unitary matrices U and V so that
    X = U*S*V'.
    S = SVD(X) returns a vector containing the singular values.
    [U,S,V] = SVD(X,0) produces the "economy size"
    decomposition. If X is m-by-n with m > n, then only the
    first n columns of U are computed and S is n-by-n.

    See also SVDS, GSVD.

 Overloaded methods
    help sym/svd.m
```

If you do not remember the exact name of a function you want to learn more about use command **lookfor** followed by the incomplete name of a function in the **Command Window**. In the following example we use a "word" **sv**

```
lookfor sv

ISVMS  True for the VMS version of MATLAB.
HSV2RGB Convert hue-saturation-value colors to red-green-blue.
RGB2HSV Convert red-green-blue colors to hue-saturation-value.
GSVD   Generalized Singular Value Decomposition.
SVD    Singular value decomposition.
SVDS   Find a few singular values and vectors.
HSV    Hue-saturation-value color map.
JET    Variant of HSV.
CSVREAD Read a comma separated value file.
CSVWRITE Write a comma separated value file.
ISVARNAME Check for a valid variable name.
RANDSVD  Random matrix with pre-assigned singular values.
Trusvibs.m: % Example: trusvibs
SVD    Symbolic singular value decomposition.
RANDSVD Random matrix with pre-assigned singular values.
```

The **helpwin** command, invoked without arguments, opens a new window on the screen. To find an information you need double click on the name of the subdirectory and next double click on a function to see the help text for that function. You can go directly to the help text of your function invoking **helpwin** command followed by an argument. For instance, executing the following command

```
helpwin zeros

ZEROS  Zeros array.
    ZEROS(N) is an N-by-N matrix of zeros.
    ZEROS(M,N) or ZEROS([M,N]) is an M-by-N matrix of zeros.
    ZEROS(M,N,P,...) or ZEROS([M N P ...]) is an M-by-N-by-P-by-...
    array of zeros.
    ZEROS(SIZE(A)) is the same size as A and all zeros.

    See also ONES.
```

generates an information about MATLAB's function **zeros**.

MATLAB also provides the browser-based help. In order to access these help files click on **Help** and next select **Help Desk (HTML)**. This will launch your Web browser. To access an information you need click on a highlighted link or type a name of a function in the text box. In order for the Help Desk to work properly on your computer the appropriate help files, in the HTML or PDF format, must be installed on your computer. You should be aware that these files require a significant amount of the disk space.

1.5 Demos

To learn more about MATLAB capabilities you can execute the **demo** command in the **Command Window** or click on **Help** and next select **Examples and Demos** from the pull-down menu. Some of the MATLAB demos use both the **Command** and the **Figure** windows.

To learn about matrices in MATLAB open the demo window using one of the methods described above. In the left pane select **Matrices** and in the right pane select **Basic matrix operations** then click on **Run Basic matrix …** . Click on the **Start >>** button to begin the show.

If you are familiar with functions of a complex variable I recommend another demo. Select **Visualization** and next **3-D Plots of complex functions**. You can generate graphs of simple power functions by selecting an appropriate button in the current window.

1.6 Interrupting a running program

To interrupt a running program press simultaneously the **Ctrl-c** keys. Sometimes you have to repeat pressing these keys a couple of times to halt execution of your program. This is not a recommended way to exit a program, however, in certain circumstances it is a necessity. For instance, a poorly written computer code can put MATLAB in the infinite loop and this would be the only option you will have left.

1.7 Long command lines

To enter a statement that is too long to be typed in one line, use three periods, **…** , followed by **Enter** or **Return**. For instance,

```
x = sin(1) - sin(2) + sin(3) - sin(4) + sin(5) -...
      sin(6) + sin(7) - sin(8) + sin(9) - sin(10)

x =
    0.7744
```

You can suppress output to the screen by adding a semicolon after the statement

```
u = 2 + 3;
```

1.8 MATLAB resources on the Internet

If your computer has an access to the Internet you can learn more about MATLAB and also download user supplied files posted in the public domain. We provide below some pointers to information related to MATLAB.

- The MathWorks Web site: http://www.mathworks.com/

  The MathWorks, the makers of MATLAB, maintains an important Web site. Here you can find information about new products, MATLAB related books, user supplied files and much more.
- The MATLAB newsgroup: news://saluki-news.siu.edu/comp.soft-sys.matlab/

  If you have an access to the Internet News, you can read messages posted in this newsgroup. Also, you can post your own messages. The link shown above would work only for those who have access to the news server in Southern Illinois University at Carbondale.

- http://dir.yahoo.com/science/mathematics/software/matlab/

  A useful source of information about MATLAB and good starting point to other Web sites.

- http://www.cse.uiuc.edu/cse301/matlab.html

  Thus Web site, maintained by the University of Illinois at Champaign-Urbana, provides several links to MATLAB resources on the Internet.

- The Mastering Matlab Web site: http://www.eece.maine.edu/mm

  Recommended link for those who are familiar with the book Mastering Matlab 5. A Comprehensive Tutorial and Reference, by D. Hanselman and B. Littlefield (see [2].)

References

[1] Getting Started with MATLAB, Version 5, The MathWorks, Inc., 1996.

[2] D. Hanselman and B. Littlefield, Mastering MATLAB 5. A Comprehensive Tutorial and Reference, Prentice Hall, Upper Saddle River, NJ, 1998.

[3] K. Sigmon, MATLAB Primer, Fifth edition, CRC Press, Boca Raton, 1998.

[4] Using MATLAB, Version 5, The MathWorks, Inc., 1996.

Tutorial 2

Programming in MATLAB

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This tutorial is intended for those who want to learn basics of MATLAB programming language. Even with a limited knowledge of this language a beginning programmer can write his/her own computer code for solving problems that are complex enough to be solved by other means. Numerous examples included in this text should help a reader to learn quickly basic programming tools of this language. Topics discussed include the m-files, inline functions, control flow, relational and logical operators, strings, cell arrays, rounding numbers to integers and MATLAB graphics.

2.1 The m-files

Files that contain a computer code are called the *m-files*. There are two kinds of m-files: the *script files* and the *function files*. Script files do not take the input arguments or return the output arguments. The function files may take input arguments or return output arguments.

To make the m-file click on **File** next select **New** and click on **M-File** from the pull-down menu. You will be presented with the **MATLAB Editor/Debugger** screen. Here you will type your code, can make changes, etc. Once you are done with typing, click on **File**, in the **MATLAB Editor/Debugger** screen and select **Save As…** . Chose a name for your file, e.g., **firstgraph.m** and click on **Save**. Make sure that your file is saved in the directory that is in MATLAB's search path.

If you have at least two files with duplicated names, then the one that occurs first in MATLAB's search path will be executed.

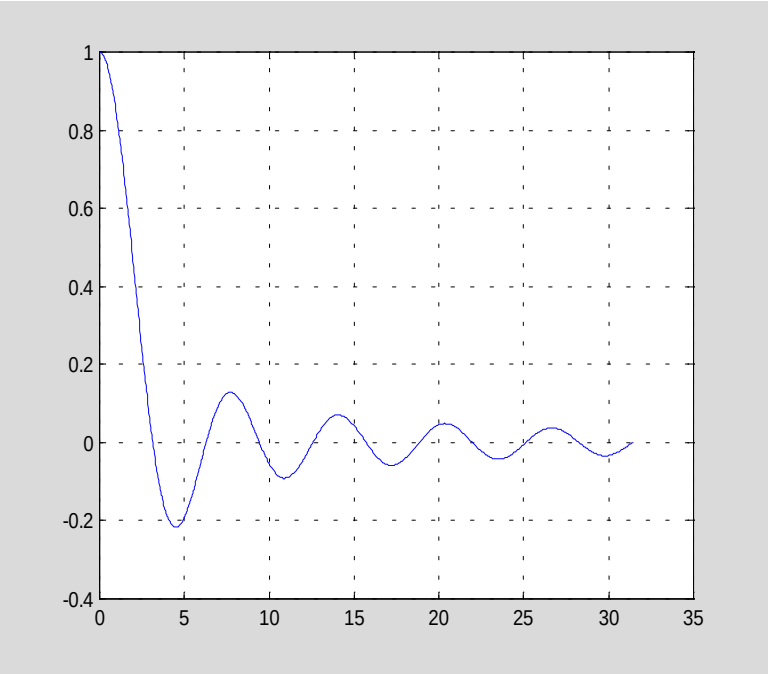
To open the m-file from within the **Command Window** type **edit firstgraph** and then press **Enter** or **Return** key.

Here is an example of a small script file

```
% Script file firstgraph.

x = pi/100:pi/100:10*pi;
y = sin(x)./x;
plot(x,y)
grid
```

Let us analyze contents of this file. First line begins with the percentage sign **%**. This is a *comment.* All comments are ignored by MATLAB. They are added to improve readability of the code. In the next two lines arrays **x** and **y** are created. Note that the semicolon follows both commands. This suppresses display of the content of both vectors to the screen (see Tutorial 1, page 5 for more details). Array **x** holds 1000 evenly spaced numbers in the interval **[π/100 10π]** while the array **y** holds the values of the *sinc function* **y = sin(x)/x** at these points. Note use of the *dot operator* **.** before the *right division operator* **/**. This tells MATLAB to perform the *componentwise division* of two arrays **sin(x)** and **x**. Special operators in MATLAB and operations on one- and two dimensional arrays are discussed in detail in Tutorial 3, Section 3.2. The command **plot** creates the graph of the sinc function using the points generated in two previous lines. For more details about command **plot** see Section 2.8.1 of this tutorial. Finally, the command **grid** is executed. This adds a *grid* to the graph. We invoke this file by typing its name in the **Command Window** and next pressing the **Enter** or **Return** key

```
firstgraph
```



Here is an example of the function file

```
function [b, j] = descsort(a)

% Function descsort sorts, in the descending order, a real array a.
% Second output parameter j holds a permutation used to obtain
% array b from the array a.
```

```
[b ,j] = sort(-a);
b = -b;
```

This function takes one input argument, the array of real numbers, and returns a sorted array together with a permutation used to obtain array **b** from the array **a**. MATLAB built-in function **sort** is used here. Recall that this function sort numbers in the ascending order. A simple trick used here allows us to sort an array of numbers in the descending order.

To demonstrate functionality of the function under discussion let

```
a = [pi –10 35 0.15];

[b, j] = descsort(a)

b =
   35.0000    3.1416    0.1500  -10.0000
j =
     3     1     4     2
```

You can execute function **descsort** without output arguments. In this case an information about a permutation used will be lost

```
descsort(a)

ans =
   35.0000    3.1416    0.1500  -10.0000
```

Since no output argument was used in the call to function **descorder** a sorted array **a** is assigned to the default variable **ans**.

2.2 Inline functions and the feval command

Sometimes it is handy to define a function that will be used during the current MATLAB session only. MATLAB has a command **inline** used to define the so-called *inline functions* in the **Command Window**.

Let

```
f = inline('sqrt(x.^2+y.^2)','x','y')

f =
     Inline function:
     f(x,y) = sqrt(x.^2+y.^2)
```

You can evaluate this function in a usual way

```
f(3,4)

ans =
     5
```

Note that this function also works with arrays. Let

```
A = [1 2;3 4]

A =
     1     2
     3     4
```

and

```
B = ones(2)

B =
     1     1
     1     1
```

Then

```
C = f(A, B)

C =
    1.4142    2.2361
    3.1623    4.1231
```

For the later use let us mention briefly a concept of the *string* in MATLAB. The *character string* is a text surrounded by single quotes. For instance,

```
str =  'programming in MATLAB is fun'

str =
programming in MATLAB is fun
```

is an example of the string. Strings are discussed in Section 2.5 of this tutorial.

In the previous section you have learned how to create the function files. Some functions take as the input argument a name of another function, which is specified as a strin*g*. In order to execute function specified by string you should use the command **feval** as shown below

feval('functname', input parameters of function functname)

Consider the problem of computing the *least common multiple* of two integers. MATLAB has a built-in function **lcm** that computes the number in question. Recall that the least common multiple and the *greatest common divisor* (**gcd**) satisfy the following equation

$$ab = lcm(a, b)gcd(a, b)$$

MATLAB has its own function, named **gcd**, for computing the greatest common divisor.

To illustrate the use of the command **feval** let us take a closer look at the code in the m-file **mylcm**

```
function c = mylcm(a, b)

% The least common multiple c of two integers a and b.

if feval('isint',a) & feval('isint',b)
   c = a.*b./gcd(a,b);
else
   error('Input arguments must be integral numbers')
end
```

Command **feval** is used twice in line two (I do do not count the comment lines and the blank lines). It checks whether or not both input arguments are integers. The *logical and operator* **&** used here is discussed in Section 2.4. If this condition is satisfied, then the least common multiple is computed using the formula mentioned earlier, otherwise the error message is generated. Note use of the command **error,** which takes as the argument a string. The conditional **if - else - end** used here is discussed in Section 2.4 of this tutorial. Function that is executed twice in the body of the function **mylcm** is named **isint**

```
function k = isint(x);

% Check whether or not x is an integer number.
% If it is, function isint returns 1 otherwise it returns 0.

if abs(x - round(x)) < realmin
   k = 1;
else
   k = 0;
end
```

New functions used here are the absolute value function (**abs**) and the round function (**round**). The former is the classical math function while the latter takes a number and rounds it to the closest integer. Other functions used to round real numbers to integers are discussed in Section 2.7. Finally, **realmin** is the smallest positive real number on your computer

```
format long

realmin

ans =
    2.225073858507201e-308

format short
```

The Trapezoidal Rule with the correction term is often used to numerical integration of functions that are differentiable on the interval of integration

$$\int_a^b f(x)dx \approx \frac{h}{2}[f(a)+f(b)]-\frac{h^2}{12}[f'(a)-f'(b)]$$

where **h = b – a**. This formula is easy to implement in MATLAB

```
function y = corrtrap(fname, fpname, a, b)

% Corrected trapezoidal rule y.
% fname - the m-file used to evaluate the integrand,
% fpname - the m-file used to evaluate the first derivative
% of the integrand,
% a,b - endpoinds of the interval of integration.

h = b - a;
y = (h/2).*(feval(fname,a) + feval(fname,b))+ (h.^2)/12.*( ...
   feval(fpname,a) - feval(fpname,b));
```

The input parameters **a** and **b** can be arrays of the same dimension. This is possible because the dot operator proceeds certain arithmetic operations in the command that defines the variable **y**.

In this example we will integrate the sine function over two intervals whose end points are stored in the arrays **a** and **b**, where

```
a = [0 0.1];
b = [pi/2 pi/2 + 0.1];

y = corrtrap('sin', 'cos', a, b)

y =
    0.9910    1.0850
```

Since the integrand and its first order derivative are both the built-in functions, there is no need to define these functions in the m-files.

2.3 Control flow

To control the flow of commands, the makers of MATLAB supplied four devices a programmer can use while writing his/her computer code

- the **for** loops
- the **while** loops
- the **if-else-end** constructions
- the **switch-case** constructions

2.3.1 Repeating with for loops

Syntax of the **for** loop is shown below

```
for k = array
   commands
end
```

The commands between the **for** and **end** statements are executed for all values stored in the **array**.

Suppose that one-need values of the sine function at eleven evenly spaced points **πn/10**, for **n = 0, 1, …, 10**. To generate the numbers in question one can use the **for** loop

```
for n=0:10
   x(n+1) = sin(pi*n/10);
end
```

```
x
```

```
x =
  Columns 1 through 7
         0    0.3090    0.5878    0.8090    0.9511    1.0000    0.9511
  Columns 8 through 11
    0.8090    0.5878    0.3090    0.0000
```

The **for** loops can be nested

```
H = zeros(5);
for k=1:5
   for l=1:5
       H(k,l) = 1/(k+l-1);
   end
end
```

```
H
```

```
H =
    1.0000    0.5000    0.3333    0.2500    0.2000
    0.5000    0.3333    0.2500    0.2000    0.1667
    0.3333    0.2500    0.2000    0.1667    0.1429
    0.2500    0.2000    0.1667    0.1429    0.1250
    0.2000    0.1667    0.1429    0.1250    0.1111
```

Matrix **H** created here is called the *Hilbert matrix*. First command assigns a space in computer's memory for the matrix to be generated. This is added here to reduce the overhead that is required by loops in MATLAB.

The **for** loop should be used only when other methods cannot be applied. Consider the following problem. Generate a 10-by-10 matrix $\mathbf{A} = [a_{kl}]$, where $a_{kl} = \sin(k)\cos(l)$. Using nested loops one can compute entries of the matrix **A** using the following code

```
A = zeros(10);
for k=1:10
   for l=1:10
      A(k,l) = sin(k)*cos(l);
   end
end
```

A loop free version might look like this

```
k = 1:10;
A = sin(k)'*cos(k);
```

First command generates a row array **k** consisting of integers 1, 2, … , 10. The command **sin(k)'** creates a column vector while **cos(k)** is the row vector. Components of both vectors are the values of the two trig functions evaluated at **k**. Code presented above illustrates a powerful feature of MATLAB called *vectorization*. This technique should be used whenever it is possible.

2.3.2 Repeating with while loops

Syntax of the **while** loop is

```
while expression
  statements
end
```

This loop is used when the programmer does not know the number of repetitions a priori.

Here is an almost trivial problem that requires a use of this loop. Suppose that the number π is divided by 2. The resulting quotient is divided by 2 again. This process is continued till the current quotient is less than or equal to 0.01. What is the largest quotient that is greater than 0.01? To answer this question we write a few lines of code

```
q = pi;
while q > 0.01
   q = q/2;
end

q
```

```
q =
    0.0061
```

2.3.3 The if-else-end constructions

Syntax of the simplest form of the construction under discussion is

```
if expression
  commands
end
```

This construction is used if there is one alternative only. Two alternatives require the construction

```
if expression
  commands (evaluated if expression is true)
else
  commands (evaluated if expression is false)
end
```

Construction of this form is used in functions **mylcm** and **isint** (see Section 2.3).

If there are several alternatives one should use the following construction

```
if expression1
  commands (evaluated if expression 1 is true)
elseif expression 2
  commands (evaluated if expression 2 is true)
elseif …
.
.
.
else
  commands (executed if all previous expressions evaluate to false)
end
```

Chebyshev polynomials $\mathbf{T_n(x)}$, $\mathbf{n = 0, 1, \ldots}$ of the first kind are of great importance in numerical analysis. They are defined recursively as follows

$$\mathbf{T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x), \quad n = 2, 3, \ldots , \quad T_0(x) = 1, T_1(x) = x.}$$

Implementation of this definition is easy

```
function T = ChebT(n)

% Coefficients T of the nth Chebyshev polynomial of the first kind.
% They are stored in the descending order of powers.

t0 = 1;
t1 = [1 0];
if n == 0
   T = t0;
elseif n == 1;
   T = t1;
else
   for k=2:n
      T = [2*t1 0] - [0 0 t0];
      t0 = t1;
      t1 = T;
   end
end
```

Coefficients of the cubic Chebyshev polynomial of the first kind are

```
coeff = ChebT(3)

coeff =
     4     0    -3     0
```

Thus $T_3(x) = 4x^3 - 3x$.

2.3.4 The switch-case construction

Syntax of the **switch-case** construction is

```
switch expression (scalar or string)
  case value1 (executes if expression evaluates to value1)
     commands
  case value2 (executes if expression evaluates to value2)
     commands
.
.
.
  otherwise
    statements
end
```

Switch compares the input expression to each case value. Once the match is found it executes the associated commands.

In the following example a random integer number x from the set {1, 2, … , 10} is generated. If x = 1 or x = 2, then the message Probability = 20% is displayed to the screen. If x = 3 or 4 or 5, then the message Probability = 30% is displayed, otherwise the message Probability = 50% is generated. The script file **fswitch** utilizes a switch as a tool for handling all cases mentioned above

```
% Script file fswitch.

x = ceil(10*rand);  % Generate a random integer in {1, 2, ... , 10}
switch x
   case {1,2}
      disp('Probability = 20%');
   case {3,4,5}
      disp('Probability = 30%');
   otherwise
      disp('Probability = 50%');
end
```

Note use of the curly braces after the word **case**. This creates the so-called *cell array* rather than the one-dimensional array, which requires use of the square brackets.

Here are new MATLAB functions that are used in file **fswitch**

rand – uniformly distributed random numbers in the interval (0, 1)
ceil – round towards plus infinity infinity (see Section 2.5 for more details)
disp – display string/array to the screen

Let us test this code ten times

```
for k = 1:10
   fswitch
end
```

```
Probability = 50%
Probability = 30%
Probability = 50%
Probability = 50%
Probability = 50%
Probability = 30%
Probability = 20%
Probability = 50%
Probability = 30%
Probability = 50%
```

2.4 Relations and logical operators

Comparisons in MATLAB are performed with the aid of the following operators

Operator	Description
<	Less than
<=	Less than or equal to
>	Greater
>=	Greater or equal to
==	Equal to
~=	Not equal to

Operator == compares two variables and returns ones when they are equal and zeros otherwise.

Let

```
a = [1 1 3 4 1]
```

```
a =
     1     1     3     4     1
```

Then

```
ind = (a == 1)

ind =
     1     1     0     0     1
```

You can extract all entries in the array **a** that are equal to 1 using

```
b = a(ind)

b =
     1     1     1
```

This is an example of so-called *logical addressing* in MATLAB. You can obtain the same result using function **find**

```
ind = find(a == 1)

ind =
     1     2     5
```

Variable **ind** now holds indices of those entries that satisfy the imposed condition. To extract all ones from the array **a** use

```
b = a(ind)

b =
     1     1     1
```

There are three *logical operators* available in MATLAB

Logical operator	Description
|	And
&	Or
~	Not

Suppose that one wants to select all entries **x** that satisfy the inequalities **x** ρ **1** or **x** < **-0.2** where

```
x = randn(1,7)

x =
   -0.4326   -1.6656    0.1253    0.2877   -1.1465    1.1909    1.1892
```

is the array of normally distributed random numbers. We can solve easily this problem using operators discussed in this section

```
ind = (x  >= 1) | (x < -0.2)

ind =
     1     1     0     0     1     1     1

y = x(ind)
```

```
y =
   -0.4326   -1.6656   -1.1465    1.1909    1.1892
```

Solve the last problem without using the logical addressing.

In addition to relational and logical operators MATLAB has several logical functions designed for performing similar tasks. These functions return **1** (true) if a specific condition is satisfied and **0** (false) otherwise. A list of these functions is too long to be included here. The interested reader is referred to [1], pp. 85-86 and [4], Chapter 10, pp. 26-27. Names of the most of these functions begin with the prefix **is**. For instance, the following command

```
isempty(y)

ans =
     0
```

returns **0** because the array **y** of the last example is not empty. However, this command

```
isempty([ ])

ans =
     1
```

returns **1** because the argument of the function used is the empty array **[ ]**.

Here is another example that requires use of the **isempty** command

```
function dp = derp(p)

% Derivative dp of an algebraic polynomial that is
% represented by its coefficients p. They must be stored
% in the descending order of powers.

n = length(p) - 1;
p = p(:)';                  % Make sure p is a row array.
dp = p(1:n).*(n:-1:1);      % Apply the Power Rule.
k = find(dp ~= 0);
if ~isempty(k)
   dp = dp(k(1):end);       % Delete leading zeros if any.
else
   dp = 0;
end
```

In this example $\mathbf{p(x) = x^3 + 2x^2 + 4}$. Using a convention for representing polynomials in MATLAB as the array of their coefficients that are stored in the descending order of powers, we obtain

```
dp = derp([1 2 0 4])

dp =
     3     4     0
```

2.5 Strings

String is an array of characters. Each character is represented internally by its ASCII value.

This is an example of a string

```
str = 'I am learning MATLAB this semester.'

str =
I am learning MATLAB this semester.
```

To see its ASCII representation use function **double**

```
str1 = double(str)

str1 =
  Columns 1 through 12
    73    32    97   109    32   108   101    97   114   110   105
110
  Columns 13 through 24
   103    32    77    65    84    76    65    66    32   116   104
105
  Columns 25 through 35
   115    32   115   101   109   101   115   116   101   114    46
```

You can convert array **str1** to its character form using function **char**

```
str2 = char(str1)

str2 =
I am learning MATLAB this semester.
```

Application of the string conversion is used in Tutorial 3, Section 3.11 to uncode and decode messages.

To compare two strings for equality use function **strcmp**

```
iseq = strcmp(str, str2)

iseq =
     1
```

Two strings can be concatenated using function **ctrcat**

```
strcat(str,str2)

ans =
I am learning MATLAB this semester.I am learning MATLAB this semester.
```

Note that the concatenated strings are not separated by the blank space.

You can create two-dimensional array of strings. To this aim the *cell array* rather than the two-dimensional array must be used. This is due to the fact that numeric array must have the same number of columns in each row.

This is an example of the cell array

```
carr = {'first name'; 'last name'; 'hometown'}

carr =
    'first name'
    'last name'
    'hometown'
```

Note use of the curly braces instead of the square brackets. Cell arrays are discussed in detail in the next section of this tutorial.

MATLAB has two functions to categorize characters: **isletter** and **isspace**. We will run both functions on the string **str**

```
isletter(str)

ans =
  Columns 1 through 12
     1     0     1     1     0     1     1     1     1     1     1
1
  Columns 13 through 24
     1     0     1     1     1     1     1     1     0     1     1
1
  Columns 25 through 35
     1     0     1     1     1     1     1     1     1     1     0


isspace(str)

ans =
  Columns 1 through 12
     0     1     0     0     1     0     0     0     0     0     0
0
  Columns 13 through 24
     0     1     0     0     0     0     0     0     1     0     0
0
  Columns 25 through 35
     0     1     0     0     0     0     0     0     0     0     0
```

The former function returns 1 if a character is a letter and 0 otherwise while the latter returns 1 if a character is whitespace (blank, tab, or new line) and 0 otherwise.

We close this section with two important functions that are intended for conversion of numbers to strings. Functions in question are named **int2str** and **num2str**. Function **int2str** rounds its argument (matrix) to integers and converts the result into a string matrix.

Let

```
A = randn(3)

A =
   -0.4326    0.2877    1.1892
   -1.6656   -1.1465   -0.0376
    0.1253    1.1909    0.3273
```

Then

```
B = int2str(A)

B =
 0  0  1
-2 -1  0
 0  1  0
```

Function **num2str** takes an array and converts it to the array string. Running this function on the matrix **A** defined earlier, we obtain

```
C = num2str(A)

C =
-0.43256     0.28768      1.1892
 -1.6656     -1.1465   -0.037633
 0.12533      1.1909     0.32729
```

Function under discussion takes a second optional argument - a number of decimal digits. This feature allows a user to display digits that are far to the right of the decimal point. Using matrix A again, we get

```
D = num2str(A, 18)

D =
-0.43256481152822068      0.28767642035854885       1.1891642016521031
  -1.665584378238097      -1.1464713506814637    -0.037633276593317645
 0.12533230647483068       1.1909154656429988      0.32729236140865414
```

For comparison, changing format to **long**, we obtain

```
format long

A

A =
  -0.43256481152822   0.28767642035855   1.18916420165210
  -1.66558437823810  -1.14647135068146  -0.03763327659332
   0.12533230647483   1.19091546564300   0.32729236140865


format short
```

Function **num2str** his is often used for labeling plots with the **title**, **xlabel**, **ylabel**, and **text** commands.

2.6 Cell arrays

Two data types the *cell arrays* and *structures* make MATLAB a powerful tool for applications. They hold other MATLAB arrays. In this section we discuss the cell arrays only. To learn about structures the interested reader is referred to [4], Chapter 13 and [1], Chapter 12.

To create the cell array one can use one of the two techniques called the *cell indexing* and the *content indexing*. The following example reveals differences between these two techniques. Suppose one want to save the string 'John Brown' and his SSN 123-45-6789 (without dashes) in the cell array.

1. Cell indexing

```
A(1,1) = {'John Brown'};

A(1,2) = {[1 2 3 4 5 6 7 8 9]};
```

2. Content indexing

```
B{1,1} = 'John Brown';

B{1,2} = [1 2 3 4 5 6 7 8 9];
```

A condensed form of the cell array **A** is

```
A

A =
    'John Brown'    [1x9 double]
```

To display its full form use function **celldisp**

```
celldisp(A)

A{1} =
John Brown
A{2} =
     1     2     3     4     5     6     7     8     9
```

To access data in a particular cell use content indexing on the right-hand side. For instance,

```
B{1,1}

ans =
John Brown
```

To delete a cell use the empty matrix operator **[ ]**. For instance, this operation

```
B(1) = []

B =
    [1x9 double]
```

deletes cell **B(1, 1)** of the cell array **B**.

This command

```
C = {A B}

C =
    {1x2 cell}    {1x1 cell}
```

creates a new cell array

```
celldisp(C)

C{1}{1} =
John Brown
C{1}{2} =
     1     2     3     4     5     6     7     8     9
C{2}{1} =
     1     2     3     4     5     6     7     8     9
```

How would you delete cell **C(2,1)**?

2.7 Rounding to integers. Functions ceil, floor, fix and round

We have already used two MATLAB functions **round** and **ceil** to round real numbers to integers. They are briefly described in the previous sections of this tutorial. A full list of functions designed for rounding numbers is provided below

Function	Description
floor	Round towards minus infinity
ceil	Round towards plus infinity
fix	Round towards zero
round	Round towards nearest integer

To illustrate differences between these functions let us create first a two-dimensional array of random numbers that are normally distributed (mean = 0, variance = 1) using another MATLAB function **randn**

```
randn('seed', 0)      % This sets the seed of the random numbers generator to zero

T = randn(5)
```

```
T =
    1.1650    1.6961   -1.4462   -0.3600   -0.0449
    0.6268    0.0591   -0.7012   -0.1356   -0.7989
    0.0751    1.7971    1.2460   -1.3493   -0.7652
    0.3516    0.2641   -0.6390   -1.2704    0.8617
   -0.6965    0.8717    0.5774    0.9846   -0.0562
```

```
A = floor(T)

A =
     1     1    -2    -1    -1
     0     0    -1    -1    -1
     0     1     1    -2    -1
     0     0    -1    -2     0
    -1     0     0     0    -1
```

```
B = ceil(T)

B =
     2     2    -1     0     0
     1     1     0     0     0
     1     2     2    -1     0
     1     1     0    -1     1
     0     1     1     1     0
```

```
C = fix(T)

C =
     1     1    -1     0     0
     0     0     0     0     0
     0     1     1    -1     0
     0     0     0    -1     0
     0     0     0     0     0
```

```
D = round(T)

D =
     1     2    -1     0     0
     1     0    -1     0    -1
     0     2     1    -1    -1
     0     0    -1    -1     1
    -1     1     1     1     0
```

It is worth mentioning that the following identities

floor(x) = fix(x) for **x ρ 0**

and

ceil(x) = fix(x) for **x β 0**

hold true.

In the following m-file functions **floor** and **ceil** are used to obtain a certain representation of a nonnegative real number

```
function [m, r] = rep4(x)

% Given a nonnegative number x, function rep4 computes an integer m
% and a real number r, where 0.25 <= r < 1, such that x = (4^m)*r.

if x == 0
   m = 0;
   r = 0;
   return
end
u = log10(x)/log10(4);
if u < 0
   m = floor(u)
else
   m = ceil(u);
end
r = x/4^m;
```

Command **return** causes a return to the invoking function or to the keyboard. Function **log10** is the decimal logarithm.

```
[m, r] = rep4(pi)

m =
     1
r =
    0.7854
```

We check this result

```
format long

(4^m)*r

ans =
   3.14159265358979


format short
```

2.8 MATLAB graphics

MATLAB has several high-level graphical routines. They allow a user to create various graphical objects including two- and three-dimensional graphs, graphical user interfaces (GUIs), movies, to mention the most important ones. For the comprehensive presentation of the MATLAB graphics the interested reader is referred to [2].

Before we begin discussion of graphical tools that are available in MATLAB I recommend that you will run a couple of demos that come with MATLAB. In the **Command Window** click on **Help** and next select **Examples and Demos**. Chose**Visualization**, and next select **2-D Plots**. You will be presented with several buttons. Select **Line** and examine the m-file below the graph. It should give you some idea about computer code needed for creating a simple graph. It is recommended that you examine carefully contents of all m-files that generate the graphs in this demo.

2.8.1 2-D graphics

Basic function used to create 2-D graphs is the **plot** function. This function takes a variable number of input arguments. For the full definition of this function type **help plot** in the **Command Window**.

In this example the graph of the rational function $f(x) = \frac{x}{1+x^2}$, -2 β x β 2, will be plotted using a variable number of points on the graph of f(x)

```
% Script file graph1.

% Graph of the rational function y = x/(1+x^2).

for n=1:2:5
   n10 = 10*n;
   x = linspace(-2,2,n10);
   y = x./(1+x.^2);
   plot(x,y,'r')
   title(sprintf('Graph %g.  Plot based upon n = %g points.' ...
   , (n+1)/2, n10))
   axis([-2,2,-.8,.8])
   xlabel('x')
   ylabel('y')
   grid
   pause(3)
end
```

Let us analyze contents of this file. The loop **for** is executed three times. Therefore, three graphs of the same function will be displayed in the **Figure Window**. A MATLAB function **linspace(a, b, n)** generates a one-dimensional array of **n** evenly spaced numbers in the interval **[a b]**. The y-ordinates of the points to be plotted are stored in the array **y**. Command **plot** is called with three arguments: two arrays holding the x- and the y-coordinates and the string **'r'**, which describes the color (red) to be used to paint a plotted curve. You should notice a difference between three graphs created by this file. There is a significant difference between smoothness of graphs 1 and 3. Based on your visual observation you should be able to reach the following conclusion: "more points you supply the smoother graph is generated by the function **plot**". Function **title** adds a descriptive information to the graphs generated by this m-file and is followed by the command **sprintf**. Note that **sprintf** takes here three arguments: the string and names of two variables printed in the title of each graph. To specify format of printed numbers we use here the construction **%g**, which is recommended for printing integers. The command **axis** tells MATLAB what the dimensions of the box holding the plot are. To add more information to

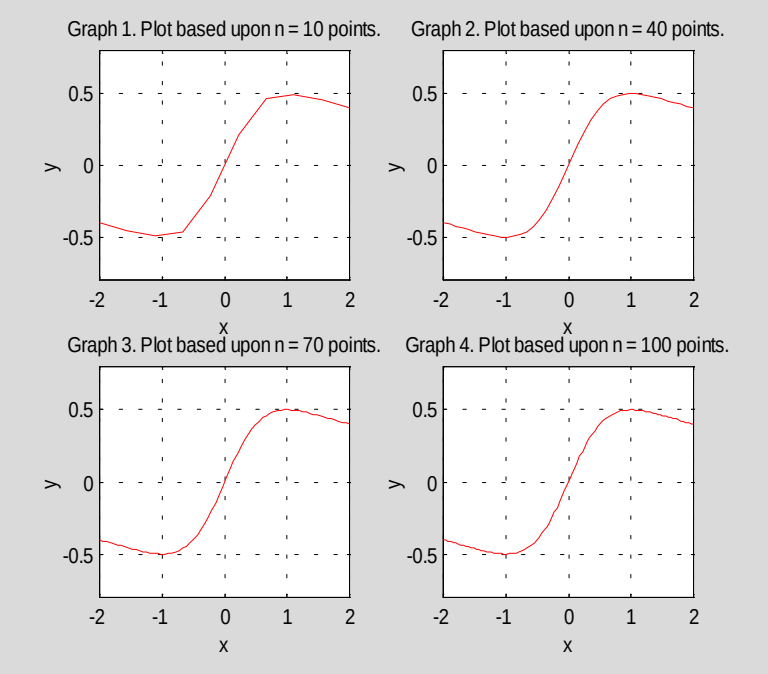
the graphs created here, we label the x- and the y-axes using commands **xlabel** and the **ylabel**, respectively. Each of these commands takes a string as the input argument. Function **grid** adds the grid lines to the graph. The last command used before the closing end is the **pause** command. The command **pause(n)** holds on the current graph for **n** seconds before continuing, where **n** can also be a fraction. If **pause** is called without the input argument, then the computer waits to user response. For instance, pressing the **Enter** key will resume execution of a program.

Function **subplot** is used to plot of several graphs in the same **Figure Window**. Here is a slight modification of the m-file **graph1**

```
% Script file graph2.

% Several plots of the rational function y = x/(1+x^2)
% in the same window.

k = 0;
for n=1:3:10
   n10 = 10*n;
   x = linspace(-2,2,n10);
   y = x./(1+x.^2);
   k = k+1;
   subplot(2,2,k)
   plot(x,y,'r')
   title(sprintf('Graph %g. Plot based upon n = %g points.' ...
      , k, n10))
   xlabel('x')
   ylabel('y')
   axis([-2,2,-.8,.8])
   grid
   pause(3);
end
```

The command **subplot** is called here with three arguments. The first two tell MATLAB that a 2-by-2 array consisting of four plots will be created. The third parameter is the running index telling MATLAB which subplot is currently generated.

```
graph2
```



Using command **plot** you can display several curves in the same **Figure Window**.

We will plot two ellipses

$$\frac{(x-3)^2}{36}+\frac{(y+2)^2}{81}=1 \quad \text{and} \quad \frac{(x-7)^2}{4}+\frac{(y-8)^2}{36}=1$$

using command **plot**

```
% Script file graph3.
% Graphs of two ellipses
%                x(t) = 3 + 6cos(t), y(t) = -2 + 9sin(t)
% and
%                x(t) = 7 + 2cos(t), y(t) = 8 + 6sin(t).

t = 0:pi/100:2*pi;
x1 = 3 + 6*cos(t);
y1 = -2 + 9*sin(t);
x2 = 7 + 2*cos(t);
y2 = 8 + 6*sin(t);
h1 = plot(x1,y1,'r',x2,y2,'b');
set(h1,'LineWidth',1.25)
axis('square')
xlabel('x')
```

```
h = get(gca,'xlabel');
set(h,'FontSize',12)
set(gca,'XTick',-4:10)
ylabel('y')
h = get(gca,'ylabel');
set(h,'FontSize',12)
set(gca,'YTick',-12:2:14)
title('Graphs of (x-3)^2/36+(y+2)^2/81 = 1 and (x-7)^2/4+(y-8)^2/36 =
1.')
h = get(gca,'Title');
set(h,'FontSize',12)
grid
```

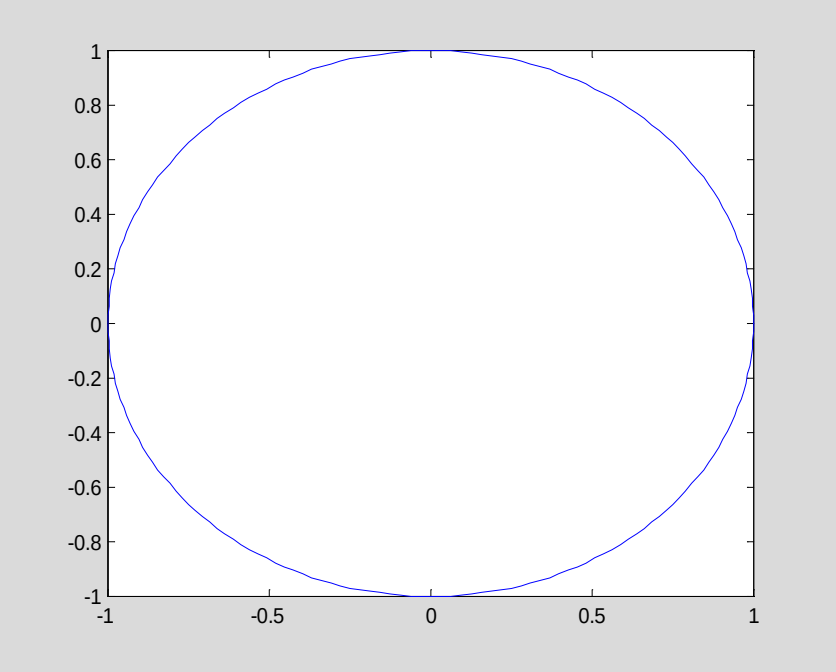
In this file we use several new MATLAB commands. They are used here to enhance the readability of the graph. Let us now analyze the computer code contained in the m-file **graph3**. First of all, the equations of ellipses in rectangular coordinates are transformed to parametric equations. This is a convenient way to plot graphs of equations in the implicit form. The points to be plotted, and smoothed by function **plot**, are defined in the first five lines of the file. I do not count here the comment lines and the blank lines. You can plot both curves using a single **plot** command. Moreover, you can select colors of the curves. They are specified as strings (see line 6). MATLAB has several colors you can use to plot graphs:

y yellow
m magenta
c cyan
r red
g green
b blue
w white
k black

Note that the command in line 6 begins with **h1 = plot**… Variable **h1** holds an information about the graph you generate and is called the *handle graphics*. Command **set** used in the next line allows a user to manipulate a plot. Note that this command takes as the input parameter the variable **h1**. We change thickness of the plotted curves from the default value to a width of our choice, namely 1.25. In the next line we use command **axis** to customize plot. We chose option **'square'** to force axes to have square dimensions. Other available options are: **'equal'**, **'normal'**, **'ij'**, **'xy'**, and **'tight'**. To learn more about these options use MATLAB's help.

If function **axis** is not used, then the circular curves are not necessarily circular. To justify this let us plot a graph of the unit circle of radius 1 with center at the origin

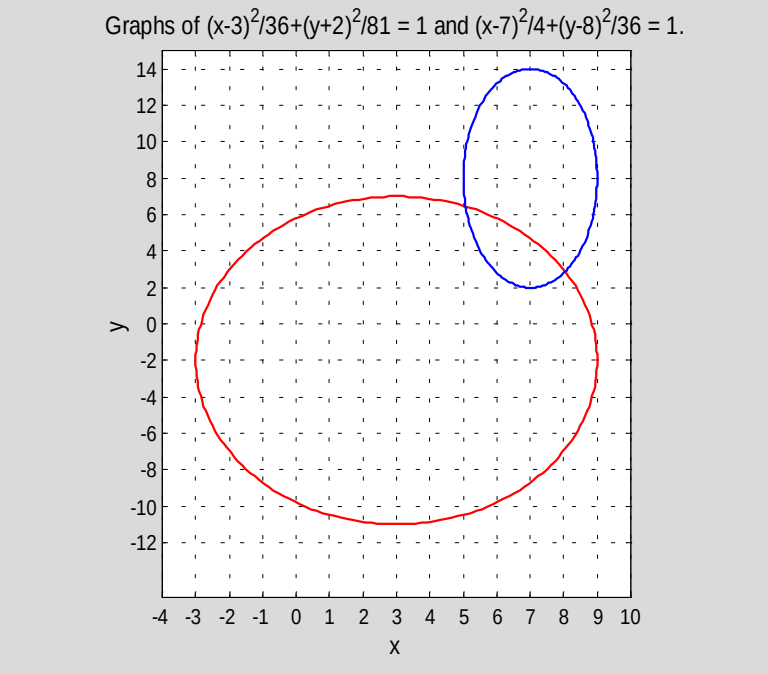
```
t = 0:pi/100:2*pi;
x = cos(t);
y = sin(t);
plot(x,y)
```



Another important MATLAB function used in the file under discussion is named **get** (see line 10). It takes as the first input parameter a variable named **gca** = *get current axis*. It should be obvious to you, that the axis targeted by this function is the x-axis. Variable **h = get(gca, …)** is the *graphics handle* of this axis. With the information stored in variable **h**, we change the font size associated with the x-axis using the **'FontSize'** string followed by a size of the font we wish to use. Invoking function **set** in line 12, we will change the tick marks along the x-axis using the **'XTick'** string followed by the array describing distribution of marks. You can comment out temporarily line 12 by adding the percent sign **%** before the word **set** to see the difference between the default tick marks and the marks generated by the command in line 12. When you are done delete the percent sign you typed in line 12 and click on **Save** from the **File** menu in the **MATLAB Editor/Debugger**. Finally, you can also make changes in the title of your plot. For instance, you can choose the font size used in the title. This is accomplished here by using function **set**. It should be obvious from the short discussion presented here that two MATLAB functions **get** and **set** are of great importance in manipulating graphs.

Graphs of the ellipses in question are shown on the next page

graph3



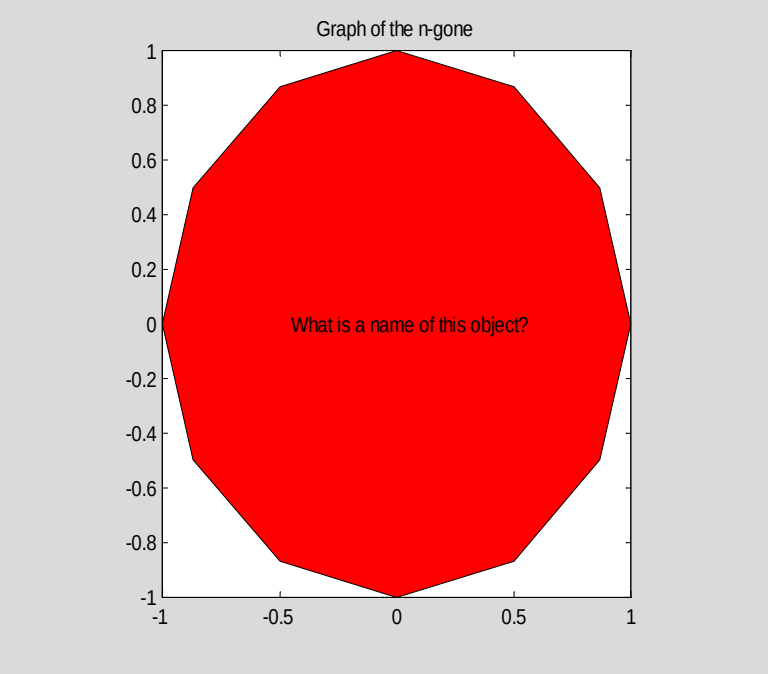
MATLAB has several functions designed for plotting specialized 2-D graphs. A partial list of these functions is included here **fill**, **polar**, **bar**, **barh**, **pie**, **hist**, **compass**, **errorbar**, **stem**, and **feather**.

In this example function **fill** is used to create a well-known object

```
n = -6:6;
x = sin(n*pi/6);
y = cos(n*pi/6);
fill(x, y, 'r')
axis('square')
title('Graph of the n-gone')
text(-0.45,0,'What is a name of this object?')
```

Function in question takes three input parameters - two arrays, named here **x** and **y**. They hold the x- and y-coordinates of vertices of the polygon to be filled. Third parameter is the user-selected color to be used to paint the object. A new command that appears in this short code is the **text** command. It is used to annotate a text. First two input parameters specify text location. Third input parameter is a text, which will be added to the plot.

Graph of the filled object that is generated by this code is displayed below



2.8.2 3-D graphics

MATLAB has several built-in functions for plotting three-dimensional objects. In this subsection we will deal mostly with functions used to plot curves in space (**plot3**), mesh surfaces (**mesh**), surfaces (**surf**) and contour plots (**contour**). Also, two functions for plotting special surfaces, **sphere** and **cylinder** will be discussed briefly. I recommend that any time you need help with the 3-D graphics you should type **help graph3d** in the **Command Window** to learn more about various functions that are available for plotting three-dimensional objects.

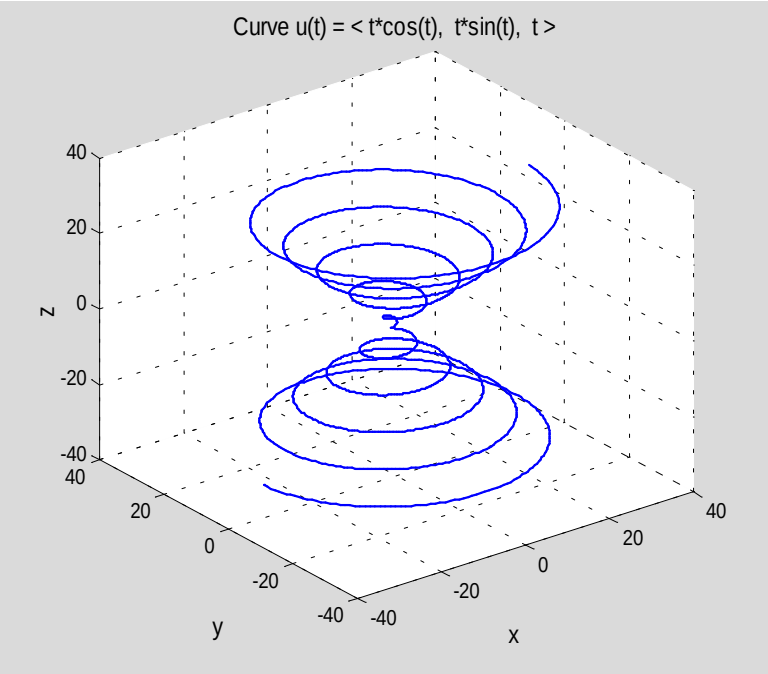
Let **r(t) = < t cos(t), t sin(t), t >**, **-10π β t β 10π**, be the space curve. We plot its graph over the indicated interval using function **plot3**

```
% Script file graph4.

% Curve r(t) = < t*cos(t), t*sin(t), t >.

t = -10*pi:pi/100:10*pi;
x = t.*cos(t);
y = t.*sin(t);
h = plot3(x,y,t);
set(h,'LineWidth',1.25)
title('Curve u(t) = < t*cos(t),  t*sin(t),  t >')
h = get(gca,'Title');
set(h,'FontSize',12)
xlabel('x')
h = get(gca,'xlabel');
set(h,'FontSize',12)
ylabel('y')
h = get(gca,'ylabel');
```

```
set(h,'FontSize',12)
zlabel('z')
h = get(gca,'zlabel');
set(h,'FontSize',12)
grid
```

Function **plot3** is used in line 4. It takes three input parameters – arrays holding coordinates of points on the curve to be plotted. Another new command in this code is the **zlabel** command (see line 4 from the bottom). Its meaning is self-explanatory.

```
graph4
```



Function **mesh** is intended for plotting graphs of the 3-D mesh surfaces. Before we begin to work with this function, another function **meshgrid** should be introduced. This function generates two two-dimensional arrays for 3-D plots. Suppose that one wants to plot a mesh surface over the grid that is defined as the Cartesian product of two sets

```
x = [0 1 2];
y = [10 12 14];
```

The **meshgrid** command applied to the arrays **x** and **y** creates two matrices

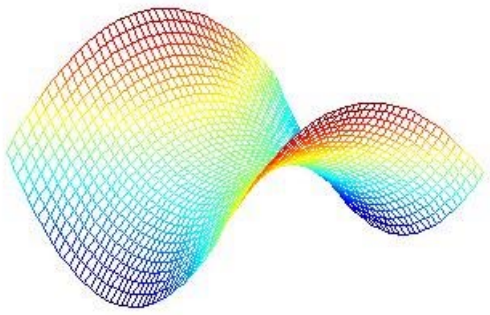
```
[xi, yi] = meshgrid(x,y)
```

```
xi =
     0     1     2
     0     1     2
     0     1     2
yi =
    10    10    10
    12    12    12
    14    14    14
```

Note that the matrix **xi** contains replicated rows of the array **x** while **yi** contains replicated columns of **y**. The z-values of a function to be plotted are computed from arrays **xi** and **yi**.

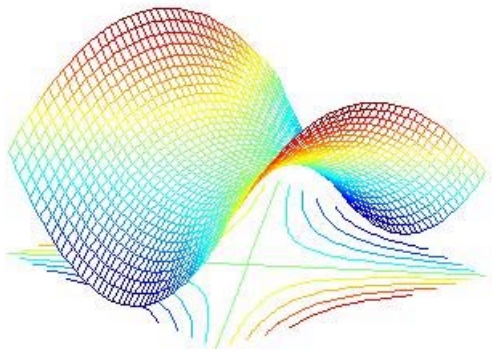
In this example we will plot the hyperbolic paraboloid $z = y^2 - x^2$ over the square $-1 \beta x \beta 1$, $-1 \beta y \beta 1$

```
x = -1:0.05:1;
y = x;
[xi, yi] = meshgrid(x,y);
zi = yi.^2 - xi.^2;
mesh(xi, yi, zi)
axis off
```



To plot the graph of the mesh surface together with the contour plot beneath the plotted surface use function **meshc**

```
meshc(xi, yi, zi)
axis off
```



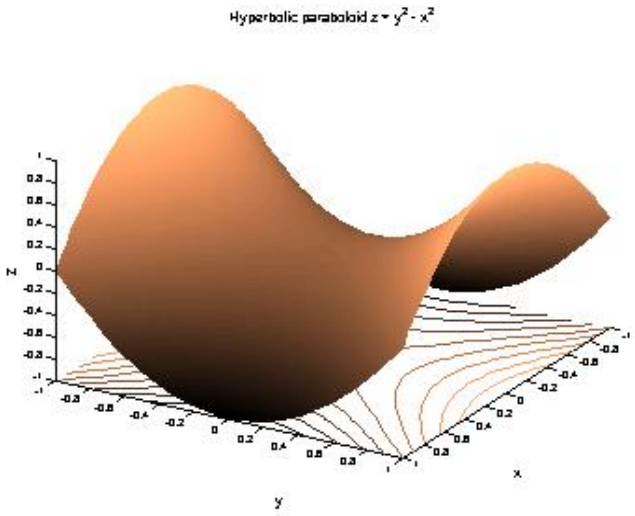
Function **surf** is used to visualize data as a shaded surface.

Computer code in the m-file **graph5** should help you to learn some finer points of the 3-D graphics in MATLAB

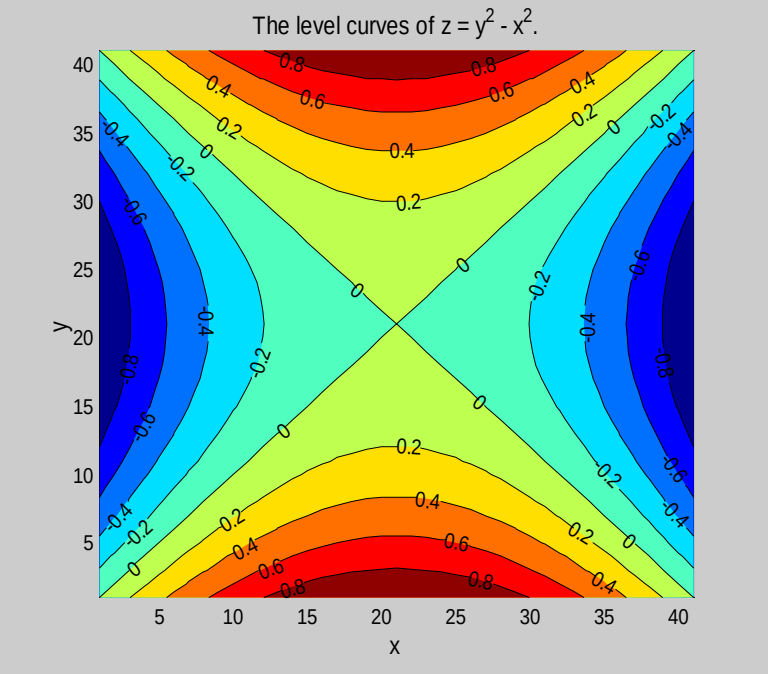
```
% Script file graph5.

% Surface plot of the hyperbolic paraboloid z = y^2 - x^2
% and its level curves.

x = -1:.05:1;
y = x;
[xi,yi] = meshgrid(x,y);
zi = yi.^2 - xi.^2;
surfc(xi,yi,zi)
colormap copper
shading interp
view([25,15,20])
grid off
title('Hyperbolic paraboloid z = y^2 – x^2')
h = get(gca,'Title');
set(h,'FontSize',12)
xlabel('x')
h = get(gca,'xlabel');
set(h,'FontSize',12)
ylabel('y')
h = get(gca,'ylabel');
set(h,'FontSize',12)
zlabel('z')
h = get(gca,'zlabel');
set(h,'FontSize',12)
pause(5)
figure
contourf(zi), hold on, shading flat
[c,h] = contour(zi,'k-'); clabel(c,h)
title('The level curves of z = y^2 - x^2.')
h = get(gca,'Title');
set(h,'FontSize',12)
```

```
xlabel('x')
h = get(gca,'xlabel');
set(h,'FontSize',12)
ylabel('y')
h = get(gca,'ylabel');
set(h,'FontSize',12)
```

```
graph5
```



A second graph is shown on the next page.



There are several new commands used in this file. On line 5 (again, I do not count the blank lines and the comment lines) a command **surfc** is used. It plots a surface together with the level lines beneath. Unlike the command **surfc** the command **surf** plots a surface only without the level curves. Command **colormap** is used in line 6 to paint the surface using a user-supplied colors. If the command **colormap** is not added, MATLAB uses default colors. Here is a list of color maps that are available in MATLAB

hsv - hue-saturation-value color map
hot - black-red-yellow-white color map
gray - linear gray-scale color map
bone - gray-scale with tinge of blue color map
copper - linear copper-tone color map
pink - pastel shades of pink color map
white - all white color map
flag - alternating red, white, blue, and black color map
lines - color map with the line colors
colorcube - enhanced color-cube color map
vga - windows colormap for 16 colors
jet - variant of HSV
prism - prism color map
cool - shades of cyan and magenta color map
autumn - shades of red and yellow color map
spring - shades of magenta and yellow color map
winter - shades of blue and green color map
summer - shades of green and yellow color map

Command **shading** (see line 7) controls the color shading used to paint the surface. Command in question takes one argument. The following

shading flat sets the shading of the current graph to flat
shading interp sets the shading to interpolated
shading faceted sets the shading to faceted, which is the default.

are the shading options that are available in MATLAB.

Command **view** (see line 8) is the 3-D graph viewpoint specification. It takes a three-dimensional vector, which sets the view angle in Cartesian coordinates.

We will now focus attention on commands on lines 23 through 25. Command **figure** prompts MATLAB to create a new **Figure Window** in which the level lines will be plotted. In order to enhance the graph, we use command **contourf** instead of **contour**. The former plots filled contour lines while the latter doesn't. On the same line we use command **hold on** to hold the current plot and all axis properties so that subsequent graphing commands add to the existing graph. First command on line 25 returns matrix **c** and graphics handle **h** that are used as the input parameters for the function **clabel**, which adds height labels to the current contour plot.

Due to the space limitation we cannot address here other issues that are of interest for programmers dealing with the 3-D graphics in MATLAB. To learn more on this subject the interested reader is referred to [1-3] and [5].

2.8.3 Animation

In addition to static graphs discussed so far one can put a sequence of graphs in motion. In other words, you can make a movie using MATLAB graphics tools. To learn how to create a movie, let us analyze the m-file **firstmovie**

```
 % Script file firstmovie.

% Graphs of y = sin(kx) over the interval [0, pi],
% where k = 1, 2, 3, 4, 5.

m = moviein(5);
x = 0:pi/100:pi;
for i=1:5
   h1_line = plot(x,sin(i*x));
   set(h1_line,'LineWidth',1.5,'Color','m')
   grid
   title('Sine functions sin(kx), k = 1, 2, 3, 4, 5')
   h = get(gca,'Title');
   set(h,'FontSize',12)
   xlabel('x')
   k = num2str(i);
   if i > 1
      s = strcat('sin(',k,'x)');
   else
      s = 'sin(x)';
   end
   ylabel(s)
   h = get(gca,'ylabel');
   set(h,'FontSize',12)
   m(:,i) = getframe;
   pause(2)
```

```
end
movie(m)
```

I suggest that you will play this movie first. To this aim type **firstmovie** in the **Command Window** and press the **Enter** or **Return** key. You should notice that five frames are displayed and at the end of the "show" frames are played again at a different speed.

There are very few new commands one has to learn in order to animate graphics in MATLAB. We will use the m-file **firstmovie** as a starting point to our discussion. Command **moviein**, on line 1, with an integral parameter, tells MATLAB that a movie consisting of five frames is created in the body of this file. Consecutive frames are generated inside the loop **for**. Almost all of the commands used there should be familiar to you. The only new one inside the loop is **getframe** command. Each frame of the movie is stored in the column of the matrix **m**. With this remark a role of this command should be clear. The last command in this file is **movie(m)**. This tells MATLAB to play the movie just created and saved in columns of the matrix **m**.

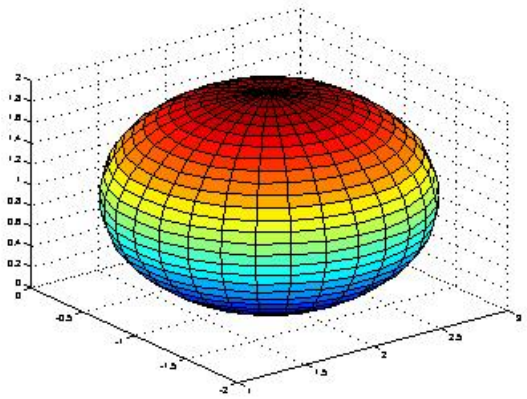
Warning. File **firstmovie** cannot be used with the **Student Edition of MATLAB, version 4.2**. This is due to the matrix size limitation in this edition of MATLAB. Future release of the **Student Edition of MATLAB, version 5.3** will allow large size matrices. According to MathWorks, Inc., the makers of MATLAB, this product will be released in September 1999.

2.8.4 Commands sphere and cylinder

MATLAB has some functions for generating special surfaces. We will be concerned mostly with two functions- **sphere** and **cylinder**.

The command **sphere(n)** generates a unit sphere with center at the origin using $(n+1)^2$ points. If function **sphere** is called without the input parameter, MATLAB uses the default value n = 20. You can translate the center of the sphere easily. In the following example we will plot graph of the unit sphere with center at (2, -1, 1)

```
[x,y,z] = sphere(30);
surf(x+2, y-1, z+1)
```

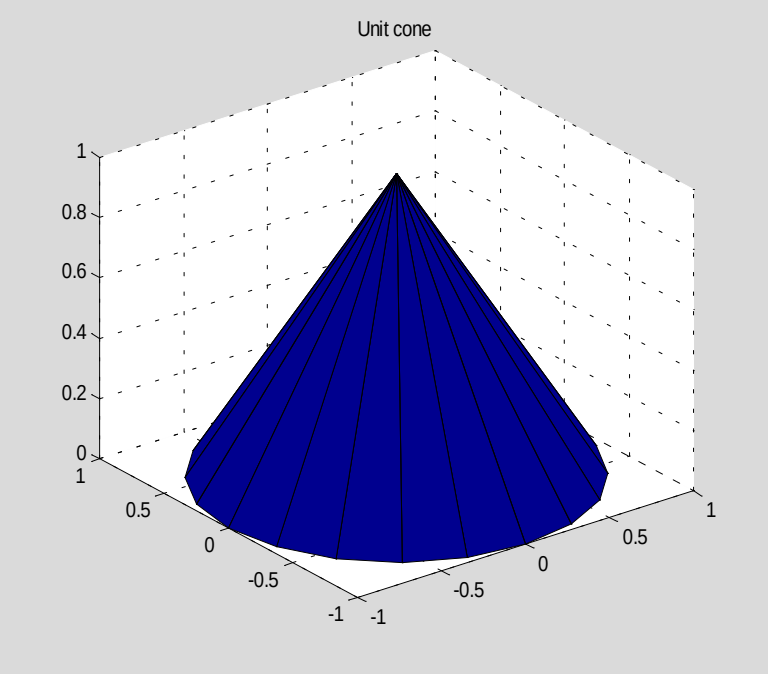


Function **sphere** together with function **surf** or **mesh** can be used to plot graphs of spheres of arbitrary radii. Also, they can be used to plot graphs of ellipsoids. See Problems 25 and 26.

Function **cylinder** is used for plotting a surface of revolution. It takes two (optional) input parameters. In the following command **cylinder(r, n)** parameter **r** stands for the vector that defines the radius of cylinder along the z-axis and **n** specifies a number of points used to define circumference of the cylinder. Default values of these parameters are **r = [1 1]** and **n = 20**. A generated cylinder has a unit height.

The following command

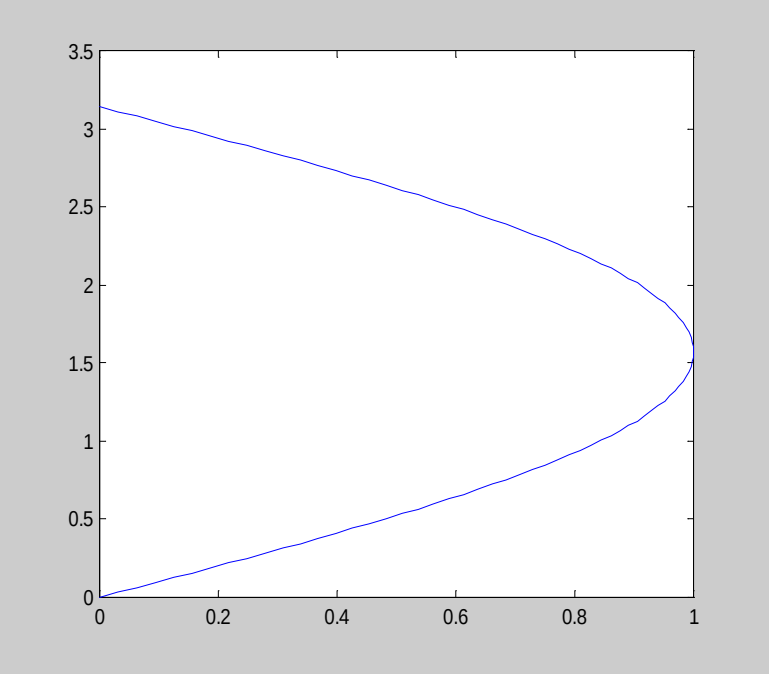
```
cylinder([1 0])
title('Unit cone')
```



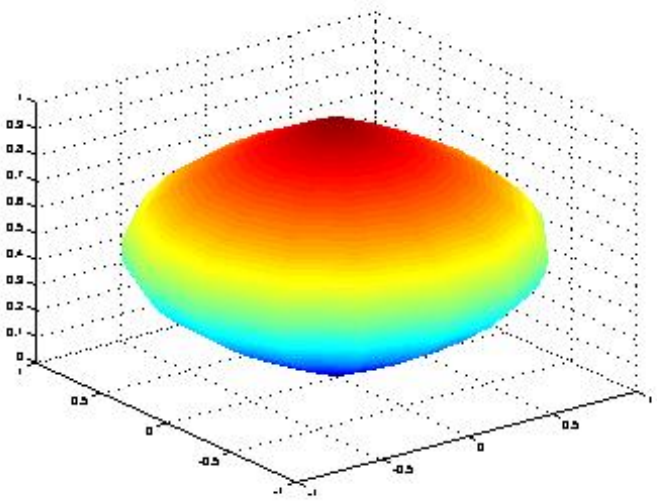
plots a cone with the base radius equal to one and the unit height.

In this example we will plot a graph of the surface of revolution obtained by rotating the curve **r(t) = < sin(t), t >**, **0 β t β π** about the y-axis. Graphs of the generating curve and the surface of revolution are created using a few lines of the computer code

```
t = 0:pi/100:pi;
r = sin(t);
plot(r,t)
```




```
cylinder(r,15)
shading interp
```



2.8.5 Printing MATLAB graphics

In this section we deal with printing MATLAB graphics. To send a current graph to the printer click on **File** and next select **Print** from the pull down menu. Once this menu is open you may

wish to preview a graph to be printed be selecting the option **PrintPreview…** first. You can also send your graph to the printer using the **print** command as shown below

```
x = 0:0.01:1;
plot(x, x.^2)
print
```

You can print your graphics to an m- file using built-in device drivers. A fairly incomplete list of these drivers is included here:

-depsc Level 1 color Encapsulated PostScript
-deps2 Level 2 black and white Encapsulated PostScript
-depsc2 Level 2 color Encapsulated PostScript

For a complete list of available device drivers see [5], Chapter 7, pp. 8-9.

Suppose that one wants to print a current graph to the m-file Figure1 using level 2 color Encapsulated PostScript. This can be accomplished by executing the following command

```
print –depsc2 Figure1
```

You can put this command either inside your m-file or execute it from within the **Command Window**.

References

[1] D. Hanselman and B. Littlefield, Mastering MATLAB 5. A Comprehensive Tutorial and Reference, Prentice Hall, Upper Saddle River, NJ, 1998.

[2] P. Marchand, Graphics and GUIs with MATLAB, Second edition, CRC Press, Boca Raton, 1999.

[3] K. Sigmon, MATLAB Primer, Fifth edition, CRC Press, Boca Raton, 1998.

[4] Using MATLAB, Version 5, The MathWorks, Inc., 1996.

[5] Using MATLAB Graphics, Version 5, The MathWorks, Inc., 1996.

Problems

In **Problems 1- 4** you cannot use loops **for** or **while**.

1. Write MATLAB function **sigma = ascsum(x)** that takes a one-dimensional array **x** of real numbers and computes their sum **sigma** in the ascending order of magnitudes.
   Hint: You may wish to use MATLAB functions **sort**, **sum**, and **abs**.

2. In this exercise you are to write MATLAB function **d = dsc(c)** that takes a one-dimensional array of numbers **c** and returns an array **d** consisting of all numbers in the array **c** with all neighboring duplicated numbers being removed. For instance, if **c = [1 2 2 2 3 1]**, then **d = [1 2 3 1]**.

3. Write MATLAB function **p = fact(n)** that takes a nonnegative integer **n** and returns value of the factorial function n! = 1*2* … *n. Add an error message to your code that will be executed when the input parameter is a negative number.

4. Write MATLAB **function [in, fr] = infr(x)** that takes an array **x** of real numbers and returns arrays **in** and **fr** holding the integral and fractional parts, respectively, of all numbers in the array **x**.

5. Given an array **b** and a positive integer **m** create an array **d** whose entries are those in the array **b** each replicated m-times. Write MATLAB **function d = repel(b, m)** that generates array **d** as described in this problem.

6. In this exercise you are to write MATLAB **function d = rep(b, m)** that has more functionality than the function **repel** of Problem 5. It takes an array of numbers **b** and the array **m** of positive integers and returns an array **d** whose each entry is taken from the array **b** and is duplicated according to the corresponding value in the array **m**. For instance, if **b = [1 2]** and **m = [2 3]**, then **d = [1 1 2 2 2]**.

7. A *checkerboard* matrix is a square block diagonal matrix, i.e., the only nonzero entries are in the square blocks along the main diagonal. In this exercise you are to write MATLAB **function A = mysparse(n)** that takes an odd number **n** and returns a checkerboard matrix as shown below

```
A = mysparse(3)

A =
     1     0     0
     0     1     2
     0     3     4


A = mysparse(5)
```

```
A =
     1     0     0     0     0
     0     1     2     0     0
     0     3     4     0     0
     0     0     0     2     3
     0     0     0     4     5
```

```
A = mysparse(7)
```

```
A =
     1     0     0     0     0     0     0
     0     1     2     0     0     0     0
     0     3     4     0     0     0     0
     0     0     0     2     3     0     0
     0     0     0     4     5     0     0
     0     0     0     0     0     3     4
     0     0     0     0     0     5     6
```

First block in the upper-left corner is the 1-by-1 matrix while the remaining blocks are all 2-by-2.

8. The Legendre polynomials **$P_n(x)$**, **n = 0, 1, …** are defined recursively as follows

$$nP_n(x) = (2n-1)xP_{n-1} - (n-1)P_{n-2}(x), \quad n = 2, 3, \ldots, \; P_0(x) = 1, \; P_1(x) = x.$$

Write MATLAB **function P = LegendP(n)** that takes an integer **n** – the degree of **$P_n(x)$** and returns its coefficient stored in the descending order of powers.

9. In this exercise you are to implement Euclid's Algorithm for computing the *greatest common divisor* (**gcd**) of two integer numbers **a** and **b**:

$$gcd(a, 0) = a, \quad gcd(a, b) = gcd(b, rem(a, b)).$$

Here **rem(a, b)** stands for the remainder in dividing **a** by **b**. MATLAB has function **rem**. Write MATLAB **function gcd = mygcd(a,b)** that implements Euclid's Algorithm.

10. The Pascale triangle holds coefficients in the series exapansion of **$(1 + x)^n$**, where **n = 0, 1, 2, …** . The top of this triangle, for **n = 0, 1, 2**, is shown here

```
     1
   1   1
 1   2   1
```

Write MATLAB **function t = pasctri(n)** that generates the Pascal triangle **t** up to the level **n**. **Remark**. Two-dimensional arrays in MATLAB must have the same number of columns in each row. In order to aviod error messages you have to add a certain number of zero entries to the right of last nonzero entry in each row of **t** but one. This

```
t = pasctri(2)
```

```
t =
     1     0     0
     1     1     0
     1     2     1
```

is an example of the array **t** for **n = 2**.

11. This is a continuation of Problem 10. Write MATLAB **function t = binexp(n)** that computes an array **t** with row **k+1** holding coefficients in the series expansion of **(1-x)^k**, **k = 0, 1, ... , n**, in the ascending order of powers. You may wish to make a call from within your function to the function **pasctri** of Problem 10. Your output sholud look like this (case **n = 3**)

```
t = binexp(3)

t =
     1     0     0     0
     1    -1     0     0
     1    -2     1     0
     1    -3     3    -1
```

12. MATLAB come with the built-in function **mean** for computing the *unweighted arithmetic mean* of real numbers. Let $x = \{x_1, x_2, \dots , x_n\}$ be an array of n real numbers. Then

$$mean(x) = \frac{1}{n}\sum_{k=1}^{n} x_n$$

In some problems that arise in mathematical statistics one has to compute the *weighted arithmetic mean* of numbers in the array x. The latter, abbreviated here as **wam**, is defined as follows

$$wam(x, w) = \frac{\sum_{k=1}^{n} w_k x_k}{\sum_{k=1}^{n} w_k}$$

Here $w = \{w_1, w_2, \dots , w_n\}$ is the array of weights associated with variables x. The weights are all nonnegative with $w_1 + w_2 + \dots + w_n > 0$.
In this exercise you are to write MATLAB **function y = wam(x, w)** that takes the arrays of variables and weights and returns the weighted arithmetic mean as defined above. Add three error messages to terminate prematurely execution of this file in the case when:

- arrays **x** and **w** are of different lengths
- at least one number in the array w is negative
- sum of all weights is equal to zero.

13. Let w = {w_1, w_2, … , w_n} be an array of positive numbers. The *weighted geometric mean*, abbreviated as **wgm**, of the nonnegative variables x = {x_1, x_2, … , x_n} is defined as follows

$$wgm(x,w) = x_1^{w_1} x_2^{w_2} \ldots x_n^{w_n}$$

Here we assume that the weights **w** sum up to one.
Write MATLAB **function y = wgm(x, w)** that takes arrays **x** and **w** and returns the weighted geometric mean **y** of **x** with weights stored in the array **w**. Add three error messages to terminate prematurely execution of this file in the case when:

- arrays **x** and **w** are of different lengths
- at least one variable in the array **x** is negative
- at least one weight in the array **w** is less than or equal to zero

Also, normalize the weights **w**, if necessary, so that they will sum up to one.

14. Write MATLAB **function [nonz, mns] = matstat(A)** that takes as the input argument a real matrix A and returns all nonzero entries of A in the column vector **nonz**. Second output parameter **mns** holds values of the unweighted arithmetic means of all columns of **A**.

15. Solving triangles requires a bit of knowledge of trigonometry. In this exercise you are to write MATLAB **function [a, B, C] = sas(b, A, c)** that is intended for solving triangles given two sides **b** and **c** and the angle **A** between these sides. Your function should determine remaining two angels and the third side of the triangle to be solved. All angles should be expressed in the degree measure.

16. Write MATLAB **function [A, B, C] = sss(a, b, c)** that takes three positive numbers **a**, **b**, and **c**. If they are sides of a triangle, then your function should return its angles **A**, **B**, and **C**, in the degree measure, otherwise an error message should be displayed to the screen.

17. In this exercise you are to write MATLAB **function dms(x)** that takes a nonnegative number **x** that represents an angle in the degree measure and converts it to the form **x deg. y min. z sec.**. Display a result to the screen using commands **disp** and **sprintf**. Example:

```
dms(10.2345)

Angle = 10 deg. 14 min. 4 sec.
```

18. Complete elliptic integral of the first kind in the Legendre form $K(k^2)$, $0 < k^2 < 1$,

$$K(k^2) = \int_0^{\pi/2} \frac{dt}{\sqrt{1 - k^2 \sin^2(t)}}$$

cannot be evaluated in terms of the elementary functions. The following algorithm, due to C. F. Gauss, generates a sequence of the arithmetic means {a_n} and a sequence of the geometric means {b_n}, where

$$a_0 = 1, \quad b_0 = \sqrt{1-k^2}$$

$$a_n = (a_{n-1} + b_{n-1})/2, \quad b_n = \sqrt{a_{n-1}b_{n-1}} \quad n = 1, 2, \ldots .$$

It is known that both sequences have a common limit g and that $a_n \geq b_n$, for all n. Moreover,

$$K(k^2) = \frac{\pi}{2g}$$

Write MATLAB **function K = compK(k2)** which implements this algorithm. The input parameter **k2** stands for k^2. Use the loop **while** to generate consecutive members of both sequences, but do not save all numbers generated in the course of computations. Continue execution of the **while** loop as long as $a_n - b_n \geq$ **eps**, where **eps** is the *machine epsilon*

```
eps

ans =
     2.2204e-016
```

Add more functionality to your code by allowing the input parameter **k2** to be an array. Test your m-file and compare your results with those included here

```
format long

compK([.1 .2 .3 .7 .8 .9])

ans =
   1.61244134872022
   1.65962359861053
   1.71388944817879
   2.07536313529247
   2.25720532682085
   2.57809211334794


format short
```

19. In this exercise you are to model one of the games in the Illinois State Lottery. Three numbers, with duplicates allowed, are selected randomly from the set {0,1,2,3,4,5,6,7,8,9} in the game Pick3 and four numbers are selected in the Pick4 game. Write MATLAB **function winnumbs = lotto(n)** that takes an integer n as its input parameter and returns an array **winnumbs** consisting of **n** numbers from the set of integers described in this problem. Use MATLAB function **rand** together with other functions to generate a set of winning numbers. Add an error message that is displayed to the screen when the input parameter is out of range.

20. Write MATLAB **function t = isodd(A)** that takes an array **A** of nonzero integers and returns **1** if all entries in the array **A** are odd numbers and **0** otherwise. You may wish to use MATLAB function **rem** in your file.

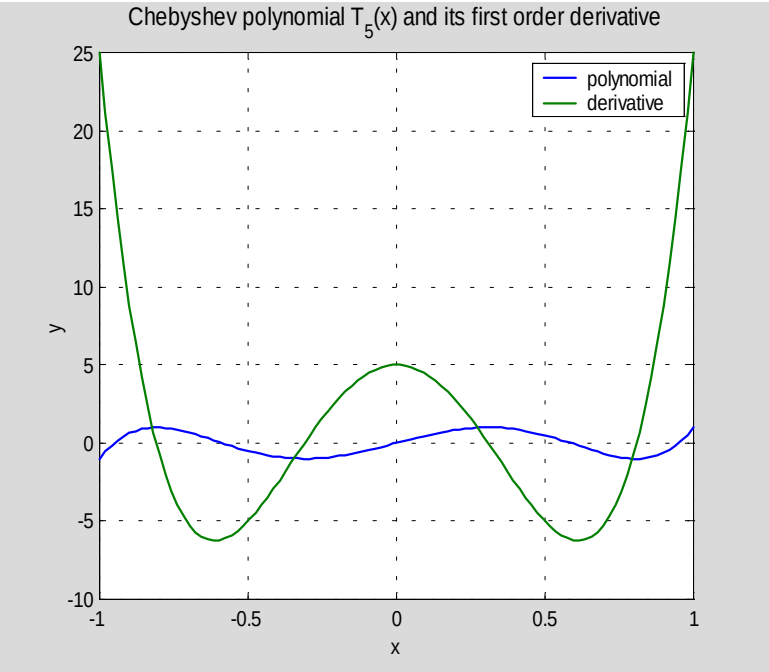
21. Given two one-dimensional arrays **a** and **b**, not necessarily of the same length. Write MATLAB **function c = interleave(a, b)** which takes arrays **a** and **b** and returns an array **c** obtained by interleaving entries in the input arrays. For instance, if **a = [1, 3, 5, 7]** and **b = [-2, –4]**, then **c = [1, –2, 3, –4, 5, 7]**. Your program should work for empty arrays too. You cannot use loops **for** or **while**.

22. Write a script file **Problem22** to plot, in the same window, graphs of two parabolas $y = x^2$ and $x = y^2$, where $-1 \beta x \beta 1$. Label the axes, add a title to your graph and use command **grid**. To improve readability of the graphs plotted add a legend. MATLAB has a command **legend**. To learn more about this command type **help legend** in the **Command Window** and press **Enter** or **Return** key.

23. Write MATLAB **function eqtri(a, b)** that plots the graph of the equilateral triangle with two vertices at **(a,a)** and **(b,a)**. Third vertex lies above the line segment that connects points **(a, a)** and **(b, a)**. Use function **fill** to paint the triangle using a color of your choice.

24. In this exercise you are to plot graphs of the Chebyshev polynomial $T_n(x)$ and its first order derivative over the interval **[-1, 1]**. Write MATLAB **function plotChT(n)** that takes as the input parameter the degree **n** of the Chebyshev polynomial. Use functions **ChebT** and **derp**, included in Tutorial 2, to compute coefficients of $T_n(x)$ and $T'_n(x)$, respectively. Evaluate both, the polynomial and its first order derivative at **x = linspace(-1, 1)** using MATLAB function **polyval**. Add a meaningful title to your graph. In order to improve readability of your graph you may wish to add a descriptive legend. Here is a sample output

```
plotChT(5)
```

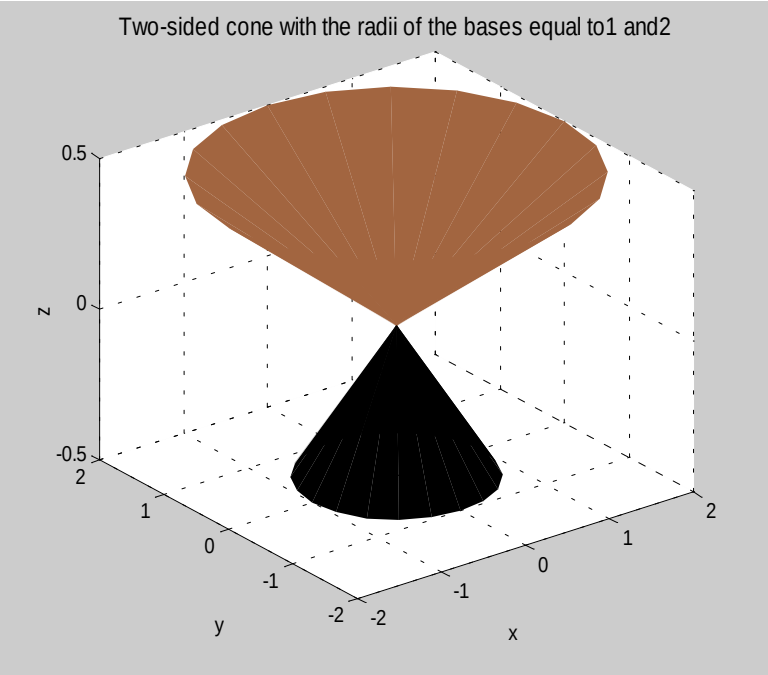


25. Use function **sphere** to plot the graph of a sphere of radius **r** with center at **(a, b, c)**. Use MATLAB function **axis** with an option **'equal'**. Add a title to your graph and save your computer code as the MATLAB function **sph(r, a, b, c)**.

26. Write MATLAB **function ellipsoid(x0, y0, z0, a, b, c)** that takes coordinates **(x0, y0, z0)** of the center of the ellipsoid with semiaxes **(a, b, c)** and plots its graph. Use MATLAB functions **sphere** and **surf**. Add a meaningful title to your graph and use function **axis('equal')**.

27. In this exercise you are to plot a graph of the two-sided cone, with vertex at the origin, and the-axis as the axis of symmetry. Write MATLAB **function cone(a, b)**, where the input parameters **a** and **b** stand for the radius of the lower and upper base, respectively. Use MATLAB functions **cylinder** and **surf** to plot a cone in question. Add a title to your graph and use function **shading** with an argument of your choice. A sample output is shown below

```
cone(1, 2)
```



28. The space curve **r(t) = < cos(t)sin(4t), sin(t)sin(4t), cos(4t) >, 0 β t β 2π**, lies on the surface of the unit sphere **$x^2 + y^2 + z^2 = 1$**. Write MATLAB script file **curvsph** that plots both the curve and the sphere in the same window. Add a meaningful title to your graph. Use MATLAB functions **colormap** and **shading** with arguments of your choice. Add the **view([150 125 50])** command.

29. This problem requires that the professional version 5.x of MATLAB is installed.
In this exercise you are to write the m-file **secondmovie** that crates five frames of the surface **z = sin(kx)cos(ky)**, where **0 β x, y β π** and **k = 1, 2, 3, 4, 5**. Make a movie consisting of the

frames you generated in your file. Use MATLAB functions **colormap** and **shading** with arguments of your choice. Add a title, which might look like this
Graphs of z = sin(kx)*cos(ky), 0 <= x, y <= π, k =1, 2, 3, 4, 5. Greek letters can be printed in the title of a graph using TeX convention, i.e., the following \pi is used to print the Greek letter π. Similarly, the string \alpha will be printed as α.

Tutorial 3

Using MATLAB in Linear Algebra

Math 221

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One of the nice features of MATLAB is its ease of computations with vectors and matrices. In this tutorial the following topics are discussed: vectors and matrices in MATLAB, solving systems of linear equations, the inverse of a matrix, determinants, vectors in n-dimensional Euclidean space, linear transformations, real vector spaces and the matrix eigenvalue problem. Applications of linear algebra to the curve fitting, message coding and computer graphics are also included.

3.1 Special characters and MATLAB functions used in Tutorial 3

For the reader's convenience we include lists of special characters and MATLAB functions that are used in this tutorial.

Special characters	
;	Semicolon operator
'	Conjugated transpose
.'	Transpose
*****	Times
.	Dot operator
^	Power operator
[ ]	Emty vector operator
:	Colon operator
=	Assignment
==	Equality
****	Backslash or left division
/	Right division
i, j	Imaginary unit
~	Logical not
~=	Logical not equal
&	Logical and
|	Logical or
{ }	Cell

Function	Description
acos	Inverse cosine
axis	Control axis scaling and appearance
char	Create character array
chol	Cholesky factorization
cos	Cosine function
cross	Vector cross product
det	Determinant
diag	Diagonal matrices and diagonals of a matrix
double	Convert to double precision
eig	Eigenvalues and eigenvectors
eye	Identity matrix
fill	Filled 2-D polygons
fix	Round towards zero
fliplr	Flip matrix in left/right direction
flops	Floating point operation count
grid	Grid lines
hadamard	Hadamard matrix
hilb	Hilbert matrix
hold	Hold current graph
inv	Matrix inverse
isempty	True for empty matrix
legend	Graph legend
length	Length of vector
linspace	Linearly spaced vector
logical	Convert numerical values to logical
magic	Magic square
max	Largest component
min	Smallest component
norm	Matrix or vector norm
null	Null space
num2cell	Convert numeric array into cell array
num2str	Convert number to string
ones	Ones array
pascal	Pascal matrix
plot	Linear plot
poly	Convert roots to polynomial
polyval	Evaluate polynomial
rand	Uniformly distributed random numbers
randn	Normally distributed random numbers
rank	Matrix rank
reff	Reduced row echelon form
rem	Remainder after division
reshape	Change size
roots	Find polynomial roots
sin	Sine function
size	Size of matrix
sort	Sort in ascending order

subs	Symbolic substitution
sym	Construct symbolic bumbers and variables
tic	Start a stopwatch timer
title	Graph title
toc	Read the stopwatch timer
toeplitz	Tioeplitz matrix
tril	Extract lower triangular part
triu	Extract upper triangular part
vander	Vandermonde matrix
varargin	Variable length input argument list
zeros	Zeros array

3.2 Vectors and matrices in MATLAB

The purpose of this section is to demonstrate how to create and transform vectors and matrices in MATLAB.

This command creates a row vector

```
a = [1 2 3]

a =
     1     2     3
```

Column vectors are inputted in a similar way, however, semicolons must separate the components of a vector

```
b = [1;2;3]

b =
     1
     2
     3
```

The *quote operator* **'** is used to create the *conjugate transpose* of a vector (matrix) while the *dot-quote operator* **.'** creates the *transpose* vector (matrix). To illustrate this let us form a complex vector **a + i*b'** and next apply these operations to the resulting vector to obtain

```
(a+i*b')'

ans =
   1.0000 - 1.0000i
   2.0000 - 2.0000i
   3.0000 - 3.0000i
```

while

```
(a+i*b').'

ans =
   1.0000 + 1.0000i
   2.0000 + 2.0000i
   3.0000 + 3.0000i
```

Command **length** returns the number of components of a vector

```
length(a)

ans =
     3
```

The *dot operator*. plays a specific role in MATLAB. It is used for the componentwise application of the operator that follows the dot operator

```
a.*a

ans =
     1     4     9
```

The same result is obtained by applying the *power operator* ^ to the vector **a**

```
a.^2

ans =
     1     4     9
```

Componentwise division of vectors **a** and **b** can be accomplished by using the *backslash operator* \ together with the dot operator .

```
a.\b'

ans =
     1     1     1
```

For the purpose of the next example let us change vector **a** to the column vector

```
a = a'

a =
     1
     2
     3
```

The *dot product* and the *outer product* of vectors **a** and **b** are calculated as follows

```
dotprod = a'*b
```

```
dotprod =
    14
outprod = a*b'

outprod =
     1     2     3
     2     4     6
     3     6     9
```

The *cross product* of two three-dimensional vectors is calculated using command **cross**. Let the vector **a** be the same as above and let

```
b = [-2 1 2];
```

Note that the semicolon after a command avoids display of the result. The cross product of **a** and **b** is

```
cp = cross(a,b)

cp =
     1    -8     5
```

The cross product vector **cp** is perpendicular to both **a** and **b**

```
[cp*a cp*b']

ans =
     0     0
```

We will now deal with operations on matrices. Addition, subtraction, and scalar multiplication are defined in the same way as for the vectors.

This creates a 3-by-3 matrix

```
A = [1 2 3;4 5 6;7 8 10]

A =
     1     2     3
     4     5     6
     7     8    10
```

Note that the *semicolon operator* **;** separates the rows. To extract a submatrix **B** consisting of rows 1 and 3 and columns 1 and 2 of the matrix **A** do the following

```
B = A([1 3], [1 2])

B =
     1     2
     7     8
```

To interchange rows 1 and 3 of **A** use the vector of row indices together with the colon operator

```
C = A([3 2 1],:)
```

```
C =
     7     8    10
     4     5     6
     1     2     3
```

The *colon operator* **:** stands for *all columns* or *all rows*. For the matrix **A** from the last example the following command

```
A(:)

ans =
     1
     4
     7
     2
     5
     8
     3
     6
    10
```

creates a vector version of the matrix **A**. We will use this operator on several occasions.

To delete a row (column) use the *empty vector operator* **[ ]**

```
A(:, 2) = []

A =
     1     3
     4     6
     7    10
```

Second column of the matrix **A** is now deleted. To insert a row (column) we use the technique for creating matrices and vectors

```
A = [A(:,1) [2 5 8]' A(:,2)]
A =
     1     2     3
     4     5     6
     7     8    10
```

Matrix **A** is now restored to its original form.

Using MATLAB commands one can easily extract those entries of a matrix that satisfy an impsed condition. Suppose that one wants to extract all entries of that are greater than one. First, we define a new matrix **A**

```
A = [-1 2 3;0 5 1]

A =
    -1     2     3
     0     5     1
```

Command **A > 1** creates a matrix of zeros and ones

```
A > 1

ans =
     0     1     1
     0     1     0
```

with ones on these positions where the entries of **A** satisfy the imposed condition and zeros everywhere else. This illustrates *logical addressing* in MATLAB. To extract those entries of the matrix **A** that are greater than one we execute the following command

```
A(A > 1)

ans =
     2
     5
     3
```

The dot operator **.** works for matrices too. Let now

```
A = [1 2 3; 3 2 1] ;
```

The following command

```
A.*A

ans =
     1     4     9
     9     4     1
```

computes the entry-by-entry product of **A** with **A**. However, the following command

```
A*A

¨??? Error using ==> *
Inner matrix dimensions must agree.
```

generates an error message.

Function **diag** will be used on several occasions. This creates a *diagonal matrix* with the diagonal entries stored in the vector **d**

```
d = [1 2 3];

D = diag(d)

D =
     1     0     0
     0     2     0
     0     0     3
```

To extract the main diagonal of the matrix **D** we use function **diag** again to obtain

```
d = diag(D)
```

```
d =
     1
     2
     3
```

What is the result of executing of the following command?

```
diag(diag(d));
```

In some problems that arise in linear algebra one needs to calculate a *linear combination* of several matrices of the same dimension. In order to obtain the desired combination both the coefficients and the matrices must be stored in *cells*. In MATLAB a cell is inputted using curly braces**{ }**. This

```
c = {1,-2,3}
```

```
c =
    [1]    [-2]    [3]
```

is an example of the cell. Function **lincomb** will be used later on in this tutorial.

```
function M = lincomb(v,A)

% Linear combination M of several matrices of the same size.
% Coefficients v = {v1,v2,…,vm} of the linear combination and the
% matrices A = {A1,A2,...,Am} must be inputted as cells.

m = length(v);
[k, l] = size(A{1});
M = zeros(k, l);
for i = 1:m
   M = M + v{i}*A{i};
end
```

3.3 Solving systems of linear equations

MATLAB has several tool needed for computing a solution of the system of linear equations.

Let **A** be an m-by-n matrix and let **b** be an m-dimensional (column) vector. To solve the linear system **Ax = b** one can use the *backslash operator* **** , which is also called the *left division*.

1. Case m = n

In this case MATLAB calculates the exact solution (modulo the roundoff errors) to the system in question.

Let

```
A = [1 2 3;4 5 6;7 8 10]
```

```
A =
     1     2     3
     4     5     6
     7     8    10
```

and let

```
b = ones(3,1);
```

Then

```
x = A\b
```

```
x =
   -1.0000
    1.0000
    0.0000
```

In order to verify correctness of the computed solution let us compute the *residual vector* **r**

```
r = b - A*x
```

```
r =
  1.0e-015 *
    0.1110
    0.6661
    0.2220
```

Entries of the computed residual **r** theoretically should all be equal to zero. This example illustrates an effect of the roundoff erros on the computed solution.

2. Case m > n

If **m > n**, then the system **Ax = b** is *overdetermined* and in most cases system is inconsistent. A solution to the system **Ax = b**, obtained with the aid of the backslash operator ****, is the *least-squares solution*.

Let now

```
A = [2 –1; 1 10; 1 2];
```

and let the vector of the right-hand sides will be the same as the one in the last example. Then

```
x = A\b
x =
    0.5849
    0.0491
```

The residual **r** of the computed solution is equal to

```
r = b - A*x

r =
   -0.1208
   -0.0755
    0.3170
```

Theoretically the residual **r** is orthogonal to the *column space* of **A**. We have

```
r'*A

ans =
  1.0e-014 *
    0.1110
    0.6994
```

3. Case m < n

If the number of unknowns exceeds the number of equations, then the linear system is *underdetermined*. In this case MATLAB computes a *particular solution* provided the system is consistent. Let now

```
A = [1 2 3; 4 5 6];
b = ones(2,1);
```

Then

```
x = A\b

x =
   -0.5000
         0
    0.5000
```

A *general solution* to the given system is obtained by forming a linear combination of **x** with the columns of the *null space* of **A**. The latter is computed using MATLAB function **null**

```
z = null(A)

z =
    0.4082
   -0.8165
    0.4082
```

Suppose that one wants to compute a solution being a linear combination of **x** and **z**, with coefficients **1** and **–1**. Using function **lincomb** we obtain

```
w = lincomb({1,-1},{x,z})
```

```
w =
   -0.9082
    0.8165
    0.0918
```

The residual **r** is calculated in a usual way

```
r = b - A*w
```

```
r =
  1.0e-015 *
   -0.4441
    0.1110
```

3.4 Function rref and its applications

The built-in function **rref** allows a user to solve several problems of linear algebra. In this section we shall employ this function to compute a solution to the system of linear equations and also to find the rank of a matrix. Other applications are discussed in the subsequent sections of this tutorial.

Function **rref** takes a matrix and returns the *reduced row echelon form* of its argument. Syntax of the **rref** command is

B = rref(A) or **[B, pivot] = rref(A)**

The second output parameter **pivot** holds the indices of the pivot columns.

Let

```
A = magic(3); b = ones(3,1);
```

A solution **x** to the linear system **Ax = b** is obtained in two steps. First the augmented matrix of the system is transformed to the *reduced echelon form* and next its last column is extracted

```
[x, pivot] = rref([A b])
```

```
x =
    1.0000         0         0    0.0667
         0    1.0000         0    0.0667
         0         0    1.0000    0.0667
pivot =
     1     2     3
```

```
x = x(:,4)

x =
    0.0667
    0.0667
    0.0667
```

The residual of the computed solution is

```
b - A*x

ans =
     0
     0
     0
```

Information stored in the output parameter **pivot** can be used to compute the rank of the matrix **A**

```
length(pivot)

ans =
     3
```

3.5 The inverse of a matrix

MATLAB function **inv** is used to compute the inverse matrix.

Let the matrix **A** be defined as follows

```
A = [1 2 3;4 5 6;7 8 10]

A =
     1     2     3
     4     5     6
     7     8    10
```

Then

```
B = inv(A)

B =
   -0.6667   -1.3333    1.0000
   -0.6667    3.6667   -2.0000
    1.0000   -2.0000    1.0000
```

In order to verify that B is the inverse matrix of A it sufficies to show that **A*B = I** and **B*A = I**, where **I** is the 3-by-3 identity matrix. We have

```
A*B

ans =
    1.0000         0   -0.0000
         0    1.0000         0
         0         0    1.0000
```

In a similar way one can check that **B*A = I**.

The *Pascal matrix*, named in MATLAB **pascal**, has several interesting properties. Let

```
A = pascal(3)

A =
     1     1     1
     1     2     3
     1     3     6
```

Its inverse **B**

```
B = inv(A)

B =
     3    -3     1
    -3     5    -2
     1    -2     1
```

is the matrix of integers. The *Cholesky triangle* of the matrix **A** is

```
S = chol(A)

S =
     1     1     1
     0     1     2
     0     0     1
```

Note that the upper triangular part of **S** holds the binomial coefficients. One can verify easily that **A = S'*S**.

Function **rref** can also be used to compute the inverse matrix. Let **A** is the same as above. We create first the augmented matrix **B** with **A** being followed by the identity matrix of the same size as **A**. Running function **rref** on the augmented matrix and next extracting columns four through six of the resulting matrix, we obtain

```
B = rref([A eye(size(A))]);

B = B(:, 4:6)

B =
     3    -3     1
    -3     5    -2
     1    -2     1
```

To verify this result, we compute first the product **A** ***B**

```
A*B

ans =
     1     0     0
     0     1     0
     0     0     1
```

and next **B***A**

```
B*A

ans =
     1     0     0
     0     1     0
     0     0     1
```

This shows that **B** is indeed the inverse matrix of **A**.

3.6 Determinants

In some applications of linear algebra knowledge of the determinant of a matrix is required. MATLAB built-in function **det** is designed for computing determinants.

Let

```
A = magic(3);
```

Determinant of **A** is equal to

```
det(A)

ans =
  -360
```

One of the classical methods for computing determinants utilizes a *cofactor expansion*. For more details, see e.g., [2], pp. 103-114.

Function **ckl = cofact(A, k, l)** computes the cofactor **ckl** of the $\mathbf{a_{kl}}$ entry of the matrix **A**

```
function ckl = cofact(A,k,l)

% Cofactor ckl of the a_kl entry of the matrix A.

[m,n] = size(A);
if m ~= n
   error('Matrix must be square')
```

```
end
B = A([1:k-1,k+1:n],[1:l-1,l+1:n]);
ckl = (-1)^(k+l)*det(B);
```

Function **d = mydet(A)** implements the method of cofactor expansion for computing determinants

```
function d = mydet(A)

% Determinant d of the matrix A. Function cofact must be
% in MATLAB's search path.

[m,n] = size(A);
if m ~= n
   error('Matrix must be square')
end
a = A(1,:);
c = [];
for l=1:n
   c1l = cofact(A,1,l);
   c = [c;c1l];
end
d = a*c;
```

Let us note that function **mydet** uses the cofactor expansion along the row **1** of the matrix **A**. Method of cofactors has a high computational complexity. Therefore it is not recommended for computations with large matrices. Its is included here for pedagogical reasons only. To measure a computational complexity of two functions **det** and **mydet** we will use MATLAB built-in function **flops**. It counts the number of *floating-point operations* (additions, subtractions, multiplications and divisions). Let

```
A = rand(25);
```

be a 25-by-25 matrix of uniformly distributed random numbers in the interval **(0, 1)**. Using function **det** we obtain

```
flops(0)
det(A)

ans =
   -0.1867

flops

ans =
       10100
```

For comparison, a number of flops used by function **mydet** is

```
flops(0)
```

```
mydet(A)

ans =
   -0.1867

flops

ans =
      223350
```

The *adjoint matrix* **adj(A)** of the matrix **A** is also of interest in linear algebra (see, e.g., [2], p.108).

```
function B = adj(A)

% Adjoint matrix B of the square matrix A.

[m,n] = size(A);
if m ~= n
   error('Matrix must be square')
end
B = [];
for k = 1:n
   for l=1:n
      B = [B;cofact(A,k,l)];
   end
end
B = reshape(B,n,n);
```

The adjoint matrix and the inverse matrix satisfy the equation

$$\mathbf{A^{-1} = adj(A)/det(A)}$$

(see [2], p.110). Due to the high computational complexity this formula is not recommended for computing the inverse matrix.

3.7 Vectors in $\mathbb{R}^n$

The *2-norm* (*Euclidean norm*) of a vector is computed in MATLAB using function **norm**.

Let

```
a = -2:2

a =
    -2    -1     0     1     2
```

The 2-norm of **a** is equal to

```
twon = norm(a)
```

```
twon =
    3.1623
```

With each nonzero vector one can associate a *unit vector* that is parallel to the given vector. For instance, for the vector **a** in the last example its unit vector is

```
unitv = a /twon

unitv =
   -0.6325   -0.3162         0    0.3162    0.6325
```

The angle θ between two vectors **a** and **b** of the same dimension is computed using the formula

$$\theta = \arccos(\mathbf{a}.\mathbf{b}/||\mathbf{a}||\,||\mathbf{b}||),$$

where **a.b** stands for the dot product of **a** and **b,** ||**a**|| is the norm of the vector **a** and **arccos** is the inverse cosine function.

Let the vector **a** be the same as defined above and let

```
b = (1:5)'

b =
     1
     2
     3
     4
     5
```

Then

```
angle = acos((a*b)/(norm(a)*norm(b)))

angle =
    1.1303
```

Concept of the cross product can be generalized easily to the set consisting of **n -1** vectors in the n-dimensional Euclidean space $\mathbb{R}^n$. Function **crossprod** provides a generalization of the MATLAB function **cross**.

```
function cp = crossprod(A)

% Cross product cp of a set of vectors that are stored in columns of A.

[n, m] = size(A);
if n ~= m+1
   error('Number of columns of A must be one less than the number of
rows')
```

```
end
if rank(A) < min(m,n)
   cp = zeros(n,1);
else
   C = [ones(n,1) A]';
   cp = zeros(n,1);
   for j=1:n
      cp(j) = cofact(C,1,j);
   end
end
```

Let

```
A = [1 -2 3; 4 5 6; 7 8 9; 1 0 1]
```

```
A =
     1    -2     3
     4     5     6
     7     8     9
     1     0     1
```

The cross product of column vectors of **A** is

```
cp = crossprod(A)
```

```
cp =
    -6
    20
   -14
    24
```

Vector **cp** is orthogonal to the column space of the matrix **A**. One can easily verify this by computing the vector-matrix product

```
cp'*A
```

```
ans =
     0     0     0
```

3.8 Linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$

Let $\mathbf{L}: \mathbb{R}^n \to \mathbb{R}^m$ be *a linear transformation*. It is well known that any linear transformation in question is represented by an m-by-n matrix **A**, i.e., $\mathbf{L(x) = Ax}$ holds true for any $\mathbf{x} \in \mathbb{R}^n$. Matrices of some linear transformations including those of *reflections* and *rotations* are discussed in detail in Tutorial 4, Section 4.3.

With each matrix one can associate four subspaces called the *four fundamental subspaces*. The subspaces in question are called the *column space*, the *nullspace*, the *row space*, and the *left*

nullspace. First two subspaces are tied closely to the linear transformations on the finite-dimensional spaces.

Throughout the sequel the symbols **$\mathcal{R}$(L)** and **$\mathcal{N}$(L)** will stand for the *range* and the *kernel* of the linear transformation **L**, respectively. Bases of these subspaces can be computed easily. Recall that **$\mathcal{R}$(L) = column space of A** and **$\mathcal{N}$(L) = nullspace of A**. Thus the problem of computing the bases of the range and the kernel of a linear transformation **L** is equivalent to the problem of finding bases of the column space and the nullspace of a matrix that represents transformation **L**.

Function **fourb** uses two MATLAB functions **rref** and **null** to campute bases of four fundamental subspaces associated with a matrix **A**.

```
function [cs, ns, rs, lns] = fourb(A)

% Bases of four fundamental vector spaces associated
% with the matrix A.
% cs-  basis of the column space of A
% ns-  basis of the nullspace of A
% rs-  basis of the row space of A
% lns- basis of the left nullspace of A

[V, pivot] = rref(A);
r = length(pivot);
cs = A(:,pivot);
ns = null(A,'r');
rs = V(1:r,:)';
lns = null(A','r');
```

In this example we will find bases of four fundamental subspaces associated with the random matrix of zeros and ones.
This set up the *seed* of the **randn** function to **0**

```
randn('seed',0)
```

Recall that this function generates normally distributed random numbers. Next a 3-by-5 random matrix is generated using function **randn**

```
A = randn(3,5)
```

```
A =
    1.1650    0.3516    0.0591    0.8717    1.2460
    0.6268   -0.6965    1.7971   -1.4462   -0.6390
    0.0751    1.6961    0.2641   -0.7012    0.5774
```

The following trick creates a matrix of zeros and ones from the random matrix **A**

```
A = A >= 0
```

```
A =
     1     1     1     1     1
     1     0     1     0     0
     1     1     1     0     1
```

Bases of four fundamental subspaces of matrix **A** are now computed using function **fourb**

```
[cs, ns, rs, lns] = fourb(A)

cs =
     1     1     1
     1     0     0
     1     1     0
ns =
    -1     0
     0    -1
     1     0
     0     0
     0     1
rs =
     1     0     0
     0     1     0
     1     0     0
     0     0     1
     0     1     0
lns =
   Empty matrix: 3-by-0
```

Vectors that form bases of the subspaces under discussion are saved as the column vectors. The *Fundamental Theorem of Linear Algebra* states that the row space of **A** is orthogonal to the nullspace of **A** and also that the column space of **A** is orthogonal to the left nullspace of **A** (see [6]). For the bases of the subspaces in this example we have

```
rs'*ns

ans =
     0     0
     0     0
     0     0


cs'*lns

ans =
   Empty matrix: 3-by-0
```

3.9 Real vector spaces

In this section we discuss some computational tools that can be used in studies of real vector spaces. Focus is on linear span, linear independence, transition matrices and the Gram-Schmidt orthogonalization.

Linear span

Concept of the *linear span* of a set of vectors in a vector space is one of the most important ones in linear algebra. Using MATLAB one can determine easily whether or not given vector is in the span of a set of vectors. Function **span** takes a vector, say **v**, and an unspecified numbers of vectors that form a span. All inputted vectors must be of the same size. On the output a message is displayed to the screen. It says that either **v** is in the span or that **v** is not in the span.

```
function span(v, varargin)

% Test whether or not vector v is in the span of a set
% of vectors.

A = [];
n = length(varargin);
for i=1:n
   u = varargin{i};
   u = u';
   A = [A u(:)];
end
v = v';
v = v(:);
if rank(A) == rank([A v])
   disp(' Given vector is in the span.')
else
   disp(' Given vector is not in the span.')
end
```

The key fact used in this function is a well-known result regarding existence of a solution to the system of linear equations. Recall that the system of linear equations **Ax = b** possesses a solution iff **rank(A) = rank([A b])**. MATLAB function **varargin** used here allows a user to enter a variable number of vectors of the span.

To test function **span** we will run this function on matrices. Let

```
v = ones(3);
```

and choose matrices

```
A = pascal(3);
```

and

```
B = rand(3);
```

to determine whether or not **v** belongs to the span of **A** and **B**. Executing function **span** we obtain

```
span(v, A, B)

 Given vector is not in the span.
```

Linear independence

Suppose that one wants to check whether or not a given set of vectors is *linearly independent*. Utilizing some ideas used in function **span** one can write his/her function that will take an uspecified number of vectors and return a message regarding linear independence/dependence of the given set of vectors. We leave this task to the reader (see Problem 32).

Transition matrix

Problem of finding the *transition matrix* from one vector space to another vector space is interest in linear algebra. We assume that the ordered bases of these spaces are stored in columns of matrices **T** and **S**, respectively. Function **transmat** implements a well-known method for finding the transition matrix.

```
function V = transmat(T, S)

% Transition matrix V from a vector space having the ordered
% basis T to another vector space having the ordered basis S.
% Bases of the vector spaces are stored in columns of the
% matrices T and S.

[m, n] = size(T);
[p, q] = size(S);
   if (m ~= p) | (n ~= q)
         error('Matrices must be of the same dimension')
   end
V = rref([S T]);
V = V(:,(m + 1):(m + n));
```

Let

```
T = [1 2;3 4]; S = [0 1;1 0];
```

be the ordered bases of two vector spaces. The transition matrix **V** form a vector space having the ordered basis **T** to a vector space whose ordered basis is stored in columns of the matrix **S** is

```
V = transmat(T, S)

V =
     3     4
     1     2
```

We will use the transition matrix **V** to compute a coordinate vector in the basis **S**. Let

$$[\mathbf{x}]_T = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

be the coordinate vector in the basis **T**. Then the coordinate vector $[\mathbf{x}]_S$, is

```
 xs = V*[1;1]
```

```
xs =
     7
     3
```

Gram-Schmidt orthogonalization

Problem discussed in this subsection is formulated as follows. Given a basis $\mathbf{A} = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m\}$ of a nonzero subspace $\mathbf{W}$ of $\mathbb{R}^n$. Find an orthonormal basis $\mathbf{V} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$ for $\mathbf{W}$. Assume that the basis $\mathbf{S}$ of the subspace $\mathbf{W}$ is stored in columns of the matrix $\mathbf{A}$, i.e., $\mathbf{A} = [\mathbf{u}_1; \mathbf{u}_2; \ldots; \mathbf{u}_m]$, where each $\mathbf{u}_k$ is a column vector. Function **gs(A)** computes an orthonormal basis $\mathbf{V}$ for $\mathbf{W}$ using a classical method of Gram and Schmidt.

```
function V = gs(A)

% Gram-Schmidt orthogonalization of vectors stored in
% columns of the matrix A. Orthonormalized vectors are
% stored in columns of the matrix V.

[m,n] = size(A);
for k=1:n
   V(:,k) = A(:,k);
   for j=1:k-1
      R(j,k) = V(:,j)'*A(:,k);
      V(:,k) = V(:,k) - R(j,k)*V(:,j);
   end
   R(k,k) = norm(V(:,k));
   V(:,k) = V(:,k)/R(k,k);
end
```

Let $\mathbf{W}$ be a subspace of $\mathbb{R}^3$ and let the columns of the matrix $\mathbf{A}$, where

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix}$$

form a basis for $\mathbf{W}$. An orthonormal basis $\mathbf{V}$ for $\mathbf{W}$ is computed using function **gs**

```
V = gs([1 1;2 1;3 1])

V =
    0.2673    0.8729
    0.5345    0.2182
    0.8018   -0.4364
```

To verify that the columns of $\mathbf{V}$ form an orthonormal set it sufficies to check that $\mathbf{V}^T\mathbf{V} = \mathbf{I}$. We have

```
V'*V

ans =
    1.0000    0.0000
    0.0000    1.0000
```

We will now use matrix **V** to compute the coordinate vector $[\mathbf{v}]_V$, where

```
v = [1 0 1];
```

We have

```
v*V

ans =
    1.0690    0.4364
```

3.10 The matrix eigenvalue problem

MATLAB function **eig** is designed for computing the eigenvalues and the eigenvectors of the matrix **A**. Its syntax is shown below

[V, D] = eig(A)

The eigenvalues of **A** are stored as the diagonal entries of the diagonal matrix **D** and the associated eigenvectors are stored in columns of the matrix **V**.

Let

```
A = pascal(3);
```

Then

```
[V, D] = eig(A)

V =
    0.5438   -0.8165    0.1938
   -0.7812   -0.4082    0.4722
    0.3065    0.4082    0.8599
D =
    0.1270         0         0
         0    1.0000         0
         0         0    7.8730
```

Clearly, matrix **A** is *diagonalizable*. The *eigenvalue-eigenvector decomposition* $\mathbf{A} = \mathbf{VDV}^{-1}$ of **A** is calculated as follows

```
V*D/V
```

```
ans =
    1.0000    1.0000    1.0000
    1.0000    2.0000    3.0000
    1.0000    3.0000    6.0000
```

Note the use of the *right division operator* **/** instead of using the inverse matrix function **inv**. This is motivated by the fact that computation of the inverse matrix takes longer than the execution of the right division operation.

The *characteristic polynomial* of a matrix is obtained by invoking the function **poly**.
Let

```
A = magic(3);
```

be the *magic square*. In this example the vector **chpol** holds the coefficients of the *characteristic polynomial* of the matrix **A**. Recall that a polynomial is represented in MATLAB by its coefficients that are ordered by descending powers

```
chpol = poly(A)
```

```
chpol =
    1.0000  -15.0000  -24.0000  360.0000
```

The eigenvalues of **A** can be computed using function **roots**

```
eigenvals = roots(chpol)
```

```
eigenvals =
   15.0000
    4.8990
   -4.8990
```

This method, however, is not recommended for numerical computing the eigenvalues of a matrix. There are several reasons for which this approach is not used in numerical linear algebra. An interested reader is referred to Tutorial 4.

The *Caley-Hamilton Theorem* states that each matrix satisfies its characteristic equation, i.e., **chpol(A) = 0**, where the last zero stands for the matrix of zeros of the appropriate dimension. We use function **lincomb** to verify this result

```
Q = lincomb(num2cell(chpol), {A^3, A^2, A, eye(size(A))})
```

```
Q =
  1.0e-012 *
   -0.5684   -0.5542   -0.4832
   -0.5258   -0.6253   -0.4547
   -0.5116   -0.4547   -0.6821
```

3.11 Applications of Linear Algebra

List of applications of methods of linear algebra is long and impressive. Areas that relay heavily on the methods of linear algebra include the data fitting, mathematical statistics, linear programming, computer graphics, cryptography, and economics, to mention the most important ones. Applications discussed in this section include the data fitting, coding messages, and computer graphics.

Data fitting

In many problems that arise in science and engineering one wants to fit a discrete set of points in the plane by a smooth curve or function. A typical choice of a smoothing function is a polynomial of a certain degree. If the smoothing criterion requires minimization of the 2-norm, then one has to solve the *least-squares approximation problem*. Function **fit** takes three arguments, the degree of the approximating polynomial, and two vectors holding the x- and the y- coordinates of points to be approximated. On the output, the coefficients of the least-squares polynomials are returned. Also, its graph and the plot of the data points are generated.

```
function c = fit(n, t, y)

% The least-squares approximating polynomial of degree n (n>=0).
% Coordinates of points to be fitted are stored in the column vectors
% t and y. Coefficients of the approximating polynomial are stored in
% the vector c. Graphs of the data points and the least-squares
% approximating polynomial are also generated.

if ( n >= length(t))
   error('Degree is too big')
end
v = fliplr(vander(t));
v = v(:,1:(n+1));
c = v\y;
c = fliplr(c');
x = linspace(min(t),max(t));
w = polyval(c, x);
plot(t,y,'ro',x,w);
title(sprintf('The least-squares polynomial of degree n = %2.0f',n))
legend('data points','fitting polynomial')
```

To demonstrate functionality of this code we generate first a set of points in the plane. Our goal is to fit ten evenly spaced points with the y-ordinates being the values of the function **y = sin(2t)** at these points

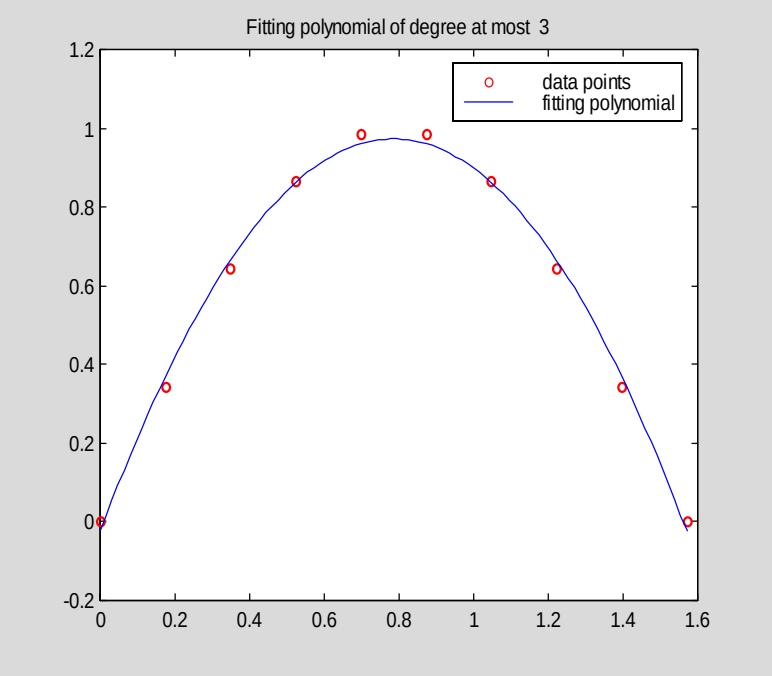
```
t = linspace(0, pi/2, 10); t = t';

y = sin(2*t);
```

We will fit the data by a polynomial of degree at most three

```
c = fit(3, t, y)

c =
   -0.0000   -1.6156    2.5377   -0.0234
```



Coded messages

Some elementary tools of linear algebra can be used to code and decode messages. A typical message can be represented as a string. The following **'coded message'** is an example of the string in MATLAB. Strings in turn can be converted to a sequence of positive integers using MATLAB's function **double**. To code a transformed message multiplication by a nonsingular matrix is used. Process of decoding messages can be viewed as the inverse process to the one described earlier. This time multiplication by the inverse of the coding matrix is applied and next MATLAB's function **char** is applied to the resulting sequence to recover the original message. Functions **code** and **decode** implement these steps.

```
function B = code(s, A)

% String s is coded using a nonsingular matrix A.
% A coded message is stored in the vector B.

p = length(s);
[n,n] = size(A);
b = double(s);
r = rem(p,n);
if r ~= 0
   b = [b zeros(1,n-r)]';
end
b = reshape(b,n,length(b)/n);
B = A*b;
B = B(:)';
```

```
function s = dcode(B, A)

% Coded message, stored in the vector B, is
% decoded with the aid of the nonsingular matrix A
% and is stored in the string s.

[n,n]= size(A);
p = length(B);
B = reshape(B,n,p/n);
d = A\B;
s = char(d(:)');
```

A message to be coded is

```
s = 'Linear algebra is fun';
```

As a coding matrix we use the Pascal matrix

```
A = pascal(4);
```

This codes the message **s**

```
B = code(s,A)
```

```
B =
  Columns 1 through 6
         392        1020        2061        3616         340
809
  Columns 7 through 12
        1601        2813         410        1009        2003
3490
  Columns 13 through 18
         348         824        1647        2922         366
953
  Columns 19 through 24
        1993        3603         110         110         110
110
```

To decode this message we have to work with the same coding matrix **A**

```
dcode(B,A)
```

```
ans =
Linear algebra is fun
```

Computer graphics

Linear algebra provides many tools that are of interest for computer programmers especially for those who deal with the computer graphics. Once the graphical object is created one has to transform it to another object. Certain plane and/or space transformations are linear. Therefore they can be realized as the matrix-vector multiplication. For instance, the reflections, translations,

rotations all belong to this class of transformations. A computer code provided below deals with the plane rotations in the counterclockwise direction. Function **rot2d** takes a planar object represented by two vectors **x** and **y** and returns its image. The angle of rotation is supplied in the degree measure.

```
function [xt, yt] = rot2d(t, x, y)

% Rotation of a two-dimensional object that is represented by two
% vectors x and y. The angle of rotation t is in the degree measure.
% Transformed vectors x and y are saved in xt and yt, respectively.

t1 = t*pi/180;
r = [cos(t1) -sin(t1);sin(t1) cos(t1)];
x = [x x(1)];
y = [y y(1)];
hold on
grid on
axis equal
fill(x, y,'b')
z = r*[x;y];
xt = z(1,:);
yt = z(2,:);
fill(xt, yt,'r');
title(sprintf('Plane rotation through the angle of %3.2f degrees',t))
hold off
```

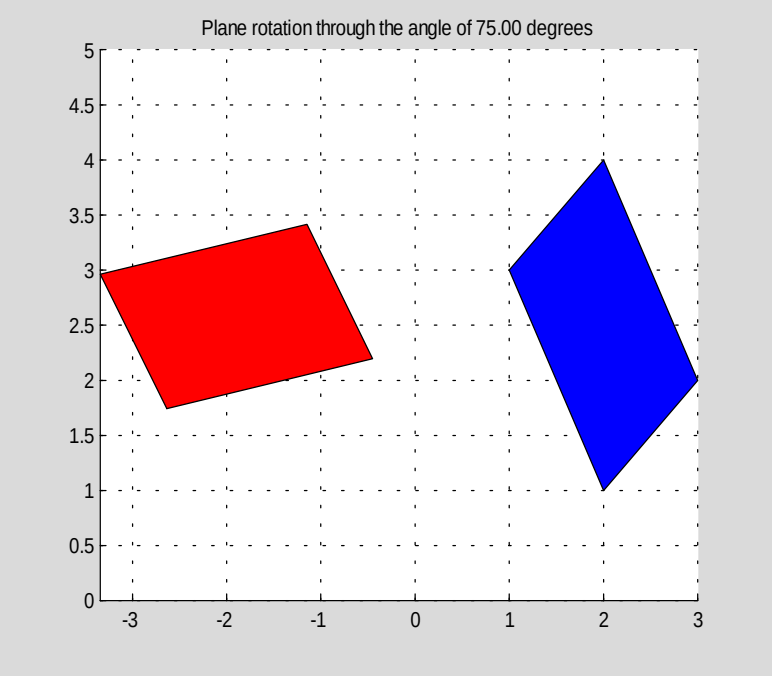
Vectors **x** and **y**

```
x = [1 2 3 2]; y = [3 1 2 4];
```

are the vertices of the parallelogram. We will test function **rot2d** on these vectors using as the angle of rotation **t = 75**.

```
[xt, yt] = rot2d(75, x, y)
```

```
xt =
   -2.6390   -0.4483   -1.1554   -3.3461   -2.6390
yt =
    1.7424    2.1907    3.4154    2.9671    1.7424
```



The right object is the original parallelogram while the left one is its image.

References

[1] B.D. Hahn, Essential MATLAB for Scientists and Engineers, John Wiley & Sons, New York, NY, 1997.

[2] D.R. Hill and D.E. Zitarelli, Linear Algebra Labs with MATLAB, Second edition, Prentice Hall, Upper Saddle River, NJ, 1996.

[3] B. Kolman, Introductory Linear Algebra with Applications, Sixth edition, Prentice Hall, Upper Saddle River, NJ, 1997.

[4] R.E. Larson and B.H. Edwards, Elementary Linear Algebra, Third edition, D.C. Heath and Company, Lexington, MA, 1996.

[5] S.J. Leon, Linear Algebra with Applications, Fifth edition, Prentice Hall, Upper Saddle River, NJ, 1998.

[6] G. Strang, Linear Algebra and Its Applications, Second edition, Academic Press, Orlando, FL, 1980.

Problems

In Problems 1 – 12 you cannot use loops **for** and/or **while**.
Problems 40 - 42 involve symbolic computations. In order to do these problems you have to use the **Symbolic Math Toolbox**.

1. Create a ten-dimensional row vector whose all components are equal **2**. You cannot enter number **2** more than once.

2. Given a row vector **a = [1 2 3 4 5]**. Create a column vector **b** that has the same components as the vector **a** but they must bestored in the reversed order.

3. MATLAB built-in function **sort(a)** sorts components of the vector **a** in the ascending order. Use function **sort** to sort components of the vector **a** in the descending order.

4. To find the largest (smallest) entry of a vector you can use function **max** (**min**). Suppose that these functions are not available. How would you calculate

   (a) the largest entry of a vector ?
   (b) the smallest entry of a vector?

5. Suppose that one wants to create a vector **a** of ones and zeros whose length is equal to **2n** (**n = 1, 2, …**). For instance, when **n = 3**, then **a = [1 0 1 0 1 0]**. Given value of **n** create a vector **a** with the desired property.

6. Let **a** be a vector of integers.

   (a) Create a vector **b** whose all components are the even entries of the vector **a**.
   (b) Repeat part (a) where now **b** consists of all odd entries of the vector **a**.

   Hint: Function **logical** is often used to logical tests. Another useful function you may consider to use is **rem(x, y)** - the remainder after division of **x** by **y**.

7. Given two nonempty row vectors **a** and **b** and two vectors **ind1**and **ind2** with **length(a) = length(ind1)** and **length(b) = length(ind2)**. Components of **ind1** and **ind2** are positive integers. Create a vector **c** whose components are those of vectors **a** and **b**. Their indices are determined by vectors **ind1** and **ind2**, respectively.

8. Using function **rand**, generate a vector of random integers that are uniformly distributed in the interval **(2, 10)**. In order to insure that the resulting vector is not empty begin with a vector that has a sufficient number of components.
   Hint: Function **fix** might be helpful. Type **help fix** in the **Command Window** to learn more about this function.

9. Let **A** be a square matrix. Create a matrix **B** whose entries are the same as those of **A** except the entries along the main diagonal. The main diagonal of the matrix **B** should consist entierly of ones.

10. Let **A** be a square matrix. Create a tridiagonal matrix **T** whose subdiagonal, main diagonal, and the superdiagonal are taken from the matrix **A**.
    Hint: You may wish to use MATLAB functions **triu** and **tril**. These functions take a second optional argument. To learn more about these functions use MATLAB's help.

11. In this exercise you are to test a square matrix **A** for symmetry. Write MATLAB function **s = issymm(A)** that takes a matrix **A** and returns a number **s**. If **A** is symmetric, then **s = 1**, otherwise **s = 0**.

12. Let **A** be an m-by-n and let **B** be an n-by-p matrices. Computing the product **C = AB** requires **mnp** multiplications. If either **A** or **B** has a special structure, then the number of multiplications can be reduced drastically. Let **A** be a full matrix of dimension m-by-n and let **B** be an upper triangular matrix of dimension n-by-n whose all nonzero entries are equal to one. The product **AB** can be calculated without using a single multiplicationa. Write an algorithm for computing the matrix product **C = A*B** that does not require multiplications. Test your code with the following matrices **A = pascal(3)** and **B = triu(ones(3))**.

13. Given square invertible matrices **A** and **B** and the column vector **b**. Assume that the matrices **A** and **B** and the vector **b** have the same number of rows. Suppose that one wants to solve a linear system of equations **ABx = b**. Without computing the matrix-matrix product **A*B**, find a solution **x** to to this system using the backslash operator \.

14. Find all solutions to the linear system **Ax = b**, where the matrix **A** consists of rows one through three of the 5-by-5 magic square

```
A = magic(5);

A = A(1:3,: )

A =
    17    24     1     8    15
    23     5     7    14    16
     4     6    13    20    22
```

    and **b = ones(3; 1)**.

15. Determine whether or not the system of linear equations **Ax = b**, where

```
A = ones(3, 2);  b = [1; 2; 3];
```

    possesses an exact solution **x**.

16. The purpose of this exercise is to demonstrate that for some matrices the computed solution to **Ax = b** can be poor. Define

```
A = hilb(50);   b = rand(50,1);
```

    Find the 2-norm of the residual **r = A*x – b**. How would you explain a fact that the computed norm is essentially bigger than zero?

17. In this exercise you are to compare computational complexity of two methods for finding a solution to the linear system **Ax = b** where **A** is a square matrix. First method utilizes the backslash operator **** while the second method requires a use of the function **rref**. Use MATLAB function **flops** to compare both methods for various linear systems of your choice. Which of these methods require, in general, a smaller number of flops?

18. Repeat an experiment described in Problem 17 using as a measure of efficiency a time needed to compute the solution vector. MATLAB has a pair of functions **tic** and **toc** that can be used in this experiment. This illustrates use of the above mentioned functions **tic; x = A\b; toc**. Using linear systems of your choice compare both methods for speed. Which method is a faster one? Experiment with linear systems having at least ten equations.

19. Let **A** be a real matrix. Use MATLAB function **rref** to extract all

    (a) columns of **A** that are linearly independent
    (b) rows of **A** that are linearly independent

20. In this exercise you are to use MATLAB function **rref** to compute the rank of the following matrices:

    (a) A = magic(3)
    (b) A = magic(4)
    (c) A = magic(5)
    (d) A = magic(6)

    Based on the results of your computations what hypotheses would you formulate about the **rank(magic(n))**, when **n** is odd, when **n** is even?

21. Use MATLAB to demonstrate that **det(A + B) ≠ det(A) + det(B)** for matrices of your choice.

22. Let **A = hilb(5)**. Hilbert matrix is often used to test computer algorithms for reliability. In this exercise you will use MATLAB function **num2str** that converts numbers to strings, to see that contrary to the well-known theorem of Linear Algebra the computed determinant **det(A*A')** is not necessarily the same as **det(A)*det(A').** You can notice a difference in computed quantities by executing the following commands: **num2str(det(A*A'), 16)** and **num2str(det(A)*det(A'), 16)**.

23. The inverse matrix of a symmetric nonsingular matrix is a symmetric matrix. Check this property using function **inv** and a symmetric nonsingular matrix of your choice.

24. The following matrix

```
A = ones(5) + eye(5)

A =
     2     1     1     1     1
     1     2     1     1     1
     1     1     2     1     1
     1     1     1     2     1
     1     1     1     1     2
```

is a special case of the *Pei matrix*. Normalize columns of the matrix **A** so that all columns of the resulting matrix, say **B**, have the Euclidean norm (2-norm) equal to one.

25. Find the angles between consecutive columns of the matrix **B** of Problem 24.

26. Find the cross product vector **cp** that is perpendicular to columns one through four of the Pei matrix of Problem 24.

27. Let **L** be a linear transformation from $\mathbb{R}^5$ to $\mathbb{R}^5$ that is represented by the Pei matrix of Problem 24. Use MATLAB to determine the range and the kernel of this transformation.

28. Let $\mathbb{P}_n$ denote a space of algebraic polynomials of degree at most **n**. Transformation **L** from $\mathbb{P}_n$ to $\mathbb{R}^3$ is defined as follows

$$\mathbf{L(p)} = \begin{bmatrix} \int_0^1 \mathbf{p(t)dt} \\ \mathbf{p(0)} \\ \mathbf{0} \end{bmatrix}$$

    (a) Show that **L** is a linear transformation.
    (b) Find a matrix that represents transformation **L** with respect to the ordered basis $\mathbf{\{t^n, t^{n-1}, \ldots 1\}}$.
    (c) Use MATLAB to compute bases of the range and the kernel of **L**. Perform your experiment for the following values of **n = 2, 3, 4**.

29. Transformation **L** from $\mathbb{P}_n$ to $\mathbb{P}_{n-1}$ is defined as follows **L(p) = p'(t)**. Symbol $\mathbb{P}_n$, is introduced in Problem 28. Answer questions (a) through (c) of Problem 28 for the transformation **L** of this problem.

30. Given vectors **a = [1; 2; 3]** and **b = [-3; 0; 2]**. Determine whether or not vector **c = [4; 1;1]** is in the span of vectors **a** and **b**.

31. Determine whether or not the Toeplitz matrix

```
A = toeplitz( [1 0 1 1 1] )

A =
     1     0     1     1     1
     0     1     0     1     1
     1     0     1     0     1
     1     1     0     1     0
     1     1     1     0     1
```

    is in the span of matrices **B = ones(5)** and **C = magic(5)**.

32. Write MATLAB function **linind(varargin)** that takes an arbitrary number of vectors (matrices) of the same dimension and determines whether or not the inputted vectors (matrices) are linearly independent. You may wish to reuse some lines of code that are contained in the function **span** presented in Section 3.9 of this tutorial.

33. Use function **linind** of Problem 32 to show that the columns of the matrix **A** of Problem 31 are linearly independent.

34. Let $\mathbf{[a]_A}$ **= ones(5,1)** be the coordinate vector with respect to the basis **A** – columns of the matrix **A** of Problem 31. Find the coordinate vector $\mathbf{[a]_P}$, where **P** is the basis of the vector space spanned by the columns of the matrix **pascal(5)**.

35. Let **A** be a real symmetric matrix. Use the well-known fact from linear algebra to determine the interval containing all the eigenvalues of **A**. Write MATLAB function **[a, b] = interval(A)** that takes a symmetric matrix **A** and returns the endpoints **a** and **b** of the interval that contains all the eigenvalues of **A**.

36. Without solving the matrix eigenvalue problem find the sum and the product of all eigenvalues of the following matrices:

    (a) **P = pascal(30)**
    (b) **M= magic(40)**
    (c) **H = hilb(50)**
    (d) **H = hadamard(64)**

37. Find a matrix **B** that is similar to **A = magic(3)**.

38. In this exercise you are to compute a power of the diagonalizable matrix **A**. Let **A = pascal(5)**. Use the eigenvalue decomposition of **A** to calculate the ninth power of **A**. You cannot apply the power operator ^ to the matrix **A**.

39. Let **A** be a square matrix. A matrix **B** is said to be the *square root* of **A** if **B^2 = A**. In MATLAB the square root of a matrix can be found using the power operator ^. In this exercise you are to use the eigenvalue-eigenvector decomposition of a matrix find the square root of **A = [3 3;-2 -2]**.

40. Declare a variable **k** to be a symbolic variable typing **syms k** in the **Command Window**. Find a value of **k** for which the following symbolic matrix **A = sym([1 k^2 2; 1 k -1; 2 –1 0])** is not invertible.

41. Let the matrix **A** be the same as in Problem 40.

    (a) Without solving the matrix eigenvalue problem, determine a value of **k** for which all the eigenvalues of **A** are real.
    (b) Let **v** be a number you found in part (a). Convert the symbolic matrix **A** to a numeric matrix **B** using the substitution command **subs**, i.e., **B = subs(A, k, v)**.
    (c) Determine whether or not the matrix **B** is diagonalizable. If so, find a diagonal matrix **D** that is similar to **B**.

(d) If matrix **B** is diagonalizable use the results of part (c) to compute all the eigenvectors of the matrix **B**. Do not use MATLAB's function **eig**.

42. Given a symbolic matrix **A = sym([1 0 k; 2 2 0; 3 3 3])**.

(a) Find a nonzero value of **k** for which all the eigenvalues of **A** are real.
(b) For what value of **k** two eigenvalues of **A** are complex and the remaining one is real?

Tutorial 4

Numerical Linear Algebra

Math 475/CS 475

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This tutorial is devoted to discussion of the computational methods used in numerical linear algebra. Topics discussed include, matrix multiplication, matrix transformations, numerical methods for solving systems of linear equations, the linear least squares, orthogonality, singular value decomposition, the matrix eigenvalue problem, and computations with sparse matrices.

4.1 MATLAB functions used in Tutorial 4

The following MATLAB functions will be used in this tutorial.

Function	Description
abs	Absolute value
chol	Cholesky factorization
cond	Condition number
det	Determinant
diag	Diagonal matrices and diagonals of a matrix
diff	Difference and approximate derivative
eps	Floating point relative accuracy
eye	Identity matrix
fliplr	Flip matrix in left/right direction
flipud	Flip matrix in up/down direction
flops	Floating point operation count
full	Convert sparse matrix to full matrix
funm	Evaluate general matrix function
hess	Hessenberg form
hilb	Hilbert matrix
imag	Complex imaginary part
inv	Matrix inverse
length	Length of vector
lu	LU factorization
max	Largest component

min	Smallest component
norm	Matrix or vector norm
ones	Ones array
pascal	Pascal matrix
pinv	Pseudoinverse
qr	Orthogonal-triangular decomposition
rand	Uniformly distributed random numbers
randn	Normally distributed random numbers
rank	Matrix rank
real	Complex real part
repmat	Replicate and tile an array
schur	Schur decomposition
sign	Signum function
size	Size of matrix
sqrt	Square root
sum	Sum of elements
svd	Singular value decomposition
tic	Start a stopwatch timer
toc	Read the stopwach timer
trace	Sum of diagonal entries
tril	Extract lower triangular part
triu	Extract upper triangular part
zeros	Zeros array

4.2 The matrix-matrix product

Computation of the product of two or more matrices is one of the basic operations in the numerical linear algebra. Number of flops needed for computing a product of two matrices **A** and **B** can be decreased drastically if a special structure of matrices **A** and **B** is utilized properly. For instance, if both **A** and **B** are upper (lower) triangular, then the product of **A** and **B** is an upper (lower) triangular matrix.

```
function C = prod2t(A, B)

% Product C = A*B of two upper triangular matrices A and B.

[m,n] = size(A);
[u,v] = size(B);
if (m ~= n) | (u ~= v)
   error('Matrices must be square')
end
if n ~= u
   error('Inner dimensions must agree')
end
C = zeros(n);
 for i=1:n
   for j=i:n
       C(i,j) = A(i,i:j)*B(i:j,j);
   end
end
```

In the following example a product of two random triangular matrices is computed using function **prod2t**. Number of flops is also determined.

```
A = triu(randn(4));  B = triu(rand(4));
flops(0)
C = prod2t(A, B)
nflps = flops

C =
   -0.4110   -1.2593   -0.6637   -1.4261
         0    0.9076    0.6371    1.7957
         0         0   -0.1149   -0.0882
         0         0         0    0.0462
nflps =
    36
```

For comparison, using MATLAB's "general purpose" matrix multiplication operator ***,**
the number of flops needed for computing the product of matrices **A** and **B** is

```
flops(0)
A*B;
flops

ans =
   128
```

Product of two Hessenberg matrices **A** and **B**, where **A** is a lower Hessenberg and **B** is an upper Hessenberg can be computed using function **Hessprod**.

```
function C = Hessprod(A, B)

% Product C = A*B, where A and B are the lower and
% upper Hessenberg matrices, respectively.

[m, n] = size(A);
C = zeros(n);
for i=1:n
   for j=1:n
      if( j<n )
         l = min(i,j)+1;
      else
         l = n;
      end
         C(i,j) = A(i,1:l)*B(1:l,j);
      end
   end
```

We will run this function on Hessenberg matrices obtained from the Hilbert matrix **H**

```
H = hilb(10);
```

```
A = tril(H,1);   B = triu(H,-1);

flops(0)

C = Hessprod(A,B);

nflps = flops

nflps =
        1039
```

Using the multiplication operator ***** the number of flops used for the same problem is

```
flops(0)

C = A*B;

nflps = flops

nflps =
        2000
```

For more algorithms for computing the matrix-matrix products see the subsequent sections of this tutorial.

4.3 Matrix transformations

The goal of this section is to discuss important matrix transformations that are used in numerical linear algebra.

On several occasions we will use function **ek(k, n)** – the kth coordinate vector in the n-dimensional Euclidean space

```
function v = ek(k, n)

% The k-th coordinate vector in the n-dimensional Euclidean space.

v = zeros(n,1);
v(k) = 1;
```

4.3.1 Gauss transformation

In many problems that arise in applied mathematics one wants to transform a matrix to an upper triangular one. This goal can be accomplished using the *Gauss transformation* (synonym: *elementary matrix*).

Let $\mathbf{m}, \mathbf{e}_k \in \mathbb{R}^n$. The Gauss transformation $\mathbf{M}_k \equiv \mathbf{M}$ is defined as $\mathbf{M} = \mathbf{I} - \mathbf{m}\mathbf{e}_k^T$. Vector **m** used here is called the *Gauss vector* and **I** is the n-by-n identity matrix. In this section we present two functions for computations with this transformation. For more information about this transformation the reader is referred to [3].

```
function m = Gaussv(x, k)

% Gauss vector m from the vector x and the position
% k (k > 0)of the pivot entry.

if x(k) == 0
   error('Wrong vector')
end;
n = length(x);
x = x(:);
if ( k > 0 & k < n )
   m = [zeros(k,1);x(k+1:n)/x(k)];
else
   error('Index k is out of range')
end
```

Let **M** be the Gauss transformation. The matrix-vector product **M*b** can be computed without forming the matrix **M** explicitly. Function **Gaussprod** implements a well-known formula for the product in question.

```
function c = Gaussprod(m, k, b)

% Product c = M*b, where M is the Gauss transformation
% determined by the Gauss vector m and its column
% index k.

n = length(b);
if ( k < 0 | k > n-1 )
   error('Index k is out of range')
end
b = b(:);
c = [b(1:k);-b(k)*m(k+1:n)+b(k+1:n)];
```

Let

```
x = 1:4; k = 2;
m = Gaussv(x,k)

m =
         0
         0
    1.5000
    2.0000
```

Then

```
c = Gaussprod(m, k, x)

c =
     1
     2
     0
     0
```

4.3.2 Householder transformation

The *Householder transformation* **H**, where $\mathbf{H = I - 2uu^T}$, also called the *Householder reflector*, is a frequently used tool in many problems of numerical linear algebra. Here **u** stands for the real unit vector. In this section we give several functions for computations with this matrix.

```
function u = Housv(x)

% Householder reflection unit vector u from the vector x.

m = max(abs(x));
u = x/m;
if  u(1) == 0
   su = 1;
else
   su = sign(u(1));
end
u(1) = u(1)+su*norm(u);
u = u/norm(u);
u = u(:);
```

Let

```
x = [1 2 3 4]';
```

Then

```
u  = Housv(x)

u =
    0.7690
    0.2374
    0.3561
    0.4749
```

The Householder reflector **H** is computed as follows

```
H = eye(length(x))-2*u*u'

H =
   -0.1826   -0.3651   -0.5477   -0.7303
   -0.3651    0.8873   -0.1691   -0.2255
   -0.5477   -0.1691    0.7463   -0.3382
   -0.7303   -0.2255   -0.3382    0.5490
```

An efficient method of computing the matrix-vector or matrix-matrix products with Householder matrices utilizes a special form of this matrix.

```
function P = Houspre(u, A)

% Product P = H*A, where H is the Householder reflector
% determined by the vector u and A is a matrix.

[n, p] = size(A);
m = length(u);
if m ~= n
   error('Dimensions of u and A must agree')
end
v = u/norm(u);
v = v(:);
P = [];
for j=1:p
   aj = A(:,j);
   P = [P aj-2*v*(v'*aj)];
end
```

Let

```
A =  pascal(4);
```

and let

```
u = Housv(A(:,1))
```

```
u =
    0.8660
    0.2887
    0.2887
    0.2887
```

Then

```
P = Houspre(u, A)
```

```
P =
   -2.0000   -5.0000  -10.0000  -17.5000
   -0.0000   -0.0000   -0.6667   -2.1667
   -0.0000    1.0000    2.3333    3.8333
   -0.0000    2.0000    6.3333   13.8333
```

In some problems that arise in numerical linear algebra one has to compute a product of several Householder transformations. Let the Householder transformations are represented by their normalized reflection vectors stored in columns of the matrix **V**. The product in question, denoted by **Q**, is defined as

Q = V(:, 1)*V(:, 2)* … *V(:, n)

where **n** stands for the number of columns of the matrix **V**.

```
function Q = Housprod(V)

% Product Q of several Householder transformations
% represented by their reflection vectors that are
% saved in columns of the matrix V.

[m, n] = size(V);
Q = eye(m)-2*V(:,n)*V(:,n)';
for i=n-1:-1:1
   Q = Houspre(V(:,i),Q);
end
```

Among numerous applications of the Householder transformation the following one: reduction of a square matrix to the upper Hessenberg form and reduction of an arbitrary matrix to the upper bidiagonal matrix, are of great importance in numerical linear algebra. It is well known that any square matrix **A** can always be transformed to an upper Hessenberg matrix **H** by orthogonal similarity (see [7] for more details). Householder reflectors are used in the course of computations. Function **Hessred** implements this method

```
function [A, V] = Hessred(A)

% Reduction of the square matrix A to the upper
% Hessenberg form using Householder reflectors.
% The reflection vectors are stored in columns of
% the matrix V. Matrix A is overwritten with its
% upper Hessenberg form.

[m,n] =size(A);
if A == triu(A,-1)
   V = eye(m);
   return
end
V = [];
for k=1:m-2
   x = A(k+1:m,k);
   v = Housv(x);
   A(k+1:m,k:m) = A(k+1:m,k:m) - 2*v*(v'*A(k+1:m,k:m));
   A(1:m,k+1:m) = A(1:m,k+1:m) - 2*(A(1:m,k+1:m)*v)*v';
   v = [zeros(k,1);v];
   V = [V v];
end
```

Householder reflectors used in these computations can easily be reconstructed from the columns of the matrix **V**. Let

```
A = [0 2 3;2 1 2;1 1 1];
```

To compute the upper Hessenberg form **H** of the matrix **A** we run function **Hessred** to obtain

```
[H, V] = Hessred(A)
```

```
H =
         0   -3.1305    1.7889
   -2.2361    2.2000   -1.4000
         0   -0.4000   -0.2000
V =
         0
    0.9732
    0.2298
```

The only Householder reflector **P** used in the course of computations is shown below

```
P = eye(3)-2*V*V'

P =
    1.0000         0         0
         0   -0.8944   -0.4472
         0   -0.4472    0.8944
```

To verify correctness of these results it suffices to show that **P*H*P = A**. We have

```
P*H*P

ans =
         0    2.0000    3.0000
    2.0000    1.0000    2.0000
    1.0000    1.0000    1.0000
```

Another application of the Householder transformation is to transform a matrix to an upper bidiagonal form. This reduction is required in some algorithms for computing the *singular value decomposition* (SVD) of a matrix. Function **upbid** works with square matrices only

```
function [A, V, U] = upbid(A)

% Bidiagonalization of the square matrix A using the
% Golub- Kahan method. The reflection vectors of the
% left Householder matrices are saved in columns of
% the matrix V, while the reflection vectors of the
% right Householder reflections are saved in columns
% of the matrix U. Matrix A is overwritten with its
% upper bidiagonal form.

[m, n] = size(A);
if  m ~= n
   error('Matrix must be square')
end
if  tril(triu(A),1) == A
   V = eye(n-1);
   U = eye(n-2);
end
V = [];
U = [];
```

```
for k=1:n-1
   x = A(k:n,k);
   v = Housv(x);
   l = k:n;
   A(l,l) = A(l,l) - 2*v*(v'*A(l,l));
   v = [zeros(k-1,1);v];
   V = [V v];
   if  k < n-1
      x = A(k,k+1:n)';
      u = Housv(x);
      p = 1:n;
      q = k+1:n;
      A(p,q) = A(p,q) - 2*(A(p,q)*u)*u';
      u = [zeros(k,1);u];
      U = [U u];
   end
end
```

Let (see [1], Example 10.9.2, p.579)

```
A = [1 2 3;3 4 5;6 7 8];
```

Then

```
[B, V, U] = upbid(A)

B =
   -6.7823   12.7620   -0.0000
    0.0000    1.9741   -0.4830
    0.0000    0.0000   -0.0000
V =
    0.7574         0
    0.2920   -0.7248
    0.5840    0.6889
U =
         0
   -0.9075
   -0.4201
```

Let the matrices **V** and **U** be the same as in the last example and let

```
Q = Housprod(V); P = Housprod(U);
```

Then

```
Q'*A*P

ans =
   -6.7823   12.7620   -0.0000
    0.0000    1.9741   -0.4830
    0.0000   -0.0000    0.0000
```

which is the same as the bidiagonal form obtained earlier.

4.3.3 Givens transformation

Givens transformation (synonym: Givens rotation) is an orthogonal matrix used for zeroing a selected entry of the matrix. See [1] for details. Functions included here deal with this transformation.

```
function J = GivJ(x1, x2)

% Givens plane rotation J = [c s;-s c]. Entries c and s
% are computed using numbers x1 and x2.

if x1 == 0 & x2 == 0
   J = eye(2);
   return
end
if abs(x2) >= abs(x1)
   t = x1/x2;
   s = 1/sqrt(1+t^2);
   c = s*t;
else
   t = x2/x1;
   c = 1/sqrt(1+t^2);
   s = c*t;
end
J = [c s;-s c];
```

Premultiplication and postmultiplication by a Givens matrix can be performed without computing a Givens matrix explicitly.

```
function A = preGiv(A, J, i, j)

% Premultiplication of A by the Givens rotation
% which is represented by the 2-by-2 planar rotation
% J. Integers i and j describe position of the
% Givens parameters.

A([i j],:) = J*A([i j],:);
```

Let

```
A = [1 2 3;-1 3 4;2 5 6];
```

Our goal is to zeroe the (2,1) entry of the matrix **A**. First the Givens matrix **J** is created using function **GivJ**

```
J = GivJ(A(1,1), A(2,1))

J =
   -0.7071    0.7071
   -0.7071   -0.7071
```

Next, using function **preGiv** we obtain

```
A = preGiv(A,J,1,2)
```

```
A =
   -1.4142    0.7071    0.7071
         0   -3.5355   -4.9497
    2.0000    5.0000    6.0000
```

Postmultiplication by the Givens rotation can be accomplished using function **postGiv**

```
function A = postGiv(A, J, i, j)

% Postmultiplication of A by the Givens rotation
% which is represented by the 2-by-2 planar rotation
% J. Integers i and j describe position of the
% Givens parameters.

A(:,[i j]) = A(:,[i j])*J;
```

An important application of the Givens transformation is to compute the QR factorization of a matrix.

```
function [Q, A] = Givred(A)

% The QR factorization A = Q*R of the rectangular
% matrix A using Givens rotations. Here Q is the
% orthogonal matrix. On the output matrix A is
% overwritten with the matrix R.

[m, n] = size(A);
if m == n
   k = n-1;
elseif m > n
   k = n;
else
   k = m-1;
end
Q = eye(m);
for j=1:k
   for i=j+1:m
      J = GivJ(A(j,j),A(i,j));
      A = preGiv(A,J,j,i);
      Q = preGiv(Q,J,j,i);
   end
end
Q = Q';
```

Let

```
A = pascal(4)
```

```
A =
     1     1     1     1
     1     2     3     4
     1     3     6    10
     1     4    10    20
```

Then

```
[Q, R] = Givred(A)

Q =
    0.5000   -0.6708    0.5000   -0.2236
    0.5000   -0.2236   -0.5000    0.6708
    0.5000    0.2236   -0.5000   -0.6708
    0.5000    0.6708    0.5000    0.2236
R =
    2.0000    5.0000   10.0000   17.5000
    0.0000    2.2361    6.7082   14.0872
    0.0000         0    1.0000    3.5000
   -0.0000         0   -0.0000    0.2236
```

A relative error in the computed QR factorization of the matrix **A** is

```
norm(A-Q*R)/norm(A)

ans =
  1.4738e-016
```

4.4 Solving systems of linear equations

A good numerical algorithm for solving a system of linear equations should, among other things, minimize computational complexity. If the matrix of the system has a special structure, then this fact should be utilized in the design of the algorithm. In this section, we give an overview of MATLAB's functions for computing a solution vector **x** to the linear system **Ax = b**. To this end, we will assume that the matrix **A** is a square matrix.

4.4.1 Triangular systems

If the matrix of the system is either a lower triangular or upper triangular, then one can easily design a computer code for computing the vector **x**. We leave this task to the reader (see Problems 2 and 3).

4.4.2 The LU factorization

MATLAB's function **lu** computes the LU factorization **PA = LU** of the matrix **A** using a partial pivoting strategy. Matrix **L** is unit lower triangular, **U** is upper triangular, and **P** is the permutation matrix. Since **P** is orthogonal, the linear system **Ax = b** is equivalent to **LUx =P^Tb**. This method is recommended for solving linear systems with multiple right hand sides.

Let

```
A = hilb(5);   b = [1 2 3 4 5]';
```

The following commands are used to compute the LU decomposition of **A**, the solution vector **x**, and the upper bound on the relative error in the computed solution

```
[L, U, P] = lu(A);

x = U\(L\(P'*b))

x =
  1.0e+004 *
    0.0125
   -0.2880
    1.4490
   -2.4640
    1.3230

rl_err = cond(A)*norm(b-A*x)/norm(b)

rl_err =
  4.3837e-008
```

Number of *decimal digits of accuracy* in the computed solution **x** is defined as the negative decimal logarithm of the relative error (see e.g., [6]). Vector **x** of the last example has

```
dda = -log10(rl_err)

dda =
    7.3582
```

about seven decimal digits of accuracy.

4.4.3 Cholesky factorization

For linear systems with symmetric positive definite matrices the recommended method is based on the Cholesky factorization $\mathbf{A} = \mathbf{H}^T\mathbf{H}$ of the matrix **A**. Here **H** is the upper triangular matrix with positive diagonal entries. MATLAB's function **chol** calculates the matrix **H** from **A** or generates an error message if **A** is not positive definite. Once the matrix H is computed, the solution **x** to **Ax = b** can be found using the trick used in 4.4.2.

4.5 The least squares solution and orthogonalization

In some problems of applied mathematics one seeks a solution to the *overdetermined linear system* **Ax = b**. In general, such a system is inconsistent. The least squares solution to this system is a vector **x** that minimizes the Euclidean norm of the residual **r = b – Ax**. Vector **x** always exists, however it is not necessarily unique. For more details, see e.g., [7], p. 81. In this section we discuss methods for computing the least squares solution.

4.5.1 Using MATLAB built-in functions

MATLAB's backslash operator \ can be used to find the least squares solution **x = A\b**. For the rank deficient systems a warning message is generated during the course of computations.
A second MATLAB's function that can be used for computing the least squares solution is the **pinv** command. The solution is computed using the following command **x = pinv(A)*b**. Here **pinv** stands for the *pseudoinverse* matrix. This method however, requires more flops than the backslash method does. For more information about the pseudoinverses, see Section 4.7 of this tutorial.

4.5.2 Normal equations

This classical method, which is due to C.F. Gauss, finds a vector **x** that satisfies the *normal equations* $\mathbf{A^TAx = A^Tb}$. The method under discussion is adequate when the condition number of **A** is small.

```
function [x, dist] = lsqne(A, b)

% The least-squares solution x to the overdetermined
% linear system Ax = b. Matrix A must be of full column
% rank.

% Input:
%       A- matrix of the system
%       b- the right-hand sides
% Output:
%       x- the least-squares solution
%       dist- Euclidean norm of the residual b - Ax

[m, n] = size(A);
if (m <= n)
   error('System is not overdetermined')
end
if (rank(A) < n)
   error('Matrix must be of full rank')
end
H = chol(A'*A);
x = H\(H'\(A'*b));
r = b - A*x;
dist = norm(r);
```

Throughout the sequel the following matrix **A** and the vector **b** will be used to test various methods for solving the least squares problem

```
format long

A = [.5 .501;.5 .5011;0 0;0 0]; b = [1;-1;1;-1];
```

Using the method of normal equations we obtain

```
[x,dist] = lsqne(A,b)
```

```
x =
  1.0e+004 *
   2.00420001218025
  -2.00000001215472
dist =
   1.41421356237310
```

One can judge a quality of the computed solution by verifying orthogonality of the residual to the column space of the matrix **A**. We have

```
err = A'*(b - A*x)

err =
  1.0e-011 *
   0.18189894035459
   0.24305336410179
```

4.5.3 Methods based on the QR factorization of a matrix

Most numerical methods for finding the least squares solution to the overdetermined linear systems are based on the orthogonal factorization of the matrix **A = QR**. There are two variants of the QR factorization method: the *full* and the *reduced* factorization. In the full version of the QR factorization the matrix **Q** is an m-by-m orthogonal matrix and **R** is an m-by-n matrix with an n-by-n upper triangular matrix stored in rows 1 through n and having zeros everywhere else. The reduced factorization computes an m-by-n matrix **Q** with orthonormal columns and an n-by-n upper triangular matrix **R**. The QR factorization of **A** can be obtained using one of the following methods:

(i) Householder reflectors
(ii) Givens rotations
(iii) Modified Gram-Schmidt orthogonalization

Householder QR factorization

MATLAB function **qr** computes matrices **Q** and **R** using Householder reflectors. The command **[Q, R] = qr(A)** generates a full form of the QR factorization of **A** while **[Q, R] = qr(A, 0)** computes the reduced form. The least squares solution **x** to **Ax = b** satisfies the system of equations $\mathbf{R^T R x = A^T b}$. This follows easily from the fact that the associated residual **r = b – Ax** is orthogonal to the column space of **A**. Thus no explicit knowledge of the matrix **Q** is required. Function **mylsq** will be used on several occasions to compute a solution to the overdetermined linear system **Ax = b** with known QR factorization of **A**

```
function x = mylsq(A, b, R)

% The least squares solution x to the overdetermined
% linear system Ax = b. Matrix R is such that R = Q'A,
% where Q is a matrix whose columns are orthonormal.

m = length(b);
[n,n] = size(R);
```

```
if m < n
   error('System is not overdetermined')
end
x = R\(R'\(A'*b));
```

Assume that the matrix **A** and the vector **b** are the same as above. Then

```
[Q,R] = qr(A,0);        % Reduced QR factorization of A

x = mylsq(A,b,R)

x =
  1.0e+004 *
   2.00420000000159
  -2.00000000000159
```

Givens QR factorization

Another method of computing the QR factorization of a matrix uses Givens rotations rather than the Householder reflectors. Details of this method are discussed earlier in this tutorial. This method, however, requires more flops than the previous one. We will run function **Givred** on the overdetermined system introduced earlier in this chapter

```
[Q,R]= Givred(A);

x = mylsq(A,b,R)

x =
  1.0e+004 *
   2.00420000000026
  -2.00000000000026
```

Modified Gram-Schmidt orthogonalization

The third method is a variant of the classical Gram-Schmidt orthogonalization. A version used in the function **mgs** is described in detail in [4]. Mathematically the Gram-Schmidt and the modified Gram-Schmidt method are equivalent, however the latter is more stable. This method requires that matrix **A** is of a full column rank

```
function [Q, R] = mgs(A)

% Modified Gram-Schmidt orthogonalization of the
% matrix A = Q*R, where Q is orthogonal and R upper
% is an upper triangular matrix. Matrix A must be
% of a full column rank.

[m, n] = size(A);
for i=1:n
   R(i,i) = norm(A(:,i));
   Q(:,i) = A(:,i)/R(i,i);
   for j=i+1:n
```

```
      R(i,j) = Q(:,i)'*A(:,j);
      A(:,j) = A(:,j) - R(i,j)*Q(:,i);
   end
end
```

Running function **mgs** on our test system we obtain

```
[Q,R] = mgs(A);

x = mylsq(A,b,R)

x =
  1.0e+004 *
   2.00420000000022
  -2.00000000000022
```

This small size overdetermined linear system was tested using three different functions for computing the QR factorization of the matrix **A**. In all cases the least squares solution was found using function **mylsq**. The flop count and the check of orthogonality of **Q** are contained in the following table. As a measure of closeness of the computed **Q** to its exact value is determined by **errorQ = norm(Q'*Q – eye(k))**, where **k = 2** for the reduced form and **k = 4** for the full form of the QR factorization

Function	Flop count	errorQ
qr(, 0)	138	2.6803e-016
Givred	488	2.2204e-016
mgs	98	2.2206e-012

For comparison the number of flops used by the backslash operator was equal to 122 while the **pinv** command found a solution using 236 flops.

Another method for computing the least squares solution finds first the QR factorization of the augmented matrix **[A b]** i.e., **QR = [A b]** using one of the methods discussed above. The least squares solution **x** is then found solving a linear system **Ux = Qb**, where **U** is an n-by- n principal submatrix of **R** and **Qb** is the $n+1^{st}$ column of the matrix **R**. See e.g., [7] for more details. Function **mylsqf** implements this method

```
function x = mylsqf(A, b, f, p)

% The least squares solution x to the overdetermined
% linear system Ax = b using the QR factorization.
% The input parameter f is the string holding the
% name of a function used to obtain the QR factorization.
% Fourth input parameter p is optional and should be
% set up to 0 if the reduced form of the qr function
% is used to obtain the QR factorization.

[m, n] = size(A);
if m <= n
```

```
   error('System is not overdetermined')
end
if nargin == 4
   [Q, R] = qr([A b],0);
else
   [Q, R] = feval(f,[A b]);
end
Qb = R(1:n,n+1);
R = R(1:n,1:n);
x = R\Qb;
```

A choice of a numerical algorithm for solving a particular problem is often a complex task. Factors that should be considered include numerical stability of a method used and accuracy of the computed solution, to mention the most important ones. It is not our intention to discuss these issues in this tutorial. The interested reader is referred to [5] and [3].

4.6 Singular value decomposition of a matrix

Many properties of a matrix can be derived from its *singular value decomposition* (SVD). The SVD is motivated by the following fact: the image of the unit sphere under the m-by-n matrix is a hyperellipse. Function **SVDdemo** takes a 2-by-2 matrix and generates two graphs: the original circle together with two perpendicular vectors and their images under the transformation used. In the example that follows the function under discussion a unit circle **C** with center at the origin is transformed using a 2-by-2 matrix **A**.

```
function SVDdemo(A)

% This illustrates a geometric effect of the application
% of the 2-by-2 matrix A to the unit circle C.

t = linspace(0,2*pi,200);
x = sin(t);
y = cos(t);
[U,S,V] = svd(A);
vx = [0 V(1,1) 0 V(1,2)];
vy = [0 V(2,1) 0 V(2,2)];
axis equal
h1_line = plot(x,y,vx,vy);
set(h1_line(1),'LineWidth',1.25)
set(h1_line(2),'LineWidth',1.25,'Color',[0 0 0])
grid
title('Unit circle C and right singular vectors v_i')
pause(5)
w = [x;y];
z = A*w;
U = U*S;
udx = [0 U(1,1) 0 U(1,2)];
udy = [0 U(2,1) 0 U(2,2)];
figure
h1_line = plot(udx,udy,z(1,:),z(2,:));
set(h1_line(2),'LineWidth',1.25,'Color',[0 0 1])
set(h1_line(1),'LineWidth',1.25,'Color',[0 0 0])
grid
```

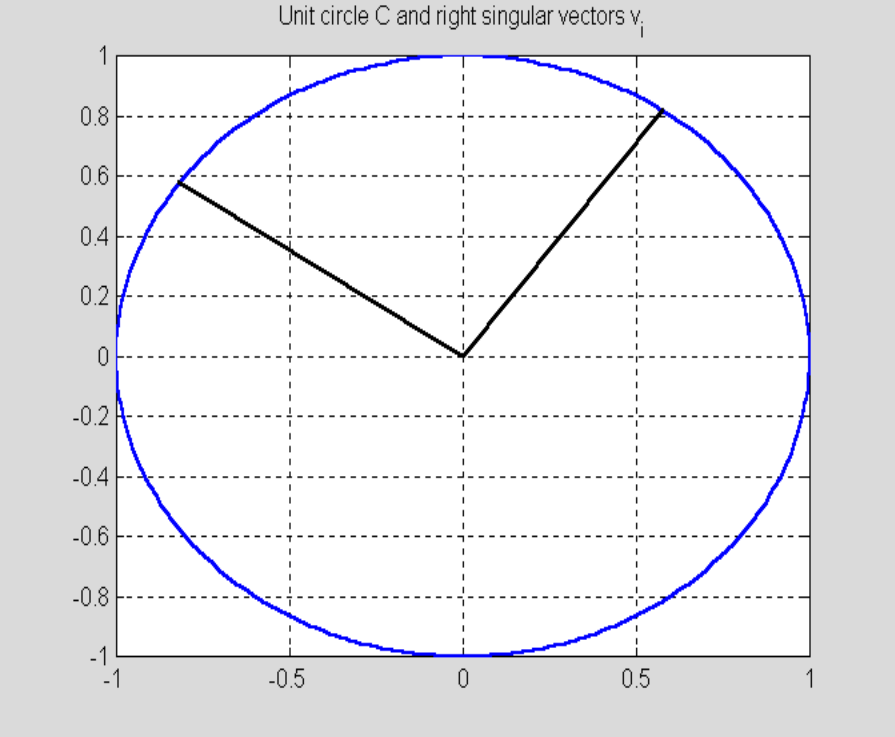
```
title('Image A*C of C and vectors \sigma_iu_i')
```

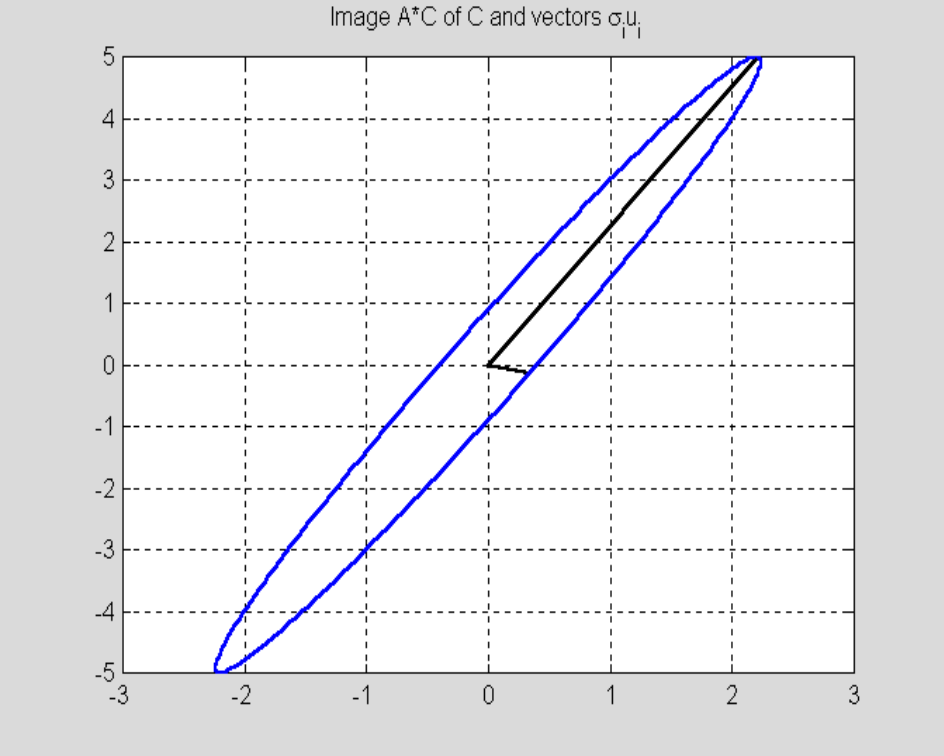
Define a matrix

```
A = [1 2;3 4];
```

Then

```
SVDdemo(A)
```





The *full form* of the singular value decomposition of the m-by-n matrix **A** (real or complex) is the factorization of the form A **= USV***, where **U** and **V** are unitary matrices of dimensions m and n, respectively and **S** is an m-by-n diagonal matrix with nonnegative diagonal entries stored in the nonincreasing order. Columns of matrices **U** and **V** are called the *left singular vectors* and the *right singular vectors*, respectively. The diagonal entries of **S** are the *singular values* of the matrix **A**. MATLAB's function **svd** computes matrices of the SVD of **A** by invoking the command **[U, S, V] = svd(A)**. The *reduced form* of the SVD of the matrix **A** is computed using function **svd** with a second input parameter being set to zero **[U, S, V] = svd(A, 0)**. If **m > n**, then only the first n columns of **U** are computed and **S** is an n-by-n matrix.

Computation of the SVD of a matrix is a nontrivial task. A common method used nowadays is the *two-phase method*. Phase one reduces a given matrix **A** to an upper bidiagonal form using the Golub-Kahan method. Phase two computes the SVD of **A** using a variant of the QR factorization. Function **mysvd** implements a method proposed in Problem 4.15 in [4]. This code works for the 2-by-2 real matrices only.

```
function [U, S, V] = mysvd(A)

% Singular value decomposition A = U*S*V'of a
% 2-by-2 real matrix A. Matrices U and V are orthogonal.
% The left and the right singular vectors of A are stored
% in columns of matrices U and V,respectively. Singular
% values of A are stored, in the nonincreasing order, on
% the main diagonal of the diagonal matrix S.
```

```
if A == zeros(2)
   S = zeros(2);
   U = eye(2);
   V = eye(2);
   return
end
[S, G] = symmat(A);
[S, J] = diagmat(S);
U = G'*J;
V = J;
d = diag(S);
s = sign(d);
for j=1:2
   if s(j) < 0
      U(:,j) = -U(:,j);
   end
end
d = abs(d);
S = diag(d);
if d(1) < d(2)
   d = flipud(d);
   S = diag(d);
   U = fliplr(U);
   V = fliplr(V);
end
```

In order to run this function two other functions **symmat** and **diagmat** must be in MATLAB's search path

```
function [S, G] = symmat(A)

% Symmetric 2-by-2 matrix S from the matrix A. Matrices
% A, S, and G satisfy the equation G*A = S, where G
% is the Givens plane rotation.

if A(1,2) == A(2,1)
   S = A;
   G = eye(2);
   return
end
t = (A(1,1) + A(2,2))/(A(1,2) - A(2,1));
s = 1/sqrt(1 + t^2);
c = -t*s;
G(1,1) = c;
G(2,2) = c;
G(1,2)= s;
G(2,1) = -s;
S = G*A;


function [D, G] = diagmat(A);

% Diagonal matrix D obtained by an application of the
% two-sided Givens rotation to the matrix A. Second output
% parameter G is the Givens rotation used to diagonalize
% matrix A, i.e., G.'*A*G = D.
```

```
if A ~= A'
   error('Matrix must be symmetric')
end
if abs(A(1,2)) < eps & abs(A(2,1)) < eps
   D = A;
   G = eye(2);
   return
end
r = roots([-1 (A(1,1)-A(2,2))/A(1,2) 1]);
[t, k] = min(abs(r));
t = r(k);
c = 1/sqrt(1+t^2);
s = c*t;
G = zeros(size(A));
G(1,1) = c;
G(2,2) = c;
G(1,2) = s;
G(2,1) = -s;
D = G.'*A*G;
```

Let

```
A = [1 2;3 4];
```

Then

```
[U,S,V] = mysvd(A)

U =
    0.4046   -0.9145
    0.9145    0.4046
S =
    5.4650         0
         0    0.3660
V =
    0.5760    0.8174
    0.8174   -0.5760
```

To verify this result we compute

```
AC = U*S*V'

AC =
    1.0000    2.0000
    3.0000    4.0000
```

and the relative error in the computed SVD decomposition

```
 norm(AC-A)/norm(A)

ans =
  1.8594e-016
```

Another algorithm for computing the least squares solution **x** of the overdetermined linear system **Ax = b** utilizes the singular value decomposition of **A**. Function **lsqsvd** should be used for ill-conditioned or rank deficient matrices.

```
function x = lsqsvd(A, b)

% The least squares solution x to the overdetermined
% linear system Ax = b using the reduced singular
% value decomposition of A.

[m, n] = size(A);
if m <= n
   error('System must be overdetermined')
end
[U,S,V] = svd(A,0);
d = diag(S);
r = sum(d > 0);
b1 = U(:,1:r)'*b;
w = d(1:r).\b1;
x = V(:,1:r)*w;
re = b - A*x;        % One step of the iterative
b1 = U(:,1:r)'*re;   % refinement
w = d(1:r).\b1;
e = V(:,1:r)*w;
x = x + e;
```

The linear system with

```
A = ones(6,3); b = ones(6,1);
```

is ill-conditioned and rank deficient. Therefore the least squares solution to this system is not unique

```
x = lsqsvd(A,b)

x =
    0.3333
    0.3333
    0.3333
```

4.7 The pseudoinverse of a matrix

Another application of the SVD is for computing the *pseudoinverse* of a matrix. Singular or rectangular matrices always possess the pseudoinverse matrix. Let the matrix **A** be defined as follows

```
A = [1 2 3;4 5 6]

A =
     1     2     3
     4     5     6
```

Its pseudoinverse is

```
B = pinv(A)

B =
   -0.9444    0.4444
   -0.1111    0.1111
    0.7222   -0.2222
```

The pseudoinverse **B** of the matrix **A** satisfy the *Penrose conditions*

$$\mathbf{ABA = A,\ \ BAB = B,\ \ (AB)^T = AB,\ \ (BA)^T = BA}$$

We will verify the first condition only

```
norm(A*B*A-A)

ans =
  3.6621e-015
```

and leave it to the reader to verify the remaining ones.

4.8 The matrix eigenvalue problem

The matrix eigenvalue problem, briefly discussed in Tutorial 3, is one of the central problems in the numerical linear algebra. It is formulated as follows.

Given a square matrix $\mathbf{A = [a_{ij}]}$, $\mathbf{1 \leqslant i, j \leqslant n}$, find a nonzero vector $\mathbf{x} \in \mathbb{R}^n$ and a number λ that satisfy the equation $\mathbf{Ax} = \lambda\mathbf{x}$. Number λ is called the *eigenvalue* of the matrix **A** and **x** is the associated *right eigenvector* of **A**.

In this section we will show how to localize the eigenvalues of a matrix using celebrated Gershgorin's Theorem. Also, we will present MATLAB's code for computing the dominant eigenvalue and the associated eigenvector of a matrix. The *QR iteration* for computing all eigenvalues of the symmetric matrices is also discussed.

Gershgorin Theorem states that each eigenvalue λ of the matrix **A** satisfies at least one of the following inequalities $|\lambda - \mathbf{a_{kk}}| \leqslant \mathbf{r_k}$, where $\mathbf{r_k}$ is the sum of all off-diagonal entries in row **k** of the matrix **|A|** (see, e.g., [1], pp.400-403 for more details). Function **Gershg** computes the centers and the radii of the Gershgorin circles of the matrix **A** and plots all Gershgorin circles. The eigenvalues of the matrix **A** are also displayed.

```
function [C] = Gershg(A)

% Gershgorin's circles C of the matrix A.

d = diag(A);
cx = real(d);
cy = imag(d);
B = A - diag(d);
```

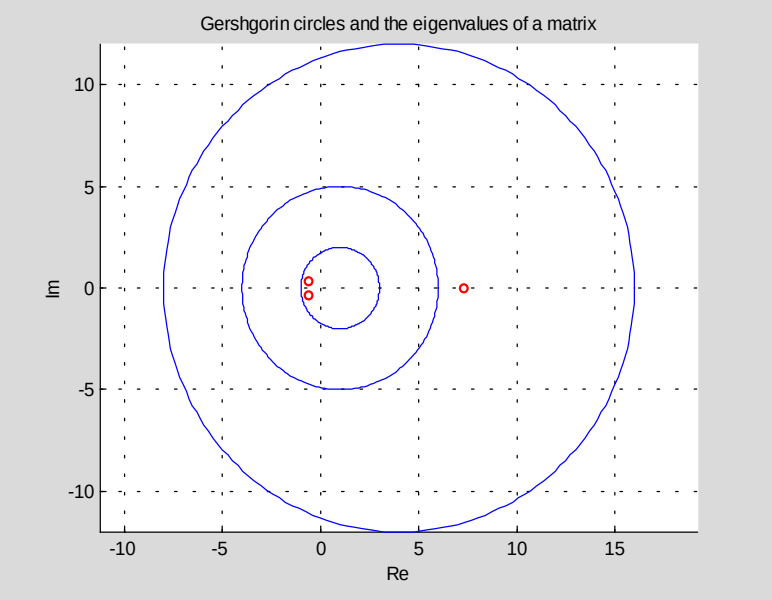
```
[m, n] = size(A);
r = sum(abs(B'));
C = [cx cy r(:)];
t = 0:pi/100:2*pi;
c = cos(t);
s = sin(t);
[v,d] = eig(A);
d = diag(d);
u1 = real(d);
v1 = imag(d);
hold on
grid on
axis equal
xlabel('Re')
ylabel('Im')
h1_line = plot(u1,v1,'or');
set(h1_line,'LineWidth',1.5)
for i=1:n
x = zeros(1,length(t));
y = zeros(1,length(t));
   x = cx(i) + r(i)*c;
   y = cy(i) + r(i)*s;
   h2_line = plot(x,y);
   set(h2_line,'LineWidth',1.2)
end
hold off
title('Gershgorin circles and the eigenvalues of a')
```

To illustrate functionality of this function we define a matrix **A**, where

```
A = [1 2 3;3 4 9;1 1 1];
```

Then

```
C = Gershg(A)

C =
     1     0     5
     4     0    12
     1     0     2
```



Information about each circle (coordinates of the origin and its radius) is contained in successive rows of the matrix **C**.

It is well known that the eigenvalues are sensitive to small changes in the entries of the matrix (see, e.g., [3]). The *condition number of the simple eigenvalue* λ of the matrix **A** is defined as follows

$$\mathbf{Cond}(\lambda) = 1/|\mathbf{y}^T\mathbf{x}|$$

where **y** and **x** are the left and right eigenvectors of **A**, respectively with $\|\mathbf{x}\|_2 = \|\mathbf{y}\|_2 = 1$. Recall that a nonzero vector **y** is said to be a *left eigenvector* of **A** if $\mathbf{y}^T\mathbf{A} = \lambda\mathbf{y}^T$. Clearly $\mathbf{Cond}(\lambda) \geq 1$. Function **eigsen** computes the condition number of all eigenvalues of a matrix.

```
function s = eigsen(A)

% Condition numbers s of all eigenvalues of the diagonalizable
% matrix A.

[n,n] = size(A);
[v1,la1] = eig(A);
[v2,la2] = eig(A');
[d1, j] = sort(diag(la1));
v1 = v1(:,j);
[d2, j] = sort(diag(la2));
v2 = v2(:,j);
s = [];
for i=1:n
   v1(:,i) = v1(:,i)/norm(v1(:,i));
   v2(:,i) = v2(:,i)/norm(v2(:,i));
   s = [s;1/abs(v1(:,i)'*v2(:,i))];
end
```

In this example we will illustrate sensitivity of the eigenvalues of the celebrated Wilkinson's matrix **W**. Its is an upper bidiagonal 20-by-20 matrix with diagonal entries 20, 19, … , 1. The superdiagonal entries are all equal to 20. We create this matrix using some MATLAB functions that are discussed in Section 4.9.

```
W =  spdiags([(20:-1:1)', 20*ones(20,1)],[0 1], 20,20);

format long

s = eigsen(full(W))

s =
  1.0e+012 *
   0.00008448192546
   0.00145503286853
   0.01206523295175
   0.06389158525507
   0.24182386727359
   0.69411856608888
   1.56521713930244
   2.83519277292867
   4.18391920177580
   5.07256664475500
   5.07256664475500
   4.18391920177580
   2.83519277292867
   1.56521713930244
   0.69411856608888
   0.24182386727359
   0.06389158525507
   0.01206523295175
   0.00145503286853
   0.00008448192546
```

Clearly all eigenvalues of the Wilkinson's matrix are sensitive.

Let us perturb the $w_{20,1}$ entry of **W**

```
W(20,1)=1e-5;
```

and next compute the eigenvalues of the perturbed matrix

```
eig(full(W))

ans =
 -1.00978219090288
 -0.39041284468158 + 2.37019976472684i
 -0.39041284468158 - 2.37019976472684i
  1.32106082150033 + 4.60070993953446i
  1.32106082150033 - 4.60070993953446i
  3.88187526711025 + 6.43013503466255i
  3.88187526711025 - 6.43013503466255i
  7.03697639135041 + 7.62654906220393i
```

```
  7.03697639135041 - 7.62654906220393i
 10.49999999999714 + 8.04218886506797i
 10.49999999999714 - 8.04218886506797i
 13.96302360864989 + 7.62654906220876i
 13.96302360864989 - 7.62654906220876i
 17.11812473289285 + 6.43013503466238i
 17.11812473289285 - 6.43013503466238i
 19.67893917849915 + 4.60070993953305i
 19.67893917849915 - 4.60070993953305i
 21.39041284468168 + 2.37019976472726i
 21.39041284468168 - 2.37019976472726i
 22.00978219090265
```

Note a dramatic change in the eigenvalues.

In some problems only selected eigenvalues and associated eigenvectors are needed. Let the eigenvalues $\{\lambda_k\}$ be rearranged so that $|\lambda_1| > |\lambda_2| \geq \ldots \geq |\lambda_n|$. The *dominant eigenvalue* λ_1 and/or the associated eigenvector can be found using one of the following methods: power iteration, inverse iteration, and Rayleigh quotient iteration. Functions **powerit** and **Rqi** implement the first and the third method, respectively.

```
function [la, v] = powerit(A, v)

% Power iteration with the Rayleigh quotient.
% Vector v is the initial estimate of the eigenvector of
% the matrix A. Computed eigenvalue la and the associated
% eigenvector v satisfy the inequality% norm(A*v - la*v,1) < tol,
% where tol = length(v)*norm(A,1)*eps.

if norm(v) ~= 1
   v = v/norm(v);
end
la = v'*A*v;
tol = length(v)*norm(A,1)*eps;
while norm(A*v - la*v,1) >= tol
   w = A*v;
   v = w/norm(w);
   la = v'*A*v;
end


function [la, v] = Rqi(A, v, iter)

% The Rayleigh quotient iteration.
% Vector v is an approximation of the eigenvector associated with the
% dominant eigenvalue la of the matrix A. Iterative process is
% terminated either if norm(A*v - la*v,1) < norm(A,1)*length(v)*eps
% or if the number of performed iterations reaches the allowed number
% of iterations iter.

if norm(v) > 1
   v = v/norm(v);
end
la = v'*A*v;
tol = norm(A,1)*length(v)*eps;
for k=1:iter
```

```
   if norm(A*v - la*v,1) < tol
      return
   else
      w = (A - la*eye(size(A)))\v;
      v = w/norm(w);
      la = v'*A*v;
   end
end
```

Let ([7], p.208, Example 27.1)

```
A = [2 1 1;1 3 1;1 1 4];  v = ones(3,1);
```

Then

```
format long

 flops(0)

[la, v] = powerit(A, v)

la =
   5.21431974337753
v =
   0.39711254978701
   0.52065736843959
   0.75578934068378

flops

ans =
        3731
```

Using function **Rqi**, for computing the dominant eigenpair of the matrix **A**, we obtain

```
flops(0)

[la, v] = Rqi(A,ones(3,1),5)

la =
   5.21431974337754
v =
   0.39711254978701
   0.52065736843959
   0.75578934068378

flops

ans =
   512
```

Once the dominant eigenvalue (eigenpair) is computed one can find another eigenvalue or eigenpair by applying a process called *deflation*. For details the reader is referred to [4], pp. 127-128.

```
function [l2, v2, B] = defl(A, v1)

% Deflated matrix B from the matrix A with a known eigenvector v1 of A.
% The eigenpair (l2, v2) of the matrix A is computed.
% Functions Housv, Houspre, Housmvp and Rqi are used
% in the body of the function defl.

n = length(v1);
v1 = Housv(v1);
C = Houspre(v1,A);
B = [];
for i=1:n
   B = [B Housmvp(v1,C(i,:))];
end
l1 = B(1,1);
b = B(1,2:n);
B = B(2:n,2:n);
[l2, y] = Rqi(B, ones(n-1,1),10);
if l1 ~= l2
   a = b*y/(l2-l1);
   v2 = Housmvp(v1,[a;y]);
else
   v2 = v1;
end
```

Let **A** be an 5-by-5 *Pei matrix*, i.e.,

```
A = ones(5)+diag(ones(5,1))
```

```
A =
     2     1     1     1     1
     1     2     1     1     1
     1     1     2     1     1
     1     1     1     2     1
     1     1     1     1     2
```

Its dominant eigenvalue is $\lambda_1 = 6$ and all the remaining eigenvalues are equal to one. To compute the dominant eigenpair of **A** we use function **Rqi**

```
[l1,v1] = Rqi(A,rand(5,1),10)
```

```
l1 =
   6.00000000000000
v1 =
   0.44721359549996
   0.44721359549996
   0.44721359549996
   0.44721359549996
   0.44721359549996
```

and next apply function **defl** to compute another eigenpair of **A**

```
[l2,v2] = defl(A,v1)

l2 =
   1.00000000000000
v2 =
  -0.89442719099992
   0.22360679774998
   0.22360679774998
   0.22360679774998
   0.22360679774998
```

To check these results we compute the norms of the "residuals"

```
[norm(A*v1-l1*v1);norm(A*v2-l2*v2)]

ans =
  1.0e-014 *
   0.07691850745534
   0.14571016336181
```

To this end we will deal with the *symmetric eigenvalue problem*. It is well known that the eigenvalues of a symmetric matrix are all real. One of the most efficient algorithms is the QR iteration with or without shifts. The algorithm included here is the two-phase algorithm. Phase one reduces a symmetric matrix **A** to the symmetric tridiagonal matrix **T** using MATLAB's function **hess**. Since **T** is orthogonally similar to **A**, **sp(A) = sp(T)**. Here **sp** stands for the *spectrum* of a matrix. During the phase two the off diagonal entries of **T** are annihilated. This is an iterative process, which theoretically is an infinite one. In practice, however, the off diagonal entries approach zero fast. For details the reader is referred to [2] and [7].

Function **qrsft** computes all eigenvalues of the symmetric matrix **A**. Phase two uses *Wilkinson's shift*. The latter is computed using function **wsft**.

```
function [la, v] = qrsft(A)

% All eigenvalues la of the symmetric matrix A.
% Method used: the QR algorithm with Wilkinson's shift.
% Function wsft is used in the body of the function qrsft.

[n, n] = size(A);
A = hess(A);
la = [];
i = 0;
while i < n
   [j, j] = size(A);
   if j == 1
      la = [la;A(1,1)];
      return
   end
   mu = wsft(A);
   [Q, R] = qr(A - mu*eye(j));
   A = R*Q + mu*eye(j);
```

```
   if abs(A(j,j-1))< 10*(abs(A(j-1,j-1))+abs(A(j,j)))*eps
      la = [la;A(j,j)];
      A = A(1:j-1,1:j-1);
      i = i + 1;
   end
end
```

```
function mu = wsft(A)

% Wilkinson's shift mu of the symmetric matrix A.

[n, n] = size(A);
if A == diag(diag(A))
   mu = A(n,n);
   return
end
mu = A(n,n);
if n > 1
   d = (A(n-1,n-1)-mu)/2;
   if d ~= 0
      sn = sign(d);
   else
      sn = 1;
   end
  bn = A(n,n-1);
  mu = mu - sn*bn^2/(abs(d) + sqrt(d^2+bn^2));
end
```

We will test function **qrsft** on the matrix **A** used earlier in this section

```
A = [2 1 1;1 3 1;1 1 4];

la = qrsft(A)

la =
   5.21431974337753
   2.46081112718911
   1.32486912943335
```

Function **eigv** computes both the eigenvalues and the eigenvectors of a symmetric matrix provided the eigenvalues are distinct. A method for computing the eigenvectors is discussed in [1], Algorithm 8.10.2, pp. 452-454

```
function [la, V] = eigv(A)

% Eigenvalues la and eigenvectors V of the symmetric
% matrix A with distinct eigenvalues.

V = [];
[n, n] = size(A);
[Q,T] = schur(A);
la = diag(T);
```

```
if nargout == 2
   d = diff(sort(la));
   for k=1:n-1
      if d(k) < 10*eps
         d(k) = 0;
      end
   end
   if ~all(d)
      disp('Eigenvalues must be distinct')
   else
      for k=1:n
         U = T - la(k)*eye(n);
         t = U(1:k,1:k);
         y1 = [];
         if k>1
            t11 = t(1:k-1,1:k-1);
            s = t(1:k-1,k);
            y1 = -t11\s;
         end
         y = [y1;1];
         z = zeros(n-k,1);
         y = [y;z];
         v = Q*y;
         V = [V v/norm(v)];
      end
   end
end
```

We will use this function to compute the eigenvalues and the eigenvectors of the matrix **A** of the last example

```
[la, V] = eigv(A)

la =
   1.32486912943335
   2.46081112718911
   5.21431974337753
V =
   0.88765033882045  -0.23319197840751   0.39711254978701
  -0.42713228706575  -0.73923873953922   0.52065736843959
  -0.17214785894088   0.63178128111780   0.75578934068378
```

To check these results let us compute the residuals **Av - λv**

```
A*V-V*diag(la)

ans =
  1.0e-014 *
                  0  -0.09992007221626   0.13322676295502
  -0.02220446049250  -0.42188474935756   0.44408920985006
                  0   0.11102230246252  -0.13322676295502
```

4.9 Sparse matrices in MATLAB

MATLAB has several built-in functions for computations with sparse matrices. A partial list of these functions is included here.

Function	Description
condest	Condition estimate for sparse matrix
eigs	Few eigenvalues
find	Find indices of nonzero entries
full	Convert sparse matrix to full matrix
issparse	True for sparse matrix
nnz	Number of nonzero entries
nonzeros	Nonzero matrix entries
sparse	Create sparse matrix
spdiags	Sparse matrix formed from diagonals
speye	Sparse identity matrix
spfun	Apply function to nonzero entries
sprand	Sparse random matrix
sprandsym	Sparse random symmetric matrix
spy	Visualize sparsity pattern
svds	Few singular values

Function **spy** works for matrices in full form as well.

Computations with sparse matrices

The following MATLAB functions work with sparse matrices: **chol**, **det**, **inv**, **jordan**, **lu**, **qr**, **size**, \.

Command **sparse** is used to create a *sparse* form of a matrix.

Let

```
A = [0 0 1 1; 0 1 0 0; 0 0 0 1];
```

Then

```
B = sparse(A)
```

```
B =
   (2,2)        1
   (1,3)        1
   (1,4)        1
   (3,4)        1
```

Command **full** converts a sparse form of a matrix to the full form

```
C = full(B)

C =
     0     0     1     1
     0     1     0     0
     0     0     0     1
```

Command **sparse** has the following syntax

```
sparse(k,l,s,m,n)
```

where **k** and **l** are arrays of row and column indices, respectively, **s** ia an array of nonzero numbers whose indices are specified in **k** and **l**, and **m** and **n** are the row and column dimensions, respectively.

Let

```
S = sparse([1 3 5 2], [2 1 3 4], [1 2 3 4], 5, 5)

S =
   (3,1)        2
   (1,2)        1
   (5,3)        3
   (2,4)        4

F = full(S)

F =
     0     1     0     0     0
     0     0     0     4     0
     2     0     0     0     0
     0     0     0     0     0
     0     0     3     0     0
```

To create a sparse matrix with several diagonals parallel to the main diagonal one can use the command **spdiags**. Its syntax is shown below

```
spdiags(B, d, m, n)
```

The resulting matrix is an m-by-n sparse matrix. Its diagonals are the columns of the matrix **B**. Location of the diagonals are described in the vector **d**.

Function **mytrid** creates a sparse form of the tridiagonal matrix with constant entries along the diagonals.

```
function T = mytrid(a,b,c,n)

% The n-by-n tridiagonal matrix T with constant entries
% along diagonals. All entries on the subdiagonal, main
% diagonal,and the superdiagonal are equal a, b, and c,
% respectively.
```

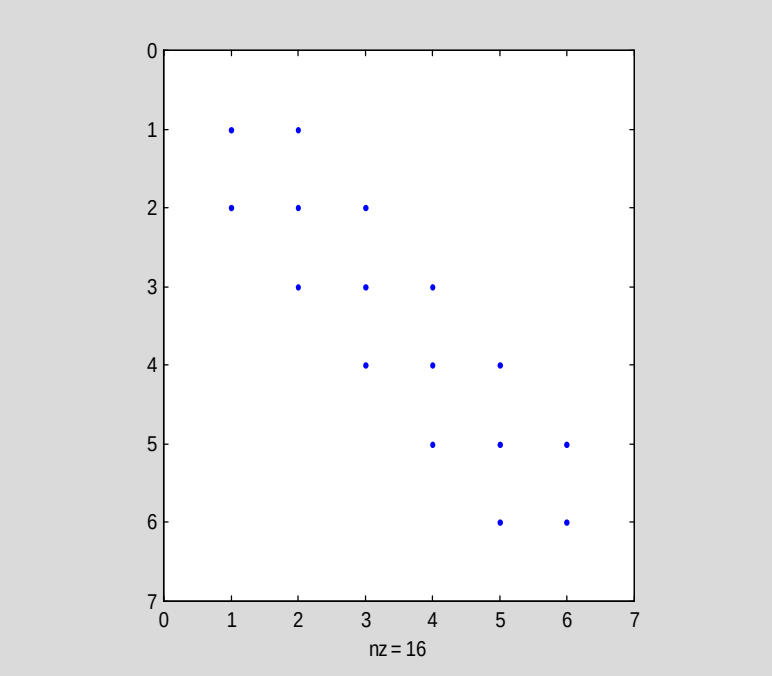
```
e = ones(n,1);
T = spdiags([a*e b*e c*e],-1:1,n,n);
```

To create a symmetric 6-by-6-tridiagonal matrix with all diagonal entries are equal 4 and all subdiagonal and superdiagonal entries are equal to one execute the following command

```
T = mytrid(1,4,1,6);
```

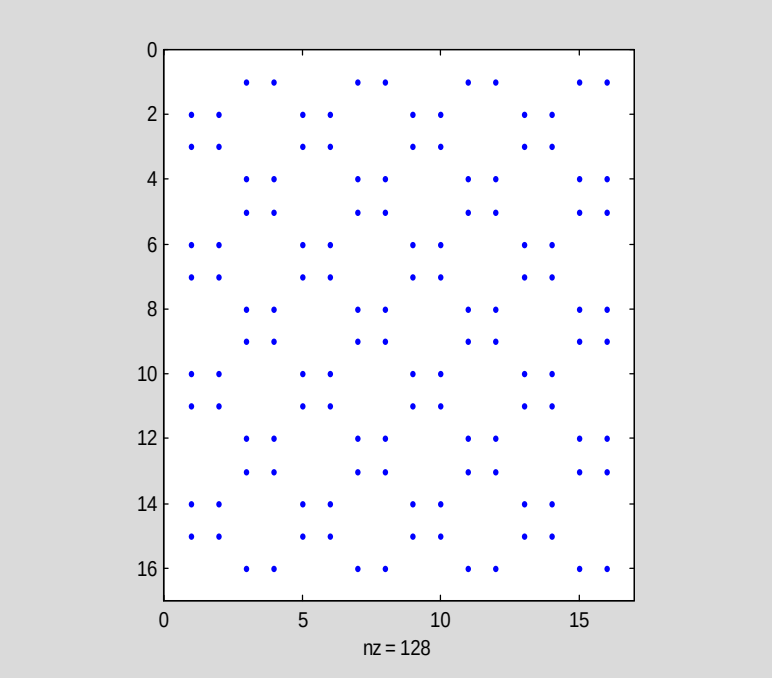
Function **spy** creates a graph of the matrix. The nonzero entries are displayed as the dots.

```
spy( T )
```



The following is the example of a sparse matrix created with the aid of the nonsparse matrix **magic**

```
spy(rem(magic(16),2))
```



Using a sparse form rather than the full form of a matrix one can reduce a number of flops used. Let

```
A = sprand(50,50,.25);
```

The above command generates a 50-by-50 random sparse matrix **A** with a density of about 25%. We will use this matrix to solve a linear system **Ax = b** with

```
b = ones(50,1);
```

Number of flops used is determined in usual way

```
flops(0)

A\b;

flops
```

```
ans =
       54757
```

Using the full form of **A** the number of flops needed to solve the same linear system is

```
flops(0)

full(A)\b;

flops
```

```
ans =
       72014
```

References

[1] B.N. Datta, Numerical Linear Algebra and Applications, Brooks/Cole Publishing Company, Pacific Grove, CA, 1995.

[2] J.W. Demmel, Applied Numerical Linear Algebra, SIAM, Philadelphia, PA, 1997.

[3] G.H. Golub and Ch.F. Van Loan, Matrix Computations, Second edition, Johns Hopkins University Press, Baltimore, MD, 1989.

[4] M.T. Heath, Scientific Computing: An Introductory Survey, McGraw-Hill, Boston, MA, 1997.

[5] N.J. Higham, Accuracy and Stability of Numerical Algorithms, SIAM, Philadelphia, PA, 1996.

[6] R.D. Skeel and J.B. Keiper, Elementary Numerical Computing with *Mathematica*, McGraw-Hill, New York, NY, 1993.

[7] L.N. Trefethen and D. Bau III, Numerical Linear Algebra, SIAM, Philadelphia, PA, 1997.

Problems

1. Let **A** by an n-by-n matrix and let **v** be an n-dimensional vector. Which of the following methods is faster?

   (i) **(v*v')*A**
   (ii) **v*(v'*A)**

2. Suppose that $\mathbf{L} \in \mathbb{R}^{n \times n}$ is lower triangular and $\mathbf{b} \in \mathbb{R}^{n}$. Write MATLAB function **x = ltri(L, b)** that computes a solution **x** to the linear system **Lx = b**.

3. Repeat Problem 2 with **L** being replaced by the upper triangular matrix **U**. Name your function **utri(U, b)**.

4. Let $\mathbf{A} \in \mathbb{R}^{n \times n}$ be a triangular matrix. Write a function **dettri(A)** that computes the determinant of the matrix **A**.

5. Write MATLAB function **MA = Gausspre(A, m, k)** that overwrites matrix $\mathbf{A} \in \mathbb{R}^{n \times p}$ with the product **MA**, where $\mathbf{M} \in \mathbb{R}^{n \times n}$ is the Gauss transformation which is determined by the Gauss vector **m** and its column index **k**.
   Hint: You may wish to use the following formula **MA = A – m(e_k^TA)**.

6. A system of linear equations **Ax = b**, where **A** is a square matrix, can be solved applying successively Gauss transformations to the augmented matrix **[A, b]**. A solution **x** then can be found using back substitution, i.e., solving a linear system with an upper triangular matrix. Using functions **Gausspre** of Problem 5, **Gaussv** described in Section 4.3, and **utri** of Problem 3, write a function **x = sol(A, b)** which computes a solution **x** to the linear system **Ax = b**.

7. Add a few lines of code to the function **sol** of Problem 6 to compute the determinant of the matrix **A**. The header of your function might look like this function **[x, d] = sol(A, b)**. The second output parameter **d** stands for the determinant of **A**.

8. The purpose of this problem is to test function **sol** of Problem 6.

   (i) Construct at least one matrix **A** for which function **sol** fails to compute a solution. Explain why it happened.
   (ii) Construct at least one matrix **A** for which the computed solution **x** is poor. For comparison of a solution you found using function **sol** with an acceptable solution you may wish to use MATLAB's backslash operator ****. Compute the relative error in **x**. Compare numbers of flops used by function **sol** and MATLAB's command ****. Which of these methods is faster in general?

9. Given a square matrix **A**. Write MATLAB function **[L, U] = mylu(A)** that computes the LU decomposition of **A** using partial pivoting.

10. Change your working format to **format long e** and run function **mylu** of Problem 11on the following matrices

    (i) A = [eps 1; 1 1]
    (ii) A = [1 1; eps 1]
    (iii) A = hilb(10)
    (iv) A = rand(10)

    In each case compute the error $\|\mathbf{A} - \mathbf{LU}\|_F$.

11. Let **A** be a tridiagonal matrix that is either diagonally dominant or positive definite. Write MATLAB's function **[L, U] = trilu(a, b, c)** that computes the LU factorization of **A**. Here **a**, **b**, and **c** stand for the subdiagonal, main diagonal, and superdiagonal of **A**, respectively.

12. The following function computes the Cholesky factor **L** of the symmetric positive definite matrix **A**. Matrix **L** is lower triangular and satisfies the equation $\mathbf{A} = \mathbf{L}\mathbf{L}^T$.

```
function L = mychol(A)

% Cholesky factor L of the matrix A; A = L*L'.

[n, n] = size(A);
for j=1:n
   for k=1:j-1
      A(j:n,j) = A(j:n,j) - A(j:n,k)*A(j,k);
   end
   A(j,j) = sqrt(A(j,j));
   A(j+1:n,j) = A(j+1:n,j)/A(j,j);
end
L = tril(A);
```

    Add a few lines of code that generates the error messages when **A** is neither

    - symmetric nor
    - positive definite

    Test the modified function **mychol** on the following matrices

    (i) **A = [1 2 3; 2 1 4; 3 4 1]**
    (ii) **A = rand(5)**

13. Prove that any 2-by-2 Householder reflector is of the form **H = [cos θ sin θ; sin θ -cos θ]**. What is the Householder reflection vector **u** of **H**?

14. Find the eigenvalues and the associated eigenvectors of the matrix **H** of Problem 13.

15. Write MATLAB function **[Q, R] = myqr(A)** that computes a full QR factorization **A = QR** of $\mathbf{A} \in \mathbb{R}^{m \times n}$ with $\mathbf{m} \geq \mathbf{n}$ using Householder reflectors. The output matrix **Q** is an m-by-m orthogonal matrix and **R** is an m-by-n upper triangular with zero entries in rows n+1 through m.

Hint: You may wish to use function **Housprod** in the body of the function **myqr**.

16. Let **A** be an n-by-3 random matrix generated by the MATLAB function **rand**. In this exercise you are to plot the error $\|\mathbf{A} - \mathbf{QR}\|_F$ versus n for n = 3, 5, … , 25. To compute the QR factorization of **A** use the function **myqr** of Problem 15. Plot the graph of the computed errors using MATLAB's function **semilogy** instead of the function **plot**. Repeat this experiment several times. Does the error increase as n does?

17. Write MATLAB function **V = Vandm(t, n)** that generates Vandermonde's matrix **V** used in the polynomial least-squares fit. The degree of the approximating polynomial is **n** while the x-coordinates of the points to be fitted are stored in the vector **t**.

18. In this exercise you are to compute coefficients of the least squares polynomials using four methods, namely the normal equations, the QR factorization, modified Gram-Schmidt orthogonalization and the singular value decomposition.
Write MATLAB function **C = lspol(t, y, n)** that computes coefficients of the approximating polynomials. They should be saved in columns of the matrix $\mathbf{C} \in \mathbb{R}^{(n+1) \times 4}$. Here **n** stands for the degree of the polynomial, **t** and **y** are the vectors holding the x- and the y-coordinates of the points to be approximated, respectively.
Test your function using **t = linspace(1.4, 1.8)**, **y = sin(tan(t)) – tan(sin(t))**, **n = 2, 4, 8**.
Use **format long** to display the output to the screen.
Hint: To create the Vandermonde matrix needed in the body of the function **lspol** you may wish to use function **Vandm** of Problem 17.

19. Modify function **lspol** of Problem 18 adding a second output parameter **err** so that the header of the modified function should look like this function **[C, err] = lspol(t, y, n)**. Parameter **err** is the least squares error in the computed solution **c** to the overdetermined linear system $\mathbf{Vc} \approx \mathbf{y}$. Run the modified function on the data of Problem 18. Which of the methods used seems to produce the least reliable numerical results? Justify your answer.

20. Write MATLAB function **[r, c] = nrceig(A)** that computes the number of real and complex eigenvalues of the real matrix **A**. You cannot use MATLAB function **eig**. Run function **nrceig** on several random matrices generated by the functions **rand** and **randn**.
Hint: You may wish to use the following MATLAB functions **schur**, **diag**, **find**. Note that the **diag** function takes a second optional argument.

21. Assume that an eigenvalue of a matrix is sensitive if its condition number is greater than 10^3. Construct an n-by-n matrix ($\mathbf{5 \leqslant n \leqslant 10}$) whose all eigenvalues are real and sensitive.

22. Write MATLAB function **A = pent(a, b, c, d, e, n)** that creates the full form of the n-by-n pentadiagonal matrix **A** with constant entries **a** along the second subdiagonal, constant entries **b** along the subdiagonal, etc.

23. Let **A = pent(1, 26, 66, 26, 1, n)** be an n-by-n symmetric pentadiagonal matrix generated by function **pent** of Problem 22. Find the eigenvalue decomposition $\mathbf{A} = \mathbf{Q}\Lambda\mathbf{Q}^T$ of **A** for various values of **n**. Repeat this experiment using random numbers in the band of the matrix **A**. Based on your observations, what conjecture can be formulated about the eigenvectors of **A**?

24. Write MATLAB function **[la, x] = smeig(A, v)** that computes the smallest

(in magnitude) eigenvalue of the nonsingular matrix **A** and the associated eigenvector **x**. The input parameter **v** is an estimate of the eigenvector of **A** that is associated with the largest (in magnitude) eigenvalue of **A**.

25. In this exercise you are to experiment with the eigenvalues and eigenvectors of the partitioned matrices. Begin with a square matrix **A** with known eigenvalues and eigenvectors. Next construct a matrix **B** using MATLAB's built-in function **repmat** to define the matrix **B** as **B = repmat(A, 2, 2)**. Solve the matrix eigenvalue problem for the matrix **B** and compare the eigenvalues and eigenvectors of matrices **A** and **B**. You may wish to continue in this manner using larger values for the second and third parameters in the function **repmat**. Based on the results of your experiment, what conjecture about the eigenvalues and eigenvectors of **B** can be formulated?

Tutorial 5

Numerical Analysis with MATLAB

Math 475/CS 475

MATLAB has many tools that make this package well suited for numerical computations. This tutorial deals with the rootfinding, interpolation, numerical differentiation and integration and numerical solutions of the ordinary differential equations. Numerical methods of linear algebra are discussed in Tutorial 4.

5.1 MATLAB functions used in Tutorial 5

Function	Description
abs	Absolute value
dblquad	Numerically evaluate double integral
erf	Error function
feval	Execute function specified by string
fzero	Scalar nonlinear zero finding
gamma	Gamma function
inline	Construct INLINE object
interp1	One-dimensional interpolation
interp2	Two-dimensional interpolation
linspace	Evenly spaced vector
meshgrid	X and Y arrays for 3-D plots
norm	Matrix or vector norm
ode23	Solve non-stiff differential equations
ode45	Solve non-stiff differential equations
ode113	Solve non-stiff differential equations
ode15s	Solve stiff differential equations
ode23s	Solve stiff differential equations
poly	Convert roots to polynomial
polyval	Evaluate polynomial
ppval	Evaluate piecewise polynomial
quad	Numerically evaluate integral, low order method
quad8	Numerically evaluate integral, higher order method
rcond	Reciprocal condition estimator
roots	Find polynomial roots
spline	Cubic spline data interpolation
surf	3-D colored surface
unmkpp	Supply details about piecewise polynomial

5.2 Rootfinding

A central problem discussed in this section is formulated as follows. Given a real-valued function **f: $\mathbb{R}^n \to \mathbb{R}^n$, n ⩾ 1**, find a vector **r** so that **f(r) = 0**. Vector **r** is called the *root* or *zero* of **f**.

5.2.1 Computing roots of the univariate polynomials

Polynomials are represented in MATLAB by their coefficients in the descending order of powers. For instance, the cubic polynomial **$p(x) = 3x^3 + 2x^2 - 1$** is represented as

```
p = [3 2 0 1];
```

Its roots can be found using function **roots**

```
format long

r = roots(p)

r =
 -1.00000000000000
  0.16666666666667 + 0.55277079839257i
  0.16666666666667 - 0.55277079839257i
```

To check correctness of this result we evaluate **p(x)** at **r** using function **polyval**

```
err = polyval(p, r)

err =
  1.0e-014 *
  0.22204460492503
                  0 + 0.01110223024625i
                  0 - 0.01110223024625i
```

To reconstruct a polynomial from its roots one can use function **poly**. Using the roots **r** computed earlier we obtain

```
poly(r)

ans =
   1.00000000000000   0.66666666666667   0.00000000000000
0.33333333333333
```

Let us note that these are the coefficients of **p(x)** all divided by 3. The coefficients of **p(x)** can be recovered easily

```
3*ans

ans =
   3.00000000000000   2.00000000000000   0.00000000000000
1.00000000000000
```

Numerical computation of roots of a polynomial is the *ill-conditioned* problem. Consider the fifth degree polynomial $\mathbf{p(x) = x^5 - 10x^4 + 40x^3 - 80x^2 + 80x - 32}$. Let us note that $\mathbf{p(x) = (x-2)^5}$. Using function **roots** we find

```
format short

p = [1 -10 40 -80 80 -32];

x = roots(p)

x =
   2.0017
   2.0005 + 0.0016i
   2.0005 - 0.0016i
   1.9987 + 0.0010i
   1.9987 - 0.0010i
```

These results are not satisfactory. We will return to the problem of finding the roots of **p(x)** in the next section.

5.2.2 Finding zeros of the univariate functions using MATLAB function fzero

Let now **f** be a transcendental function from $\mathbb{R}$ to $\mathbb{R}$. MATLAB function **fzero** computes a zero of the function **f** using user supplied initial guess of a zero sought.

In the following example let **f(x) = cos(x) – x**. First we define a function **y = f1(x)**

```
function y = f1(x)

% A univariate function with a simple zero.

y = cos(x) - x;
```

To compute its zero we use MATLAB function **fzero**

```
r = fzero('f1', 0.5)

r =
   0.73908513321516
```

Name of the function whose zero is computed is entered as a string. Second argument of function **fzero** is the initial approximation of **r**. One can check last result using function **feval**

```
err = feval('f1', r)

err =
     0
```

In the case when a zero of function is bracketed a user can enter a two-element vector that designates a starting interval. In our example we choose **[0 1]** as a starting interval to obtain

```
r = fzero('f1', [0 1])

r =
   0.73908513321516
```

However, this choice of the designated interval

```
fzero('f1', [1 2])

¨??? Error using ==> fzero
The function values at the interval endpoints must differ in sign.
```

generates the error message.

By adding the third input parameter **tol** you can force MATLAB to compute the zero of a function with the relative error tolerance **tol**. In our example we let **tol = 10^{-3}** to obtain

```
rt = fzero('f1', .5, 1e-3)

rt =
   0.73886572291538
```

A relative error in the computed zero **rt** is

```
rel_err = abs(rt-r)/r

rel_err =
    2.969162630892787e-004
```

Function **fzero** takes fourth optional parameter. If it is set up to 1, then the iteration information is displayed. Using function **f1**, with **x0 = 0.5**, we obtain

```
format short

rt = fzero('f1', .5, eps, 1)

 Func evals      x           f(x)          Procedure
    1             0.5      0.377583        initial
    2        0.485858      0.398417        search
    3        0.514142      0.356573        search
    4            0.48      0.406995        search
    5            0.52      0.347819        search
    6        0.471716      0.419074        search
    7        0.528284      0.335389        search
    8            0.46      0.436052        search
    9            0.54      0.317709        search
   10        0.443431      0.459853        search
   11        0.556569      0.292504        search
   12            0.42      0.493089        search
   13            0.58      0.256463        search
   14        0.386863      0.539234        search
   15        0.613137       0.20471        search
   16            0.34      0.602755        search
   17            0.66      0.129992        search
   18        0.273726      0.689045        search
```

```
   19        0.726274     0.0213797        search
   20            0.18      0.803844        search
   21            0.82     -0.137779        search

   Looking for a zero in the interval [0.18, 0.82]

   22        0.726355     0.0212455        interpolation
   23        0.738866    0.00036719        interpolation
   24        0.739085 -6.04288e-008        interpolation
   25        0.739085  2.92788e-012        interpolation
   26        0.739085             0        interpolation
rt =
    0.7391
```

We have already seen that MATLAB function **roots** had faild to produce satisfactory results when computing roots of the polynomial $\mathbf{p(x) = (x - 2)^5}$. This time we will use function **fzero** to find a multiple root of **p(x)**. We define a new function named **f2**

```
function y = f2(x)

y = (x - 2)^5;
```

and next change format to

```
format long
```

Running function **fzero** we obtain

```
rt = fzero('f2', 1.5)

rt =
   2.00000000000000
```

This time the result is as expected.

Finally, we will apply function **fzero** to compute the multiple root of **p(x)** using an expanded form of the polynomial **p(x)**

```
function y = f3(x)

y = x^5 - 10*x^4 + 40*x^3 -80*x^2 + 80*x - 32;

rt = fzero('f3', 1.5)

rt =
   1.99845515925755
```

Again, the computed approximation of the root of **p(x)** has a few correct digits only.

5.2.3 The Newton-Raphson method for systems of nonlinear equations

This section deals with the problem of computing zeros of the vector-valued function $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$, $n \geq 1$. Assume that the first order partial derivatives of $\mathbf{f}$ are continuous on an open domain holding all zeros of $\mathbf{f}$. A method discussed below is called the *Newton-Raphson method*. To present details of this method let us introduce more notation. Using MATLAB's convention for representing vectors we write $\mathbf{f}$ as a column vector $\mathbf{f} = [f_1; \ldots; f_n]$, where each f_k is a function from $\mathbb{R}^n$ to $\mathbb{R}$. Given an initial approximation $\mathbf{x}^{(0)} \in \mathbb{R}^n$ of $\mathbf{r}$ this method generates a sequence of vectors $\{\mathbf{x}^{(k)}\}$ using the iteration

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} - J_f(\mathbf{x}^{(k)})^{-1} \mathbf{f}(\mathbf{x}^{(k)}), \qquad k = 0, 1, \ldots .$$

Here J_f stands for the *Jacobian matrix* of $\mathbf{f}$, i.e., $J_f(\mathbf{x}) = [\partial f_i(\mathbf{x})/\partial x_j]$, $1 \leq i, j \leq n$. For more details the reader is referred to [6] and [9].

Function **NR** computes a zero of the system of nonlinear equations.

```
function [r, niter] = NR(f, J, x0, tol, rerror, maxiter)

% Zero r of the nonlinear system of equations f(x) = 0.
% Here J is the Jacobian matrix of f and x0 is the initial
% approximation of the zero r.
% Computations are interrupted either if the norm of
% f at current approximation is less (in magnitude)
% than the number tol,or if the relative error of two
% consecutive approximations is smaller than the prescribed
% accuracy rerror, or if the number of allowed iterations
% maxiter is attained.
% The second output parameter niter stands for the number
% of performed iterations.

Jc = rcond(feval(J,x0));
if Jc < 1e-10
   error('Try a new initial approximation x0')
end
   xold = x0(:);
   xnew = xold - feval(J,xold)\feval(f,xold);
for k=1:maxiter
   xold = xnew;
   niter = k;
   xnew = xold - feval(J,xold)\feval(f,xold);
   if (norm(feval(f,xnew)) < tol) |...
         norm(xold-xnew,'inf')/norm(xnew,'inf') < tol|...
         (niter == maxiter)
      break
   end
end
r = xnew;
```

The following nonlinear system

$$f_1(\mathbf{x}) = x_1 + 2x_2 - 2,$$
$$f_2(\mathbf{x}) = x_1^2 + 4x_2^2 - 4$$

has the exact zeros **r = [0 1]^T** and **r = [2 0]^T** (see [6], p. 166). Functions **fun1** and **J1** define the system of equations and its Jacobian, respectively

```
function z = fun1(x)

z = zeros(2,1);
z(1) = x(1) + 2*x(2) - 2;
z(2) = x(1)^2 + 4*x(2)^2 - 4;


function s = J1(x)

s = [1 2;2*x(1) 8*x(2)];
```

Let

```
x0 = [0 0];
```

Then

```
[r, iter] = NR('fun1', 'J1', x0, eps, eps, 10)

¨??? Error using ==> nr
Try a new initial approximation x0
```

For **x0** as chosen above the associated Jacobian is singular. Let's try another initial guess for **r**

```
x0 = [1 0];

[r, niter] = NR('fun1', 'J1', x0, eps, eps, 10)

r =
   2.00000000000000
  -0.00000000000000
niter =
     5
```

Consider another nonlinear system

$f_1(x) = x_1 + x_2 - 1$
$f_2(x) = \sin(x_1^2 + x_2^2) - x_1.$

The m-files needed for computing its zeros are named **fun2** and **J2**

```
function w = fun2(x);

w(1) = x(1) + x(2) - 1;
w(2) = sin(x(1)^2 + x(2)^2) - x(1);
w = w(:);
```

```
function s = J2(x)

s = [1 1;
       2*x(1)*cos(x(1)^2 + x(2)^2)-1 2*x(2)*cos(x(1)^2 + x(2)^2)];
```

With the initial guess

```
x0 = [0 1];
```

the zero **r** is found to be

```
 [r, niter] = NR('fun2', 'J2', x0, eps, eps, 10)

r =
   0.48011911689839
   0.51988088310161
niter =
     5
```

while the initial guess

```
x0 = [1 1];

[r, iter] = NR('fun2', 'J2', x0, eps, eps, 10)

r =
  -0.85359545600207
   1.85359545600207
iter =
    10
```

gives another solution. The value of function **fun2** at the computed zero **r** is

```
feval('fun2', r)

ans =
  1.0e-015 *
                    0
  -0.11102230246252
```

Implementation of other classical methods for computing the zeros of scalar equations, including the fixed-point iteration, the secant method and the Schroder method are left to the reader (see Problems 3, 6, and 12 at the end of this tutorial).

5.3 One Dimensional Interpolation

Interpolation of functions is one of the classical problems in numerical analysis. A one dimensional interpolation problem is formulated as follows.

Given set of **n+1** points $\{x_k , y_k\}$, $0 \leq k \leq n$, with $x_0 < x_1 < \ldots < x_n$, find a function **f(x)** whose graph interpolates the data points, i.e., $f(x_k) = y_k$, for $k = 0, 1, \ldots, n$.

In this section we will use as the interpolating functions algebraic polynomials and *spline functions*.

5.3.1 MATLAB function interp1

The general form of the function **interp1** is **yi = interp1(x, y, xi, method)**, where the vectors **x** and **y** are the vectors holding the x- and the y- coordinates of points to be interpolated, respectively, **xi** is a vector holding points of evaluation, i.e., **yi = f(xi)** and **method** is an optional string specifying an interpolation method. The following methods work with the function **interp1**

- Nearest neighbor interpolation, method = **'nearest'**. Produces a locally piecewise constant interpolant.
- Linear interpolation method = **'linear'**. Produces a piecewise linear interpolant.
- Cubic spline interpolation, method = **'spline'**. Produces a cubic spline interpolant.
- Cubic interpolation, method = **'cubic'**. Produces a piecewise cubic polynomial.

In this example, the following points $(x_k, y_k) = (k\pi/5, \sin(2x_k))$, $k = 0, 1, \ldots, 5$,

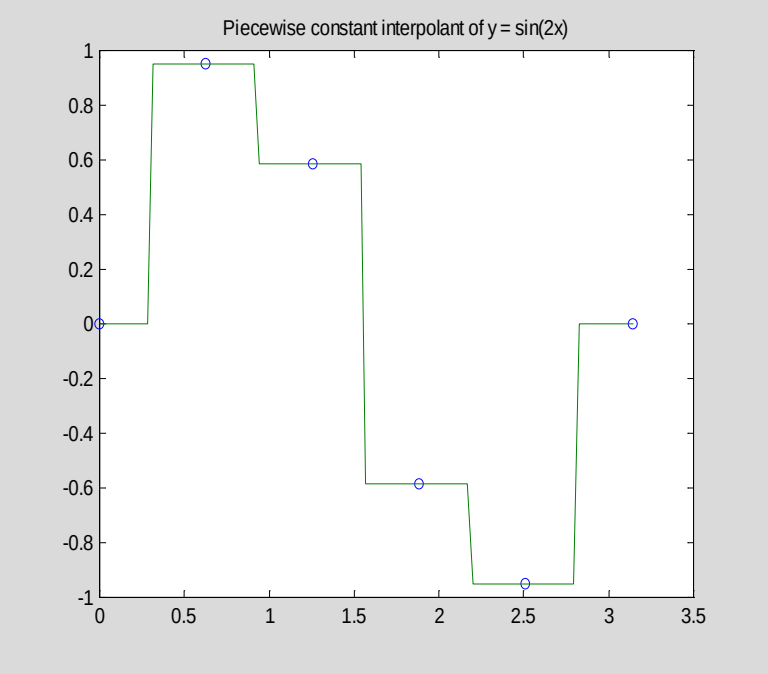
```
x = 0:pi/5:pi;

y = sin(2.*x);
```

are interpolated using two methods of interpolation **'nearest'** and **'cubic'** . The interpolant is evaluated at the following points

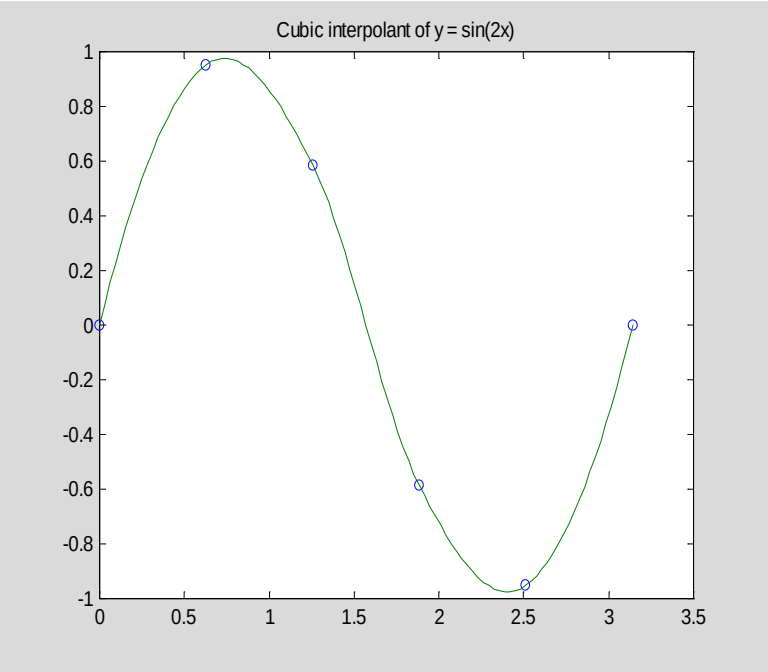
```
xi = 0:pi/100:pi;

yi = interp1(x, y, xi, 'nearest');
```

Points of interpolation together with the resulting interpolant are displayed below

```
plot(x, y, 'o', xi, yi), title('Piecewise constant interpolant of y =
sin(2x)')
```




```
yi = interp1(x, y, xi, 'cubic');
plot(x, y, 'o', xi, yi), title('Cubic interpolant of y = sin(2x)')
```



5.3.2 Interpolation by algebraic polynomials

Assume now that the interpolating function is an algebraic polynomial $\mathbf{p_n(x)}$ of degree at most **n**, where **n** = number of points of interpolation – 1. It is well known that the interpolating polynomial $\mathbf{p_n}$ always exists and is unique (see e.g., [6], [9]). To determine the polynomial interpolant one can use either the Vandermonde's method or Lagrange form or Newton's form or Aitken's method. We shall describe briefly the Newton's method.

We begin writing $\mathbf{p(x)}$ as

(5.3.1) $$\mathbf{p_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \ldots + a_n(x - x_0)(x - x_1) \ldots (x - x_{n-1})}$$

Coefficients $\mathbf{a_0}$, $\mathbf{a_1}$, … , $\mathbf{a_n}$ are called the *divided differences* and they can be computed recursively. Representation (5.3.1) of $\mathbf{p_n(x)}$ is called the *Newton's form* of the interpolating polynomial. The k-th order divided difference based on points $\mathbf{x_0}$, … $\mathbf{x_k}$, denoted by $\mathbf{[x_0, \ldots, x_k]}$, is defined recursively as

$$\mathbf{[x_m] = y_m} \quad \text{if } \mathbf{k = 0}$$

$$\mathbf{[x_0, \ldots, x_k] = ([x_1, \ldots, x_k] - [x_0, \ldots, x_{k-1}])/(x_k - x_0)} \quad \text{if } \mathbf{k > 0}.$$

Coefficients $\mathbf{\{a_k\}}$ in representation (5.3.1) and the divided differences are related in the following way

$$\mathbf{a_k = [x_0, \ldots, x_k]}.$$

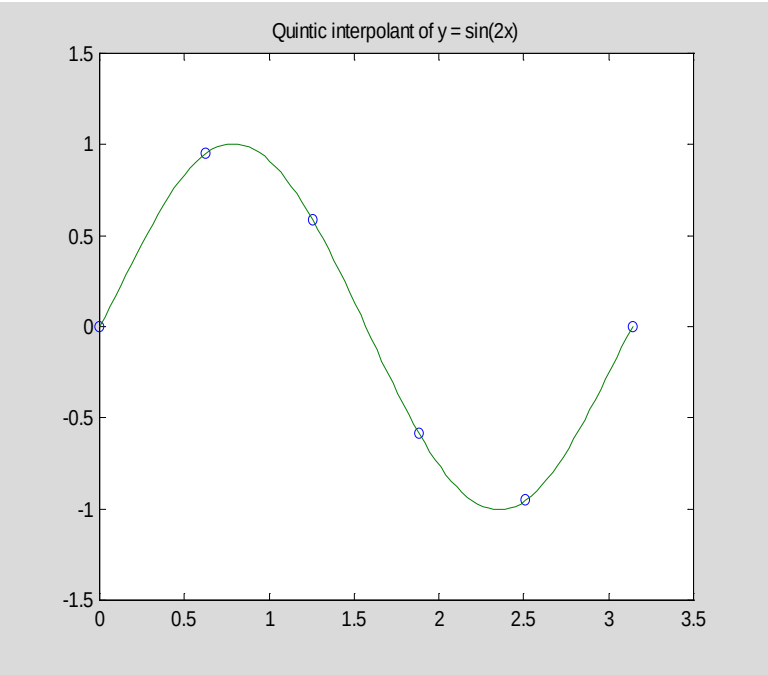
Function **Newtonpol** evaluates an interpolating polynomial at the user supplied points.

```
function [yi, a] = Newtonpol(x, y, xi)

% Values yi of the interpolating polynomial at the points xi.
% Coordinates of the points of interpolation are stored in
% vectors x and y. Horner's method is used to evaluate
% a polynomial. Second output parameter a holds coeeficients
% of the interpolating polynomial in Newton's form.

a = divdiff(x, y);
n = length(a);
val = a(n);
for m = n-1:-1:1
   val = (xi - x(m)).*val + a(m);
end
yi = val(:);


function a = divdiff(x, y)

% Divided differences based on points stored in arrays x and y.

n = length(x);
for k=1:n-1
```

```
   y(k+1:n) = (y(k+1:n) - y(k))./(x(k+1:n) - x(k));
end
a = y(:);
```

For the data of the last example, we will evaluate Newton's interpolating polynomial of degree at most five, using function **Newtonpol**. Also its graph together with the points of interpolation will be plotted.

```
[yi, a] = Newtonpol(x, y, xi);

plot(x, y, 'o', xi, yi), title('Quintic interpolant of y = sin(2x)')
```



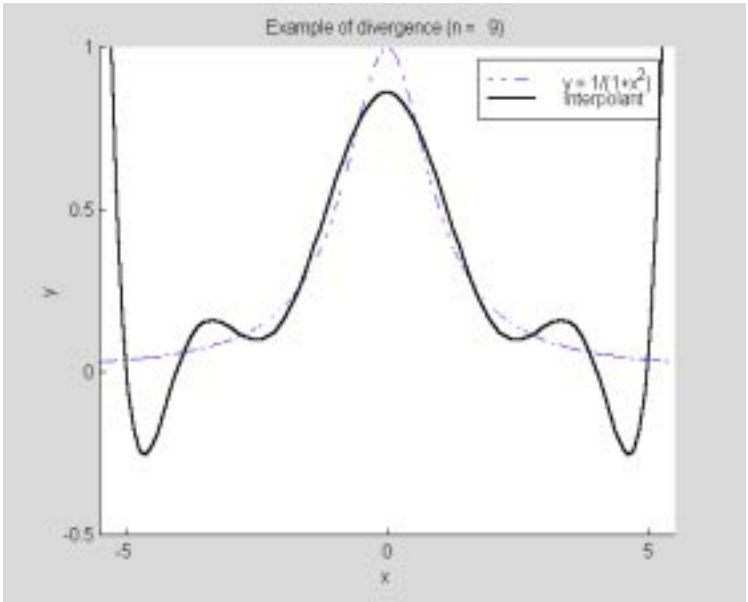
Interpolation process not always produces a sequence of polynomials that converge uniformly to the interpolated function as degree of the interpolating polynomial tends to infinity. A famous example of divergence, due to Runge, illustrates this phenomenon. Let $g(x) = 1/(1 + x^2)$, $-5 \leqslant x \leqslant 5$, be the function that is interpolated at $n + 1$ evenly spaced points $x_k = -5 + 10k/n$, $k = 0, 1, \ldots, n$.

Script file **showint** creates graphs of both, the function $g(x)$ ant its interpolating polynomial $p_n(x)$.

```
% Script showint.m
% Plot of the function 1/(1 + x^2) and its
% interpolating polynomial of degree n.

m = input('Enter number of interpolating polynomials ');
```

```
for k=1:m
   n = input('Enter degree of the interpolating polynomial ');
   hold on
   x = linspace(-5,5,n+1);
   y = 1./(1 + x.*x);
   z = linspace(-5.5,5.5);
   t = 1./(1 + z.^2);
   h1_line = plot(z,t,'-.');
   set(h1_line, 'LineWidth',1.25)
   t = Newtonpol(x,y,z);
   h2_line = plot(z,t,'r');
   set(h2_line,'LineWidth',1.3,'Color',[0 0 0])
   axis([-5.5 5.5 -.5 1])
   title(sprintf('Example of divergence (n =  %2.0f)',n))
   xlabel('x')
   ylabel('y')
   legend('y = 1/(1+x^2)','interpolant')
   hold off
end
```

Typing **showint** in the Command Window you will be prompted to enter value for the parameter **m** = number of interpolating polynomials you wish to generate and also you have to enter value(s) of the degree of the interpolating polynomial(s). In the following example **m = 1** and **n = 9**



Divergence occurs at points that are close enough to the endpoints of the interval of interpolation **[-5, 5]**.

We close this section with the *two-point Hermite interpolaion* problem by cubic polynomials. Assume that a function **y= g(x)** is differentiable on the interval **[a, b]**. We seek a cubic polynomial $\mathbf{p_3(x)}$ that satisfies the following interpolatory conditions

(5.3.2) $p_3(a) = g(a), \quad p_3(b) = g(b), \quad p_3'(a) = g'(a), \quad p_3'(b) = g'(b)$

Interpolating polynomial $p_3(x)$ always exists and is represented as follows

(5.3.3) $$p_3(x) = (1 + 2t)(1 - t)^2 g(a) + (3 - 2t)t^2 g(b) + h[t(1 - t)^2 g'(a) + t^2(t - 1)g'(b)] ,$$

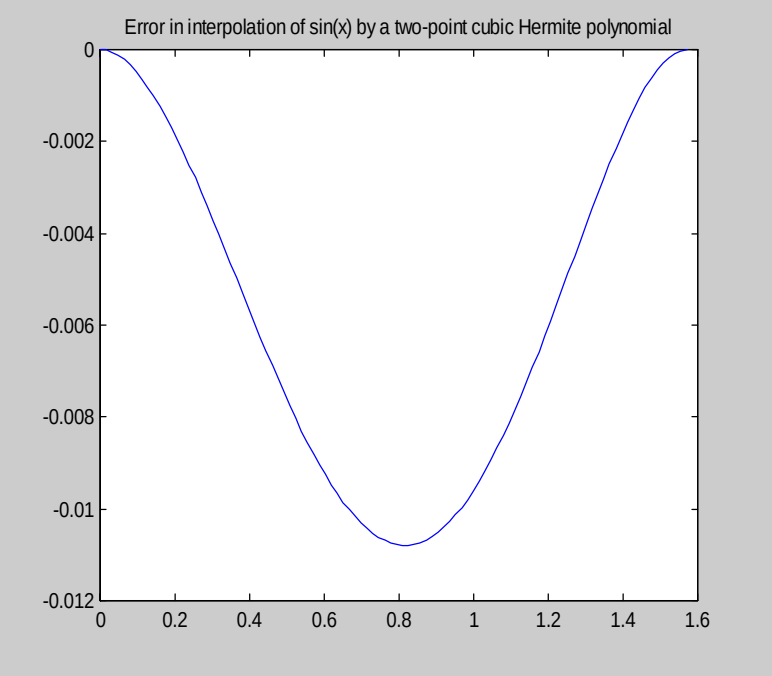
where $t = (x - a)/(b - a)$ and $h = b - a$.

Function **Hermpol** evaluates the Hermite interpolant at the points stored in the vector **xi**.

```
function yi = Hermpol(ga, gb, dga, dgb, a, b, xi)

% Two-point cubic Hermite interpolant. Points of interpolation
% are a and b. Values of the interpolant and its first order
% derivatives at a and b are equal to ga, gb, dga and dgb,
% respectively.
% Vector yi holds values of the interpolant at the points xi.

h = b – a;
t = (xi - a)./h;
t1 = 1 - t;
t2 = t1.*t1;
yi = (1 + 2*t).*t2*ga + (3 - 2*t).*(t.*t)*gb +…
h.*(t.*t2*dga + t.^2.**(t - 1)*dgb);
```

In this example we will interpolate function $g(x) = \sin(x)$ using a two-point cubic Hermite interpolant with $a = 0$ and $b = \pi/2$

```
xi = linspace(0, pi/2);

yi = Hermpol(0, 1, 1, 0, 0, pi/2, xi);

zi = yi – sin(xi);

plot(xi, zi), title('Error in interpolation of sin(x) by a two-point
cubic Hermite polynomial')
```



5.3.3 Interpolation by splines

In this section we will deal with interpolation by polynomial splines. In recent decades splines have attracted attention of both researchers and users who need a versatile approximation tools. We begin with the definition of the polynomial spline functions and the spline space.

Given an interval **[a, b]**. A *partition* **Δ** of the interval **[a, b]** with the *breakpoints* $\{x_l\}_1^m$ is defined as $\Delta = \{a = x_1 < x_2 < \ldots < x_m = b\}$, where $m > 1$. Further, let **k** and **n**, $k < n$, be nonnegative integers. Function **s(x)** is said to be a *spline function of degree* **n** *with smoothness* **k** if the following conditions are satisfied:

(i) On each subinterval $[x_l, x_{l+1}]$ **s(x)** coincides with an algebraic polynomial of degree at most **n**.
(ii) **s(x)** and its derivatives up to order **k** are all continuous on the interval **[a, b]**

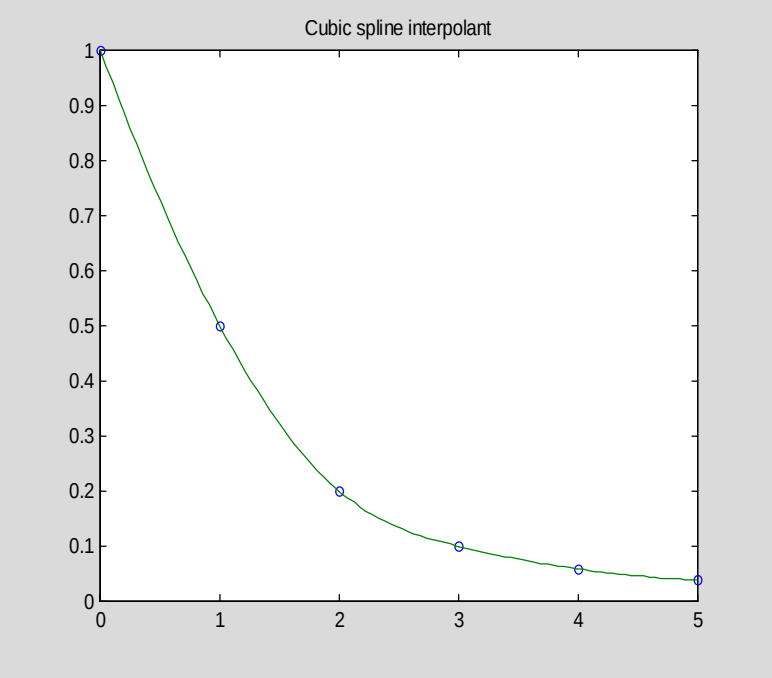
Throughout the sequel the symbol **Sp(n, k, Δ)** will stand for the *space of the polynomial splines* of degree **n** with smoothness **k** , and the breakpoints **Δ**. It is well known that Sp(n, k, Δ) is a linear subspace of dimension $(n + 1)(m - 1) - (k + 1)(m - 2)$. In the case when $k = n - 1$, we will write **Sp(n, Δ)** instead of **Sp(n, n – 1, Δ)**.

MATLAB function **spline** is designed for computations with the cubic splines ($n = 3$) that are twice continuously differentiable ($k = 2$) on the interval $[x_1, x_m]$. Clearly
$\dim Sp(3, \Delta) = m + 2$. The spline interpolant **s(x)** is determined uniquely by the interpolatory conditions $s(x_l) = y_l$, $l = 1, 2, \ldots, m$ and two additional boundary conditions, namely that $s'''(x)$ is continuous at $x = x_2$ and $x = x_{m-1}$. These conditions are commonly referred to as the *not-a-knot* end conditions.

MATLAB's command **yi = spline(x, y, xi)** evaluates cubic spline **s(x)** at points stored in the array **xi**. Vectors **x** and **y** hold coordinates of the points to be interpolated. To obtain the piecewise polynomial representation of the spline interpolant one can execute the command **pp = spline(x, y)**. Command **zi = ppval(pp, xi)** evaluates the piecewise polynomial form of the spline interpolant. Points of evaluation are stored in the array **xi**. If a spline interpolant has to be evaluated for several vectors **xi**, then the use of function **ppval** is strongly recommended.

In this example we will interpolate Runge's function **$g(x) = 1/(1 + x^2)$** on the interval **[0, 5]** using six evenly spaced breakpoints

```
x = 0:5;

y = 1./(1 + x.^2);

xi = linspace(0, 5);

yi = spline(x, y, xi);

plot(x, y, 'o', xi, yi), title('Cubic spline interpolant')
```



The maximum error on the set **xi** in approximating Runge's function by the cubic spline we found is

```
err = norm(abs(yi-1./(1+xi.^2)),'inf')

err =
    0.0859
```

Detailed information about the piecewise polynomial representation of the spline interpolant can be obtained running function **spline** with two input parameters **x** and **y**

```
pp = spline(x, y);
```

and next executing command **unmkpp**

```
[brpts, coeffs, npol, ncoeff] = unmkpp(pp)

brpts =
     0     1     2     3     4     5
coeffs =
    0.0074    0.0777   -0.5852    1.0000
    0.0074    0.1000   -0.4074    0.5000
   -0.0371    0.1223   -0.1852    0.2000
   -0.0002    0.0110   -0.0519    0.1000
   -0.0002    0.0104   -0.0306    0.0588
npol =
     5
ncoeff =
     4
```

The output parameters **brpts**, **coeffs**, **npol**, and **ncoeff** represent the breakpoints of the spline interpolant, coefficients of **s(x)** on successive subintervals, number of polynomial pieces that constitute spline function and number of coefficients that represent each polynomial piece, respectively. On the subinterval $[x_l, x_{l+1}]$ the spline interpolant is represented as

$$s(x) = c_{l1}(x - x_l)^3 + c_{l2}(x - x_l)^2 + c_{l3}(x - x_l) + c_{l4}$$

where $[c_{l1}\ c_{l2}\ c_{l3}\ c_{l4}]$ is the lth row of the matrix **coeffs**. This form is called the *piecewise polynomial form* (*pp–form*) of the spline function.

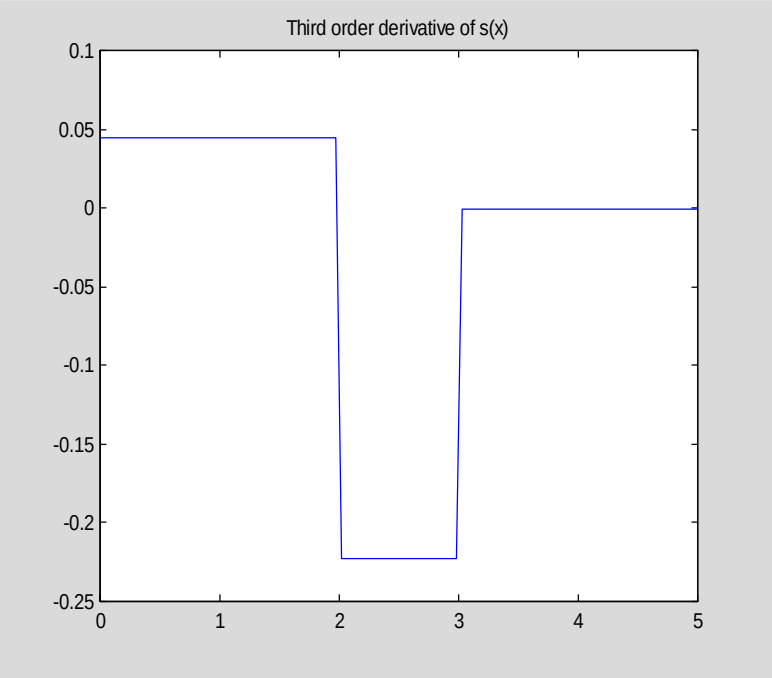
Differentiation of the spline function s**(x)** can be accomplished running function **splder**. In order for this function to work properly another function **pold** (see Problem 19) must be in MATLAB's search path.

```
function p = splder(k, pp, x)

% Piecewise polynomial representation of the derivative
% of order k (0 <= k <= 3) of a cubic spline function in the
% pp form with the breakpoints stored in the vector x.

m = pp(3);
lx4 = length(x) + 4;
n = pp(lx4);
c = pp(1 + lx4:length(pp))';
c = reshape(c, m, n);
b = pold(c, k);
b = b(:)';
```

```
p = pp(1:lx4);
p(lx4) = n - k;
p = [p b];
```

The third order derivative of the spline function of the last example is shown below

```
p = splder(3, pp, x);

yi = ppval(p, xi);

plot(xi, yi), title('Third order derivative of s(x)')
```



Note that **s'''(x)** is continuous at the breakpoints $x_2 = 1$ and $x_5 = 4$. This is due to the fact that the not-a-knot boundary conditions were imposed on the spline interpolant.

Function **evalppf** is the utility tool for evaluating the piecewise polynomial function **s(x)** at the points stored in the vector **xi**. The breakpoints $x = \{x_1 < x_2 < \ldots < x_m\}$ of **s(x)** and the points of evaluation **xi** must be such that $x_1 = xi_1$ and $x_m = xi_p$, where **p** is the index of the largest number in **xi**. Coefficients of the polynomial pieces of **s(x)** are stored in rows of the matrix **A** in the descending order of powers.

```
function [pts, yi] = evalppf(x, xi, A)

% Values yi of the piecewise polynomial function (pp-function)
% evaluated at the points xi. Vector x holds the breakpoints
% of the pp-function and matrix A holds the coefficients of the
% pp-function. They are stored in the consecutive rows in
```

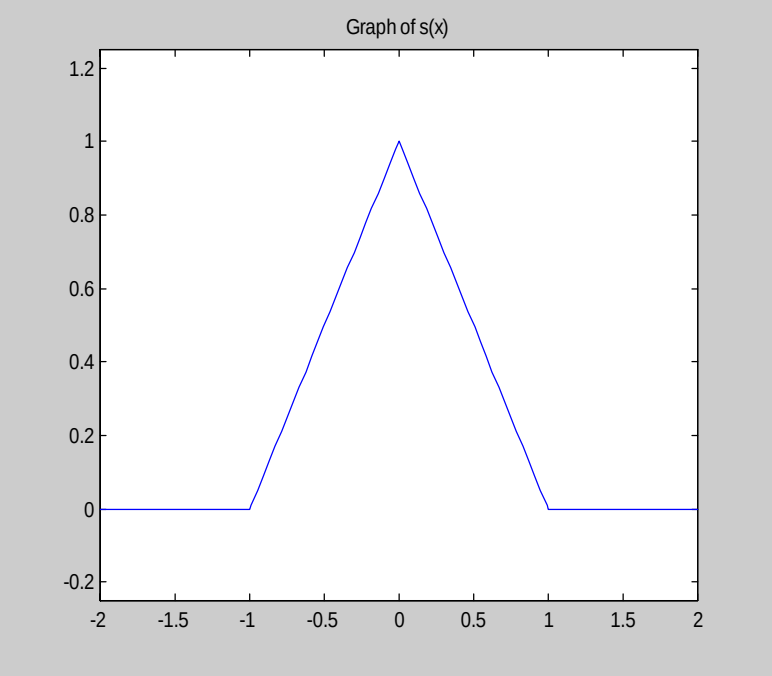
```
% the descending order of powers.The output parameter pts holds
% the points of the union of two sets x and xi.


n = length(x);
[p, q] = size(A);
if n-1 ~= p
   error('Vector t and matrix A must be "compatible"')
end
yi = [];
pts = union(x, xi);
for m=1:p
   l = find(pts == x(m));
   r = find(pts == x(m+1));
   if m < n-1
      yi = [yi polyval(A(m,:), pts(l:r-1))];
   else
      yi = [yi polyval(A(m,:), pts(l:r))];
   end
end
```

In this example we will evaluate and plot the graph of the piecewise linear function **s(x)** that is defined as follows

s(x) = 0, if **|x| ⩾ 1**
s(x) = 1 + x, if **-1 ⩽ x ⩽ 0**
s(x) = 1 – x, if **0 ⩽ x ⩽ 1**

Let

```
x = -2:2;

xi = linspace(-2, 2);

A = [0 0;1 1;1 –1;0 0];

[pts, yi] = evalppf(x, xi, A);

plot(pts, yi), title('Graph of s(x)'), axis([-2 2 -.25 1.25])
```



5.3.4 Two Dimensional Interpolation

The interpolation problem discussed in this section is formulated as follows.

Given a rectangular grid **$\{x_k, y_l\}$** and the associated set of numbers **z_{kl}**, **$1 \leqslant k \leqslant m$**, **$1 \leqslant l \leqslant n$**, find a bivariate function **$z = f(x, y)$** that interpolates the data, i.e., **$f(x_k, y_l) = z_{kl}$** for all values of **k** and **l**. The grid points must be sorted monotonically, i.e. **$x_1 < x_2 < \ldots < x_m$** with a similar ordering of the y-ordinates.

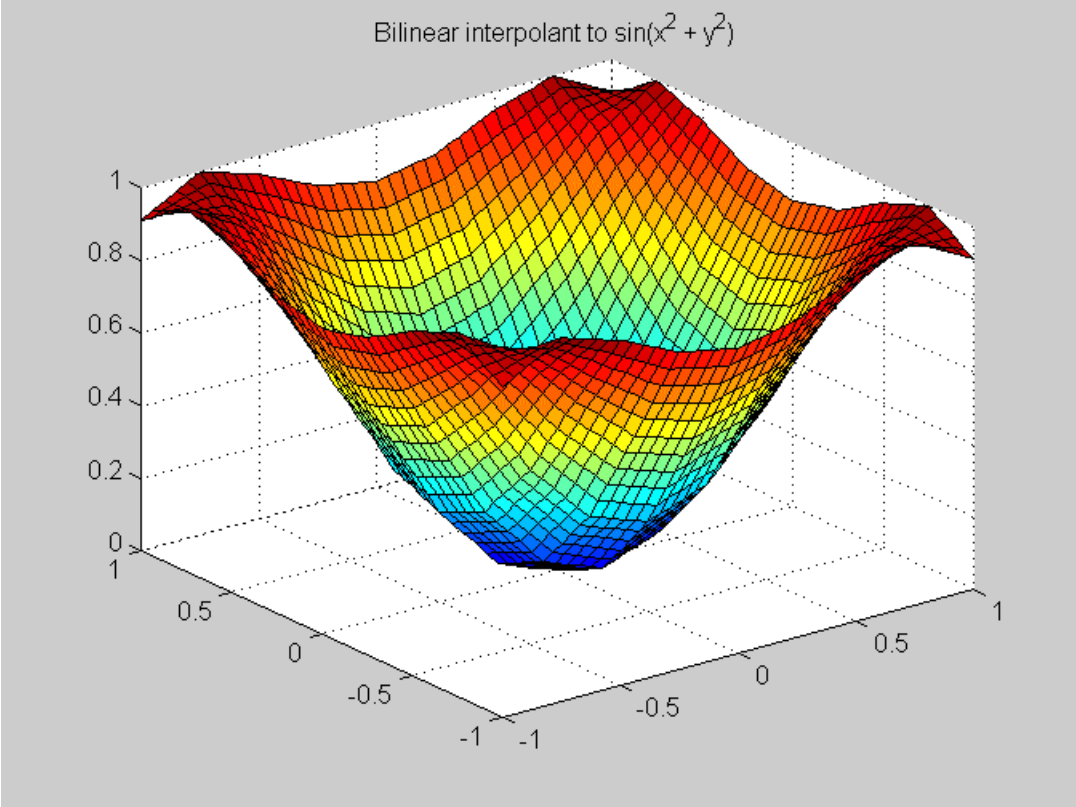
MATLAB's built-in function **zi = interp2(x, y, z, xi, yi, 'method')** generates a bivariate interpolant on the rectangular grids and evaluates it in the points specified in the arrays **xi** and **yi**. Sixth input parameter **'method'** is optional and specifies a method of interpolation. Available methods are:

- **'nearest'** - nearest neighbor interpolation
- **'linear'** - bilinear interpolation
- **'cubic'** - bicubic interpolation
- **'spline'** - spline interpolation

In the following example a bivariate function **$z = \sin(x^2 + y^2)$** is interpolated on the square **$-1 \leqslant x \leqslant 1$**, **$-1 \leqslant y \leqslant 1$** using the 'linear' and the 'cubic' methods.

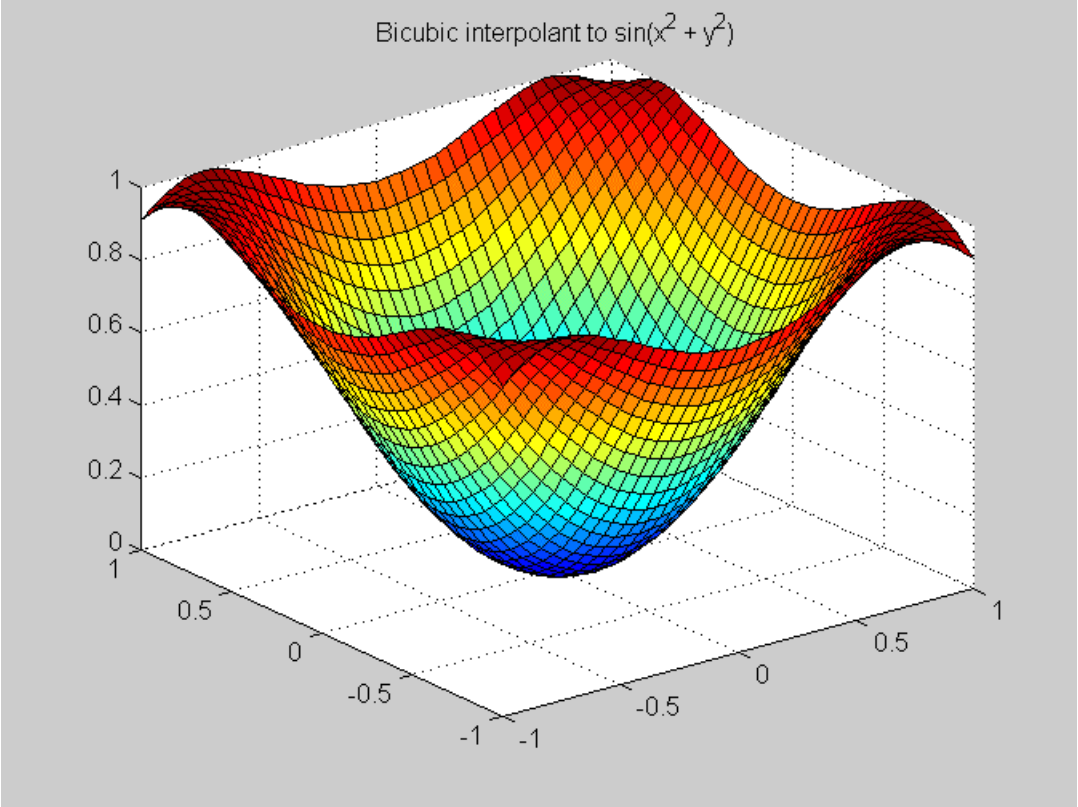
```
[x, y] = meshgrid(-1:.25:1);

z = sin(x.^2 + y.^2);

[xi, yi] = meshgrid(-1:.05:1);
```

```
zi = interp2(x, y, z, xi, yi, 'linear');
surf(xi, yi, zi), title('Bilinear interpolant to sin(x^2 + y^2)')
```



The bicubic interpolant is obtained in a similar fashion

```
zi = interp2(x, y, z, xi, yi, 'cubic');
```



5.4 Numerical Integration and Differentiation

A classical problem of the numerical integration is formulated as follows.

Given a continuous function **f(x)**, **a ⩽ x ⩽ b**, find the coefficients **{w_k}** and the nodes **{x_k}**, **1 ⩽ k ⩽ n**, so that the *quadrature formula*

(5.4.1) $$\int_a^b f(x)dx \approx \sum_{k=1}^{n} w_k f(x_k)$$

is exact for polynomials of a highest possible degree.

For the evenly spaced nodes **{x_k}** the resulting family of the quadrature formulas is called the *Newton-Cotes formulas*. If the coefficients **{w_k}** are assumed to be all equal, then the quadrature formulas are called the *Chebyshev quadrature formulas*. If both, the coefficients **{w_k}** and the nodes **{x_k}** are determined by requiring that the formula (5.4.1) is exact for polynomials of the highest possible degree, then the resulting formulas are called the *Gauss quadrature formulas*.

5.4.1 Numerical integration using MATLAB functions quad and quad8

Two MATLAB functions **quad('f ', a, b, tol, trace, p1, p2, …)** and **quad8('f ', a, b, tol, trace, p1, p2, …)** are designed for numerical integration of the univariate functions. The input parameter **'f'** is a string containing the name of the function to be integrated from **a** to **b**. The fourth input parameter **tol** is optional and specifies user's chosen relative error in the computed integral. Parameter **tol** can hold both the relative and the absolute errors supplied by the user. In this case a two-dimensional vector tol **= [rel_tol, abs_tol]** must be included. Parameter **trace** is optional and traces the function evaluations with a point plot of the integrand. To use default values for **tol** or **trace** one may pass in the empty matrix **[ ]**. Parameters **p1**, **p2**, … are also optional and they are supplied only if the integrand depends on **p1**, **p2**, … .

In this example a simple rational function

$$\mathbf{f(x)} = \frac{\mathbf{a + bx}}{\mathbf{1 + cx^2}}$$

```
function y = rfun(x, a, b, c)

% A simple rational function that depends on three
% parameters a, b and c.

y = (a + b.*x)./(1 + c.*x.^2);
y = y';
```

is integrated numerically from **0** to **1** using both functions **quad** and **quad8**. The assumed relative and absolute errors are stored in the vector **tol**

```
tol = [1e-5 1e-3];

format long

[q, nfev] = quad('rfun', 0, 1, tol, [], 1, 2, 1)

q =
   1.47856630183943
nfev =
     9
```

Using function **quad8** we obtain

```
[q8,nfev] = quad8('rfun', 0, 1, tol, [], 1, 2, 1)

q8 =
   1.47854534395683
nfev =
    33
```

Second output parameter **nfev** gives an information about the number of function evaluations needed in the course of computation of the integral.

The exact value of the integral in question is

```
exact = log(2) + pi/4
```

```
exact =
   1.47854534395739
```

The relative errors in the computed approximations **q** and **q8** are

```
rel_errors = [abs(q – exact)/exact; abs(q8 – exact)/exact]
```

```
rel_errors =
  1.0e-004 *
   0.14174663036002
   0.00000000380400
```

5.4.2 Newton – Cotes quadrature formulas

One of the oldest method for computing the approximate value of the definite integral over the interval **[a, b]** was proposed by Newton and Cotes. The nodes of the Newton – Cotes formulas are chosen to be evenly spaced in the interval of integration. There are two types of the Newton – Cotes formulas the closed and the open formulas. In the first case the endpoints of the interval of integration are included in the sets of nodes whereas in the open formulas they are not. The weights $\{w_k\}$ are determined by requiring that the quadrature formula is exact for polynomials of a highest possible degree.

Let us discuss briefly the Newton – Cotes formulas of the closed type. The nodes of the n – point formula are defined as follows $x_k = a + (k – 1)h$, $k = 1, 2, \dots, n$, where $h = (b – a)/(n – 1)$, $n > 1$. The weights of the quadrature formula are determined from the conditions that the following equations are satisfied for the monomials $f(x) = 1, x, \dots x^{n-1}$

$$\int_a^b f(x)dx = \sum_{k=1}^{n} w_k f(x_k)$$

```
function [s, w, x] = cNCqf(fun, a, b, n, varargin)

% Numerical approximation s of the definite integral of
% f(x). fun is a string containing the name of the integrand f(x).
% Integration is over the interval [a, b].
% Method used:
% n-point closed Newton-Cotes quadrature formula.
% The weights and the nodes of the quadrature formula
% are stored in vectors w and x, respectively.

if n < 2
   error(' Number of nodes must be greater than 1')
end
x = (0:n-1)/(n-1);
```

```
f = 1./(1:n);
V = Vander(x);
V = rot90(V);
w = V\f';
w = (b-a)*w;
x = a + (b-a)*x;
x = x';
s = feval(fun,x,varargin{:});
s = w'*s;
```

In this example the *error function* **Erf(x)** , where

$$\mathbf{Erf(x)} = \frac{2}{\sqrt{\pi}} \int_0^x \mathbf{e}^{-t^2}\,\mathbf{dt}$$

will be approximated at **x = 1** using the closed Newton – Cotes quadrature formulas wit **n = 2** (Trapezoidal Rule), **n = 3** (Simpson's Rule), and **n = 4** (Boole's Rule). The integrand of the last integral is evaluated using function **exp2**

```
function w = exp2(x)

% The weight function w of the Gauss-Hermite quadrarure formula.

w = exp(-x.^2);
```

```
approx_v = [];

for n =2:4
   approx_v = [approx_v; (2/sqrt(pi))*cNCqf('exp2', 0, 1, n)];
end
```

```
approx_v

approx_v =
   0.77174333225805
   0.84310283004298
   0.84289057143172
```

For comparison, using MATLAB's built - in function **erf** we obtain the following approximate value of the error function at **x = 1**

```
exact_v = erf(1)

exact_v =
   0.84270079294971
```

5.4.3 Gauss quadature formulas

This class of numerical integration formulas is constructed by requiring that the formulas are exact for polynomials of the highest possible degree. The Gauss formulas are of the type

$$\int_a^b p(x)f(x)dx \approx \sum_{k=1}^{n} w_k f(x_k)$$

where p(x) denotes the *weight function*. Typical choices of the weight functions together with the associated intervals of integration are listed below

Weight p(x)	Interval [a, b]	Quadrature name
1	**[-1, 1]**	Gauss-Legendre
$1/\sqrt{1-x^2}$	**[-1, 1]**	Gauss-Chebyshev
e^{-x}	**$[0,\infty)$**	Gauss-Laguerre
e^{-x^2}	**$(-\infty,\infty)$**	Gauss-Hermite

It is well known that the weights of the Gauss formulas are all positive and the nodes are the roots of the class of polynomials that are orthogonal, with respect to the given weight function **p(x)**, on the associated interval.

Two functions included below**, Gquad1** and **Gquad2** are designed for numerical computation of the definite integrals using Gauss quadrature formulas. A method used here is described in [3], pp. 93 – 94.

```
function [s, w, x] = Gquad1(fun, a, b, n, type, varargin)

% Numerical integration using either the Gauss-Legendre (type = 'L')
% or the Gauss-Chebyshev (type = 'C') quadrature with n (n > 0) nodes.
% fun is a string representing the name of the function that is
% integrated from a to b. For the Gauss - Chebyshev quadrature
% it is assumed that a = -1 and b = 1.
% The output parameters s, w, and x hold the computed approximation
% of the integral, list of weights, and the list of nodes,
% respectively.

d = zeros(1,n-1);
if type == 'L'
   k = 1:n-1;
   d = k./(2*k - 1).*sqrt((2*k - 1)./(2*k + 1));
   fc = 2;
   J = diag(d,-1) + diag(d,1);
   [u,v] = eig(J);
   [x,j] = sort(diag(v));
   w = (fc*u(1,:).^2)';
   w = w(j)';
```

```
   w = 0.5*(b - a)*w;
   x = 0.5*((b - a)*x + a + b);
else
   x = cos((2*(1:n) - (2*n + 1))*pi/(2*n))';
   w(1:n) = pi/n;
end
f = feval(fun,x,varargin{:});
s = w*f(:);
w = w';
```

In this example we will approximate the error function **Erf(1)** using Gauss-Legendre formulas with **n = 2, 3, … , 8**.

```
approx_v = [];

for n=2:8
   approx_v = [approx_v; (2/sqrt(pi))*Gquad1('exp2', 0, 1, n, 'L')];
end

approx_v
```

```
approx_v =
   0.84244189252255
   0.84269001848451
   0.84270117131620
   0.84270078612733
   0.84270079303742
   0.84270079294882
   0.84270079294972
```

Recall that using MATLAB's function **erf** we have already found that

```
exact_v = erf(1)
```

```
exact_v =
   0.84270079294971
```

If the interval of integration is either semi-infinite or bi-infinite then one may use function **Gquad2**. Details of a method used in this function are discussed in [3], pp. 93 – 94.

```
function [s, w, x] = Gquad2(fun, n, type, varargin)

% Numerical integration using either the Gauss-Laguerre
% (type = 'L') or the Gauss-Hermite (type = 'H') with n (n > 0) nodes.
% fun is a string containing the name of the function that is
% integrated.
% The output parameters s, w, and x hold the computed approximation
% of the integral, list of weights, and the list of nodes,
% respectively.

if type == 'L'
   d = -(1:n-1);
```

```
   f = 1:2:2*n-1;
   fc = 1;
else
   d = sqrt(.5*(1:n-1));
   f = zeros(1,n);
   fc = sqrt(pi);
end
J = diag(d,-1) + diag (f) + diag(d,1);
[u,v] = eig(J);
[x,j] = sort(diag(v));
w = (fc*u(1,:).^2)';
w = w(j);
f = feval(fun,x,varargin{:});
s = w'*f(:);
```

The Euler's gamma function

$$\Gamma(t) = \int_0^\infty e^{-x} x^{t-1} dx \qquad (t > -1)$$

can be approximated using function **Gquad2** with type being set to **'L'** (Gauss-Laguerre quadratures). Let us recall that $\Gamma(n) = (n-1)!$ for $n = 1, 2, \ldots$. Function **mygamma** is designed for computing numerical approximation of the gamma function using Gauss-Laguerre quadratures.

```
function y = mygamma(t)

% Value(s) y of the Euler's gamma function evaluated at t (t > -1).

td = t - fix(t);
if td == 0
   n = ceil(t/2);
else
   n = ceil(abs(t)) + 10;
end
y = Gquad2('pow',n,'L',t-1);
```

The following function

```
function z = pow(x, e)

% Power function z = x^e

z = x.^e;
```

is called from within function **mygamma**.

In this example we will approximate the gamma function for **t = 1, 1.1, … , 2** and compare the results with those obtained by using MATLAB's function **gamma**. A script file **testmyg** computes approximate values of the gamma function using two functions **mygamma** and **gamma**

```
% Script testmyg.m

format long
disp('        t               mygamma              gamma')
disp(sprintf('\n
_____________________________________________________________'))

for t=1:.1:2
   s1 = mygamma(t);
   s2 = gamma(t);
   disp(sprintf('%1.14f   %1.14f   %1.14f',t,s1,s2))
end


testmyg

        t               mygamma              gamma

 _____________________________________________________________
1.00000000000000   1.00000000000000   1.00000000000000
1.10000000000000   0.95470549811706   0.95135076986687
1.20000000000000   0.92244757458893   0.91816874239976
1.30000000000000   0.90150911731168   0.89747069630628
1.40000000000000   0.89058495940663   0.88726381750308
1.50000000000000   0.88871435840715   0.88622692545276
1.60000000000000   0.89522845323377   0.89351534928769
1.70000000000000   0.90971011289336   0.90863873285329
1.80000000000000   0.93196414951082   0.93138377098024
1.90000000000000   0.96199632935381   0.96176583190739
2.00000000000000   1.00000000000000   1.00000000000000
```

5.4.4 Romberg's method

Two functions, namely **quad** and **qauad8**, discussed earlier in this tutorial are based on the adaptive methods. Romberg (see, e.g., [2]), proposed another method, which does not belong to this class of methods. This method is the two-phase method. Phase one generates a sequence of approximations using the *composite trapezoidal rule*. Phase two improves approximations found in phase one using *Richardson's extrapolation*. This process is a recursive one and the number of performed iterations depends on the value of the integral parameter **n**. In many cases a modest value for **n** suffices to obtain a satisfactory approximation.

Function **Romberg(fun, a, b, n, varargin)** implements Romberg's algorithm

```
function [rn, r1] = Romberg(fun, a, b, n, varargin)

% Numerical approximation rn of the definite integral from a to b
% that is obtained with the aid of Romberg's method with n rows
% and n columns. fun is a string that names the integrand.
% If integrand depends on parameters, say p1, p2, ... , then
```

```
% they should be supplied just after the parameter n.
% Second output parameter r1 holds approximate values of the
% computed integral obtained with the aid of the composite
% trapezoidal rule using 1, 2, ... ,n subintervals.

h = b - a;
d = 1;
r = zeros(n,1);
r(1) = .5*h*sum(feval(fun,[a b],varargin{:}));
for i=2:n
   h = .5*h;
   d = 2*d;
   t = a + h*(1:2:d);
   s = feval(fun, t, varargin{:});
   r(i) = .5*r(i-1) + h*sum(s);
end
r1 = r;
d = 4;
for j=2:n
   s = zeros(n-j+1,1);
   s = r(j:n) + diff(r(j-1:n))/(d - 1);
   r(j:n) = s;
   d = 4*d;
end
rn = r(n);
```

We will test function **Romberg** integrating the rational function introduced earlier in this tutorial (see the m-file **rfun**). The interval of integration is **[a, b] = [0, 1]**, **n= 10**, and the values of the parameters **a**, **b**, and **c** are set to **1**, **2**, and **1**, respectively.

```
[rn, r1] = Romberg('rfun', 0 , 1, 10, 1, 2, 1)

rn =
   1.47854534395739
r1 =
   1.25000000000000
   1.42500000000000
   1.46544117647059
   1.47528502049722
   1.47773122353730
   1.47834187356141
   1.47849448008531
   1.47853262822223
   1.47854216503816
   1.47854454922849
```

The absolute and relative errors in **rn** are

```
[abs(exact - rn); abs(rn - exact)/exact]

ans =
     0
```

5.4.4 Numerical integration of the bivariate functions using MATLAB function dblquad

Function **dblquad** computes a numerical approximation of the double integral

$$\iint_D f(x,y)dxdy$$

where **D = {(x, y): a ⩽ x ⩽ b, c ⩽ y ⩽ d}** is the domain of integration. Syntax of the function **dblquad** is **dblquad (fun, a, b, c, d, tol)**, where the parameter **tol** has the same meaning as in the function **quad**.

Let **f(x, y) = e^{-xy} sin(xy)** , **-1 ⩽ x ⩽ 1, 0 ⩽ y ⩽ 1**. The m-file **esin** is used to evaluate function **f**

```
function z = esin(x,y);

z = exp(-x*y).*sin(x*y);
```

Integrating function **f** , with the aid of the function **dblquad**, over the indicated domain we obtain

```
result = dblquad('esin', -1, 1, 0, 1)

result =
  -0.22176646183245
```

5.4.5 Numerical differentiation

Problem discussed in this section is formulated as follows. Given a univariate function **f(x)** find an approximate value of **f '(x)**. The algorithm presented below computes a sequence of the approximate values to derivative in question using the following finite difference approximation of **f '(x)**

$$f'(x) \approx \frac{f(x+h)-f(x-h)}{2h}$$

where **h** is the initial stepsize. Phase one of this method computes a sequence of approximations to f'(x) using several values of **h**. When the next approximation is sought the previous value of **h** is halved. Phase two utilizes Richardson's extrapolation. For more details the reader is referred to [2], pp. 171 – 180.

Function **numder** implements the method introduced in this section.

```
function der = numder(fun, x, h, n, varargin)

% Approximation der of the first order derivative, at the point x,
% of a function named by the string fun. Parameters h and n
% are user supplied values of the initial stepsize and the number
% of performed iterations in the Richardson extrapolation.
% For fuctions that depend on parameters their values must follow
% the parameter n.

d = [];
for i=1:n
   s = (feval(fun,x+h,varargin{:})-feval(fun,x-h,varargin{:}))/(2*h);
   d = [d;s];
   h = .5*h;
end
l = 4;
for j=2:n
   s = zeros(n-j+1,1);
   s = d(j:n) + diff(d(j-1:n))/(l - 1);
   d(j:n) = s;
   l = 4*l;
end
der = d(n);
```

In this example numerical approximations of the first order derivative of the function $\mathbf{f(x) = e^{-x^2}}$ are computed using function **numder** and they are compared against the exact values of **f '(x)** at **x = 0.1, 0.2, … , 1.0**. The values of the input parameters **h** and **n** are **0.01** and **10**, respectively.

```
function testnder(h, n)

% Test file for the function numder. The initial stepsize is h and
% the number of iterations is n. Function to be tested is
% f(x) = exp(-x^2).

format long
disp('        x               numder               exact')
disp(sprintf('\n
_____________________________________________________________'))

for x=.1:.1:1
   s1 = numder('exp2', x, h, n);
   s2 = derexp2(x);
   disp(sprintf('%1.14f   %1.14f   %1.14f',x,s1,s2))
end



function y = derexp2(x)

% First order derivative of f(x) = exp(-x^2).

y = -2*x.*exp(-x.^2);
```

The following results are obtained with the aid of function **testndr**

```
testnder(0.01, 10)
```

```
        x                numder                exact

  ____________________________________________________________

0.10000000000000   -0.19800996675001   -0.19800996674983
0.20000000000000   -0.38431577566308   -0.38431577566093
0.30000000000000   -0.54835871116311   -0.54835871116274
0.40000000000000   -0.68171503117430   -0.68171503117297
0.50000000000000   -0.77880078306967   -0.77880078307140
0.60000000000000   -0.83721159128436   -0.83721159128524
0.70000000000000   -0.85767695185699   -0.85767695185818
0.80000000000000   -0.84366787846708   -0.84366787846888
0.90000000000000   -0.80074451919839   -0.80074451920129
```

5.5 Numerical Methods for the Ordinary Differential Equations

Many problems that arise in science and engineering require a knowledge of a function **y = y(t)** that satisfies the *first order differential equation* **y' = f(t, y)** and the *initial condition* **y(a) = y_0**, where **a** and **y_0** are given real numbers and **f** is a bivariate function that satisfies certain smoothness conditions. A more general problem is formulated as follows. Given function **f** of **n** variables, find a function **y = y(t)** that satisfies the *nth order* ordinary differential equation **$y^{(n)}$ = f(t, y, y', … , $y^{(n-1)}$)** together with the initial conditions **y(a) = y_0**, **y'(a) = y_0'**, … , **$y^{(n-1)}$ (a) = $y_0^{(n-1)}$**. The latter problem is often transformed into the problem of solving a system of the first order differential equations. To this end a term "ordinary differential equations" will be abbreviated as ODEs.

5.5.1 Solving the initial value problems using MATLAB built-in functions

MATLAB has several functions for computing a numerical solution of the initial value problems for the ODEs. They are listed in the following table

Function	Application	Method used
ode23	Nonstiff ODEs	Explicit Runge-Kutta (2, 3) formula
ode45	Nonstiff ODEs	Explicit Runge-Kutta (4, 5) formula
ode113	Nonstiff ODEs	Adams-Bashforth-Moulton solver
ode15s	Stiff ODEs	Solver based on the numerical differentiation formulas
ode23s	Stiff ODEs	Solver based on a modified Rosenbrock formula of order 2

A simplest form of the syntax for the MATLAB ODE solvers is

[t, y] = *solver*(fun, tspan, y0], where **fun** is a string containing name of the ODE m-file that describes the differential equation, **tspan** is the interval of integration, and **y0** is the vector holding the initial value(s). If **tspan** has more than two elements, then solver returns computed values of **y** at these points. The output parameters **t** and **y** are the vectors holding the points of evaluation and the computed values of **y** at these points.

In the following example we will seek a numerical solution **y** at **t = 0, .25, .5, .75, 1** to the following initial value problem $y' = -2ty^2$, with the initial condition **y(0) = 1**. We will use both the **ode23** and the **ode45** solvers. The exact solution to this problem is $y(t) = 1/(1 + t^2)$ (see, e.g., [6], p.289). The ODE m-file needed in these computations is named **eq1**

```
function dy = eq1(t,y)

% The m-file for the ODE y' = -2ty^2.

dy = -2*t.*y(1).^2;
```

```
format long

tspan = [0 .25 .5 .75 1]; y0 = 1;

[t1 y1] = ode23('eq1', tspan, y0);
[t2 y2] = ode45('eq1', tspan, y0);
```

To compare obtained results let us create a three-column table holding the points of evaluation and the y-values obtained with the aid of the **ode23** and the **ode45** solvers

```
[t1 y1 y2]

ans =
                   0   1.00000000000000   1.00000000000000
   0.25000000000000   0.94118221525751   0.94117646765650
   0.50000000000000   0.80002280597122   0.79999999678380
   0.75000000000000   0.64001788410487   0.63999998775736
   1.00000000000000   0.49999658522366   0.50000000471194
```

Next example deals with the system of the *first order* ODEs

$$y_1'(t) = y_1(t) - 4y_2(t),\ y_2'(t) = -y_1(t) + y_2(t),$$
$$y_1(0) = 1;\ y_2(0) = 0.$$

Instead of writing the ODE m – file for this system, we will use MATLAB **inline** function

```
dy = inline('[1 -4;-1 1]*y', 't', 'y')

dy =
     Inline function:
     dy(t,y) = [1 -4;-1 1]*y
```

The inline functions are created in the Command Window. Interval over wich numerical solution is computed and the initial values are stored in the vectors **tspan** and **y0**, respectively

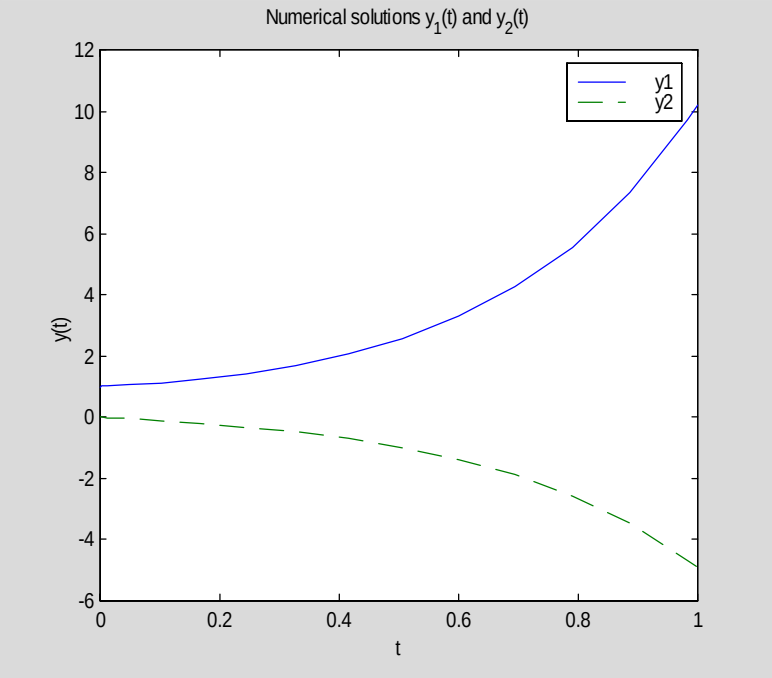
```
tspan = [0 1];  y0 = [1 0];
```

Numerical solution to this system is obtained using the **ode23** function

```
[t,y] = ode23(dy, tspan, y0);
```

Graphs of $y_1(t)$ (solid line) and $y_2(t)$ (dashed line) are shown below

```
plot(t,y(:,1),t,y(:,2),'--'), legend('y1','y2'), xlabel('t'),
ylabel('y(t)'), title('Numerical solutions y_1(t) and y_2(t)')
```



The exact solution $(y_1(t), y_2(t))$ to this system is

```
y1, y2

y1 =
1/2*exp(-t)+1/2*exp(3*t)
y2 =
-1/4*exp(3*t)+1/4*exp(-t)
```

Functions **y1** and **y2** were found using command **dsolve** which is available in the **Symbolic Math Toolbox**.

Last example in this section deals with the *stiff ODE*. Consider

$$y'(t) = -1000(y - \log(1 + t)) + \frac{1}{1+t},$$
$$y(0) = 1.$$

```
dy = inline('-1000*(y - log(1 + t)) + 1/(1 + t)', 't', 'y')

dy =
     Inline function:
     dy(t,y) = -1000*(y - log(1 + t)) + 1/(1 + t)
```

Using the **ode23s** function on the interval

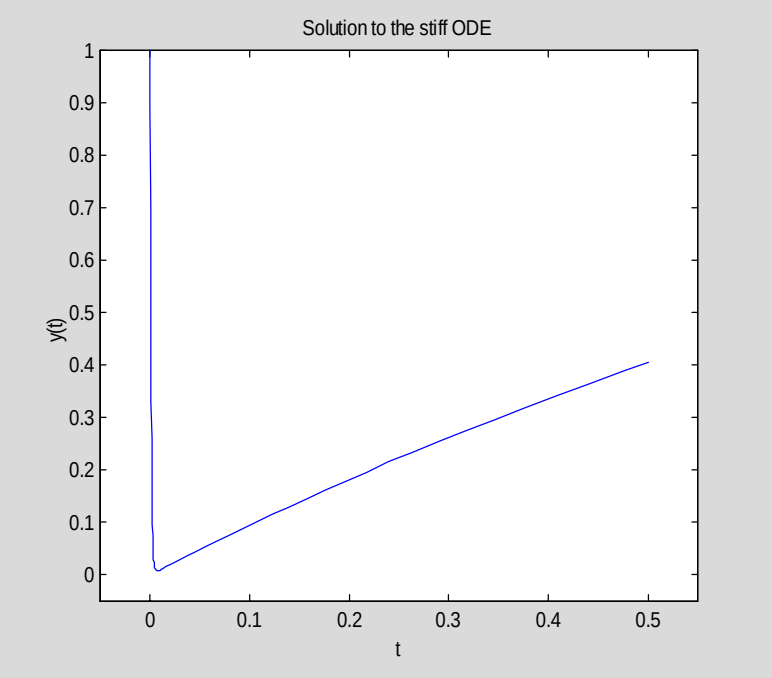
```
tspan = [0 0.5];
```

we obtain

```
[t, y] = ode23s(dy, tspan, 1);
```

To illustrate the effect of stiffness of the differential equation in question, let us plot the graph of the computed solution

```
plot(t, y), axis([-.05 .55 -.05 1] ), xlabel('t'), ylabel('y(t)'),
title('Solution to the stiff ODE')
```



The exact solution to this problem is **y(t) = log(1+t) + exp(-1000*t)**. Try to plot this function on the interval **[-0.05, 0.5]**.

5.5.2 The two – point boundary value problem for the second order ODE's

The purpose of this section is to discuss a numerical method for the two – point boundary value problem for the second order ODE

$$y''(t) = f(t, y, y')$$
$$y(a) = ya,\ \ y(b) = yb.$$

A method in question is the *finite difference method*. Let us assume that the function **f** is of the form $f(t, y, y') = g_0(t) + g_1(t)y + g_2(t)y'$. Thus the function **f** is linear in both **y** and **y'**. Using standard second order approximations for **y'** and y'' one can easily construct a linear system of equations for computing approximate values of the function y on the set of evenly spaced points. Function **bvp2ode** implements this method

```
function [t, y] = bvp2ode(g0, g1, g2, tspan, bc, n)

% Numerical solution y of the boundary value problem
% y'' = g0(t) + g1(t)*y + g2(t)*y', y(a) = ya, y(b) = yb,
% at n+2 evenly spaced points t in the interval tspan = [a b].
% g0, g1, and g2 are strings representing functions g0(t),
% g1(t), and g2(t), respectively. The boundary values
% ya and yb are stored in the vector bc = [ya yb].

a = tspan(1);
b = tspan(2);
t = linspace(a,b,n+2);
t1 = t(2:n+1);
u = feval(g0, t1);
v = feval(g1, t1);
w = feval(g2, t1);
h = (b-a)/(n+1);
d1 = 1+.5*h*w(1:n-1);
d2 = -(2+v(1:n)*h^2);
d3 = 1-.5*h*w(2:n);
A = diag(d1,-1) + diag(d2) + diag(d3,1);
f = zeros(n,1);
f(1) = h^2*u(1) - (1+.5*h*w(1))*bc(1);
f(n) = h^2*u(n) - (1-.5*h*w(n))*bc(2);
f(2:n-1) = h^2*u(2:n-1)';
s = A\f;
y = [bc(1);s;bc(2)];
t = t';
```

In this example we will deal with the two-point boundary value problem

$$y''(t) = 1 + \sin(t)y + \cos(t)y'$$
$$y(0) = y(1) = 1.$$

We define three inline functions

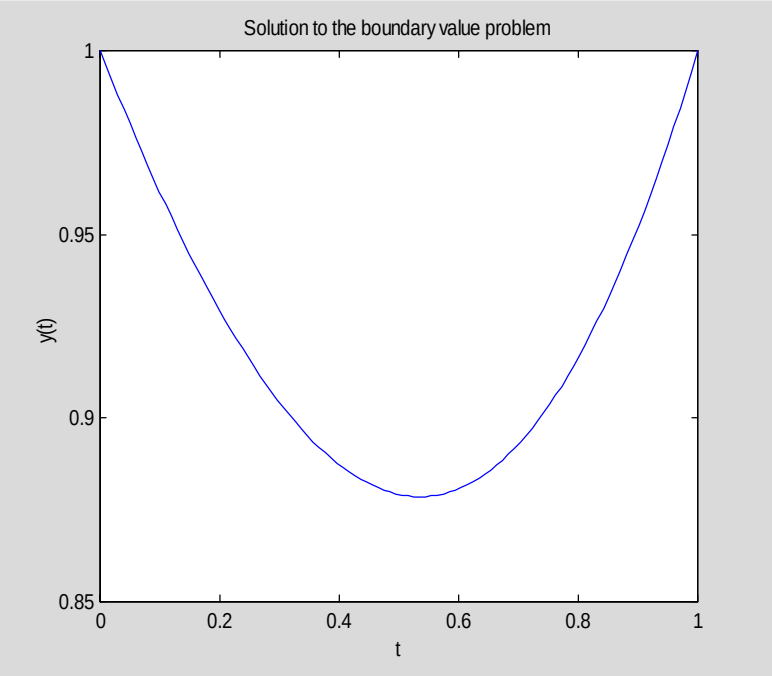
```
g0 = inline('ones(1, length(t))', 't'), g1 = inline('sin(t)', 't'), g2
= inline('cos(t)', 't')

g0 =
     Inline function:
     g0(t) = ones(1, length(t))
g1 =
     Inline function:
     g1(t) = sin(t)
g2 =
     Inline function:
     g2(t) = cos(t)
```

and next run function **bvp2ode** to obtain

```
[t, y] = bvp2ode(g0, g1, g2, [0 1],[1 1],100);
```

Graph of a function generated by **bvp2ode** is shown below

```
plot(t, y), axis([0 1 0.85 1]), title('Solution to the boundary value
problem'), xlabel('t'), ylabel('y(t)')
```



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Problems

1. Give an example of a polynomial of degree $\mathbf{n \geqslant 3}$ with real roots only for which function **roots** fails to compute a correct type of its roots.

2. All roots of the polynomial $\mathbf{p(x) = a_0 + a_1 x + \ldots + a_{n-1} x^{n-1} + x^n}$, with real coefficients $\mathbf{a_k\ (k = 0, 1, \ldots n - 1)}$, are the eigenvalues of the *companion matrix*

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{1} & \mathbf{\ldots} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{\ldots} & \mathbf{0} \\ \mathbf{.} & \mathbf{.} & \mathbf{.} & \mathbf{.} & \mathbf{.} \\ \mathbf{-a_0} & \mathbf{-a_1} & \mathbf{-a_2} & \mathbf{\ldots} & \mathbf{-a_{n-1}} \end{bmatrix}$$

   Write MATLAB function **r = polroots(a)** that takes a one-dimensional array **a** of the coefficients of the polynomial **p(x)** in the descending order of powers and returns its roots in the array **r**.
   Organize your work as follows:

   (i) Create a matrix **A**. You may wish to use MATLAB's built-in function **diag** to avoid using loops. Function **diag** takes a second argument that can be used to put a superdiagonal in the desired position.
   (ii) Use MATLAB's function **eig** to compute all eigenvalues of the companion matrix **A**. See Tutorial 4 for more details about the matrix eigenvalue problem.

3. Write MATLAB function **[r, niter] = fpiter(g, x0, maxiter)** that computes a zero **r** of **x = g(x)** using the fixed-point iteration $\mathbf{x_{n+1} = g(x_n)}$, **n = 0, 1, …** with a given initial approximation **x0** of **r**. The input parameter **maxiter** is the maximum number of allowed iterations while the output parameter **niter** stands for the number of iterations performed. Use an appropriate stopping criterion to interrupt computations when current approximation satisfies the exit condition of your choice.

4. In this exercise you are to test function **fpiter** of Problem 3.
   Recall that a convergent sequence $\mathbf{\{x^{(k)}\}}$, with the limit **r**, has the *order of convergence* μ if

$$|\mathbf{x^{(k+1)} - r}| \leqslant \mathbf{C}|\mathbf{x^{(k)} - r}|^{\mu}, \quad \text{for some } \mathbf{C > 0}.$$

   If $\mu = \mathbf{1}$, then $\mathbf{C < 1}$.

   (i) Construct at least one equation of the form **x = g(x)**, with at least one real zero, for which function **fpiter** computes a sequence of approximations $\mathbf{\{x_n\}}$ that converges to the zero of your function. Print out consecutive approximations of the zero **r** and determine the order of convergence.
   (ii) Repeat previous part where this time a sequence of approximations generated by the function **fpiter** does not converge to the zero **r**. Explain why a computed sequence diverges.

5. Derive Newton's iteration for a problem of computing the reciprocal of a nonzero number **a**.
   (i) Does your iteration always converge for any value of the initial guess x_0?
   (ii) Write MATLAB function **r = recp(a, x0)** that computes the reciprocal of **a** using Newton's method with the initial guess **x0**.
   (iii) Run function **recp** for the following following values of **(a, x_0)** : (2, 0.3) and (10, 0.15) and print out consecutive approximations generated by the function **recp** and determine the order of convergence.

6. In this exercise you are to write MATLAB function **[r, niter] = Sch(f, derf, x0, m, tol)** to compute a multiple root **r** of the function **f(x)**. Recall that **r** is a root of multiplicity **m** of **f(x)** if $f(x) = (x - r)^m g(x)$, for some function **g(x)**. Schroder (see [8]) has proposed the following iterative scheme for computing a multiple root **r** of **f(x)**

$$x_{k+1} = x_k - mf(x_k)/f'(x_k), \quad k = 0, 1, \ldots .$$

   When **m = 1**, this method becomes the Newton – Raphson method.
   The input parameters: **f** is the function with a multiple root **r**, **derf** is the first derivative of **f**, **x0** is the initial guess, **m** stands for the multiplicity of **r** and **tol** is the assumed tolerance for the computed root.
   The output parameters: **r** is the computed root and **niter** is the number of performed iterations.

7. In this exercise you are to test function **Sch** of Problem 6.

   (i) Use function **f2** defined in Section 5.2 and write function **derf2** to compute the first order derivative of a function in file **f2**.
   (ii) Use unexpanded form for the derivative. Run function **Sch** with **m = 5** then repeat this experiment letting **m = 1**. In each case choose $x_0 = 0$. Compare number of iterations performed in each case.
   (iii) Repeat the above experiment using function **f3**. You will need a function **derf3** to evaluate the first derivative in the expanded form.

8. Let **p(x)** be a cubic polynomial with three distinct real roots r_k , **k = 1, 2, 3**. Suppose that the exact values of r_1 and r_2 are available. To compute the root r_3 one wants to use function **Sch** of Problem 6 with **m = 1** and $x_0 = (r_1 + r_2)/2$. How many iterations are needed to compute r_3?

9. Based on your observations made during the numerical experiments performed when solving Problem 8 prove that only one step of the Newton-Raphson method is needed to compute the third root of **p(x)**.

10. Given a system of nonlinear equations

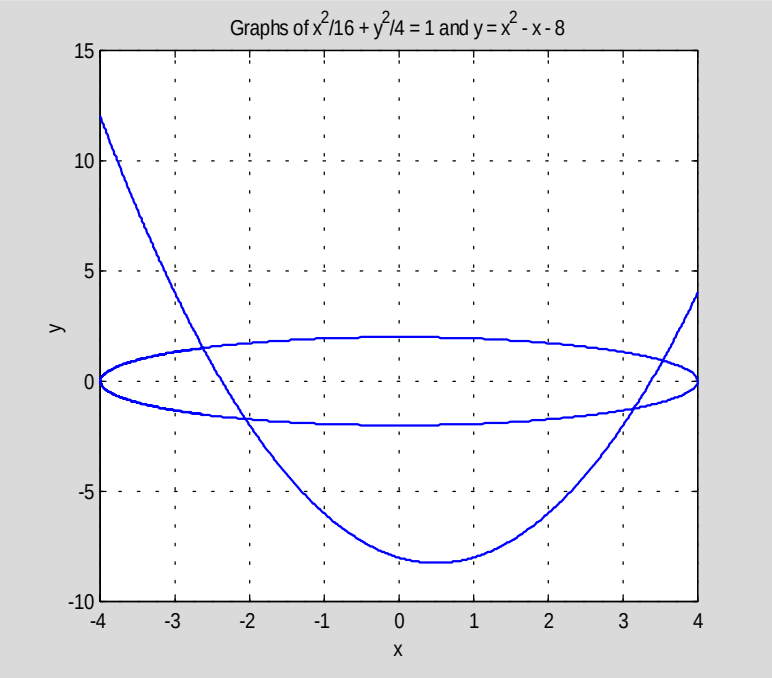
$$x^2/16 + y^2/4 = 1$$
$$x^2 - y^2 = 1$$

    Use function **NR** to compute all the zeros of this system. Compare your results with the exact values **x = ±2** and **y = ±√3**. Evaluate function **f** at the computed zeros and print your results using **format long**.

11. Using function **NR** find all the zeros of the system of nonlinear equations

$$x^2/16 + y^2/4 = 1$$
$$x^2 - x - y - 8 = 0$$

The following graph should help you to choose the initial approximations to the zeros of this system



Evaluate function **f** at the computed zeros and print out your results using **format long**.

12. Another method for computing zeros of the scalar equation **f(x) = 0** is the *secant method*. Given two initial approximations x_0 and x_1 of the zero **r** this method generates a sequence $\{x_k\}$ using the iterative scheme

$$x_{k+1} = x_k - f(x_k)\frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})}, \quad k = 1, 2, \ldots .$$

Write MATLAB function **[r, niter] = secm(f, x0, x1, tol, maxiter)** that computes the zero **r** of **f(x) = 0**. The input parameters: **f** is the name of a function whose zero is computed, **x0** and **x1** are the initial approximations of **r**, **tol** is the prescribed tolerance and **maxiter** is the maximum number of the allowed iterations. The output parameters: **r** is the computed zero of **f(x)** and **niter** is the number of the performed iterations.

13. Use the function **secm** of Problem 12 to find the smallest positive zero of **f(x)**.

(i) $f(x) = \sin(\tan(x)) - x$
(ii) $f(x) = \sin(x) + 1/(1 + e^{-x}) - 1$
(iii) $f(x) = \cos(x) - e^{-\sin(x)}$

Evaluate each function **f** at the computed zero and print out results using **format long**.

14. Another form of the interpolating polynomial is due to Lagrange and uses as the basis function the so–called *fundamental polynomials* $L_k(x)$, $0 \leq k \leq n$. The kth fundamental polynomial L_k is defined as follows: $L_k(x_k) = 1$, $L_k(x_m) = 0$ for $k \neq m$, and $\deg(L_k) \leq n$. Write MATLAB function **yi = fundpol(k, x, xi)** which evaluates the kth Lagrange fundamental polynomial at points stored in the array **xi**.

15. The Lagrange form of the interpolating polynomial $p_n(x)$ of degree at most **n** which interpolates the data (x_k , y_k), $0 \leq k \leq n$, is

$$p_n(x) = y_0L_0(x) + y_1L_1(x) + \ldots + y_nL_n(x)$$

Write MATLAB function **yi = Lagrpol(x, y, xi)** that evaluates polynomial p_n at points stored in the array **xi**. You may wish to use function **fundpol** of Problem 14.

16. In this exercise you are to interpolate function **g(x)**, $a \leq x \leq b$, using functions **Newtonpol** (see Section 5.3) and **Lagrpol** (see Problem 15). Arrays **x**, **y**, and **xi** are defined as follows $x_k = a + k(b - a)/10$, $y_k = g(x_k)$, $k = 0, 1, \ldots , 10$, and **xi = linspace(a, b).** Run both functions using the following functions **g(x)** and the associated intervals **[a, b]**

(i) $g(x) = \sin(4\pi x)$, $[a, b] = [0, 1]$

(ii) $g(x) = J_0(x)$, $[a, b] = [2, 3]$,

where J_0 stands for the Bessel function of the first kind of order zero. In MATLAB Bessel function $J_0(x)$ can be evaluated using command **besselj(0, x)**.
In each case find the values **yi** of the interpolating polynomial at **xi** and compute the error maximum **err = norm(abs(yi - g(xi)), 'inf ')**. Compare efficiency of two methods used to interpolate function **g(x)**. Which method is more efficient? Explain why.

17. Continuation of Problem 16. Using MATLAB's function **interp1**, with options **'cubic'** and **'spline'**, interpolate both functions **g(x)** of Problem 16 and answer the same questions as stated in this problem. Among four method if interpolation you have used to interpolate functions g(x) which method is the the best one as long as

(i) efficiency is considered?

(ii) accuracy is considered?

18. The *Lebesgue function* $\Lambda(x)$ of the interpolating operator is defined as follows

$$\Lambda(x) = |L_0(x)| + |L_1(x)| + \ldots + |L_n(x)|,$$

where L_k stands for the kth fundamental polynomial introduced in Problem 14. This function was investigated in depth by numerous researchers. It's global maximum over the interval of interpolation provides a useful information about the error of interpolation.
In this exercise you are to graph function $\Lambda(x)$ for various sets of the interpolating abscissa $\{x_k\}$. We will assume that the points of interpolation are symmetric with respect to the origin, i.e., $-x_k = x_{n-k}$, for $k = 0, 1, \ldots , n$. Without loss of generality, we may also assume

that $-x_0 = x_n = 1$. Plot the graph of the Lebesgue function $\Lambda(x)$ for the following choices of the points x_k

(i) $x_k = -1 + 2k/n, \quad k = 0, 1, \dots, n$

(ii) $x_k = -\cos(k\pi/n), \quad k = 0, 1, \dots, n$

In each case put $n = 1, 2, 3$ and estimate the global maximum of $\Lambda(x)$. Which set of the interpolating abscissa provides a smaller value of $\mathbf{Max}\{\Lambda(x) : x_0 \leq x \leq x_n\}$?

19. MATLAB's function **polyder** computes the first order derivative of an algebraic polynomial that is represented by its coefficients in the descending order of powers. In this exercise you are to write MATLAB function **B = pold(A, k)** that computes the kth order derivative of several polynomials of the same degree whose coefficients are stored in the consecutive rows of the matrix **A**. This utility function is useful in manipulations with splines that are represented as the piecewise polynomial functions.
Hint: You may wish to use function **polyder**.

20. The Hermite cubic spline interpolant $s(x)$ with the breakpoints $\Delta = \{x_x < x_2 < \dots < x_m\}$ is a member of $Sp(3, 1, \Delta)$ that is uniquely determined by the interpolatory conditions

(i) $s(x_l) = y_l, \quad l = 1, 2, \dots, m$

(ii) $s'(x_l) = p_l, \quad l = 1, 2, \dots, m$

On the subinterval $[x_l, x_{l+1}]$, $l = 1, 2, \dots, m - 1$, $s(x)$ is represented as follows

$$s(x) = (1 + 2t)(1 - t)^2 y_l + (3 - 2t)t^2 y_{l+1} + h_l[t(1 - t)^2 p_l + t^2(t - 1)p_{l+1}],$$

where $t = (x - x_l)/(x_{l+1} - x_l)$ and $h_l = x_{l+1} - x_l$.
Prove that the Hermite cubic spline interpolant s(x) is convex on the interval $[x_1, x_m]$ if and only if the following inequalities

$$\frac{2p_l + p_{l+1}}{3} \leq \frac{s_{l+1} - s_l}{h_l} \leq \frac{p_l + 2p_{l+1}}{3}$$

are satisfied for all $l = 1, 2, \dots, m - 1$.

21. Write MATLAB function **[pts, yi] = Hermspl(x, y, p)** that computes coefficients of the Hermite cubic spline interpolant **s(x)** described in Problem 20 and evaluates spline interpolant at points stored in the array **xi**. Parameters **x**, **y**, and **p** stand for the breakpoints, values of **s(x)**, and values of **s'(x)** at the breakpoints of **s(x)**, respectively. The output parameter **yi** is the array of values of **s(x)** at points stored in the array **pts** which is defined as the union of the arrays **linspace(x(k), x(k+1))**, **k = 1, 2, … , n – 1**, where **n = length(x)**.
Hint: You may wish to use function **Hermpol** discussed in Section 5.3.

22. The nodes $\{x_k\}$ of the Newton – Cotes formulas of the open type are defined as follows $x_k = a + (k - 1/2)h$, $k = 1, 2, \dots, n - 1$, **where h = (b – a)/(n – 1)**. Write MATLAB function **[s, w, x] = oNCqf(fun, a, b, n, varargin)** that computes an approximate value **s** of

the integral of the function that is represented by the string **fun**. Interval of integration is **[a, b]** and the method used is the n-point open formula whose weights and nodes are are stored in the arrays **w** and **x**, respectively.

23. The Fresnel integral

$$\mathbf{f(x) = \int_0^x \exp\left(\frac{i\pi t^2}{2}\right)dt}$$

is of interest in several areas of applied mathematics. Write MATLAB function **[fr1, fr2] = Fresnel(x, tol, n)** which takes a real array **x**, a two dimensional vector **tol** holding the relative and absolute tolerance for the error of the computed integral (see MATLAB help file for the function **quad8**), and a positive integer **n** used in the function **Romberg** and returns numerical approximations **fr1** and **fr2** of the Fresnel integral using each of the following methods

(i) **quad8** with tolerance **tol = [1e-8 1e-8]**
(ii) **Romberg** with **n = 10**

Compute Fresnel integrals for the following values of **x = 0: 0.1:1**.To compare the approximations **fr1** and **fr2** calculate the number of *decimal places of accuracy* **dpa = –log10(norm(fr1 – fr2, 'inf '))**. For what choices of the input parameters **tol** and **n** the number **dpa** is greater than or equal to 13? The last inequality must be satisfied for all values **x** as defined earlier.

24. Let us assume that the real-valued function **f(x)** has a convergent integral

$$\mathbf{\int_0^\infty f(x)dx}\,.$$

Explain how would you compute an approximate value of this integral using function **Gquad2** developed earlier in this chapter? Extend your idea to convergent integrals of the form

$$\mathbf{\int_{-\infty}^{\infty} f(x)dx.}$$

25. The following integral is discussed in [3], p. 317

$$\mathbf{J = \int_{-1}^{1} \frac{dx}{x^4 + x^2 + 0.9}}\,.$$

To compute an approximate value of the integral **J** use

(i) MATLAB functions **quad** and **quad8** with tolerance **tol = [1e-8 1e-8]**
(ii) functions **Romberg** and **Gquad1** with **n = 8**.

Print out numerical results using **format long**. Which method should be recommended for numerical integration of the integral **J**? Justify your answer.

26. The arc length **s** of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

from **(a, 0)** to **(x, y)** in quadrant one is equal to

$$s = b\int_0^{\theta}\sqrt{1-k^2\sin^2 t}\,dt$$

where $k^2 = 1 - (a/b)^2$, $a \le b$, and $\theta = \arccos(x/a) = \arcsin(y/b)$.
In this exercise you are to write MATLAB function **[sR, sq8] = arcell(a, b, x, n, tol)** that takes as the input parameters the semiaxes **a** and **b** and the x – coordinate of the point on the ellipse in quadrant one and returns an approximate value of the arc of ellipse from **(a, 0)** to **(x, y)** using functions **Romberg**, described in this chapter, and the MATLAB function **quad8**. The fourth input parameter **n** will be used by the function **Romberg**. The fifth input parameter **tol** is optional and it holds the user supplied tolerances for the relative and absolute errors in the computed approximation **sq8**. If **tol** is not supplied, the default values for tolerances should be assigned. For more details about using this parameter, type **help quad8** in the **Command Window**. Your program should also work for ellipses whose semiaxes are not restricted to those in the definition of the parameter k^2. Test your function for the following values of the input parameters

(i) **a = 1, b = 2, x = 1: -0.1: 0, n = 10, tol = [ ]**
(ii) **a = 2, b = 1, x = 2: -0.2: 0, n = 10, tol = [ ]**
(iii) **a = 2, b = 1, x = 0: -0.2: -2, n = 10, tol = [ ]**

Note that the terminal points **(x, y)** of the third ellipse lie in quadrant two.

27. Many of the most important special functions can be represented as the *Dirichlet average* **F** of a continuous function **f** (see [1])

$$F(b_1, b_2; a, b) = \frac{1}{B(b_1, b_2)}\int_0^1 t^{b_1-1}(1-t)^{b_2-1} f[ta+(1-t)b]\,dt,$$

where $B(b_1, b_2)$, $(b_1, b_2 > 0)$ stands for the *beta function*. Of special interest are the Dirichlet averages of elementary functions $f(t) = t^{-c}$ and $f(t) = e^t$. Former gives raise to the *hypergeometric functions* such as a celebrated *Gauss hypergeometric function* ${}_2F_1$ while the latter is used to represent the *confluent hypergeometric functions*.
In this exercise you are to implement a method for approximating the Dirichlet integral defined above using $f(t) = t^{-c}$. Write MATLAB function **y = dav(c, b1, b2, a, b)** which computes a numerical approximation of the Dirichlet average of **f**. Use a method of your choice to integrate numerically the Dirichlet integral. MATLAB has a function named **beta** designed for evaluating the beta function. Test your function for the following values of the parameter **c**:

c = 0 (exact value of the Dirichlet average **F** is equal to **1**)

$c = b_1 + b_2$ (exact value of the Dirichlet average is equal to **$1/(a^{b}_{1}\, b^{b}_{2})$**.

28. Gauss hypergeometric function **$_2F_1(a, b; c; x)$** is defined by the infinite power series as follows

$$_2F_1(a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a,n)(b,n)}{(c,n)n!} x^n, \quad |x| \le 1,$$

where **$(a, n) = a(a + 1) \ldots (a + n - 1)$** is the *Appel symbol*. Gauss hypergeometric function can be represented as the Dirichlet average of the power function **$f(t) = t^{-a}$**

$$_2F_1(a, b; c; x) = F(b, c - b; 1 - x, 1) \qquad (c > b > 0,\ |x| < 1).$$

Many of the important elementary functions are special cases of this function. For instance for **$|x| < 1$**, the following formulas

$$\arcsin x = {}_2F_1(0.5, 0.5; 1.5; x^2)$$
$$\ln(1 + x) = x\,{}_2F_1(1, 1; 1.5; x^2)$$
$$\operatorname{arctanh} x = x\,{}_2F_1(0.5, 1; 1.5; x^2)$$

hold true. In this exercise you are to use function **dav** of Problem 27 to evaluate three functions listed above for **x = -0.9 : 0.1 : 0.9**. Compare obtained approximate values with those obtained by using MATLAB functions **asin**, **log**, and **atanh**.

29 Let **a** and **b** be positive numbers. In this exercise you will deal with the four formulas for computing the mean value of **a** and **b**. Among the well – known means the *arithmetic mean* **A(a, b) = (a + b)/2** and the geometric mean **$G(a, b) = \sqrt{ab}$** are the most frequently used ones. Two less known means are the *logarithmic mean* **L** and the *identric mean* **I**

$$L(a, b) = \frac{a - b}{\ln a - \ln b}$$

$$I(a, b) = e^{-1}(a^a/b^b)^{1/(a-b)}$$

The logarithmic and identric means are of interest in some problems that arise in economics, electrostatics, to mention a few areas only. All four means described in this problem can be represented as the Dirichlet averages of some elementary functions. For the means under discussion their b – parameters are both equal to one. Let **M(a, b)** stand for any of these means. Then

$$M(a, b) = f^{-1}(F(1, 1; a, b))$$

where **f^{-1}** stands for the inverse function of **f** and **F** is the Dirichlet average of **f**.
In this exercise you will deal with the means described earlier in this problem.

(i) Prove that the arithmetic, geometric, logarithmic and identric means can be represented as the inverse function of the Dirichlet average of the following functions $\mathbf{f(t) = t}$, $\mathbf{f(t) = t^{-2}}$, $\mathbf{f(t) = t^{-1}}$ and $\mathbf{f(t) = \ln t}$, respectively.

(ii) The logarithmic mean can also be represented as

$$\mathbf{L(a,b) = \int_0^1 a^t b^{1-t} dt.}$$

Establish this formula.

(iii) Use the integral representations you found in the previous part together with the midpoint and the trapezoidal quadrature formulas (with the error terms) to establish the following inequalities: $\mathbf{G \le L \le A}$ and $\mathbf{G \le I \le A}$. For the sake of brevity the arguments **a** and **b** are omitted in these inequalities.

30. A second order approximation of the second derivative of the function **f(x)** is

$$\mathbf{f\,''(x) = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} + O(h^2).}$$

Write MATLAB function **der2 = numder2(fun, x, h, n, varargin)** that computes an approximate value of the second order derivative of a function named by the string **fun** at the point **x**. Parameters **h** and **n** are user-supplied values of the initial step size and the number of performed iterations in the Richardson extrapolation. For functions that depend on parameters their values must follow the parameter **n**.
Test your function for **f(x) = tan x** with $\mathbf{x = \pi/4}$ and **h = 0.01**. Experiment with different values for **n** and compare your results with the exact value $\mathbf{f\,''(\pi/4) = 4}$.

31. Consider the following initial – value problem

$$\mathbf{y_1'''(t) = - y_1(t)y_1''(t), \quad y_1(0) = 1, \quad y_1'(0) = -1, \quad y_1''(0) = 1.}$$

(i) Replace the differential equation by the system of three differential equations of order one.

(ii) Write a MATLAB function **dy = order3(t, y)** that evaluates the right – hand sides of the equations you found in the previous part.

(iii) Write a script file **Problem31** to solve the resulting initial – value problem using MATLAB solver **ode45** on the interval **[0 1]**.

(iv) Plot in the same window graphs of function $\mathbf{y_1(t)}$ together with its derivatives up to order two. Add a legend to your graph that clearly describes curves you are plotting.

32. In this exercise you are to deal with the following initial – value problem

$$\mathbf{x\,'(t) = -x(t) - y(t), \quad y\,'(t) = -20x(t) - 2y(t), \quad x(0) = 2, \quad y(0) = 0.}$$

(i) Determine whether or not this system is stiff.

(ii) If it is, use an appropriate MATLAB solver to find a numerical solution on the interval **[0 1]**.

(iii) Plot the graphs of functions **x(t)** and **y(t)** in the same window.

33. The Lotka – Volterra equations describe populations of two species

$$y_1'(t) = y_1(t) - y_1(t)y_2(t), \quad y_2'(t) = -15y_2(t) + y_1(t)y_2(t).$$

Write MATLAB function **LV(y10, y20, tspan)** that takes the initial values **y10** = $y_1(0)$ and **y20** = $y_2(0)$ and plots graphs of the numerical solutions **y1** and **y2** over the interval **tspan**.

34. Numerical evaluation of a definite integral

$$\int_a^b f(t)dt$$

can be accomplished by solving the ODE **y'(t) = f(t)** with the initial condition **y(a) = 0**. Then the integral in question is equal to **y(b)**. Write MATLAB function **yb = integral(a, b, fun)** which implements this method. The input parameter **fun** is the string holding the name of the integrand **f(t)** and **a** and **b** are the limits of integration. To find a numerical solution of the ODE use the MATLAB solver **ode45**. Test your function on integrals of your choice.

35. Given the two – point boundary value problem

$$y'' = -t + t^2 + e^t - ty + ty', \quad y(0) = 1, \quad y(1) = 1 + e.$$

(i) Use function **bvp2ode** included in this tutorial to find the approximate values of the function **y** for the following values of **n = 8, 18**.

(ii) Plot, in the same window, graphs of functions you found in the previous part of the problem. Also, plot the graph of function $y(t) = t + e^t$ which is the exact solution to the boundary value problem in question.

36. Another method for solving the two – point boundary value problem is the *collocation method*. Let

$$y'' = f(t, y, y'), \quad y(a) = ya, \quad y(b) = yb.$$

This method computes a polynomial **p(t)** that approximates a solution **y(t)** using the following conditions

$$p(a) = ya, p(b) = yb, \quad p''(t_k) = f(t_k, p(t_k), p'(t_k))$$

where $k = 2, 3, \ldots, n - 1$ and $a = t_1 < t_2 < \ldots < t_n = b$ are the collocation points that are evenly spaced in the given interval.
In this exercise function **f** is assumed to be of the form $f(t, y, y') = g_0(t) + g_1(t)y + g_2(t)y'$.

(i) Set up a system of linear equations that follows from the collocation conditions.

(ii) Write MATLAB function **p = colloc(g0, g1, g2, a, b, ya, yb, n)** which computes coefficients **p** of the approximating polynomial. They should be stored in the array **p** in the descending order of powers. Note that the approximating polynomial is of degree **n – 1** or less.

37. Test the function **colloc** of Problem 36 using the two – point boundary value problem of Problem 35 and plot the graph of the approximating polynomial.

Tutorial 6

Linear Programming with MATLAB

Math 472/CS 472

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This tutorial is devoted to the discussion of computational tools that are of interest in linear programming (LP). MATLAB powerful tools for computations with vectors and matrices make this package well suited for solving typical problems of linear programming. Topics discussed in this tutorial include the basic feasible solutions, extreme points and extreme directions of the constraint set, geometric solution of the linear programming problem, the Two-Phase Method, the Dual Simplex Algorithm, addition of a constraint and Gomory's cutting plane algorithm.

6.1 MATLAB functions used in Tutorial 6

Function	Description
abs	Absolute value
all	True if all elements of a vector are nonzero
any	True if any element of a vector is nonzero
axis	Control axis scaling and appearance
break	Terminate execution of for or while loop
clc	Clear Command Window
convhull	Convex hull
diff	Difference and approximate derivative
disp	Display array
eps	Floating point relative accuracy
eye	Identity matrix
find	Find indices of nonzero of nonzero elements
FontSize	Size of a font
gca	Get handle to current axis
get	Get object properties
grid	Grid lines
hold	Hold current graph
inf	Infinity

intersect	Set intersection
isempty	True for empty matrix
length	Length of vector
LineStyle	Style of a line
LineWidth	Width of a line
max	Largest component
min	Smallest component
msgbox	Message box
nchoosek	Binomial coefficient or all combinations
patch	Create patch
pause	Wait for user response
plot	Linear plot
questdlg	Question dialog box
return	Return to invoking function
set	Set object properties
size	Size of matrix
sprintf	Write formatted data to string
sqrt	Square root
strcmp	Compare strings
sum	Sum of elements
title	Graph title
union	Set union
varargin	Variable length input argument list
varargout	Variable length output argument list
warning off	Suppresses all subsequent warning messages
xlabel	X-axis label
ylabel	Y-axis label
zeros	Zeros array

To learn more about a particular MATLAB function type **help *functionname*** in the **Command Window** and next press the **Enter** key.

6.2 Notation

The following symbols will be used throughout the sequel.

- $\mathbb{R}^n$ – n-dimensional Euclidean vector space. Each member of this space is an n-dimensional column vector. Lower case letters will denote members of this space.
- $\mathbb{R}^{m \times n}$ – collection of all real matrices with m rows and n columns. Upper case letters will denote members of this space.
- $\mathbf{T}$ – operator of transposition. In MATLAB the *single quote operator* **'** is used to create transposition of a real vector or a real matrix.
- $\mathbf{x}^T\mathbf{y}$ – the inner product (dot product) of $\mathbf{x}$ and $\mathbf{y}$.
- $\mathbf{x} \geqslant \mathbf{0}$ – nonnegative vector. All components of $\mathbf{x}$ are greater than or equal to zero.

6.3 Five auxiliary MATLAB functions

Some MATLAB functions that are presented in the subsequent sections of this tutorial make calls to functions named **ver**, **delcols**, **MRT**, **MRTD** and **Br**. These functions should be saved in the directory holding other m-files that are used in this tutorial.

```
function e = vr(m,i)

% The ith coordinate vector e in the m-dimensional Euclidean space.

e = zeros(m,1);
e(i) = 1;
```

```
function d = delcols(d)

% Delete duplicated columns of the matrix d.

d = union(d',d','rows')';
n = size(d,2);
j = [];
for k =1:n
   c = d(:,k);
   for l=k+1:n
      if norm(c - d(:,l),'inf') <= 100*eps
         j = [j l];
      end
   end
end
if ~isempty(j)
   j = sort(j);
   d(:,j) = [ ];
end
```

First line of code in the body of function **delcols** is borrowed from the MATLAB's help file. Two vectors are regarded as duplicated if the corresponding entries differ by more than 100 times the *machine epsilon*. In MATLAB this number is denoted by **eps** and is approximately equal to

```
format long

eps

ans =
    2.220446049250313e-016

format short
```

In order to display more digits we have changed the default format (**short**) to **long**. To learn more about available formats in MATLAB type **help format** in the **Command Window**.

```
function [row, mi] = MRT(a, b)

% The Minimum Ratio Test (MRT) performed on vectors a and b.
% Output parameters:
% row – index of the pivot row
% mi – value of the smallest ratio.

m = length(a);
c = 1:m;
a = a(:);
b = b(:);
l = c(b > 0);
[mi, row] = min(a(l)./b(l));
row = l(row);



function col = MRTD(a, b)

% The Maximum Ratio Test performed on vectors a and b.
% This function is called from within the function dsimplex.
% Output parameter:
% col - index of the pivot column.

m = length(a);
c = 1:m;
a = a(:);
b = b(:);
l = c(b < 0);
[mi, col] = max(a(l)./b(l));
col = l(col);



function [m, j] = Br(d)

% Implementation of the Bland's rule applied to the array d.
% This function is called from within the following functions:
% simplex2p, dsimplex, addconstr, simplex and cpa.
% Output parameters:
% m - first negative number in the array d
% j - index of the entry m.

ind = find(d < 0);
if ~isempty(ind)
   j = ind(1);
   m = d(j);
else
   m = [];
   j = [];
end
```

6.4 Basic feasible solutions

The *standard form* of the linear programming problem is formulated as follows. Given matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$ and two vectors $\mathbf{c} \in \mathbb{R}^n$ and $\mathbf{b} \in \mathbb{R}^m$, $\mathbf{b} \geqslant \mathbf{0}$, find a vector $\mathbf{x} \in \mathbb{R}^n$ such that

$$\begin{aligned} \textbf{min} \quad & z = \mathbf{c}^T\mathbf{x} \\ \textbf{Subject to} \quad & \mathbf{Ax} = \mathbf{b} \\ & \mathbf{x} \geqslant \mathbf{0} \end{aligned}$$

Function **vert = feassol(A, b)** computes all basic feasible solutions, if any, to the system of constraints in standard form.

```
function vert = feassol(A, b)

% Basic feasible solutions vert to the system of constraints

%                       Ax = b, x >= 0.

% They are stored in columns of the matrix vert.

[m, n] = size(A);
warning off
b = b(:);
vert = [];
if (n >= m)
   t = nchoosek(1:n,m);
   nv = nchoosek(n,m);
   for i=1:nv
      y = zeros(n,1);
      x = A(:,t(i,:))\b;
      if all(x >= 0 & (x ~= inf & x ~= -inf))
         y(t(i,:)) = x;
         vert = [vert y];
      end
   end
else
   error('Number of equations is greater than the number of
variables.')
end
if ~isempty(vert)
   vert = delcols(vert);
else
   vert = [];
end
```

To illustrate functionality of this code consider the following system of constraints (see [1], Example 3.2, pp. 85-87):

$$\begin{aligned} x_1 + x_2 &\leqslant 6 \\ x_2 &\leqslant 3 \end{aligned}$$

$$x_1, x_2 \geq 0.$$

To put this system in standard form two *slack variables* x_3 and x_4 are added. The constraint matrix **A** and the right hand sides **b** are

```
A = [1 1 1 0; 0 1 0 1];

b = [6; 3];

vert = feassol(A, b)

vert =
     0     0     3     6
     0     3     3     0
     6     3     0     0
     3     0     0     3
```

To obtain values of the legitimate variables x_1 and x_2 it suffices to extract rows one and two of the matrix **vert**

```
vert = vert(1:2, :)

vert =
     0     0     3     6
     0     3     3     0
```

6.5 Extreme points and extreme directions of the constraint set

Problem discussed in this section is formulated as follows. Given a polyhedral set $X = \{x: Ax \leq b$ or $Ax \geq b$, $x \geq 0\}$ find all *extreme points* **t** of **X**. If **X** is *unbounded*, then in addition to finding the extreme points **t** its *extreme directions* **d** should be determined as well. To this end we will assume that the constraint set does not involve the equality constraints. If the LP problem has the equality constraint, then one can replace it by two inequality constraints. This is based on the following trivial observation: the equality $a = b$ is equivalent to the system of inequalities $a \leq b$ and $a \geq b$. Knowledge of the extreme points and extreme directions of the polyhedral set **X** is critical for a full mathematical description of this set. If set **X** is bounded, than a *convex combination* of its extreme points gives a point in this set. For the unbounded sets, however, a convex combination of its extreme points and a *linear combination*, with positive coefficients, of its extreme directions gives a point in set **X**. For more details, the interested reader is referred to [1], Chapter 2.

Extreme points of the set in question can be computed using function **extrpts**

```
function vert = extrpts(A, rel, b)

% Extreme points vert of the polyhedral set

%               X = {x: Ax <= b or Ax >= b, x >= 0}.

% Inequality signs are stored in the string rel, e.g.,
```

```
% rel = '<<>' stands for <= , <= , and >= , respectively.

[m, n] = size(A);
nlv = n;
for i=1:m
   if(rel(i) == '>')
      A = [A -vr(m,i)];
   else
      A = [A vr(m,i)];
   end
   if b(i) < 0
      A(i,:) = - A(i,:);
      b(i) = -b(i);
   end
end
warning off
[m, n]= size(A);
b = b(:);
vert = [ ];
if (n >= m)
   t = nchoosek(1:n,m);
   nv = nchoosek(n,m);
   for i=1:nv
      y = zeros(n,1);
      x = A(:,t(i,:))\b;
      if all(x >= 0 & (x ~= inf & x ~= -inf))
         y(t(i,:)) = x;
         vert = [vert y];
      end
   end
else
   error('Number of equations is greater than the number of variables')
end
vert = delcols(vert);
vert = vert(1:nlv,:);
```

Consider the polyhedral set **X** given by the inequalities

$$-x_1 + x_2 \leq 1$$
$$-0.1x_1 + x_2 \leq 2$$
$$x_1, x_2 \geq 0$$

To compute its extreme points we define

```
A = [-1 1; -0.1 1];

rel = '<<';

b = [1; 2];
```

and next run the m-file **extrpts**

```
vert = extrpts(A, rel, b)
```

```
vert =
         0         0    1.1111
         0    1.0000    2.1111
```

Recall that the extreme points are stored in columns of the matrix **vert**.

Extreme directions, if any, of the polyhedral set **X** can be computed using function **extrdir**

```
function d = extrdir(A, rel, b)

% Extreme directions d of the polyhedral set

%            X = {x: Ax <= b, or Ax >= b, x >= 0}.

% Matrix A must be of the full row rank.

[m, n] = size(A);
n1 = n;
for i=1:m
   if(rel(i) == '>')
      A = [A -vr(m,i)];
   else
      A = [A vr(m,i)];
   end
end
[m, n] = size(A);
A = [A;ones(1,n)];
b = [zeros(m,1);1];
d = feassol(A,b);
if ~isempty(d)
   d1 = d(1:n1,:);
   d = delcols(d1);
   s = sum(d);
   for i=1:n1
      d(:,i) = d(:,i)/s(i);
   end
else
   d = [];
end
```

Function **extrdir** returns the empty set operator **[ ]** for bounded polyhedral sets.

We will test this function using the polyhedral set that is defined in the last example of this section. We have

```
d = extrdir(A, rel, b)

d =
    1.0000    0.9091
         0    0.0909
```

Again, the directions in question are stored in columns of the output matrix **d**.

6.6 Solving the LP problem geometrically

A solution to the LP problem with two legitimate variables x_1 and x_2 can be found geometrically in three steps. First, the feasible region described by the constraint system $Ax \leqslant b$ or $Ax \geqslant b$ with $x \geqslant 0$ is drawn. Next, a direction of the level line $z = c_1x_1 + c_2x_2$ is determined. Finally moving the level line one can find easily vertex (extreme point) of the feasible region where the minimum or maximum value of z is attained or conclude that the objective function is unbounded. For more details see, e.g., [3], Chapter 2 and [1], Chapter 1.

Function **drawfr** implements two initial steps of this method

```
function  drawfr(c, A, rel, b)

% Graphs of the feasible region and the line level
% of the LP problem with two legitimate variables
%
%                    min (max)z = c*x
%                Subject to    Ax <= b (or Ax >= b),
%                               x >= 0

clc
[m, n] = size(A);
if n ~= 2
   str = 'Number of the legitimate variables must be equal to 2';
   msgbox(str,'Error Window','error')
   return
end
vert = extrpts(A,b,rel);
if isempty(vert)
   disp(sprintf('\n   Empty feasible region'))
   return
end
vert = vert(1:2,:);
vert = delcols(vert);
d = extrdir(A,b,rel);
if ~isempty(d)
   msgbox('Unbounded feasible region','Warning Window','warn')
   disp(sprintf('\n   Extreme direction(s) of the constraint set'))
   d
   disp(sprintf('\n   Extreme points of the constraint set'))
   vert
   return
end
t1 = vert(1,:);
t2 = vert(2,:);
z = convhull(t1,t2);
hold on
patch(t1(z),t2(z),'r')
h = .25;
mit1 = min(t1)-h;
mat1 = max(t1)+h;
mit2 = min(t2)-h;
```

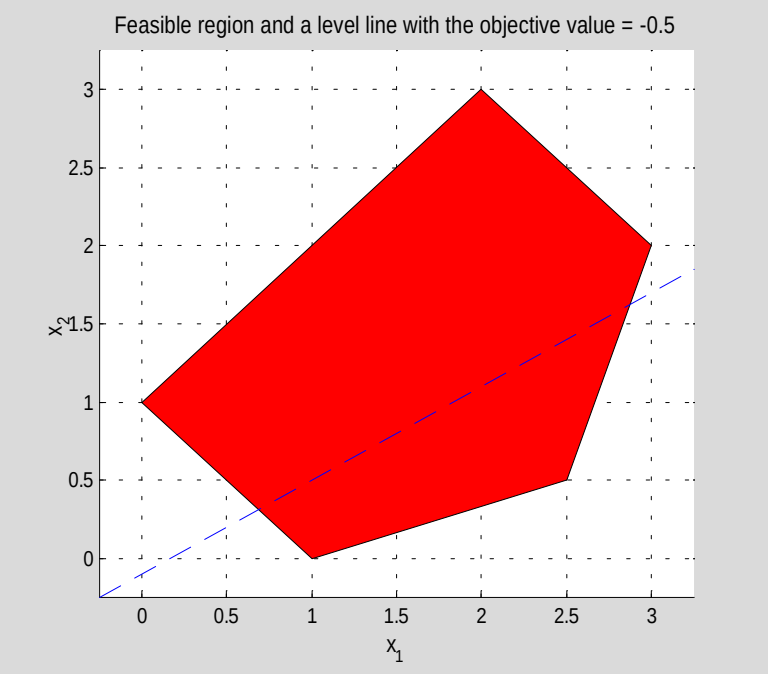
```
mat2 = max(t2)+h;
if c(1) ~= 0 & c(2) ~= 0
   sl = -c(1)/c(2);
   if sl > 0
      z = c(:)'*[mit1;mit2];
      a1 = [mit1 mat1];
      b1 = [mit2 (z-c(1)*mat1)/c(2)];
   else
      z = c(:)'*[mat1;mit2];
      a1 = [mit1 mat1];
      b1 = [(z-c(1)*mit1)/c(2) mit2];
   end
elseif c(1) == 0 & c(2) ~= 0
   z = 0;
   a1 = [mit1 mat1];
   b1 = [0,0];
else
   z = 0;
   a1 = [0 0];
   b1 = [mit2 mat2];
end
h = plot(a1,b1);
set(h,'linestyle','--')
set(h,'linewidth',2)
str = 'Feasible region and a level line with the objective value = ';
title([str,num2str(z)])
axis([mit1 mat1 mit2 mat2])
h = get(gca,'Title');
set(h,'FontSize',11)
xlabel('x_1')
h = get(gca,'xlabel');
set(h,'FontSize',11)
ylabel('x_2')
h = get(gca,'ylabel');
set(h,'FontSize',11)
grid
hold off
```

To test this function consider the LP problem with five constraints

```
c = [-3 5];

A = [-1 1;1 1;1 1;3 -1;1 -3];

rel = '<><<<';

b = [1;1;5;7;1];
```

Graphs of the feasible region and the level line are shown below

```
drawfr(c, A, rel, b)
```



Let us note that for the minimization problem the optimal solution occurs at

$$\mathbf{x} = \begin{bmatrix} 2.5 \\ 0.5 \end{bmatrix}$$

while the maximum of the objective function is attained at

$$\mathbf{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

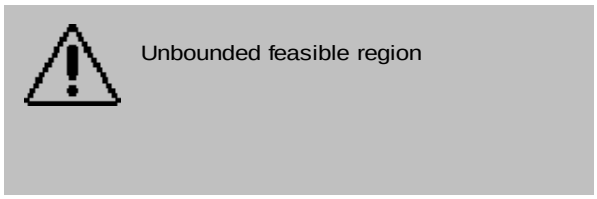
Next LP problem is defined as follows

```
c = [1 1];
A = [-1 1;-0.1 1];
rel = '<<';
b = [1;2];
```

Invoking function **drawfr** we obtain

```
drawfr(c, A, rel, b)
```

```
   Extreme direction(s) of the constraint set
d =
    1.0000    0.9091
         0    0.0909

   Extreme points of the constraint set
vert =
         0         0    1.1111
         0    1.0000    2.1111
```



Graph of the feasible region is not displayed. Extreme directions and the extreme points are shown above. Also, a warrning message is generated.

6.7 The Two-Phase Method

The Two-Phase Method is one of the basic computational tools used in linear programming. An implementation of this method presented in this section takes advantage of all features of this method. The LP problems that can be solved are either the minimization or maximization problems. Optimal solution is sought over a polyhedral set that is described by the inequality and/or equality constraints together with the nonnegativity constraints imposed on the legitimate variables.

Function **simplex2p** displays messages about the LP problem to be solved. These include:

- information about uniqueness of the optimal solution
- information about unboundedness of the objective function, if any
- information about empty feasible region

In addition to these features some built-in mechanisms allow to solve the LP problems whose constraint systems are *redundant*. Also, *Bland's Rule* to prevent *cycling* is used. For details, see [1], p. 169 and p. 174.

During the execution of function **simplex2p** a user has an option to monitor progress of computations by clicking on the **Yes** button in the message windows.

```
function  simplex2p(type, c, A, rel, b)

% The Two-Phase Method for solving the LP problem

%                     min(or max) z = c*x
%                     Subject to   Ax rel b
%                                   x >= 0

% The input parameter type holds information about type of the LP
% problem to be solved. For the minimization problem type = 'min' and
% for the maximization problem type = 'max'.
% The input parameter rel is a string holding the relation signs.
% For instance, if rel = '<=>', then the constraint system consists
% of one inequality <=, one equation, and one inequality >=.

clc
if (type == 'min')
   mm = 0;
else
   mm = 1;
   c = -c;
end
b = b(:);
c = c(:)';
[m, n] = size(A);
n1 = n;
les = 0;
neq = 0;
red = 0;
if length(c) < n
   c = [c zeros(1,n-length(c))];
end
for i=1:m
   if(rel(i) == '<')
      A = [A vr(m,i)];
      les = les + 1;
   elseif(rel(i) == '>')
      A = [A -vr(m,i)];
   else
      neq = neq + 1;
   end
end
ncol = length(A);
if les == m
   c = [c zeros(1,ncol-length(c))];
   A = [A;c];
   A = [A [b;0]];
   [subs, A, z, p1] = loop(A, n1+1:ncol, mm, 1, 1);
   disp('                    End of Phase 1')
   disp('          *********************************')
else
   A = [A eye(m) b];
   if m > 1
      w = -sum(A(1:m,1:ncol));
   else
      w = -A(1,1:ncol);
   end
```

```
   c = [c zeros(1,length(A)-length(c))];
   A = [A;c];
   A = [A;[w zeros(1,m) -sum(b)]];
   subs = ncol+1:ncol+m;
   av = subs;
   [subs, A, z, p1] = loop(A, subs, mm, 2, 1);
   if p1 == 'y'
         disp('                    End of Phase 1')
         disp('          *********************************')
      end
   nc = ncol + m + 1;
   x = zeros(nc-1,1);
   x(subs) = A(1:m,nc);
   xa = x(av);
   com = intersect(subs,av);
   if (any(xa) ~= 0)
      disp(sprintf('\n\n                      Empty feasible region\n'))
      return
   else
      if ~isempty(com)
         red = 1;
      end
   end
   A = A(1:m+1,1:nc);
   A =[A(1:m+1,1:ncol) A(1:m+1,nc)];
   [subs, A, z, p1] = loop(A, subs, mm, 1, 2);
   if p1 == 'y'
         disp('                    End of Phase 2')
         disp('          *********************************')
      end
end
if (z == inf | z == -inf)
   return
end
[m, n] = size(A);
x = zeros(n,1);
x(subs) = A(1:m-1,n);
x = x(1:n1);
if mm == 0
   z = -A(m,n);
else
   z = A(m,n);
end
disp(sprintf('\n\n           Problem has a finite optimal solution\n'))
disp(sprintf('\n Values of the legitimate variables:\n'))
for i=1:n1
   disp(sprintf(' x(%d)= %f ',i,x(i)))
end
disp(sprintf('\n Objective value at the optimal point:\n'))
disp(sprintf(' z= %f',z))
t = find(A(m,1:n-1) == 0);
if length(t) > m-1
   str = 'Problem has infinitely many solutions';
   msgbox(str,'Warning Window','warn')
end
if red == 1
   disp(sprintf('\n Constraint system is redundant\n\n'))
```

```
end


function [subs, A, z, p1]= loop(A, subs, mm, k, ph)

% Main loop of the simplex primal algorithm.
% Bland's rule to prevente cycling is used.

tbn = 0;
str1 = 'Would you like to monitor the progress of Phase 1?';
str2 = 'Would you like to monitor the progress of Phase 2?';
if ph == 1
   str = str1;
else
   str = str2;
end
question_ans = questdlg(str,'Make a choice Window','Yes','No','No');
if strcmp(question_ans,'Yes')
   p1 = 'y';
end
if p1 == 'y' & ph == 1
   disp(sprintf('\n\n                    Initial tableau'))
   A
   disp(sprintf(' Press any key to continue ...\n\n'))
   pause
end
if p1 == 'y' & ph == 2
   tbn = 1;
   disp(sprintf('\n\n                    Tableau %g',tbn))
   A
   disp(sprintf(' Press any key to continue ...\n\n'))
   pause
end
[m, n] = size(A);
[mi, col] = Br(A(m,1:n-1));
while ~isempty(mi) & mi < 0 & abs(mi) > eps
   t = A(1:m-k,col);
   if all(t <= 0)
      if mm == 0
         z = -inf;
      else
         z = inf;
      end
      disp(sprintf('\n        Unbounded optimal solution with z=
%s\n',z))
      return
   end
   [row, small] = MRT(A(1:m-k,n),A(1:m-k,col));
   if ~isempty(row)
      if abs(small) <= 100*eps & k == 1
         [s,col] = Br(A(m,1:n-1));
      end
      if p1 == 'y'
         disp(sprintf('          pivot row-> %g   pivot column->
%g',...
            row,col))
      end
```

```
      A(row,:)= A(row,:)/A(row,col);
      subs(row) = col;
      for i = 1:m
         if i ~= row
            A(i,:)= A(i,:)-A(i,col)*A(row,:);
         end
      end
      [mi, col] = Br(A(m,1:n-1));
   end
   tbn = tbn + 1;
   if p1 == 'y'
      disp(sprintf('\n\n                    Tableau %g',tbn))
      A
      disp(sprintf(' Press any key to continue ...\n\n'))
      pause
   end
end
z = A(m,n);
```

MATLAB subfunction **loop** is included in the m-file **simplex2p**. It implements the main loop of the simplex algorithm. Organization of the tableaux generated by function **simplex2p** is the same as the one presented in [3].

We will now test this function on five LP problems.

Let

```
type = 'min';

c = [-3 4];

A = [1 1;2 3];

rel = '<>';

b = [4; 18];
```

One can easily check, either by drawing the graph of the constraint set or running function **drawfr** on these data, that this problem has an empty feasible region.

```
simplex2p(type, c, A, rel, b)
```

```
                Initial tableau

A =

     1     1     1     0     1     0     4
     2     3     0    -1     0     1    18
    -3     4     0     0     0     0     0
    -3    -4    -1     1     0     0   -22

 Press any key to continue ...
```

```
          pivot row-> 1   pivot column-> 1

                    Tableau 1

A =

     1     1     1     0     1     0     4
     0     1    -2    -1    -2     1    10
     0     7     3     0     3     0    12
     0    -1     2     1     3     0   -10

 Press any key to continue ...

          pivot row-> 1   pivot column-> 2

                    Tableau 2

A =

     1     1     1     0     1     0     4
    -1     0    -3    -1    -3     1     6
    -7     0    -4     0    -4     0   -16
     1     0     3     1     4     0    -6

 Press any key to continue ...

                End of Phase 1
          *********************************

                    Empty feasible region
```

Next LP problem has an unbounded objective function. Let

```
type = 'max';

c = [3 2 1];

A = [2 –3 2;-1 1 1];

rel = '<<';

b = [3;55];
```

Running function **simplex2p** we obtain

```
simplex2p(type, c, A, rel, b)
```

```
                    Initial tableau

A =

     2    -3     2     1     0     3
    -1     1     1     0     1    55
    -3    -2    -1     0     0     0

 Press any key to continue ...


          pivot row-> 1   pivot column-> 1


                    Tableau 1

A =

    1.0000   -1.5000    1.0000    0.5000         0    1.5000
         0   -0.5000    2.0000    0.5000    1.0000   56.5000
         0   -6.5000    2.0000    1.5000         0    4.5000

 Press any key to continue ...


        Unbounded optimal solution with z= Inf

                  End of Phase 1
          *********************************
```

In this example we will deal with the LP problem that is described by a system of two equations with seven variables

```
type = 'min';

c = [3 4 6 7 1 0 0];

A = [2 –1 1 6 –5 –1 0;1 1 2 1 2 0 –1];

rel = '==';

b = [6;3];


simplex2p(type, c, A, rel, b)
```

```
                    Initial tableau

A =

     2    -1     1     6    -5    -1     0     1     0     6
     1     1     2     1     2     0    -1     0     1     3
     3     4     6     7     1     0     0     0     0     0
    -3     0    -3    -7     3     1     1     0     0    -9

 Press any key to continue ...


          pivot row-> 1   pivot column-> 1


                   Tableau 1

A =

  Columns 1 through 7

    1.0000   -0.5000    0.5000    3.0000   -2.5000   -0.5000         0
         0    1.5000    1.5000   -2.0000    4.5000    0.5000   -1.0000
         0    5.5000    4.5000   -2.0000    8.5000    1.5000         0
         0   -1.5000   -1.5000    2.0000   -4.5000   -0.5000    1.0000

  Columns 8 through 10

    0.5000         0    3.0000
   -0.5000    1.0000         0
   -1.5000         0   -9.0000
    1.5000         0         0

 Press any key to continue ...


          pivot row-> 2   pivot column-> 2
```

```
                    Tableau 2

A =

  Columns 1 through 7

    1.0000         0    1.0000    2.3333   -1.0000   -0.3333   -0.3333
         0    1.0000    1.0000   -1.3333    3.0000    0.3333   -0.6667
         0         0   -1.0000    5.3333   -8.0000   -0.3333    3.6667
         0         0         0         0         0         0         0

  Columns 8 through 10

    0.3333    0.3333    3.0000
   -0.3333    0.6667         0
    0.3333   -3.6667   -9.0000
    1.0000    1.0000         0

 Press any key to continue ...


                End of Phase 1
          ********************************


                    Tableau 1

A =

  Columns 1 through 7

    1.0000         0    1.0000    2.3333   -1.0000   -0.3333   -0.3333
         0    1.0000    1.0000   -1.3333    3.0000    0.3333   -0.6667
         0         0   -1.0000    5.3333   -8.0000   -0.3333    3.6667

  Column 8

    3.0000
         0
   -9.0000

 Press any key to continue ...


          pivot row-> 2   pivot column-> 3
```

```
                    Tableau 2

A =

  Columns 1 through 7

    1.0000   -1.0000         0    3.6667   -4.0000   -0.6667    0.3333
         0    1.0000    1.0000   -1.3333    3.0000    0.3333   -0.6667
         0    1.0000         0    4.0000   -5.0000    0.0000    3.0000

  Column 8

    3.0000
         0
   -9.0000

 Press any key to continue ...


          pivot row-> 2   pivot column-> 5


                    Tableau 3

A =

  Columns 1 through 7

    1.0000    0.3333    1.3333    1.8889         0   -0.2222   -0.5556
         0    0.3333    0.3333   -0.4444    1.0000    0.1111   -0.2222
         0    2.6667    1.6667    1.7778         0    0.5556    1.8889

  Column 8

    3.0000
         0
   -9.0000

 Press any key to continue ...


                    End of Phase 2
          *********************************


           Problem has a finite optimal solution


 Values of the legitimate variables:

 x(1)= 3.000000
 x(2)= 0.000000
 x(3)= 0.000000
 x(4)= 0.000000
 x(5)= 0.000000
 x(6)= 0.000000
 x(7)= 0.000000
```

```
 Objective value at the optimal point:

 z= 9.000000
```

The constraint system of the following LP problem

```
type = 'min';

c = [-1 2 –3];

A = [1 1 1;-1 1 2;0 2 3;0 0 1];

rel = '===<';

b = [6 4 10 2];
```

is redundant. Function **simplex2p** generates the following tableaux

```
simplex2p(type, c, A, rel, b)

                    Initial tableau

A =

     1     1     1     0     1     0     0     0     6
    -1     1     2     0     0     1     0     0     4
     0     2     3     0     0     0     1     0    10
     0     0     1     1     0     0     0     1     2
    -1     2    -3     0     0     0     0     0     0
     0    -4    -7    -1     0     0     0     0   -22

 Press any key to continue ...

          pivot row-> 2   pivot column-> 2

                    Tableau 1

A =

     2     0    -1     0     1    -1     0     0     2
    -1     1     2     0     0     1     0     0     4
     2     0    -1     0     0    -2     1     0     2
     0     0     1     1     0     0     0     1     2
     1     0    -7     0     0    -2     0     0    -8
    -4     0     1    -1     0     4     0     0    -6

 Press any key to continue ...

          pivot row-> 1   pivot column-> 1
```

```
                    Tableau 2

A =

  Columns 1 through 7

    1.0000         0   -0.5000         0    0.5000   -0.5000         0
         0    1.0000    1.5000         0    0.5000    0.5000         0
         0         0         0         0   -1.0000   -1.0000    1.0000
         0         0    1.0000    1.0000         0         0         0
         0         0   -6.5000         0   -0.5000   -1.5000         0
         0         0   -1.0000   -1.0000    2.0000    2.0000         0

  Columns 8 through 9

         0    1.0000
         0    5.0000
         0         0
    1.0000    2.0000
         0   -9.0000
         0   -2.0000

 Press any key to continue ...


          pivot row-> 4   pivot column-> 3


                    Tableau 3

A =

  Columns 1 through 7

    1.0000         0         0    0.5000    0.5000   -0.5000         0
         0    1.0000         0   -1.5000    0.5000    0.5000         0
         0         0         0         0   -1.0000   -1.0000    1.0000
         0         0    1.0000    1.0000         0         0         0
         0         0         0    6.5000   -0.5000   -1.5000         0
         0         0         0         0    2.0000    2.0000         0

  Columns 8 through 9

    0.5000    2.0000
   -1.5000    2.0000
         0         0
    1.0000    2.0000
    6.5000    4.0000
    1.0000         0

 Press any key to continue ...


                End of Phase 1
          ********************************
```

```
                    Tableau 1

A =

    1.0000         0         0    0.5000    2.0000
         0    1.0000         0   -1.5000    2.0000
         0         0         0         0         0
         0         0    1.0000    1.0000    2.0000
         0         0         0    6.5000    4.0000

 Press any key to continue ...


               End of Phase 2
          ********************************


           Problem has a finite optimal solution


 Values of the legitimate variables:

 x(1)= 2.000000
 x(2)= 2.000000
 x(3)= 2.000000
 Objective value at the optimal point:

 z= -4.000000

 Constraint system is redundant
```

If the LP problem is *degenerated*, then there is a chance that the method under discussion would never terminate. In order to avoid cycling the Bland's rule is implemented in the body of the function l**oop**. The following LP problem

```
type = 'min';

c = [-3/4 150 –1/50 6];

A = [1/4 –60 –1/25 9; 1/2 -90 –1/50 3; 0 0 1 0];

rel = '<<<';

b = [0 0 1];
```

is discussed in [2], pp. 136-138. Function **simplex2p** generates the following tableaux

```
simplex2p(type, c, A, rel, b)
```

```
                    Initial tableau

A =

  Columns 1 through 7

    0.2500  -60.0000   -0.0400    9.0000    1.0000         0         0
    0.5000  -90.0000   -0.0200    3.0000         0    1.0000         0
         0         0    1.0000         0         0         0    1.0000
   -0.7500  150.0000   -0.0200    6.0000         0         0         0

  Column 8

         0
         0
    1.0000
         0

 Press any key to continue ...


          pivot row-> 1   pivot column-> 1


                   Tableau 1

A =

  Columns 1 through 7

    1.0000 -240.0000   -0.1600   36.0000    4.0000         0         0
         0   30.0000    0.0600  -15.0000   -2.0000    1.0000         0
         0         0    1.0000         0         0         0    1.0000
         0  -30.0000   -0.1400   33.0000    3.0000         0         0

  Column 8

         0
         0
    1.0000
         0

 Press any key to continue ...


          pivot row-> 2   pivot column-> 2
```

```
                    Tableau 2

A =

  Columns 1 through 7

    1.0000         0    0.3200  -84.0000  -12.0000    8.0000         0
         0    1.0000    0.0020   -0.5000   -0.0667    0.0333         0
         0         0    1.0000         0         0         0    1.0000
         0         0   -0.0800   18.0000    1.0000    1.0000         0

  Column 8

         0
         0
    1.0000
         0

 Press any key to continue ...


          pivot row-> 1   pivot column-> 3


                    Tableau 3

A =

  Columns 1 through 7

    3.1250         0    1.0000 -262.5000  -37.5000   25.0000         0
   -0.0063    1.0000         0    0.0250    0.0083   -0.0167         0
   -3.1250         0         0  262.5000   37.5000  -25.0000    1.0000
    0.2500         0         0   -3.0000   -2.0000    3.0000         0

  Column 8

         0
         0
    1.0000
         0

 Press any key to continue ...


          pivot row-> 2   pivot column-> 4
```

```
                    Tableau 4

A =

  1.0e+004 *

  Columns 1 through 7

   -0.0062    1.0500    0.0001         0    0.0050   -0.0150         0
   -0.0000    0.0040         0    0.0001    0.0000   -0.0001         0
    0.0062   -1.0500         0         0   -0.0050    0.0150    0.0001
   -0.0000    0.0120         0         0   -0.0001    0.0001         0

  Column 8

         0
         0
    0.0001
         0

 Press any key to continue ...


          pivot row-> 3   pivot column-> 1


                    Tableau 5

A =

  Columns 1 through 7

         0         0    1.0000         0         0         0    1.0000
         0   -2.0000         0    1.0000    0.1333   -0.0667    0.0040
    1.0000 -168.0000         0         0   -0.8000    2.4000    0.0160
         0   36.0000         0         0   -1.4000    2.2000    0.0080

  Column 8

    1.0000
    0.0040
    0.0160
    0.0080

 Press any key to continue ...


          pivot row-> 2   pivot column-> 5
```

```
                    Tableau 6

A =

  Columns 1 through 7

         0         0    1.0000         0         0         0    1.0000
         0  -15.0000         0    7.5000    1.0000   -0.5000    0.0300
    1.0000 -180.0000         0    6.0000         0    2.0000    0.0400
         0   15.0000         0   10.5000         0    1.5000    0.0500

  Column 8

    1.0000
    0.0300
    0.0400
    0.0500

 Press any key to continue ...


                End of Phase 1
          ********************************


           Problem has a finite optimal solution


 Values of the legitimate variables:

 x(1)= 0.040000
 x(2)= 0.000000
 x(3)= 1.000000
 x(4)= 0.000000

 Objective value at the optimal point:

 z= -0.050000
```

The last entry in column eight, in tableaux 1 through 4, is equal to zero. Recall that this is the current value of the objective function. Simplex algorithm tries to improve a current basic feasible solution and after several attempts it succeeds. Two basic variables in tableaux 1 through 4 are both equal to zero. Based on this observation one can conclude that the problem in question is degenerated. Without a built-in mechanism that prevents cycling this algorithm would never terminate. Another example of the degenerated LP problem is discussed in [3], Appendix C, pp. 375-376.

6.8 The Dual Simplex Algorithm

Another method that is of great importance in linear programming is called the *Dual Simplex Algorithm*. The optimization problem that can be solved with the aid of this method is formulated as follows

$$\begin{aligned} \textbf{min(max)} \quad & z = c^T x \\ \textbf{Subject to} \quad & Ax \geq b \\ & x \geq 0 \end{aligned}$$

Components of the vector **b** are not required to satisfy the nonnegativity constraints. The Dual Simplex Algorithm has numerous applications to other problems of linear programming. It is used, for instance, in some implementations of the Gomory's cutting plane algorithm for solving the integer programming problems.

Function **dsimplex** computes an optimal solution to the optimization problem that is defined above. One of the nice features of this method is its ability to detect the case of the empty feasible region. Function under discussion takes four input parameters **types**, **c**, **A** and **b**. Their meaning is described in Section 6.7 of this tutorial.

```
function varargout = dsimplex(type, c, A, b)

% The Dual Simplex Algorithm for solving the LP problem

%                      min (max) z = c*x
%                    Subject to  Ax >= b
%                                 x >= 0
%

if type == 'min'
   mm = 0;
else
   mm = 1;
   c = -c;
end
str = 'Would you like to monitor the progress of computations?';
A = -A;
b = -b(:);
c = c(:)';
[m, n] = size(A);
A = [A eye(m) b];
A = [A;[c zeros(1,m+1)]];
question_ans = questdlg(str,'Make a choice Window','Yes','No','No');
if strcmp(question_ans,'Yes')
   p1 = 'y';
else
   p1 = 'n';
end
if p1 == 'y'
   disp(sprintf('\n\n                    Initial tableau'))
   A
```

```
   disp(sprintf(' Press any key to continue ...\n\n'))
   pause
end
subs = n+1:n+m;
[bmin, row] = Br(b);
tbn = 0;
while ~isempty(bmin) & bmin < 0 & abs(bmin) > eps
   if A(row,1:m+n) >= 0
      disp(sprintf('\n\n               Empty feasible region\n'))
      varargout(1)={subs(:)};
      varargout(2)={A};
      varargout(3) = {zeros(n,1)};
      varargout(4) = {0};
      return
   end
   col = MRTD(A(m+1,1:m+n),A(row,1:m+n));
   if p1 == 'y'
      disp(sprintf('          pivot row-> %g   pivot column-> %g',...
            row,col))
   end
   subs(row) = col;
   A(row,:)= A(row,:)/A(row,col);
      for i = 1:m+1
      if i ~= row
         A(i,:)= A(i,:)-A(i,col)*A(row,:);
      end
   end
   tbn = tbn + 1;
   if p1 == 'y'
      disp(sprintf('\n\n                    Tableau %g',tbn))
      A
      disp(sprintf(' Press any key to continue ...\n\n'))
      pause
   end
   [bmin, row] = Br(A(1:m,m+n+1));
end
x = zeros(m+n,1);
x(subs) = A(1:m,m+n+1);
x = x(1:n);
if mm == 0
   z = -A(m+1,m+n+1);
else
   z = A(m+1,m+n+1);
end
disp(sprintf('\n\n           Problem has a finite optimal
solution\n\n'))
disp(sprintf('\n Values of the legitimate variables:\n'))
for i=1:n
   disp(sprintf(' x(%d)= %f ',i,x(i)))
end
disp(sprintf('\n Objective value at the optimal point:\n'))
disp(sprintf(' z= %f',z))
disp(sprintf('\n Indices of basic variables in the final tableau:'))
varargout(1)={subs(:)};
varargout(2)={A};
varargout(3) = {x};
varargout(4) = {z};
```

Function **dsimplex** allows a variable number of the output parameters. Indices of the basic variables in the final tableau are always displayed. The final tableau also can be saved in a variable. For instance, execution of the following command **[subs, B] = dsimplex(type, c, A, b)** will return, among other things, indices of basic variables stored in the variable **subs** and the final tableau stored in the variable **B**.

Let

```
type = 'min';

c = [-2 3 4];

A = [-1 –1 –1;2 2 2];

b = [-1;4];
```

One can easily check that the constraint system defines an empty set. Running function **dsimplex**, we obtain

```
subs = dsimplex(type, c, A, b)


          Initial tableau

A =

     1     1     1     1     0     1
    -2    -2    -2     0     1    -4
    -2     3     4     0     0     0

 Press any key to continue ...


     pivot row-> 2   pivot column-> 1


                   Tableau 1

A =

         0         0         0    1.0000    0.5000   -1.0000
    1.0000    1.0000    1.0000         0   -0.5000    2.0000
         0    5.0000    6.0000         0   -1.0000    4.0000

 Press any key to continue ...


               Empty feasible region
```

```
subs =

     4
     1
```

In this example we will use function **dsimplex** to find an optimal solution to the problem with four variables and three constraints

```
type = 'max';

c = [1 2 3 –1];

A = [2 1 5 0;1 2 3 0;1 1 1 1];

b = [20;25;10];


subs = dsimplex(type, c, A, b)
```

```
                    Initial tableau

A =

    -2    -1    -5     0     1     0     0   -20
    -1    -2    -3     0     0     1     0   -25
    -1    -1    -1    -1     0     0     1   -10
    -1    -2    -3     1     0     0     0     0
 Press any key to continue ...


          pivot row-> 1   pivot column-> 2


                    Tableau 1

A =

     2     1     5     0    -1     0     0    20
     3     0     7     0    -2     1     0    15
     1     0     4    -1    -1     0     1    10
     3     0     7     1    -2     0     0    40

 Press any key to continue ...
```

```
           Problem has a finite optimal solution


 Values of the legitimate variables:

 x(1)= 0.000000
 x(2)= 20.000000
 x(3)= 0.000000
 x(4)= 0.000000

 Objective value at the optimal point:

 z= 40.000000

 Indices of basic variables in the final tableau:

subs =

     2
     6
     7
```

6.9 Addition of a constraint

Topic discussed in this section is of interest in the sensitivity analysis. Suppose that the LP problem has been solved using one of the methods discussed earlier in this tutorial. We assume that the final tableau **A** and the vector **subs** - vector of indices of basic variables are given. One wants to find an optimal solution to the original problem with an extra constraint $\mathbf{a}^T\mathbf{x} \leqslant \mathbf{d}$ or $\mathbf{a}^T\mathbf{x} \geqslant \mathbf{d}$ added.

Function **addconstr(type, A, subs, a, rel, d)** implements a method that is described in [3], pp. 171-174.

```
function addconstr(type, A, subs, a, rel, d)

% Adding a constraint to the final tableau A.
% The input parameter subs holds indices of basic
% variables associated with tableau A. Parameters
% lhs, rel, and rhs hold information about a costraint
% to be added:
% a - coefficients of legitimate variables
% rel - string describing the inequality sign;
% for instance, rel = '<' stands for <=.
% d - constant term of the constraint to be added.
% Parameter type is a string that describes type of
% the optimization problem. For the minimization problems
% type = 'min' and for the maximization problems type = 'max'.


clc
str = 'Would you like to monitor the progress of computations?';
question_ans = questdlg(str,'Make a choice Window','Yes','No','No');
if strcmp(question_ans,'Yes')
```

```
   p1 = 'y';
else
   p1 = 'n';
end
[m, n] = size(A);
a = a(:)';
lc = length(a);
if lc < n-1
   a = [a zeros(1,n-lc-1)];
end
if type == 'min'
   ty = -1;
else
   ty = 1;
end
x = zeros(n-1,1);
x(subs) = A(1:m-1,n);
dp = a*x;
if (dp <= d & rel == '<') | (dp >= d & rel == '>')
   disp(sprintf('\n\n            Problem has a finite optimal
solution\n'))
   disp(sprintf('\n Values of the legitimate variables:\n'))
   for i=1:n-1
      disp(sprintf(' x(%d)= %f ',i,x(i)))
   end
   disp(sprintf('\n Objective value at the optimal point:\n'))
   disp(sprintf(' z= %f',ty*A(m,n)))
   return
end
B = [A(:,1:n-1) zeros(m,1) A(:,n)];
if rel == '<'
   a = [a 1 d];
else
   a = [a -1 d];
end
for i=1:m-1
   a = a - a(subs(i))*B(i,:);
end
if a(end) > 0
   a = -a;
end
A = [B(1:m-1,:);a;B(m,:)];
if p1 == 'y'
   disp(sprintf('\n\n                    Initial tableau'))
   A
   disp(sprintf(' Press any key to continue ...\n\n'))
   pause
end
[bmin, row] = Br(A(1:m,end));
tbn = 0;
while ~isempty(bmin) & bmin < 0 & abs(bmin) > eps
   if A(row,1:n) >= 0
      disp(sprintf('\n\n               Empty feasible region\n'))
      return
   end
   col = MRTD(A(m+1,1:n),A(row,1:n));
   if p1 == 'y'
```

```
         disp(sprintf('          pivot row-> %g   pivot column->
%g',...
            row,col))
   end
   subs(row) = col;
   A(row,:)= A(row,:)/A(row,col);
      for i = 1:m+1
      if i ~= row
         A(i,:)= A(i,:)-A(i,col)*A(row,:);
      end
   end
   tbn = tbn + 1;
   if p1 == 'y'
      disp(sprintf('\n\n                    Tableau %g',tbn))
      A
      disp(sprintf(' Press any key to continue ...\n\n'))
      pause
   end
  [bmin, row] = Br(A(1:m,end));
end
x = zeros(n+1,1);
x(subs) = A(1:m,end);
x = x(1:n);
disp(sprintf('\n\n           Problem has a finite optimal
solution\n\n'))
disp(sprintf('\n Values of the legitimate variables:\n'))
for i=1:n
   disp(sprintf(' x(%d)= %f ',i,x(i)))
end
disp(sprintf('\n Objective value at the optimal point:\n'))
disp(sprintf(' z= %f',ty*A(m+1,n+1)))
```

The following examples are discussed in [3], pp. 171-173. Let

```
type = 'max';

A = [-2 0 5 1 2 –1 6; 11 1 –18 0 –7 4 4; 3 0 2 0 2 1 106];

subs = [4; 2];

a = [4 1 0 4];

rel = '>';

d = 29;


addconstr(type, A, subs, a, rel, d)
```

```
                    Initial tableau

A =

    -2     0     5     1     2    -1     0     6
    11     1   -18     0    -7     4     0     4
    -1     0     2     0     1     0     1    -1
     3     0     2     0     2     1     0   106

 Press any key to continue ...

          pivot row-> 3   pivot column-> 1

                 Tableau 1

A =

     0     0     1     1     0    -1    -2     8
     0     1     4     0     4     4    11    -7
     1     0    -2     0    -1     0    -1     1
     0     0     8     0     5     1     3   103

 Press any key to continue ...

               Empty feasible region
```

Changing the additional constraint to $3x_1 + x_2 + 3x_4 \leqslant 20$, we obtain

```
a = [3 1 0 3];

rel = '<';

d = 20;

addconstr(type, A, subs, a, rel, d)


                    Initial tableau
A =
    -2     0     5     1     2    -1     0     6
    11     1   -18     0    -7     4     0     4
    -2     0     3     0     1    -1     1    -2
     3     0     2     0     2     1     0   106
 Press any key to continue ...

          pivot row-> 3   pivot column-> 6
```

```
                    Tableau 1
A =
     0     0     2     1     1     0    -1     8
     3     1    -6     0    -3     0     4    -4
     2     0    -3     0    -1     1    -1     2
     1     0     5     0     3     0     1   104
 Press any key to continue ...


          pivot row-> 2   pivot column-> 3

                    Tableau 2
A =
  Columns 1 through 7
    1.0000    0.3333         0    1.0000         0         0    0.3333
   -0.5000   -0.1667    1.0000         0    0.5000         0   -0.6667
    0.5000   -0.5000         0         0    0.5000    1.0000   -3.0000
    3.5000    0.8333         0         0    0.5000         0    4.3333
  Column 8
    6.6667
    0.6667
    4.0000
  100.6667
 Press any key to continue ...


           Problem has a finite optimal solution


 Values of the legitimate variables:

 x(1)= 0.000000
 x(2)= 0.000000
 x(3)= 0.666667
 x(4)= 6.666667
 x(5)= 0.000000
 x(6)= 4.000000
 x(7)= 0.000000

 Objective value at the optimal point:

 z= 100.666667
```

6.10 Gomory's cutting plane algorithm

Some problems that arise in economy can be formulated as the linear programming problems with decision variables being restricted to integral values

min(max) z = c*x
Subject to Ax ⩽ b
x ⩾ 0 and integral

where the vector of the right-hand sides **b** is nonnegative. Among numerous algorithms designed for solving this problem the one, proposed by R.E. Gomory (see [3], pp. 194-203) is called the *cutting plane algorithm*. Its implementation, named here **cpa**, is included in this tutorial

```
function cpa(type, c, A, b)

% Gomory's cutting plane algorithm for solving
% the integer programming problem

%                      min(max) z = c*x
%                   Subject to: Ax <= b
%                                x >= 0 and integral


str = 'Would you like to monitor the progress of computations?';
question_ans = questdlg(str,'Make a choice Window','Yes','No','No');
if strcmp(question_ans,'Yes')
   p1 = 'y';
else
   p1 = 'n';
end
if type == 'min'
   tp = -1;
else
   tp = 1;
end
[m,n] = size(A);
nlv = n;
b = b(:);
c = c(:)';
if p1 == 'y'
   [A,subs] = simplex(type,c,A,b,p1);
else
   [A,subs] = simplex(type,c,A,b);
end
[m,n] = size(A);
d = A(1:m-1,end);
pc = fractp(d);
tbn = 0;
if p1 == 'y'
   disp(sprintf('
_____________________________________________________'))
   disp(sprintf('\n              Tableaux of the Dual Simplex Method'))
   disp(sprintf('
_____________________________________________________'))
end
while norm(pc,'inf') > eps
   [el,i] = max(pc);
   nr = A(i,1:n-1);
   nr = [-fractp(nr) 1 -el];
   B = [A(1:m-1,1:n-1) zeros(m-1,1) A(1:m-1,end)];
   B = [B;nr;[A(m,1:n-1) 0 A(end,end)]];
   A = B;
   [m,n] = size(A);
```

```
   [bmin, row] = min(A(1:m-1,end));
   while bmin < 0 & abs(bmin) > eps
      col = MRTD(A(m,1:n-1),A(row,1:n-1));
      if p1 == 'y'
         disp(sprintf('\n          pivot row-> %g   pivot column->
%g',...
               row,col))
         tbn = tbn + 1;
         disp(sprintf('\n                    Tableau %g',tbn))
         A
         disp(sprintf(' Press any key to continue ...\n'))
         pause
      end
      if isempty(col)
         disp(sprintf('\n Algorithm fails to find an optimal
solution.'))
         return
      end
      A(row,:)= A(row,:)./A(row,col);
      subs(row) = col;
      for i = 1:m
         if i ~= row
            A(i,:)= A(i,:)-A(i,col)*A(row,:);
         end
      end
     [bmin, row] = min(A(1:m-1,end));
   end
   d = A(1:m-1,end);
   pc = fractp(d);
end
if p1 == 'y'
   disp(sprintf('\n                  Final tableau'))
   A
   disp(sprintf(' Press any key to continue ...\n'))
   pause
end
x = zeros(n-1,1);
x(subs) = A(1:m-1,end);
x = x(1:nlv);
disp(sprintf('\n           Problem has a finite optimal solution\n\n'))
disp(sprintf('\n Values of the legitimate variables:\n'))
for i=1:nlv
   disp(sprintf(' x(%d)= %g ',i,x(i)))
end
disp(sprintf('\n Objective value at the optimal point:\n'))
disp(sprintf(' z= %f',tp*A(m,n)))



function y = fractp(x)

% Fractional part y of x, where x is a matrix.

y = zeros(1,length(x));
ind = find(abs(x - round(x)) >= 100*eps);
y(ind) = x(ind) - floor(x(ind));
```

The subfunction **fractp** is called from within the function **cpa**. It computes the fractional parts of real numbers that are stored in a matrix. Function **cpa** makes a call to another function called **simplex**. The latter finds an optimal solution to the problem that is formulated in the beginning of this section without the integral restrictions imposed on the legitimate variables.

```
function[A, subs, x, z] = simplex(type, c, A, b, varargin);

% The simplex algorithm for the LP problem

%                    min(max) z = c*x
%                 Subject to: Ax <= b
%                              x >= 0

% Vector b must be nonnegative.
% For the minimization problems the string type = 'min',
% otherwise type = 'max'.
% The fifth input parameter is optional. If it is set to 'y',
% then the initial and final tableaux are displayed to the
% screen.
% Output parameters:
% A - final tableau of the simplex method
% subs - indices of the basic variables in the final tableau
% x - optimal solution
% z - value of the objective function at x.

if any(b < 0)
   error(' Right hand sides of the constraint set must be
nonnegative.')
end
if type == 'min'
   tp = -1;
else
   tp = 1;
   c = -c;
end
[m, n] = size(A);
A = [A eye(m)];
b = b(:);
c = c(:)';
A = [A b];
d = [c zeros(1,m+1)];
A = [A;d];
if nargin == 5
   disp(sprintf('
_____________________________________________________'))
   disp(sprintf('\n              Tableaux of the Simplex Algorithm'))
   disp(sprintf('
_____________________________________________________'))
   disp(sprintf('\n                   Initial tableau\n'))
   A
   disp(sprintf(' Press any key to continue ...\n\n'))
   pause
end
[mi, col] = Br(A(m+1,1:m+n));
subs = n+1:m+n;
```

```
while ~isempty(mi) & mi < 0 & abs(mi) > eps
   t = A(1:m,col);
   if all(t <= 0)
      disp(sprintf('\n       Problem has unbounded objective
function'));
      x = zeros(n,1);
      if tp == -1
         z = -inf;
      else
         z = inf;
      end
      return;
   end
   row = MRT(A(1:m,m+n+1),A(1:m,col));
   if ~isempty(row)
      A(row,:)= A(row,:)/A(row,col);
      subs(row) = col;
      for i = 1:m+1
         if i ~= row
            A(i,:)= A(i,:)-A(i,col)*A(row,:);
         end
      end
   end
   [mi, col] = Br(A(m+1,1:m+n));
end
z = tp*A(m+1,m+n+1);
x = zeros(1,m+n);
x(subs) = A(1:m,m+n+1);
x = x(1:n)';
if nargin == 5
   disp(sprintf('\n\n                    Final tableau'))
   A
   disp(sprintf(' Press any key to continue ...\n'))
   pause
end
```

We will test function **cpa** on two examples from [3], pp. 195-201. Let

```
type = 'min';

c = [1 –3];

A = [1 –1;2 4];

b = [2;15];
```

Using the option to monitor computations, the following tableaux are displayed to the screen

```
cpa(type, c, A, b)
```

```
_____________________________________________________

         Tableaux of the Simplex Algorithm
_____________________________________________________


                Initial tableau

A =

     1    -1     1     0     2
     2     4     0     1    15
     1    -3     0     0     0

 Press any key to continue ...



                Final tableau

A =

    1.5000         0    1.0000    0.2500    5.7500
    0.5000    1.0000         0    0.2500    3.7500
    2.5000         0         0    0.7500   11.2500

 Press any key to continue ...


_____________________________________________________

         Tableaux of the Dual Simplex Method
_____________________________________________________

      pivot row-> 3   pivot column-> 4

                Tableau 1

A =

    1.5000         0    1.0000    0.2500         0    5.7500
    0.5000    1.0000         0    0.2500         0    3.7500
   -0.5000         0         0   -0.2500    1.0000   -0.7500
    2.5000         0         0    0.7500         0   11.2500

 Press any key to continue ...
```

```
                  Final tableau

A =

     1     0     1     0     1     5
     0     1     0     0     1     3
     2     0     0     1    -4     3
     1     0     0     0     3     9

 Press any key to continue ...


           Problem has a finite optimal solution


 Values of the legitimate variables:

 x(1)= 0
 x(2)= 3

 Objective value at the optimal point:

 z= -9.000000
```

Only the initial and final tableaux of the simplex algorithm are displayed. However, the all tableaux generated during the execution of the dual simplex method are included.

Let now

```
type = 'min';

c = [1 -2];

A = [2 1;-4 4];
b = [5;5];

cpa(type, c, A, b)
```

```
      ---------------------------------------------------

              Tableaux of the Simplex Algorithm

      ---------------------------------------------------

                  Initial tableau

A =
     2     1     1     0     5
    -4     4     0     1     5
     1    -2     0     0     0
 Press any key to continue ...
```

```
                Final tableau
A =
    1.0000         0    0.3333   -0.0833    1.2500
         0    1.0000    0.3333    0.1667    2.5000
         0         0    0.3333    0.4167    3.7500
 Press any key to continue ...
```

Tableaux of the Dual Simplex Method

```
          pivot row-> 3   pivot column-> 3

                  Tableau 1
A =
    1.0000         0    0.3333   -0.0833         0    1.2500
         0    1.0000    0.3333    0.1667         0    2.5000
         0         0   -0.3333   -0.1667    1.0000   -0.5000
         0         0    0.3333    0.4167         0    3.7500
 Press any key to continue ...

          pivot row-> 4   pivot column-> 4

                  Tableau 2
A =
    1.0000         0         0   -0.2500    1.0000         0    0.7500
         0    1.0000         0         0    1.0000         0    2.0000
         0         0    1.0000    0.5000   -3.0000         0    1.5000
         0         0         0   -0.7500         0    1.0000   -0.7500
         0         0         0    0.2500    1.0000         0    3.2500
 Press any key to continue ...



                Final tableau
A =
    1.0000         0         0         0    1.0000   -0.3333    1.0000
         0    1.0000         0         0    1.0000         0    2.0000
         0         0    1.0000         0   -3.0000    0.6667    1.0000
         0         0         0    1.0000         0   -1.3333    1.0000
         0         0         0         0    1.0000    0.3333    3.0000
 Press any key to continue ...
```

```
          Problem has a finite optimal solution

 Values of the legitimate variables:

 x(1)= 1
 x(2)= 2

Objective value at the optimal point:

 z= -3.000000
```

Function **cpa** has been tested extensively. It failed, however, to compute the optimal solution to the following integer programming problem (see [3], pp. 208-209)

$$\begin{aligned} \textbf{max} \quad & z = 3x_1 + 13x_2 \\ \textbf{Subject to} \quad & 2x_1 + 9x_2 \leqslant 40 \\ & 11x_1 - 8x_2 \leqslant 82 \\ & x_1, x_2 \geqslant 0 \text{ and integral} \end{aligned}$$

In Problem 17 you are asked to determine the cause of the premature termination of computations.

References

[1] M.S. Bazaraa, J.J. Jarvis, and H.D. Sherali, Linear Programming and Network Flows, Second edition, John Wiley & Sons, New York, 1990.

[2] S.G. Nash and A. Sofer, Linear and Nonlinear Programming, The McGraw-Hill Companies, Inc., New York, 1996.

[3] P.R. Thie, An Introduction to Linear Programming and Game Theory, Second edition, John Wiley & Sons, New York, 1988.

Problems

1. Solve the following LP problems geometrically.

(i)
$$\begin{aligned} \textbf{min } z &= -3x_1 + 9x_2 \\ \textbf{Subject to } & -x_1 + x_2 \leqslant 1 \\ & x_1 + x_2 \geqslant 1 \\ & x_1 + x_2 \leqslant 5 \\ & 3x_1 - x_2 \leqslant 7 \\ & x_1 - 3x_2 \leqslant 1 \\ & x_1, x_2 \geqslant 0 \end{aligned}$$

Let the objective function and the constraint set be the same as in Part (i). Find the optimal solution to the maximization problem.

(ii)
$$\begin{aligned} \textbf{max } z &= x_1 + x_2 \\ \textbf{Subject to } & x_1 + x_2 \leqslant 2 \\ & x_1 \leqslant 1 \\ & x_1 + x_2 \geqslant 1 \\ & -x_1 + x_2 \leqslant 1.5 \\ & x_1, x_2 \geqslant 0 \end{aligned}$$

2. Given the constraint set $\mathbf{X = \{(x_1, x_2, x_3)^T \in \mathbb{R}^3 : \ 0 \leqslant x_1 \leqslant x_2 \leqslant x_3\}}$.

(i) Write these constraints in the following form $\mathbf{Ax \leqslant b}$, $\mathbf{x \geqslant 0}$.
(ii) Find the extreme points and the extreme directions, if any, of $\mathbf{X}$.
(iii) Based on the numerical evidence from Part (ii) of this problem, formulate a hypothesis about the extreme points and the extreme directions of the more general constraint set $\mathbf{X = \{(x_1, x_2, \ldots x_n)^T \in \mathbb{R}^n : \ 0 \leqslant x_1 \leqslant x_2 \leqslant \ldots \leqslant x_n\}}$.

3. Give an example of the constraint set with at least three constraints and three variables that have empty feasible region.

4. Consider the system of constraints in the standard form $\mathbf{Ax = b}$, $\mathbf{x \geqslant 0}$ and $\mathbf{b \geqslant 0}$, where $\mathbf{A}$ is an m-by-n matrix with $\mathbf{n \geqslant m}$. Number of all basic feasible solutions to this system is not bigger than $\dfrac{n!}{m!(n-m)!}$. Construct a constraint system with two equations ($\mathbf{m = 2}$) and four variables ($\mathbf{n = 4}$) that has

(i) no basic feasible solutions
(ii) exactly one basic feasible solution
(iii) exactly three distinct basic feasible solutions

Hint: You may wish to perform numerical experiments using function **feassol** discussed in Section 6.4 of this tutorial.

5. Let **X** denote the constraint set that is described in Problem 1(i). Its graph is shown on page 10 of this tutorial. Express point $\mathbf{P} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ as a convex combination of its extreme points. Is this representation unique?
Hint: Compute the extreme points $\mathbf{P_1}, \mathbf{P_2}, \ldots, \mathbf{P_n}$ of **X** using function **extrpts**. Next form a system of linear equations

$$\lambda_1\mathbf{P_1} + \lambda_2\mathbf{P_2} + \ldots + \lambda_n\mathbf{P_n} = \mathbf{P}$$
$$\lambda_1 + \lambda_2 + \ldots + \lambda_n = 1$$

where all the λ's are nonnegative. Function **feassol** can be used to find all nonnegative solutions to this system.

6. Repeat Problem 5 with $\mathbf{P} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $\mathbf{P} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$.

7. The feasible region described by the inequalities of Problem 1(i) is bounded. An optimal solution to the minimization (maximization) problem can be found by evaluating the objective function at the extreme points of the feasible region and next finding the smallest (largest) value of the objective function. MATLAB has two functions **min** and **max** for computing the smallest and largest numbers in a set. The following command
[z, ind] = min(t) finds the smallest number in the set **t** together with the index **ind** of the element sought. In this exercise you are to find an optimal solution and the corresponding objective value of the Problem 1(i).
Remark. This method for solving the LP problems is **not** recommended in practice because of its high computational complexity. It is included here for pedagogical reasons only.

8. The following LP problem

$$\text{min (max)} = c_1x_1 + c_2x_2 + \ldots + c_nx_n$$
$$\text{Subject to} \quad x_1 + x_2 + \ldots + x_n = 1$$
$$x_1, x_2, \ldots, x_n \geq 0$$

can be solved without using advanced computational methods that are discussed in the Linear Programming class. Write MATLAB function **[x, z] = oneconstr(type, c)** which takes two input parameters **type**, where **type = 'min'** for the minimization problem and **type = 'max'** for the maximization problem and **c** – vector of the cost coefficients and returns the optimal solution **x** and the associated objective value **z**.

9. The following linear programming problem

$$\text{min (max) } z = c*x$$
$$\text{Subject to} \quad Ax \leq b$$
$$x \text{ unrestricted}$$

with $\mathbf{b} \geq \mathbf{0}$ can be solved using function **simplex** (see Section 6.10). A trick is to use a substitution $\mathbf{x = x' - x''}$, where $\mathbf{x', x'' \geq 0}$. Write MATLAB function
[x, z, A] = simplexu(type, c, A, b) which computes an optimal solution to the LP problem that is formulated above. The input and the output parameters have the same meaning as those in function **simplex**. Test your function on the following problem

$$\begin{aligned} \textbf{min } z &= 2x_1 - x_2 \\ \textbf{Subject to } & x_1 + x_2 \leqslant 2 \\ & -x_1 + x_2 \leqslant 1 \\ & x_1 - 2x_2 \leqslant 2 \\ & x_1, x_2 \textbf{ unrestricted} \end{aligned}$$

Also, find an optimal solution to the maximization problem using the same objective function and the same constraint set. Drop the first inequality in the constraint set and solve the maximization problem for the same objective function.

10. In this exercise you are to write MATLAB function **x = nnsol(A, b)** which computes a nonnegative solution **x**, if any, to the system of linear equations **Ax = b**. A trick that can be used is to add a single artificial variable to each equation and next apply a technique of Phase 1 of the Dual Simplex Algorithm with the objective function being equal to the sum of all artificial variables. See [3], Section 3.6 for more details. Recall that function **feassol**, discussed earlier in this tutorial, computes all nonnegative solutions to the constraint system that is in the standard form. A method used in this function is quite expensive and therefore is not recommended for computations with large systems.

11. Write MATLAB function **Y = rotright(X, k)** that takes a matrix **X** and a nonnegative integer **k** and returns a matrix **Y** whose columns are obtained from columns of the matrix **X** by cycling them **k** positions to the right, i.e., if $\mathbf{X = [\, x_1, x_2, \ldots, x_n]}$, then $\mathbf{Y = [x_{k+1}, \ldots, x_n, x_1, \ldots, x_k]}$, where $\mathbf{x_l}$ stands for the lth column of **X**. Function **rotright** may be used by MATLAB functions you are to write in Problems 12 and 13.

12. The following problem is of interest in interpolation by shape preserving spline functions. Given nonnegative numbers $\mathbf{b_1, b_2, \ldots, b_n}$. Find the nonnegative numbers $\mathbf{p_1, p_2, \ldots, p_{n+1}}$ that satisfy a system of linear inequalities

$$\mathbf{2p_l + p_{l+1} \leqslant b_l \leqslant p_l + 2p_{l+1}, \quad l = 1, 2, \ldots, n.}$$

This problem can be converted easily to the problem of finding a basic feasible solution of a constraint set in standard form **Ap = d, p ⩾ 0**.

(i) What is the structure of the constraint matrix **A**?
(ii) How would you compute the column vector **d**?
(iii) Write MATLAB function **p = slopec(b)** which computes a nonnegative vector **p** that satisfies the system of inequalities in this problem. You may wish to make calls from within your function to the function **nnsol** of Problem 10.
(iv) Test your function for the following vectors **b = [1; 2]** and **b = [2; 1]**.

13. Another problem that arose in interpolation theory has the same constraint set as the one in Problem 12 with additional conditions imposed on the legitimate variables $\mathbf{p_1 \leqslant p_2 \leqslant \ldots \leqslant p_{n+1}}$. Answer the questions (i) and (ii) of Problem 12 and write MATLAB function **p = slopei(b)** which computes a solution to the associated LP problem. Also, test your function using vectors **b = [1; 2; 3]** and **b = [3; 2; 1]**.

14. Some LP problems can be solved using different methods that are discussed in this tutorial. A choice of a right method is often based on the criterion of its computational complexity. MATLAB allows a user to count a number of flops (the floating point operations +, -, *, /) needed by a particular algorithm. To obtain an information about a number of flops used type the following commands:

    flops(0)
    functionname
    flops

    First command sets up the counter to zero. Next a function is executed and finally a number of flops used is displayed in the **Command Window**.
    Suppose that one wants to find an optimal solution to the following LP problem

    $$\begin{aligned} &\textbf{max (min) } z = c^*x \\ &\textbf{Subject to } Ax \leqslant b \\ &x \geqslant 0 \end{aligned}$$

    ($\mathbf{b \geqslant 0}$). Function **simplex**, included in Section 6.10, can be used to compute an optimal solution of this problem. Another option is to use function **simplex2p** (see Section 6.7). Third method that would be used is the Dual Simplex Algorithm discussed in Section 6.8. Compare computational complexity of these three methods solving the LP problems of your choice. You may begin your experiment using the LP problem in Example 1 (see [3], page 104).

15. Time of execution of a function is also considered as a decisive factor in choosing a computational method for solving a particular problem. MATLAB has a pair of functions named **tic** and **toc** used to measure a time of computations. To use these functions issue the following commands: **tic, functionname; toc**. Compare time of execution of three functions **simplex**, **simplex2p**, and **dsimplex** solving the LP problems of your choice. During the execution of functions **simplex2p** and **dsimplex** you will be asked whether or not you wish to monitor the progress of computations. In both instances click on the **No** button.

16. Consider the following LP problem with bounded variables

    $$\begin{aligned} &\textbf{min (max) } z = c^T x \\ &\textbf{Subject to } Ax \leqslant b \\ &0 \leqslant x \leqslant u \end{aligned}$$

    where **u** is a given positive vector and $\mathbf{b \geqslant 0}$. Special methods for solving this problem are discussed in [1] and [2]. For the LP problems with a small number of constraints one can add the upper bound constraints $\mathbf{x \leqslant u}$ to the original constraint set and next solve the new problem using the simplex method. In this exercise you are to write MATLAB function **[x, z, B] = simplub(type, c, A, b, u)** which computes an optimal solution **x** to the problem under discussion. Other output parameters include the objective value **z** at the optimal solution and the final tableau **B** that is generated by the simplex method. You may wish to use the function **simplex** included in Section 6.10. Test your function on the LP problems of your choice.

17. Use function **simplex2p** to solve the optimization problem discussed in [3], Appendix C, pp. 375 - 376. Explain why this problem is degenerated. Display all the intermediate tableaux generated by the function **simplex2p**.

18. Run function **cpa** on the LP problem that is discussed in [3], Example 3, pp. 208-209 (see also pages 44 - 45 of this tutorial). Analyze the intermediate tableaux generated by this function. Explain why function **cpa** terminates computations without finding an optimal solution.